QUASISYMMETRIC ROBUSTNESS OF THE COLLET-ECKMANN CONDITION IN THE QUADRATIC FAMILY

ARTUR AVILA AND CARLOS GUSTAVO MOREIRA

Abstract. We consider quasisymmetric reparametrizations of the parameter space of the quadratic family. We prove that the set of quadratic maps which are either regular or Collet-Eckmann with polynomial recurrence of the critical orbit has full Lebesgue measure.

1. Introduction

Here we consider the quadratic family, $f_a = a - x^2$, where $-1/4 \leq a \leq 2$ is the parameter. In [AM1], a thorough understanding of the dynamics of typical (with respect to Lebesgue measure) quadratic maps was obtained. More specifically, it was shown that a typical quadratic map is either regular (with a periodic attractor) or Collet-Eckmann (positive Lyapunov exponent of the critical value) with polynomial recurrence of the critical orbit. The first possibility corresponds to a hyperbolic deterministic setting, with the well known good properties of hyperbolic systems. The second is a particularly well studied case of non-uniformly hyperbolic chaotic dynamics: in the 90's such maps were shown to possess many hyperbolic-like properties like stochastic stability, exponential decay of correlations and others ([KN], [Y], [BV] and [BBM]). In particular it was possible to answer affirmatively Palis Conjecture [Pa] for the quadratic family.

It was shown in [ALM] that the parameter space of general analytic families of unimodal maps (with negative Schwarzian derivative) can be related to the parameter space of quadratic maps through a quasisymmetric 'holonomy map'. It becomes then feasible to transfer results from the quadratic family to other families, but there is one obstruction: quasisymmetric maps are not absolutely continuous.

Here we show that the set of "good" parameters has not only full Lebesgue measure, but is resistent to a quasisymmetric reparametrization:

Theorem A. *Consider a quasisymmetric reparametrization of the parameter space of the quadratic family. The set of parameters which are either regular or Collet-Eckmann:*

$$\liminf_{n\to\infty} \frac{\ln(|Df^n(f(0))|)}{n} > 0. \tag{1.1}$$

has full Lebesgue measure.

Theorem B. *Consider a quasisymmetric reparametrization of the parameter space of the quadratic family. The set of parameters which are either regular or have polynomial recurrence of the critical orbit*

$$0 < \liminf_{n\to\infty} \frac{-\ln(f^n(0))}{\ln(n)} \leq \limsup_{n\to\infty} \frac{-\ln(f^n(0))}{\ln(n)} < \infty \tag{1.2}$$

has full Lebesgue measure.

In [AM2] those results are used to obtain a proof of the Palis Conjecture for unimodal maps with negative Schwarzian derivative. Here we give a detailed proof of those results following essentially the sketch provided in [AM2] (we simplified a couple of arguments), with all estimates worked out. We refer the reader to [AM2] and [AM3] for discussions on the heuristics of the approaches in this paper, as well as in the original argument of [AM1].

Date: August 28, 2003.

2. Basic background

This section will introduce the basic language of this paper, and corresponds essentially (with minor modifications) to sections §1 and §2 and parts of §3 of [AM1].

2.1. **General definitions.**

2.1.1. *Maps of the interval.* Let $f : I \to I$ be a C^1 map defined on some interval $I \subset \mathbb{R}$. The *orbit* of a point $p \in I$ is the sequence $\{f^k(p)\}_{k=0}^\infty$. We say that p is *recurrent* if there exists a subsequence $n_k \to \infty$ such that $\lim f^{n_k}(p) = p$.

We say that p is a *periodic point of period* n of f if $f^n(p) = p$, and $n \geq 1$ is minimal with this property. In this case we say that p is *hyperbolic* if $|Df^n(p)|$ is not 0 or 1. Hyperbolic periodic orbits are *attracting* or *repelling* according to $|Df^n(p)| < 1$ or $|Df^n(p)| > 1$.

We will often consider the restriction of iterates f^n to intervals $T \subset I$, such that $f^n|_T$ is a diffeomorphism. In this case we will be interested on the *distortion* of $f^n|_T$,

$$\mathrm{dist}(f^n|_T) = \frac{\sup_T |Df^n|}{\inf_T |Df^n|}. \tag{2.1}$$

This is always a number bigger than or equal to 1, we will say that it is small if it is close to 1.

2.1.2. *Trees.* We let Ω denote the set of finite sequences of non-zero integers (including the empty sequence). Let Ω_0 denote Ω without the empty sequence. For $\underline{d} \in \Omega$, $\underline{d} = (j_1, ..., j_m)$, we let $|\underline{d}| = m$ denote its length.

We denote $\sigma^+ : \Omega_0 \to \Omega$ by $\sigma^+(j_1, ..., j_m) = (j_1, ..., j_{m-1})$ and $\sigma^- : \Omega_0 \to \Omega$ by $\sigma^-(j_1, ..., j_m) = (j_2, ..., j_m)$.

For the purposes of this paper, one should view Ω as a (directed) tree with root $\underline{d} = \emptyset$ and edges connecting $\sigma^+(\underline{d})$ to $\underline{d}$ for each $\underline{d} \in \Omega_0$. We will use Ω to label objects which are organized in a similar tree structure (for instance, certain families of intervals ordered by inclusion).

2.2. **Borel-Cantelli Lemma.**

We will repeatedly use the following version of the Borel-Cantelli Lemma (Lemma 4.1 of [AM1]).

Lemma 2.1. *Let $X \subset \mathbb{R}$ be a measurable set such that for each $x \in X$ is defined a sequence $D_n(x)$ of nested intervals converging to x such that for all $x_1, x_2 \in X$ and any n, $D_n(x_1)$ is either equal or disjoint to $D_n(x_2)$. Let Q_n be measurable subsets of $\mathbb{R}$ and $q_n(x) = |Q_n \cap D_n(x)|/|D_n(x)|$. Let Y be the set of all $x \in X$ which belong to at most finitely many Q_n. If $\sum q_n(x)$ is finite for almost any $x \in X$ then $|Y| = |X|$.*

The following obvious reformulation will be often convenient (Lemma 4.2 of [AM1]).

Lemma 2.2. *In the same context as above, assume that we are given sequences $Q_{n,m}$, $m \geq n$ of measurable sets and let Y_n be the set of x belonging to at most finitely many $Q_{n,m}$. Let $q_{n,m}(x) = |Q_{n,m} \cap D_m(x)|/|D_m(x)|$. Let $n_0(x) \in \mathbb{N} \cup \{\infty\}$ be such that $\sum_{m=n}^\infty q_{n,m}(x) < \infty$ for $n \geq n_0(x)$. Then for almost every $x \in X$, $x \in Y_n$ for $n \geq n_0(x)$.*

2.3. **Quasisymmetric maps.**

Let $k \geq 1$ be given. We say that a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is *quasisymmetric* with constant k if for all $h > 0$

$$\frac{1}{k} \leq \frac{f(x+h) - f(x)}{f(x) - f(x-h)} \leq k. \tag{2.2}$$

The space of quasisymmetric maps is a group under composition, and the set of quasisymmetric maps with constant k preserving a given interval is compact in the uniform topology of compact subsets of $\mathbb{R}$. It also follows that quasisymmetric maps are Hölder.

To describe further the properties of quasisymmetric maps, we need the concept of quasiconformal maps and dilatation so we just mention a result of Ahlfors-Beurling which connects both concepts: any quasisymmetric map extends to a quasiconformal real-symmetric map of $\mathbb{C}$ and, conversely, the restriction of a quasiconformal real-symmetric map of $\mathbb{C}$ to $\mathbb{R}$ is quasisymmetric. Furthermore, it is possible to work out upper bounds on the dilatation (of an optimal extension) depending only on k and conversely.

The constant k is awkward to work with: the inverse of a quasisymmetric map with constant k may have a larger constant. We will therefore work with a less standard constant: we will say that h is γ-quasisymmetric (γ-qs) if h admits a quasiconformal symmetric extension to $\mathbb{C}$ with dilatation bounded by γ. This definition behaves much better: if h_1 is γ_1-qs and h_2 is γ_2-qs then $h_2 \circ h_1$ is $\gamma_2\gamma_1$-qs.

If $X \subset \mathbb{R}$ and $h : X \to \mathbb{R}$ has a γ-quasisymmetric extension to $\mathbb{R}$ we will also say that h is γ-qs.

Let $QS(\gamma)$ be the set of γ-qs maps of $\mathbb{R}$.

2.3.1. *Capacities.* If $X \subset \mathbb{R}$ is measurable, let us denote $|X|$ its Lebesgue measure. Let us explicit the metric properties of γ-qs maps we will use.

To each γ, there exists a constant $k \geq 1$ such that for all $f \in QS(\gamma)$, for all $J \subset I$ intervals,

$$\frac{1}{k}\left(\frac{|J|}{|I|}\right)^k \leq \frac{|f(J)|}{|f(I)|} \leq \left(\frac{k|J|}{|I|}\right)^{1/k}. \tag{2.3}$$

Furthermore $\lim_{\gamma\to 1} k(\gamma) = 1$. So for each $\epsilon > 0$ there exists $\gamma > 1$ such that $k(2\gamma - 1) < 1 + \epsilon/5$. From now on, once a given γ close to 1 is chosen, ϵ will always denote a small number with this property.

2.3.2. *Capacities and trees.* The γ-capacity of a set X in an interval I is defined as follows:

$$p_\gamma(X|I) = \sup_{h \in QS(\gamma)} \frac{|h(X \cap I)|}{|h(I)|}. \tag{2.4}$$

This geometric quantity is well adapted to our context, since it is well behaved under tree decompositions of sets. In other words, if I^j are disjoint subintervals of I and $X \subset \cup I^j$ then

$$p_\gamma(X|I) \leq p_\gamma(\cup_j I_j|I) \sup_j p_\gamma(X|I^j). \tag{2.5}$$

2.4. **The combinatorics of real quadratic maps.**

2.4.1. *Real quadratic maps.* If $a \in \mathbb{R}$ we let $f_a : \mathbb{R} \to \mathbb{R}$ denote the quadratic map $a - x^2$. If $-1/4 \leq a \leq 2$, there exists an interval $I_a = [\beta, -\beta]$ with $\beta = \frac{-1-\sqrt{1+4a}}{2}$ such that $f_a(I_a) \subset I_a$ and $f_a(\partial I_a) \subset \partial I_a$. For such values of the parameter a, the map $f = f_a|_{I_a}$ is unimodal, that is, it is a self map of I_a with a unique turning point. To simplify the notation, we will usually drop the dependence on the parameter and let $I = I_a$.

We will now introduce objects related to the dynamics of a fixed quadratic map f.

2.4.2. *Return maps.* Given an interval $T \subset I$ we define the *first return map* $R_T : X \to T$ where $X \subset T$ is the set of points x such that there exists $n > 0$ with $f^n(x) \in T$, and $R_T(x) = f^n(x)$ for the minimal n with this property.

2.4.3. *Nice intervals.* An interval T is *nice* if it is symmetric around 0 and the iterates of ∂T never intersect $\operatorname{int} T$. Given a nice interval T we notice that the domain of the first return map R_T decomposes in a union of intervals T^j, indexed by integer numbers (if there are only finitely many intervals, some indexes will be corresponded to the empty set). If 0 belongs to the domain of R_T, we say that T is *proper*. In this case we reserve the index 0 to denote the component of the critical point: $0 \in T^0$.

If T is nice, it follows that for all $j \in \mathbb{Z}$, $R_T(\partial T^j) \subset \partial T$. In particular, $R_T|_{T^j}$ is a diffeomorphism onto T unless $0 \in T^j$ (and in particular $j = 0$ and T is proper). If T is proper, $R_T|_{T^0}$ is symmetric (even) with a unique critical point 0. As a consequence, T^0 is also a nice interval. If $R_T(0) \in T^0$, we say that R_T is *central*. If T is a proper interval then both R_T and R_{T^0} are defined, and we say that R_{T^0} is the generalized renormalization of R_T.

2.4.4. *Landing maps.* Given a proper interval T we define the *landing map* $L_T : X \to T^0$ where $X \subset T$ is the set of points x such that there exists $n \geq 0$ with $f^n(x) \in T^0$, and $L_T(x) = f^n(x)$ for the minimal n with this property. We notice that $L_T|_{T^0} = \mathrm{id}$.

2.4.5. *Trees.* We will use Ω to label iterations of non-central branches of R_T, as well as their domains. If $\underline{d} \in \Omega$, we define $T^{\underline{d}}$ inductively in the following way. We let $T^{\underline{d}} = T$ if $\underline{d}$ is empty and if $\underline{d} = (j_1, ..., j_m)$ we let $T^{\underline{d}} = (R_T|_{T^{j_1}})^{-1}(T^{\sigma^-(\underline{d})})$. We denote $R_T^{\underline{d}} = R_T^{|\underline{d}|}|_{T^{\underline{d}}}$ which is always a diffeomorphism onto T.

Notice that the family of intervals $T^{\underline{d}}$ is organized by inclusion in the same way as Ω is organized by (right side) truncation (the previously introduced tree structure).

If T is a proper interval, the first return map to T naturally relates to the first landing to T^0. Indeed, denoting $C^{\underline{d}} = (R_T^{\underline{d}})^{-1}(T^0)$, the domain of the first landing map L_T is easily seen to coincide with the union of the $C^{\underline{d}}$, and furthermore $L_T|_{C^{\underline{d}}} = R_T^{\underline{d}}$. Notice that this allows us to relate R_T and R_{T^0} since $R_{T^0} = L_T \circ R_T$.

2.4.6. *Renormalization.* We say that f is *renormalizable* if there is an interval $0 \in T$ and $m > 1$ such that $f^m(T) \subset T$ and $f^j(\operatorname{int} T) \cap \operatorname{int} T = \emptyset$ for $1 \leq j < m$. The maximal such interval is called the *renormalization interval of period* m, it has the property that $f^m(\partial T) \subset \partial T$.

The set of renormalization periods of f gives an increasing (possibly empty) sequence of numbers m_i, $i = 1, 2, ...$, each related to a unique renormalization interval $T^{(i)}$ which form a nested sequence of intervals. We include $m_0 = 1$, $T^{(0)} = I$ in the sequence to simplify the notation.

We say that f is *finitely renormalizable* if there is a smallest renormalization interval $T^{(k)}$. We say that $f \in \mathcal{F}$ if f is finitely renormalizable and 0 is recurrent but not periodic. We let $\mathcal{F}_k$ denote the set of maps f in $\mathcal{F}$ which are exactly k times renormalizable.

2.4.7. *Principal nest.* Let Δ_k denote the set of all maps f which have (at least) k renormalizations and which have an orientation reversing non-attracting periodic point of period m_k which we denote p_k (that is, p_k is the fixed point of $f^{m_k}|_{T^{(k)}}$ with $Df^{m_k}(p_k) \leq -1$). For $f \in \Delta_k$, we denote $T_0^{(k)} = [-p_k, p_k]$. We define by induction a (possibly finite) sequence $T_i^{(k)}$, such that $T_{i+1}^{(k)}$ is the component of the domain of $R_{T_i^{(k)}}$ containing 0. If this sequence is infinite, then either it converges to a point or to an interval.

If $\cap_i T_i^{(k)}$ is a point, then f has a recurrent critical point which is not periodic, and it is possible to show that f is not $k+1$ times renormalizable. Obviously in this case we have $f \in \mathcal{F}_k$, and all maps in $\mathcal{F}_k$ are obtained in this way: if $\cap_i T_i^{(k)}$ is an interval, it is possible to show that f is $k+1$ times renormalizable.

We can of course write $\mathcal{F}$ as a disjoint union $\cup_{i=0}^{\infty} \mathcal{F}_i$. For a map $f \in \mathcal{F}_k$ we refer to the sequence $\{T_i^{(k)}\}_{i=1}^{\infty}$ as the *principal nest.*

It is important to notice that the domain of the first return map to $T_i^{(k)}$ is always dense in $T_i^{(k)}$. Moreover, the next result shows that, outside a very special case, the return map has a hyperbolic structure.

Lemma 2.3. *Assume $T_i^{(k)}$ does not have a non-hyperbolic periodic orbit in its boundary. For all $T_i^{(k)}$ there exists $C > 0$, $\lambda > 1$ such that if $x, f(x), ..., f^{n-1}(x)$ do not belong to $T_i^{(k)}$ then $|Df^n(x)| > C\lambda^n$.*

This lemma is a simple consequence of a general theorem of Guckenheimer on hyperbolicity of maps of the interval without critical points and non-hyperbolic periodic orbits (Guckenheimer considers unimodal maps with negative Schwarzian derivative, so this applies directly to the case of quadratic maps, the general case is also true by Mañé's Theorem, see [MvS]). Notice that the existence of a non-hyperbolic periodic orbit in the boundary of $T_i^{(k)}$ depends on a very special combinatorial setting, in particular, all $T_j^{(k)}$ must coincide (with $[-p_k, p_k]$), and the k-th renormalization of f is in fact renormalizable of period 2.

By Lemma 2.3, the maximal invariant of $f|_{I \setminus T_i^{(k)}}$ is an expanding set, which admits a Markov partition (since $\partial T_i^{(k)}$ is preperiodic, see also the proof of Lemma 6.1): it is easy to see that it is indeed a Cantor set[1] (except if $i = 0$ or in the special period 2 renormalization case just described). It follows that the geometry of this Cantor set is well behaved: for instance, its image by any quasisymmetric map has zero Lebesgue measure.

In particular, one sees that the domain of the first return map to $T_i^{(k)}$ has infinitely many components (except in the special case above or if $i = 0$) and that its complement has well behaved geometry.

[1]Dynamically defined Cantor sets with such properties are usually called *regular Cantor sets.*

2.5. **Parameter partition.** Part of our work is to transfer information from the phase space of some map $f \in \mathcal{F}$ to a neighborhood of f in the parameter space. This is done in the following way. We consider the first landing map L_i: the complement of the domain of L_i is a hyperbolic Cantor set $K_i = I_i \setminus \cup C_i^{\underline{d}}$. This Cantor set persists in a small parameter neighborhood J_i of f, changing in a continuous way. Thus, loosely speaking, the domain of L_i induces a persistent partition of the interval I_i.

Along J_i, the first landing map is topologically the same (in a way that will be clear soon). However the critical value $R_i[g](0)$ moves relative to the partition (when g moves in J_i). This allows us to partition the parameter piece J_i in smaller pieces, each corresponding to a region where $R_i(0)$ belongs to some fixed component of the domain of the first landing map.

The relation between the partitions on the phase space and on the parameter space can be described, topologically, as follows.

Theorem 2.4 (Topological Phase-Parameter relation)**.** *Let $f \in \mathcal{F}_\kappa$. There is a sequence $\{J_i\}_{i \in \mathbb{N}}$ of nested parameter intervals (the principal parapuzzle nest of f) with the following properties.*

1. *J_i is the maximal interval containing f such that for all $g \in J_i$ the interval $I_{i+1}[g] = T_{i+1}^{(\kappa)}[g]$ is defined and changes in a continuous way. (Since the first return map to $R_i[g]$ has a central domain, the landing map $L_i[g] : \cup C_i^{\underline{d}}[g] \to I_i[g]$ is defined.)*
2. *$L_i[g]$ is topologically the same along J_i: there exists homeomorphisms $H_i[g] : I_i \to I_i[g]$, such that $H_i[g](C_i^{\underline{d}}) = C_i^{\underline{d}}[g]$. The maps $H_i[g]$ may be chosen to change continuously.*
3. *There exists a homeomorphism $\Xi_i : I_i \to J_i$ such that $\Xi_i(C_i^{\underline{d}})$ is the set of g such that $R_i[g](0)$ belongs to $C_i^{\underline{d}}[g]$.*

The formulation above is the same as Theorem 2.2 of [AM1] (the result itself was known much before).

The homeomorphisms H_i and Ξ_i are not uniquely defined, it is easy to see that we can modify them inside each $C_i^{\underline{d}}$ window keeping the above properties. However, H_i and Ξ_i are well defined maps if restricted to K_i.

With this result we can define for any $f \in \mathcal{F}_\kappa$ intervals $J_i^j = \Xi_i(I_i^j)$ and $J_i^{\underline{d}} = \Xi_i(I_i^{\underline{d}})$. From the description we gave it immediately follows that two intervals $J_{i_1}[f]$ and $J_{i_2}[g]$ associated to maps f and g are either disjoint or nested, and the same happens for intervals J_i^j or $J_i^{\underline{d}}$. Notice that if $g \in \Xi_i(C_i^{\underline{d}}) \cap \mathcal{F}_\kappa$ then $\Xi_i(C_i^{\underline{d}}) = J_{i+1}[g]$.

We will concentrate on the analysis of the regularity of Ξ_i for the special class of simple maps f: one of the good properties of the class of simple maps is better control of the phase-parameter relation. Even for simple maps, however, the regularity of Ξ_i is not great: there is too much dynamical information contained in it. A solution to this problem is to forget some dynamical information.

2.5.1. *Gape interval.* If $i > 1$, we define the *gape interval* $\tilde{I}_{i+1}$ as follows.

We have that $R_i|_{I_{i+1}} = L_{i-1} \circ R_{i-1} = R_{i-1}^{\underline{d}} \circ R_{i-1}$ for some $\underline{d}$, so that $I_{i+1} = (R_{i-1}|_{I_i})^{-1}(C_{i-1}^{\underline{d}})$. We define the gape interval $\tilde{I}_{i+1} = (R_{i-1}|_{I_i})^{-1}(I_{i-1}^{\underline{d}})$.

Notice that $I_{i+1} \subset \tilde{I}_{i+1} \subset I_i$. Furthermore, for each I_i^j, the gape interval $\tilde{I}_{i+1}$ either contains or is disjoint from I_i^j.

2.5.2. *The Phase-Parameter relation.* As we discussed before, the dynamical information contained in Ξ_i is entirely given by $\Xi_i|_{K_i}$: a map obtained by Ξ_i by modification inside a $C_i^{\underline{d}}$ window has still the same properties. Therefore it makes sense to ask about the regularity of $\Xi_i|_{K_i}$. As we anticipated before we must erase some information to obtain good results.

Let $f \in \mathcal{F}_\kappa$ and let τ_i be such that $R_i(0) \in I_i^{\tau_i}$. We define two Cantor sets, $K_i^\tau = K_i \cap I_i^{\tau_i}$ which contains refined information restricted to the $I_i^{\tau_i}$ window and $\tilde{K}_i = I_i \setminus (\cup I_i^j \cup \tilde{I}_{i+1})$, which contains global information, at the cost of erasing information inside each I_i^j window and in $\tilde{I}_{i+1}$.

Theorem 2.5 (Phase-Parameter relation)**.** *Let f be a simple map. For all $\delta > 0$ there exists i_0 such that for all $i > i_0$ we have*

PhPa1: $\Xi_i|_{K_i^\tau}$ *is $1+\delta$-qs,*
PhPa2: $\Xi_i|_{\tilde{K}_i}$ *is $1+\delta$-qs,*
PhPh1: $H_i[g]|_{K_i}$ *is $1+\delta$-qs if $g \in J_i^{\tau_i}$,*
PhPh2: *the map $H_i[g]|_{\tilde{K}_i}$ is $1+\delta$-qs if $g \in J_i$.*

This result is stated as Theorem 2.3 of [AM1], where a proof is sketched in the Appendix. A full proof is given in a more general context in [AM4].

3. Preliminary reductions and basic scheme

3.1. **Reduction to the study of simple maps.** In [L2] Lyubich has shown that almost every finitely renormalizable map is simple, and in [L3] he showed that infinitely renormalizable maps have zero Lebesgue measure. In [ALM], it is remarked that the proofs of those results actually imply the following:

Theorem 3.1. *Consider a quasisymmetric reparametrization of the parameter space of the quadratic family. The set of parameters which are either regular or simple has full Lebesgue measure.*

Thus we can concentrate on the study of simple maps.

3.2. **Language.** We will now fix, once and for all, an arbitrary quasisymmetric reparametrization of the parameter space of the quadratic family. *From now on, all mentions to the parameter space will take into account this reparametrization (unless specified otherwise).* For instance, the previous theorem would now be stated "The set of parameters which are either regular or simple has full Lebesgue measure", without any mention to the reparametrization. Our aim is to replace "simple" by "Collet-Eckmann with polynomial recurrence of the critical orbit" in this formulation.

The quasisymmetric constant of the fixed reparametrization will be denoted $\hat{\gamma}$. We will fix an arbitrary $\gamma > \hat{\gamma}$. We let a be a small constant only depending on γ (it should be smaller than $1/20$ of the Hölder constant of 2γ-qs maps), and $b = a^{-1}$.

We must change the statement of properties PhPa1 and PhPa2 of the Phase-Parameter relation (which was stated with respect to the unreparametrized parameter space). Taking into account reparametrization we replace PhPa1 and PhPa2 by

PhPa1': $\Xi_i|_{K_i^\tau}$ is γ-qs,
PhPa2': $\Xi_i|_{\tilde{K}_i}$ is γ-qs.

We shall fix also the renormalization level κ, and consider only maps in Δ_κ. Whenever we say that some property is valid "with total probability", it will mean that it is satisfied for a set of maps in $\mathcal{F}_\kappa$ of full Lebesgue measure. Since there are countably many levels, we can reformulate our aim as showing that the properties "Collet-Eckmann" and "polynomial recurrence of the critical orbit" hold with total probability.

This will not be done at once: we will show in a sequence of steps that more and more properties are valid with total probability. Sometimes when proving that a new property has total probability, we will only need to use that this property is implied by properties that had previously been shown to have total probability. Sometimes, we will need to use the previous "total probability" properties and still exclude some zero Lebesgue measure set of parameters. This will be done always via a Borel-Cantelli argument (either of Lemmas 2.1 or 2.2) coupled with the Phase-Parameter relation. The best way to introduce the argument is by going through an explicit application.

3.2.1. *Example: torrential decay of geometry.* We will illustrate the use of Lemma 2.1 and the phase-parameter relation with an estimate on the decay of geometry. More precisely, we will consider the *scaling factor*

$$c_n = \frac{|I_{n+1}|}{|I_n|}. \tag{3.1}$$

The scaling factor is a particularly important parameter in the subsequent analysis: all statistical estimates that follow will be related to c_n. This variable of course changes inside each $J_n^{\tau_n}$ window, however, not by much. From PhPh1, for instance, we get that with total probability

$$\lim_{n\to\infty} \sup_{g_1,g_2 \in J_n^{\tau_n}} \frac{\ln(c_n[g_1])}{\ln(c_n[g_2])} = 1. \tag{3.2}$$

One initial information on the scaling factors is provided by the following result of Lyubich:

Theorem 3.2 (see [L1])**.** *If f is simple then there exists $C > 0$, $\lambda < 1$ such that $c_n < C\lambda^n$.*

We will now show that, with total probability, the decay of c_n is much faster than exponential. To express this decay, let us consider the tower function defined by recursion $T(1) = 2$, $T(n+1) = 2^{T(n)}$. We will show that, with total probability, the c_n decrease torrentially to 0, that is, there exists $k > 0$ such that $c_n^{-1} > T(n-k)$ for n big enough. More precisely, we will show that c_{n+1}^{-1} is bounded from below by an exponential of a (bounded) power of c_n^{-1}.

We start with an estimate in phase space. For $x \in I_n$, let $\underline{d}^{(n)}(x) \in \Omega$ be defined by $x \in I_n^{\underline{d}^{(n)}(x)}$.

Lemma 3.3. *With total probability, for all n sufficiently big we have*

$$p_{2\gamma}(|\underline{d}^{(n)}(x)| \le k | I_n) < k c_n^{8a}, \tag{3.3}$$

$$p_{2\gamma}(|\underline{d}^{(n)}(x)| \ge k | I_n) < e^{-k c_n^{b/8}}. \tag{3.4}$$

We also have

$$p_{2\gamma}(|\underline{d}^{(n)}(x)| \le k | I_n^{\tau_n}) < k c_n^{8a}, \tag{3.5}$$

$$p_{2\gamma}(|\underline{d}^{(n)}(x)| \ge k | I_n^{\tau_n}) < e^{-k c_n^{b/8}}. \tag{3.6}$$

Proof. Let us compute the first two estimates.

Since I_n^0 is in the middle of I_n, we have as a simple consequence of the Real Schwarz Lemma (see [L1] and (4.3) in Lemma 4.1 below) that

$$\frac{c_n}{4} < \frac{|C_n^{\underline{d}}|}{|I_n^{\underline{d}}|} < 4 c_n. \tag{3.7}$$

As a consequence

$$p_{2\gamma}(|\underline{d}^{(n)}(x)| = m | x \in I_n) < (4 c_n)^{10a}. \tag{3.8}$$

We get the estimate (3.3) summing up on $0 \le m \le k$.

For the same reason, we get that

$$p_{2\gamma}(|\underline{d}^{(n)}(x)| > m | x \in I_n) < \left(1 - \left(\frac{c_n}{4}\right)^{b/10}\right) p_{2\gamma}(|\underline{d}^{(n)}(x)| \ge m | x \in I_n). \tag{3.9}$$

This implies

$$p_{2\gamma}(|\underline{d}^{(n)}(x)| \ge k | x \in I_n) \le \left(1 - \left(\frac{c_n}{4}\right)^{b/10}\right)^k. \tag{3.10}$$

Estimate (3.4) follows from

$$\left(1 - \left(\frac{c_n}{4}\right)^{b/10}\right)^k < (1 - c_n^{b/9})^k < ((1 - c_n^{b/9})^{c_n^{-b/9}})^{k c_n^{b/9}} < e^{-k c_n^{b/9}}. \tag{3.11}$$

The two remaining estimates are analogous. □

Let us now transfer this result to the parameter. Let $s_n = \underline{d}^{(n)}(R_n(0))$, so that $R_{n+1}(0) = R_n^{s_n+1}(0)$.

Lemma 3.4. *With total probability, for n sufficiently big we have*

$$c_n^{-a} < s_n < c_n^{-b}. \tag{3.12}$$

Proof. For the moment we only know that simple maps have total probability. Thus, fix a simple map and consider its principal nest J_n. By the previous lemma, we have

$$p_\gamma(|\underline{d}^{(n)}(x)| < c_n^{-3a/2} | I_n^{\tau_n}) \le c_n^{a/2}, \tag{3.13}$$

By PhPa1', the Lebesgue measure of the set of parameters in J_n such that $s_n < c_n^{-3a/2}$ is at most $c_n^{a/2}$. But $\sum c_n^{a/2} < \infty$ (c_n decays exponentially by Theorem 3.2), so we can apply Lemma 2.1 to get that for almost every simple map we have $s_n \ge c_n^{-a}$ (in the notation of Lemma 2.1, we have taken X as the set of simple maps, $D_n = J_n^{\tau_n}$, and Q_n as the set of parameters such that $s_n < c_n^{-a}$)[2]. This implies one of the estimates, the other being analogous. □

[2] We used implicitly the fact that for n large we have $c_n[g]^{-a} < c_n^{-3a/2}$, see (3.2).

From now on, whenever we need the parameter exclusion argument described above we will only say *by PhPa1'* or *by PhPa2'*, and be done with it.

We can now show torrential decay of geometry without any further parameter exclusion:

Lemma 3.5. *With total probability, for n large we have*

$$c_{n+1}^{-1} \geq e^{c_n^{-a/2}}. \tag{3.14}$$

Proof. It is easy to see (using for instance the Real Schwarz Lemma, see [L1], see also item (4.4) in Lemma 4.1 below) that there exists a constant $K > 0$ (independent of n) such that for each $\underline{d} \in \Omega$, both components of $I_n^{\sigma^+(\underline{d})} \setminus I_n^{\underline{d}}$ have size at least $(e^K - 1)|I_n^{\underline{d}}|$. In particular, by induction, if $R_n(0) \in C_n^{\underline{d}}$ we have that both gaps of $I_n \setminus C_n^{\underline{d}}$ have size at least $(e^{Ks_n} - 1)|C_n^{\underline{d}}|$. Taking the preimage by R_n, and using the Real Schwarz Lemma again, we see that $c_{n+1} < Ce^{Ks_n/2}$ for some constant $C > 0$ independent of n. We conclude that

$$\liminf \frac{\ln(c_{n+1}^{-1})}{s_n} \geq \frac{K}{2}, \tag{3.15}$$

and since $c_n \to 0$ as $n \to \infty$, the result follows from the previous lemma. □

4. Initial estimates

4.1. **Fine partitions.** We use Cantor sets K_n and $\tilde{K}_n$ to partition the phase space. In many circumstances we are directly concerned with intervals of this partition. However, sometimes we just want to exclude an interval of given size (usually a neighborhood of 0). This size does not usually correspond to a union of gaps, so we instead should consider in applications an interval which is union of gaps, with approximately the given size. The degree of relative approximation will always be torrentially good (in n), so we usually won't elaborate on this. In this section we just give some results which will imply that the partition induced by the Cantor sets are fine enough to allow torrentially good approximations.

The following lemma summarizes the situation. The proof is based on estimates of distortion using the Real Schwarz Lemma and the Koebe Principle (see [L1]) and is very simple, so we just sketch the proof.

Lemma 4.1. *The following estimates hold:*

$$\frac{|I_n^j|}{|I_n|} = O(\sqrt{c_{n-1}}), \tag{4.1}$$

$$\frac{|I_n^{\underline{d}}|}{|I_n^{\sigma^+(\underline{d})}|} = O(\sqrt{c_{n-1}}), \tag{4.2}$$

$$\frac{c_n}{4} < \frac{|C_n^{\underline{d}}|}{|I_n^{\underline{d}}|} < 4c_n, \tag{4.3}$$

$$\frac{|\tilde{I}_{n+1}|}{|I_n|} = O(e^{-s_{n-1}}). \tag{4.4}$$

Proof. (Sketch.) Since $R_n^{\underline{d}}$ has negative Schwarzian derivative, it immediately follows that the Koebe space[3] of $C_n^{\underline{d}}$ inside $I_n^{\underline{d}}$ has at least order c_n^{-1}.

It is easy to see that $R_{n-1}|_{I_n}$ can be written as $\phi \circ f$ where ϕ extends to a diffeomorphism onto I_{n-2} with negative Schwarzian derivative and thus with very small distortion. Since $R_{n-1}(I_n^j)$ is contained on some $C_{n-1}^{\underline{d}}$, we see that the Koebe space of I_n^j in I_n is at least of order $c_{n-1}^{-1/2}$ which implies (4.1).

[3]The Koebe space of an interval T' inside an interval $T \supset T'$ is the minimum of $|L|/|T'|$ and $|R|/|T'|$ where L and R are the components of $T \setminus T'$. If the Koebe space of T' inside T is big, then the Koebe Principle states that a diffeomorphism onto T' which has an extension with negative Schwarzian derivative onto T has small distortion. In this case, it follows that the Koebe space of the preimage of T' inside the preimage of T is also big.

Let us now consider an interval $I_n^{\underline{d}}$. Let I_n^j be such that $R_n^{\sigma^+(\underline{d})}(I_n^{\underline{d}}) = I_n^j$. We can pullback the Koebe space of I_n^j inside I_n by $R_n^{\sigma^+(\underline{d})}$, so (4.1) implies (4.2). Moreover, this shows by induction that the Koebe space of $I_n^{\underline{d}}$ inside I_n is at least of order $c_{n-1}^{-|\underline{d}|/2}$. Since $R_{n-1}(\tilde{I}_{n+1}) \subset I_{n-1}^{\underline{d}}$ with $|\underline{d}| = s_{n-1}$, the Koebe space of $\tilde{I}_{n+1}$ in I_n is at least $c_{n-2}^{-|\underline{d}|/4}$, which implies (4.4).

It is easy to see that $R_n^{\underline{d}}|_{I_n^{\underline{d}}}$ can be written as $\phi \circ f \circ R_n^{\sigma^+(\underline{d})}$, where ϕ has small distortion. Due to (4.1), $R_n^{\sigma^+(\underline{d})}|_{I_n^{\underline{d}}}$ also has small distortion, so a direct computation with f (which is purely quadratic) gives (4.3). □

In other words, distances in I_n can be measured with precision $\sqrt{c_{n-1}}|I_n|$ in the partition induced by $\tilde{K}_n$, due to (4.1) and (4.4) (since $e^{-s_{n-1}} \ll c_{n-1}$).

Distances can be measured much more precisely with respect to the partition induced by K_n, in fact we have good precision in each $I_n^{\underline{d}}$ scale. In other words, inside $I_n^{\underline{d}}$, the central gap $C_n^{\underline{d}}$ is of size $O(c_n|I_n^{\underline{d}}|)$ (by (4.3)) and the other gaps have size $O(\sqrt{c_{n-1}}|C_n^{\underline{d}}|)$ (by (4.2) and (4.3)).

4.2. **Initial estimates on distortion.** To deal with the distortion control we need some preliminary known results. Those estimates are based on the Koebe Principle and the estimates of Lemma 4.1. All needed arguments are already contained in the proof of Lemma 4.1, so we won't get into details.

Proposition 4.2. *The following estimates hold:*

(1) *For any j, if $R_n|_{I_n^j} = f^k$,* $\operatorname{dist}(f^{k-1}|_{f(I_n^j)}) = 1 + O(c_{n-1})$,
(2) *For any $\underline{d}$,* $\operatorname{dist}(R_n^{\sigma^+(\underline{d})}|_{I_n^{\underline{d}}}) = 1 + O(\sqrt{c_{n-1}})$.

We will use the following immediate consequence for the decomposition of certain branches.

Lemma 4.3. *With total probability,*

(1) $R_n|_{I_n^0} = \phi \circ f$ *where ϕ has torrentially small distortion,*
(2) $R_n^{\underline{d}} = \phi_2 \circ f \circ \phi_1$ *where ϕ_2 and ϕ_1 have torrentially small distortion and* $\phi_1 = R_n^{\sigma^+(\underline{d})}$.

4.3. **Estimating derivatives.**

Lemma 4.4. *With total probability, the distance between $R_n(0)$ and $\partial I_n \cup \{0\}$ is at least $|I_n|n^{-b/2}$. In particular $R_n(0) \notin \tilde{I}_{n+1}$ for all n large enough.*

Proof. This is a simple consequence of PhPa2', using summability of n^{-2} (use (4.4) to obtain the last conclusion). □

Lemma 4.5. *With total probability, for n big enough and $j \neq 0$*

$$\operatorname{dist}(f|_{I_n^j})) < n^{b/2}. \tag{4.5}$$

Proof. Denote by $P_n^{\underline{d}}$ a $|C_n^{\underline{d}}|/n^{b/2}$ neighborhood of $C_n^{\underline{d}}$. Notice that the gaps of the Cantor set K_n inside $I_n^{\underline{d}}$ which are different from $C_n^{\underline{d}}$ are torrentially (in n) smaller then $C_n^{\underline{d}}$, so we can take $P_n^{\underline{d}}$ as a union of gaps of K_n up to torrentially small error.

It is clear that if h is a γ-qs homeomorphism then

$$|h(P_n^{\underline{d}} \setminus C_n^{\underline{d}})| \leq n^{-2}|h(C_n^{\underline{d}})| \tag{4.6}$$

Notice that if $C_n^{\underline{d}}$ is contained in I_n^j with $j \neq \tau_n$, then $P_n^{\underline{d}}$ does not intersect $I_n^{\tau_n}$. Since the $C_n^{\underline{d}}$ are disjoint,

$$p_\gamma(I_n^{\tau_n} \cap \cup(P_n^{\underline{d}} \setminus C_n^{\underline{d}})|I_n^{\tau_n}) \leq n^{-2} \tag{4.7}$$

which is summable.

Transferring this estimate to the parameter using PhPa1' we see that with total probability, if n is sufficiently big, if $R_n(0)$ does not belong to $C_n^{\underline{d}}$ then $R_n(0)$ does not belong to $P_n^{\underline{d}}$ as well. In particular, if n is sufficiently big, the critical point 0 will never be in a $n^{-b/2}|I_{n+1}^j|$ neighborhood of any I_{n+1}^j with $j \neq 0$ (just take the inverse image by $R_n|_{I_{n+1}}$). □

Lemma 4.6. *With total probability, for all n sufficiently big and for all $\underline{d}$,*

$$\operatorname{dist}(R_n^{\underline{d}}) < n^b \le 2^n. \tag{4.8}$$

In particular, for n big enough, $|DR_n(x)| > 2$, $x \in \cup_{j \neq 0} I_n^j$.

Proof. Lemmas 4.3 and Lemma 4.5 imply (4.8). If $j \neq 0$, by (4.1) of Lemma 4.1 we get that $|R_n(I_n^j)|/|I_n^j| = |I_n|/|I_n^j| > c_{n-1}^{-1/3}$, so $\operatorname{dist}(R_n|_{I_n^j}) \le 2^n$ implies that for all $x \in I_n^j$, $|DR_n(x)| > c_{n-1}^{-1/3} 2^{-n} > 2$. □

Lemma 4.7. *With total probability, if n is sufficiently big and if $x \in I_n^j$, $j \neq 0$, and $R_n|_{I_n^j} = f^K$, then for $1 \le k \le K$, $|(Df^k(x))| > |x| c_{n-1}^3$.*

Proof. First notice that by Lemma 4.4 and Lemma 4.3, $R_n|_{I_n^0} = \phi \circ f$ with $|D\phi| > 1$, provided n is big enough (since ϕ has small distortion and there is a big macroscopic expansion from $f(I_n^0)$ to $R_n(I_n^0)$). Also, by Lemma 3.5, $|I_n|$ decays so fast that $\prod_{r=1}^n |I_n| > c_{n-1}^{3/2}$ for n big enough. Finally, by Lemma 4.6, for n big enough, $|DR_n(x)| > 1$ for $x \in I_n^j$, $j \neq 0$. Let n_0 be so big that if $n \ge n_0$, all the above properties hold.

From hyperbolicity of f restricted to the complement of I_{n_0} (from Lemma 2.3), there exists a constant $C > 0$ such that if s_0 is such that $f^s(x) \notin I_{n_0}^0$ for every $s_0 \le s < k$ then $|Df^{k-s_0}(f^{s_0}(x))| > C$.

Let us now consider some $n \ge n_0$. If $k = K$, we have a full return and the result follows from Lemma 4.6.

Assume now $k < K$. Let us define $d(s)$, $0 \le s \le k$ such that $f^s(x) \in I_{d(s)} \setminus I_{d(s)}^0$ (if $f^s(x) \notin I_0$ we set $d(s) = -1$). Let $m(s) = \max_{s \le t \le k} d(t)$. Let us define a finite sequence $\{k_r\}_{r=0}^l$ as follows. We let $k_0 = 0$ and supposing $k_r < k$ we let $k_{r+1} = \max\{k_r < s \le k | d(s) = m(s)\}$. Notice that $d(k_i) < n$ if $i \ge 1$, since otherwise $f^{k_i}(x) \in I_n$ so $k = k_i = K$ which contradicts our assumption.

The sequence $0 = k_0 < k_1 < \ldots < k_l = k$ satisfies $n = d(k_0) > d(k_1) > \ldots > d(k_l)$. Let θ be maximal with $d(k_\theta) \ge n_0$. We have of course

$$|Df^{k-k_\theta}(f^{k_\theta}(x))| > C|Df(f^{k_\theta}(x))|, \tag{4.9}$$

so if $\theta = 0$ then $Df^k(x) > |2Cx|$ and we are done.

Assume now $\theta > 0$. We have of course

$$|Df^{k-k_\theta}(f^{k_\theta}(x))| > C|Df(f^{k_\theta}(x))| > C|I_{d(k_\theta)+1}| \tag{4.10}$$

For $1 \le r \le \theta$, the action of $f^{k_r - k_{r-1}}$ near $f^{k_{r-1}}(x)$ is obtained by applying the central component of $R_{d(k_r)}$ followed by several non-central components of $R_{d(k_r)}$. Since $d(k_r) \ge n_0$, we can estimate

$$|Df^{k_r - k_{r-1}}(f^{k_{r-1}}(x))| > |DR_{d(k_r)}(f^{k_{r-1}}(x))| > |Df(f^{k_{r-1}}(x))|. \tag{4.11}$$

For $r = 1$, this argument gives $|Df^{k_1}(x)| \ge |Df(x)|$, while for $r > 1$ we can estimate

$$|Df^{k_r - k_{r-1}}(f^{k_{r-1}}(x))| > |Df(f^{k_{r-1}}(x))| > |I_{d(k_{r-1})+1}|. \tag{4.12}$$

Combining it all we get

$$\begin{aligned} |Df^k(x)| &= |Df^{k_1}(x)| \cdot |Df^{k-k_\theta}(f^{k_\theta}(x))| \prod_{r=2}^{\theta} |Df^{k_r - k_{r-1}}(f^{k_{r-1}}(x))| > |2x| \cdot C \cdot |I_{d(k_\theta)+1}| \prod_{r=2}^{\theta} |I_{d(k_{r-1})+1}| \\ &= |2Cx| \prod_{r=1}^{\theta} |I_{d(k_r)+1}| \ge |2Cx| \prod_{r=0}^{n} |I_r| > |x| c_{n-1}^3. \end{aligned} \tag{4.13}$$

□

5. Sequence of quasisymmetric constants and trees

5.1. **Preliminary estimates.** From now on, we will need to consider not only γ'-capacities with some $\gamma' \geq \gamma$ fixed, but different constants for different levels of the principal nest. To do so, we will make use of sequence of constants converging (decreasing) to γ. Let $\gamma_n = \frac{n+1}{n}\gamma$, $\tilde{\gamma}_n = \frac{2n+3}{2n+1}\gamma$. Notice that $\gamma_n > \tilde{\gamma}_n > \gamma_{n+1}$ and $\lim \gamma_n = \lim \tilde{\gamma}_n = \gamma$.

The generalized renormalization process relating R_n to R_{n+1} has two phases, first we go from R_n to L_n and then we go from L_n to R_{n+1}. The following remarks shows why it is useful to consider the sequence of quasisymmetric constants due to losses related to distortion.

Remark 5.1. Let S be an interval contained in $I_n^{\underline{d}}$. Using Lemma 4.3 we have $R_n^{\underline{d}}|_S = \psi_2 \circ f \circ \psi_1$, where the distortion of ψ_2 and ψ_1 are torrentially small and $\psi_1(S)$ is contained in some I_n^j, $j \neq 0$. If S is contained in I_n^0 we may as well write $R_n|_S = \phi \circ f$, and the distortion of ϕ is also torrentially small.

In either case, if we decompose S in $2km$ intervals S_i of equal length, where k is the distortion of either $R_n^{\underline{d}}|_S$ or $R_n|_S$ and m is subtorrentially big (say, $m < 2^n$), the distortion obtained restricting to any interval S_i will be bounded by $1 + m^{-1}$. Indeed, in the case $S \subset I_n^0$, we have $\operatorname{dist}(R_n|_{S_i}) \leq \operatorname{dist}(\phi) \operatorname{dist}(f|_{S_i})$. Now $k = \operatorname{dist}(R_n|_S) \geq \operatorname{dist}(\phi)^{-1} \operatorname{dist}(f|_S)$. Since f is quadratic,

$$\operatorname{dist}(f|_{S_i}) - 1 \leq \frac{|S_i|}{|S|}(\operatorname{dist}(f|_S) - 1) \leq \frac{1}{2km}(k \operatorname{dist}(\phi) - 1) \leq \frac{\operatorname{dist}(\phi)}{2m}. \tag{5.1}$$

Since $\operatorname{dist}(\phi) - 1$ is torrentially small, $\operatorname{dist}(f|_{S_i}) \leq 1 + (2/3)m^{-1}$ and $\operatorname{dist}(R_n|_{S_i}) \leq 1 + m^{-1}$. The case $S \subset I_n^{\underline{d}}$ is entirely analogous, considering $\operatorname{dist}(R_n^{\underline{d}}|_{S_i}) \leq \operatorname{dist}(\psi_2) \operatorname{dist}(f|_{\psi_1(S_i)}) \operatorname{dist}(\psi_1)$, and using torrentially small distortion of ψ_1 and ψ_2. The estimate now becomes

$$\operatorname{dist}(f|_{\psi_1(S_i)}) - 1 \leq \frac{|\psi_1(S_i)|}{|\psi_1(S)|}(\operatorname{dist}(f|_{\psi_1(S)}) - 1) \leq \frac{\operatorname{dist}(\psi_1)}{2km}(k \operatorname{dist}(\psi_1) \operatorname{dist}(\psi_2) - 1) \leq \frac{\operatorname{dist}(\psi_1)^2 \operatorname{dist}(\psi_2))}{2m} \tag{5.2}$$

and we conclude again that $\operatorname{dist}(R_n^{\underline{d}}|_{S_i}) \leq 1 + m^{-1}$.

Remark 5.2. We have the following estimate for the effect of the pullback of a subset of I_n by the central branch $R_n|_{I_n^0}$. With total probability, for all n sufficiently big, if $X \subset I_n$ satisfies

$$p_{\tilde{\gamma}_n}(X|I_n) < \delta < n^{-b^2} \tag{5.3}$$

then

$$p_{\gamma_{n+1}}((R_n|_{I_{n+1}})^{-1}(X)|I_n) < \delta^{5a^2}. \tag{5.4}$$

Indeed, let V be a $\delta^{10a}|I_{n+1}|$ neighborhood of 0. Then $R_n|_{I_{n+1}\setminus V}$ has distortion bounded by $2\delta^{-10a}$.

Let $W \subset I_n$ be an interval of size $\lambda|I_n|$. Of course

$$p_{\tilde{\gamma}_n}(X \cap W|W) < \delta\lambda^{-b/15}. \tag{5.5}$$

Let us decompose each side of $I_{n+1} \setminus V$ as a union of $n^{b/10}\delta^{-5a/2}$ intervals of equal length. Let W be such an interval. From Lemma 4.4, it is clear that the image of W covers at least $\delta^{5a}n^{-2b}|I_n|$. It is clear then that

$$p_{\tilde{\gamma}_n}(X \cap R_n(W)|R_n(W)) < \delta(\delta^{5a}n^{-2b})^{-b/15} < \delta^{1/2} \tag{5.6}$$

(using that $\delta < n^{-b^2}$). So we conclude that (since the distortion of $R_n|_W$ is bounded by $1 + n^{-3}$ by Remark 5.1)

$$p_{\gamma_{n+1}}((R_n|_{I_{n+1}})^{-1}(X) \cap W|W) < \delta^{1/2} \tag{5.7}$$

(we use the fact that the composition of a γ_{n+1}-qs map with a map with small distortion in $\tilde{\gamma}_n$-qs). Since

$$p_{\gamma_{n+1}}(V|I_{n+1}) < \delta^{10a^2}, \tag{5.8}$$

we get the required estimate.

5.2. **More on trees.** Let us see an application of the above remarks.

Lemma 5.1. *With total probability, for all n sufficiently big*

$$p_{\tilde{\gamma}_n}((R_n^{\underline{d}})^{-1}(X)|I_n^{\underline{d}}) < 2^n p_{\gamma_n}(X|I_n). \tag{5.9}$$

Proof. Decompose $I_n^{\underline{d}}$ in $n^{\ln(n)}$ intervals of equal length, say, $\{W_i\}_{i=1}^{n^{\ln(n)}}$. Then by Lemma 4.6, $|R_n^{\underline{d}}(W_i)| > n^{-2\ln n}|I_n|$, so we get

$$p_{\gamma_n}(R_n^{\underline{d}}(W_i) \cap X|R_n^{\underline{d}}(W_i)) < n^{4\ln(n)} p_{\gamma_n}(X|I_n). \tag{5.10}$$

Applying Remark 5.1, we see that

$$p_{\tilde{\gamma}_n}((R_n^{\underline{d}})^{-1}(X) \cap W_i|W_i) < n^{4\ln(n)} p_{\gamma_n}(X|I_n), \tag{5.11}$$

(we use the fact that the composition of a $\tilde{\gamma}_n$-qs map with a map with small distortion is γ_n-qs) which implies the desired estimate. □

By induction we get:

Lemma 5.2. *With total probability, for n is big enough, if $X_1, ..., X_m \subset \mathbb{Z} \setminus \{0\}$*

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) = (j_1, ..., j_m, ..., j_{|\underline{d}^{(n)}(x)|}), j_i \in X_i, 1 \le i \le m|I_n) \le 2^{mn} \prod_{i=1}^m p_{\gamma_n}(j^{(n)}(x) \in X_i|I_n). \tag{5.12}$$

The following is an obvious variation of the previous lemma fixing the start of the sequence.

Lemma 5.3. *With total probability, for n is big enough, if $X_1, ..., X_m \subset \mathbb{Z} \setminus \{0\}$, and if $\underline{d} = (j_1, ..., j_k)$ we have*

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) = (j_1, ..., j_k, j_{k+1}, ..., j_{k+m}, ..., j_{|\underline{d}^{(n)}(x)|}), j_{i+k} \in X_i, 1 \le i \le m|I_n^{\underline{d}}) \le 2^{mn} \prod_{i=1}^m p_{\gamma_n}(j^{(n)}(x) \in X_i|I_n). \tag{5.13}$$

In particular, with $\underline{d} = (\tau_n)$,

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) = (\tau_n, j_1, ..., j_m, j_{m+1}, ..., j_{|\underline{d}^{(n)}(x)|}), j_i \in X_i, 1 \le i \le m|I_n^{\tau_n}) \le 2^{mn} \prod_{i=1}^m p_{\gamma_n}(j^{(n)}(x) \in X_i|I_n). \tag{5.14}$$

The last part of the above lemma will be often necessary in order to apply PhPa1'.

Lemma 5.4. *Let $Q \subset \mathbb{Z} \setminus \{0\}$. Let $Q(m, k)$ denote the set of $\underline{d} = (j_1, ..., j_m)$ such that $\#\{1 \le i \le m,\ j_i \in Q\} \ge k$. Define $q_n(m, k) = p_{\tilde{\gamma}_n}(\cup_{\underline{d} \in Q(m,k)} I_n^{\underline{d}}|I_n)$. Let $q_n = p_{\gamma_n}(\cup_{j \in Q} I_n^j|I_n)$.*

With total probability, for n large enough,

$$q_n(m, k) \le \binom{m}{k} (2^n q_n)^k. \tag{5.15}$$

Proof. We have the following recursive estimates for $q_n(m, k)$:

(1) $q_n(1, 0) = 1$, $q_n(1, 1) \le q_n \le 2^n q_n$, and $q_n(m + 1, 0) \le 1$ for $m \ge 1$,
(2) $q_n(m + 1, k + 1) \le q_n(m, k + 1) + 2^n q_n q_n(m, k)$.

Indeed, (1) is completely obvious and if $(j_1, ..., j_{m+1}) \in Q(m+1, k+1)$ then either $(j_1, ..., j_m) \in Q(m, k+1)$ or $(j_1, ..., j_m) \in Q(m, k)$ and $j_{m+1} \in Q$, so (2) follows from Lemma 5.1. It is clear that (1) and (2) imply by induction (5.15). □

We recall that by Stirling Formula,

$$\binom{m}{qm} < \frac{m^{qm}}{(qm)!} < \left(\frac{3}{q}\right)^{qm}. \tag{5.16}$$

So we can get the following estimate. For $q \ge q_n$,

$$q_n(m, (6 \cdot 2^n)qm) < \left(\frac{1}{2}\right)^{(6 \cdot 2^n)qm}. \tag{5.17}$$

6. Estimates on time

Our aim in this section is to estimate the distribution of return times to I_n: they are concentrated around c_{n-1}^{-1} up to an exponent in a bounded range.

The basic estimate is a large deviation estimate and is proven in the next subsection (Corollary 6.5) and states that for $k \geq 1$ the set of branches with time larger then kc_n^{-2b} has capacity less then e^{-k}.

6.1. **A Large Deviation lemma for times.** Let $r_n(j)$ be such that $R_n|_{I_n^j} = f^{r_n(j)}$. We will also use the notation $r_n(x) = r_n(j^{(n)}(x))$, the n-th return time of x (there should be no confusion for the reader, since we consistently use j for an integer index and x for a point in the phase space).

Let

$$A_n(k) = p_{\gamma_n}(r_n(x) \geq k | x \in I_n) \tag{6.1}$$

Since f restricted to the complement of I_{n+1} is hyperbolic, from Lemma 2.3, it is clear that $A_n(k)$ decays exponentially with k:

Lemma 6.1. *With total probability, for all $n > 0$, there exists $C > 0$, $\lambda > 1$ such that $A_n(k) < C\lambda^k$.*

Proof. Consider a Markov partition for $f|_{I \setminus I_{n+1}}$, that is, a finite union of intervals $M_1, ..., M_m$ such that
(1) $\cup_{i=1}^m M_i = I \setminus I_{n+1}$,
(2) For every $1 \leq i \leq m$, $f|_{M_i}$ is a diffeomorphism,
(3) $f(\cup_{i=1}^m \partial M_i) \subset \cup_{i=1}^m \partial M_i$.

It is easy to see that such a Markov partition also satisfies
(4) For every $1 \leq i \leq m$, either

$$f(M_i) = \bigcup_{M_j \subset f(M_i)} M_j \quad \text{or} \quad f(M_i) = I_{n+1} \cup \bigcup_{M_j \subset f(M_i)} M_j. \tag{6.2}$$

(To construct such Markov partition, notice first that the boundary of I_{n+1} is preperiodic to a periodic orbit q (of period p). In particular we have $f^s(\partial I_{n+1}) = q$ for some integer $s > p$. Let K be the (finite) set of all x which never enter $\operatorname{int} I_{n+1}$ and such that $f^j(x) = q$ for some $j \leq s$. Since I_{n+1} is nice, $\partial I_{n+1} \subset K$, and since $s > p$, $q \in K$. In particular K is forward invariant. It is easy to see that the connected components of $I \setminus (K \cup I_{n+1})$ form a Markov partition of $I \setminus I_{n+1}$.)

It follows that if $f^j(x) \in \cup_{i=1}^m \operatorname{int} M_i$, $0 \leq j \leq k$ then there exists a unique interval $x \in M^k(x)$ such that $f^k|_{M^k(x)}$ is a diffeomorphism onto some M_j. Notice that if $k \geq 1$, $f(M^k(x)) = M^{k-1}(f(x))$.

By Lemma 2.3, if $y \in M^k(x)$, $|Df^k(y)|$ is exponentially big in k. In particular, $\sum_{j=0}^{k-1} |f^j(M^k(x))| < C'$ for some constant $C' > 0$ independent of $M^k(x)$. Since f is C^2, $\operatorname{dist}(f|_{M^k(x)})$ is uniformly bounded in k. Notice that the bounds on distortion depend on n. (An alternative to this classical argument is to obtain the bounded distortion from the negative Schwarzian derivative).

By Lemma 2.3 again, the set of points $x \in I$ which never enter I_{n+1} has empty interior: for every $T \subset I$ there is an iterate $f^r(T)$ which intersects I_{n+1} (otherwise the exponentially growing intervals $f^r(T) \subset I$ would eventually become bigger than I). So there exists $r > 0$ such that, for every M_j, there exists $x \in M_j$ and $t_j < r$ with $f^{t_j}(x) \in \operatorname{int} I_{n+1}$. It follows that there exists an interval $E_j \subset M_j$ such that $f^{t_j}(E_j) \subset \operatorname{int} I_{n+1}$.

Fixing some $M^k(x)$ with $f^k(M^k(x)) = M_j$, let $E^k(x) = (f^k|_{M^k(x)})^{-1}(E_j)$. By bounded distortion, it follows that $\frac{|E^k(x)|}{|M^k(x)|}$ is uniformly bounded from below independently of $M^k(x)$. In particular, $p_{2\gamma}(M^k(x) \setminus E^k(x) | M^k(x)) < \lambda$ for some constant $\lambda < 1$.

Let M^k be the union of the $M^k(x)$ and E^k be the union of the $E^k(x)$. Then $M^{k+r} \cap E^k = \emptyset$. In particular, $p_{2\gamma}(M^{(k+1)r}|I) < \lambda p_{2\gamma}(M^{kr}|I)$.

We conclude that $p_{2\gamma}(M^k|I_n) < C\lambda^{k/r}$ for some constant $C > 0$. If $k > r_n(0)$, then $M^k \cap I_n$ contains the set of points $x \in I_n$ such that $f^j(x) \notin I_n$, $1 \leq j \leq k$, that is, all points $x \in I_n$ with $r_n(x) > k$. Adjusting C and λ if necessary, we have $A_n(k) < C\lambda^k$. ∎

Let ζ_n be the maximum $\zeta \leq c_{n-1}$ such that for all $k \geq \zeta^{-1}$ we have

$$A_n(k) \leq e^{-\zeta k} \tag{6.3}$$

and finally let $\alpha_n = \min_{1\leq m\leq n} \zeta_m$.

Let $l_n(\underline{d})$ be such that $L_n|_{I_n^{\underline{d}}} = f^{l_n(\underline{d})}$. We will also use the notation $l_n(x) = l_n(\underline{d}^{(n)}(x))$. Let us define

$$B_n(k) = p_{\tilde{\gamma}_n}(l_n(x) > k|I_n). \tag{6.4}$$

$$B_n^{\tau_n}(k) = p_{\tilde{\gamma}_n}(l_n(x) > k + r_n(\tau_n)|I_n^{\tau_n}). \tag{6.5}$$

Lemma 6.2. *If $k > c_n^{-b/2}\alpha_n^{-b/2}$ then*

$$B_n(k) < e^{-c_n^{b/2}\alpha_n^{b/2}k}, \tag{6.6}$$

$$B_n^{\tau_n}(k) < e^{c_n^{b/2}\alpha_n^{b/2}k}. \tag{6.7}$$

Proof. Let us first show (6.6), the proof of estimate (6.7) being analogous.

Let $k > c_n^{-b/2}\alpha_n^{-b/2}$ be fixed. Let $m_0 = \alpha_n^{b/2}k$.

Notice that by Lemma 3.3

$$p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| \geq m_0|x \in I_n) \leq e^{-c_n^{b/4}\alpha_n^{b/2}k}. \tag{6.8}$$

Fix now $m < m_0$. Let us estimate

$$p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| = m, l_n(x) > k|x \in I_n). \tag{6.9}$$

For each $\underline{d} = (j_1, ..., j_m)$ we can associate a sequence of m positive integers r_i such that $r_i \leq r_n(j_i)$ and $\sum r_i = k$. The average value of r_i is at least k/m so we conclude that

$$\sum_{r_i\geq k/2m} r_i > k/2. \tag{6.10}$$

Recall also that

$$\frac{k}{2m} > \frac{1}{(2\alpha_n^{b/2})} > \alpha_n^{-1}. \tag{6.11}$$

Given a sequence of m positive integers r_i as above we can do the following estimate using Lemma 5.2

$$\begin{aligned} p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) = (j_1, ..., j_m), r_n(j_i) \geq r_i|I_n) &\leq 2^{mn}\prod_{j=1}^{m} p_{\gamma_n}(r_n(x) \geq r_j|I_n) \leq 2^{mn}\prod_{r_j\geq\alpha_n^{-1}} p_{\gamma_n}(r_n(x) \geq r_j|I_n) \\ &\leq 2^{mn}\prod_{r_j\geq k/2m} e^{-\alpha_n r_j} \leq 2^{mn}e^{-\alpha_n k/2}. \end{aligned} \tag{6.12}$$

The number of sequences of m positive integers r_i with sum k is

$$\binom{k+m-1}{m-1} \leq \frac{1}{(m-1)!}(k+m-1)^{m-1} \leq \frac{1}{m!}(k+m)^m \leq \left(\frac{2ek}{m}\right)^m. \tag{6.13}$$

Notice that

$$\begin{aligned} 2^{mn}\left(\frac{2ek}{m}\right)^m &\leq \left(\frac{2^{n+3}k}{m}\right)^{\frac{m}{k2^{n+3}}k2^{n+3}} \\ &\leq \left(\frac{2^{n+3}k}{m_0}\right)^{\frac{m_0}{k2^{n+3}}k2^{n+3}} \qquad \text{(since } x^{1/x} \text{ is decreases for } x > e) \\ &\leq \left(\frac{2^{n+3}}{\alpha_n^{b/2}}\right)^{m_0} \leq e^{\alpha_n^{b/4}k}. \end{aligned} \tag{6.14}$$

So we can finally estimate

$$p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| = m, l_n(x) \geq k | x \in I_n) \leq 2^{mn} \left(\frac{2ek}{m}\right)^m e^{-\alpha_n k/2} < e^{(\alpha_n^{(b/4)-1} - \frac{1}{2})\alpha_n k}. \tag{6.15}$$

Summing up on m we get

$$\begin{aligned} p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| < m_0, l_n(x) \geq k | x \in I_n) &\leq m_0 e^{(\alpha_n^{(b/4)-1} - \frac{1}{2})\alpha_n k} \\ &< e^{(\alpha_n^{b/2} + \alpha_n^{(b/4)-1} - \frac{1}{2})\alpha_n k} \qquad (\text{since} \frac{\ln(m_0)}{k} \leq \frac{\ln(k)}{k} \leq \alpha_n^{b/4}) \\ &\leq e^{-\alpha_n k/3}. \end{aligned} \tag{6.16}$$

As a direct consequence we get

$$B_n(k) < e^{-\alpha_n k/3} + e^{-c_n^{b/4}\alpha_n^{b/2} k} < e^{-c_n^{b/2}\alpha_n^{b/2} k}. \tag{6.17}$$

concluding the proof of (6.6). □

Let $v_n = r_n(0)$ be the return time of the critical point.

Lemma 6.3. *With total probability, for n large enough,*

$$v_{n+1} < c_n^{-3b/4}\alpha_n^{-3b/4}. \tag{6.18}$$

Proof. By the definition of α_n and PhPa2', it follows that with total probability, for n large enough,

$$r_n(\tau_n) < c_{n-1}^{-1}\alpha_n^{-1}. \tag{6.19}$$

Recall that $\underline{d}^{(n)}(0)$ is such that $R_n(0) \in C_n^{\underline{d}^{(n)}(0)}$. Using Lemma 6.2, more precisely estimate (6.7), together with PhPa1', we get with total probability, for n large enough,

$$l_n(\underline{d}^{(n)}(0)) - r_n(\tau_n) < n\alpha_n^{-b/2} c_n^{-3b/4}, \tag{6.20}$$

and thus

$$v_{n+1} < v_n + c_{n-1}^{-1}\alpha_n^{-1} + \alpha_n^{-b/2} c_n^{-3b/4} < v_n + \alpha_n^{-3b/4} c_n^{-3b/4}/4. \tag{6.21}$$

Notice that α_n decreases monotonically, thus for n_0 big enough and for $n > n_0$,

$$v_n < v_{n_0} + \sum_{k=n_0}^{n-1} \alpha_k^{-3b/4} c_k^{-3b/4}/4 < v_{n_0} + \alpha_n^{-3b/4} c_n^{-3b/4}/3. \tag{6.22}$$

which for n big enough implies $v_{n+1} < c_n^{-3b/4}\alpha_n^{-3b/4}$. □

Lemma 6.4. *With total probability, for n large enough,*

$$\alpha_{n+1} \geq \min\{\alpha_n^{2b}, c_n^{2b}\}. \tag{6.23}$$

Proof. Let $k \geq \max\{\alpha_n^{-2b}, c_n^{-2b}\}$. From Lemma 6.3 one immediately sees that if $r_{n+1}(j) \geq k$ then $R_n(I_{n+1}^j))$ is contained on some $C_n^{\underline{d}}$ with $l_n(\underline{d}) \geq k/2 \geq \alpha_n^{-b/2} c_n^{-b/2}$.

Applying Lemma 6.2 we have $B_n(k/2) < e^{-\alpha_n^{b/2} c_n^{b/2} k/2}$.

Applying Remark 5.2 we get

$$A_{n+1}(k) < e^{-k\alpha_n^{b/2} c_n^{b/2} a^2/2} < e^{-k \min\{\alpha_n^{2b}, c_n^{2b}\}}. \tag{6.24}$$

□

Since c_n decreases torrentially, we get

Corollary 6.5. *With total probability, for n large enough $\alpha_{n+1} \geq c_n^{2b}$.*

6.2. **Consequences.** The lemma below contains the basic estimates on return times that we will need (and also contains estimates already proved).

Lemma 6.6. *With total probability, for all n sufficiently large we have*

$$p_{\tilde\gamma_n}(l_n(x) < c_n^{-s} | x \in I_n) < c_n^{\frac{a}{2}-s}, \quad \text{with } s > 0, \tag{6.25}$$

$$p_{\tilde\gamma_n}(l_n(x) < c_n^{-s} | x \in I_n^{\tau_n}) < c_n^{\frac{a}{2}-s}, \quad \text{with } s > 0, \tag{6.26}$$

$$p_{\tilde\gamma_n}(l_n(x) > c_n^{-s} | x \in I_n) < e^{-c_n^{b-s}}, \quad \text{with } s > b, \tag{6.27}$$

$$p_{\tilde\gamma_n}(l_n(x) > c_n^{-s} | x \in I_n^{\tau_n}) < e^{-c_n^{b-s}}, \quad \text{with } s > b, \tag{6.28}$$

$$p_{\gamma_n}(r_n(x) > c_{n-1}^{-s} | x \in I_n) < e^{-c_{n-1}^{2b-s}}, \quad \text{with } s > 2b. \tag{6.29}$$

Moreover we also have

$$r_n(\tau_n) < c_{n-1}^{-3b}, \tag{6.30}$$

$$c_{n-1}^{-a} < v_n < c_{n-1}^{-4b/5}, \tag{6.31}$$

$$c_{n-1}^{-a/2} < \ln(c_n^{-1}) < c_{n-1}^{-b}. \tag{6.32}$$

Proof. The estimate from above in (6.31) is given by Corollary 6.5 together with Lemma 6.3, while the estimate from below is contained in Lemma 3.4 (since $v_n > s_{n-1}$). Estimate (6.29) is Corollary 6.5.

Estimates (6.25) and (6.26) are contained in Lemma 3.3 (it is enough to use that $l_n(x) \geq |\underline{d}^{(n)}(x)|$).

Estimate (6.27) follow from Lemma 6.2 and (6.29). Using also the estimate from above in (6.31) one also gets estimate (6.28).

Estimate (6.29) implies (6.30) by application of PhPa1'.

The estimate from below on (6.32) is given by Corollary 3.5. Notice that $R_n(I_{n+1}) > 2^{-n}|I_n|$ (Lemma 4.4), and since $|Df|$ is bounded (by 4) this implies $4^{v_n}|I_{n+1}| > 2^{-n}|I_n|$ which gives $c_n > 2^{-n}4^{-v_n}$. So the estimate from above in (6.32) follows from the estimate from above in (6.31). □

7. Some kinds of branches and landings

7.1. **Standard and fast landings.** Let us define the set of standard landings at time n, $LS(n) \subset \Omega$ as the set of all $\underline{d} = (j_1, ..., j_m)$ satisfying the following:

LS1: $c_n^{-a^4/2} < m < c_n^{-2b^4}$,
LS2: $r_n(j_i) < c_{n-1}^{-3b^4}$ for all i.

We also define the set of fast landings at time n, $LF(n) \subset \Omega$ by the following conditions

LF1: $m \leq c_n^{-a^4/2}$.
LF2: (=LS2) $r_n(j_i) < c_{n-1}^{-3b^4}$ for all i.

Lemma 7.1. *With total probability, for all n sufficiently big,*

$$p_{\tilde\gamma_n}(\underline{d}^{(n)}(x) \notin LS(n) | x \in I_n) < c_n^{a^4/3}/2, \tag{7.1}$$

$$p_{\tilde\gamma_n}(\underline{d}^{(n)}(x) \notin LS(n) \cup LF(n) | x \in I_n) < c_n^{n^2}/2, \tag{7.2}$$

$$p_{\tilde\gamma_n}(\underline{d}^{(n)}(x) \notin LS(n) | x \in I_n^{\tau_n}) < c_n^{a^4/3}/2, \tag{7.3}$$

$$p_{\tilde\gamma_n}(\underline{d}^{(n)}(x) \notin LS(n) \cup LF(n) | x \in I_n^{\tau_n}) < c_n^{n^2}/2. \tag{7.4}$$

Proof. Let us start with the first two estimates.

(LS1) A simple application of (6.25) and (6.27) allows to estimate $p_{\tilde\gamma_n}(|\underline{d}^{(n)}(x)| \geq c_n^{2b^4}$ or $|\underline{d}^{(n)}(x) \leq c_n^{a^4/2}|I_n)$ and shows that the set of landings violating LS1 has $\tilde\gamma_n$-capacity bounded by $c_{n-1}^{2a^4/5}$.

(LS1+LF1) An application of (6.27) allows to estimate $p_{\tilde{\gamma}_n}(|\underline{d}^{(n)}(x)| \geq c_n^{2b^4}|I_n)$ and shows that the set of landings violating both LS1 and LF1 has $\tilde{\gamma}_n$-capacity bounded by $c_n^{n^2}/10$.

(LS2=LF2) An application of (6.29) gives $p_{\tilde{\gamma}_n}(r_n(x) \geq c_{n-1}^{-3b^4}|I_n) \leq e^{-c_{n-1}^{-2b^4}}$. Using Lemma 5.1, this implies that the set of landings violating LS2 and satisfying either LS1 or LF1 (so that $|\underline{d}| < c_n^{-2b^4}$) has $\tilde{\gamma}_n$-capacity bounded by $2^n c_n^{-2b^4} p_{\tilde{\gamma}_n}(r_n(x) \geq c_{n-1}^{-3b^4}|I_n) \leq c_n^{n^2}/10$.

Putting those estimates together gives the first two estimates. To get the last two estimates, we proceed in the same way for estimating LS1 and LS1+LF1. The estimate of LS2=LF2 follows the same lines with one extra ingredient: we have to be careful since if $r_n(\tau_n)$ is very large then automatically LS2 is violated for every $\underline{d}$ which starts by τ_n. But this was taken care by estimate (6.30), and with this observation the estimates are the same. $\square$

7.2. **Very good returns, bad returns and excellent landings.** For $n_0, n \in \mathbb{N}$ such that $n \geq n_0$, define the set of very good returns, $VG(n_0, n) \subset \mathbb{Z} \setminus \{0\}$ and the set of bad returns, $B(n_0, n) \subset \mathbb{Z} \setminus \{0\}$, by induction as follows. We let $VG(n_0, n_0) = \mathbb{Z} \setminus \{0\}$,$B(n_0, n_0) = \emptyset$ and supposing $VG(n_0, n)$ and $B(n_0, n)$ defined, we then define the set of excellent landings $LE(n_0, n) \subset LS(n)$ as the set of all standard landings $\underline{d} = (j_1, ..., j_m)$ satisfying the following extra assumptions

- **LE1:** For all $c_{n-1}^{-2b^4} < k \leq m$, $\#\{1 \leq i \leq k,\ j_i \notin VG(n_0, n)\} < (6 \cdot 2^n) c_{n-1}^{a^8} k$,
- **LE2:** For all $c_n^{-1/n} < k \leq m$ $\#\{1 \leq i \leq k,\ j_i \in B(n_0, n)\} < (6 \cdot 2^n) c_{n-1}^{n} k$.

We then define $VG(n_0, n+1)$ as the set of $j \in \mathbb{Z} \setminus \{0\}$ such that $R_n(I_{n+1}^j) = C_n^{\underline{d}}$ with $\underline{d} \in LE(n_0, n)$ and such that:

- **VG:** The distance of I_{n+1}^j to 0 is bigger than $c_n^{n^2}|I_{n+1}|$.

And we define $B(n_0, n+1)$ as the set of $j \notin VG(n_0, n+1)$ such that $R_n(I_{n+1}^j) = C_n^{\underline{d}}$ with $\underline{d} \notin LF(n)$.

Lemma 7.2. *With total probability, for all n_0 sufficiently big,*

$$p_{\gamma_n}(j^{(n)}(x) \notin VG(n_0, n) | x \in I_n) < c_{n-1}^{a^8}, \tag{7.5}$$

$$p_{\gamma_n}(j^{(n)}(x) \in B(n_0, n) | x \in I_n) < c_{n-1}^{2n}, \tag{7.6}$$

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LE(n_0, n) | x \in I_n) < c_n^{2a^4/5}, \tag{7.7}$$

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LE(n_0, n) \cup LF(n) | x \in I_n) < c_n^{n^2}, \tag{7.8}$$

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LE(n_0, n) | x \in I_n^{\tau_n}) < c_n^{2a^4/5}. \tag{7.9}$$

Proof. The argument is by induction: if for a given value of n we have (7.5) and (7.6) (this holds trivially for $n = n_0$), we will show that we also have (7.7), (7.8) and (7.9), and this in turn implies that (7.5) and (7.6) hold for $n + 1$.

Assuming the validity for a given value of n of (7.5) and (7.6) we can estimate the $\tilde{\gamma}_n$-capacity of the set of landings which fail LE1 or LE2 using the techniques of §5.2 as follows:

(LE1) We use estimate (5.17). Setting $q = c_{n-1}^{a^8}$ we see that the set of landings which fail LE1 for a specific value of $k \geq c_{n-1}^{2b^4}$ is bounded by $2^{-(6 \cdot 2^n)qk}$. Summing up on $k \geq c_{n-1}^{2b^4}$ we get the upper bound $c_n^{n^2}/10$.

(LE2) We use estimate (5.17) again, setting this time $q = c_{n-1}^n$. The upper bound we get using the same argument as before is $\sum_{k > c_n^{-1/n}} 2^{-(6 \cdot 2^n)qk} \leq c_n^{n^2}/10$.

Those estimates imply (7.7) and (7.8) (it is enough to use (7.1) and (7.2)). An analogous argument shows that (7.5) and (7.6) imply (7.9).

To see that the validity of (7.7) and (7.8) for n implies the validity of (7.5) and (7.6) for $n + 1$ is just a matter of applying Remark 5.2 (notice that condition VG is quite weak: it excludes a set of branches of γ_{n+1}-capacity at most $c_n^{n^2/a}$). $\square$

This translates immediately (using the measure-theoretical argument of Lemma 2.2) to a parameter estimate using PhPa2':

Lemma 7.3. *With total probability, for all n_0 big enough, for all n big enough, $\tau_n \in VG(n_0, n)$.*

Lemma 7.4. *With total probability, for all n_0 big enough and for all $n \geq n_0$, if $j \in VG(n_0, n+1)$ then*

$$c_n^{-a^4/2} \leq m < r_{n+1}(j) < m c_{n-1}^{-4b^4} \leq c_n^{-2b^4} c_{n-1}^{-4b^4}, \tag{7.10}$$

where as usual, m is such that $R_n(I_{n+1}^j) = C_n^{\underline{d}}$ and $\underline{d} = (j_1, ..., j_m)$.

Proof. We have $c_n^{-a^4/2} \leq m \leq c_n^{-2b^4}$ by LS1, while $m < r_{n+1}(j)$ is obvious. We get $r_{n+1}(j) < m c_{n-1}^{-4b^4}$ from LS2 and (6.31). □

Lemma 7.5. *With total probability for all n_0 sufficiently big, if $n > n_0$, if $j \notin VG(n_0, n) \cup B(n_0, n)$ then $r_n(j) < c_{n-1}^{-a^4/2} c_{n-2}^{-4b^4}$.*

Proof. Indeed, if $j \notin VG(n_0, n) \cup B(n_0, n)$ then $R_{n-1}(I_n^j) \in LF(n_0, n-1)$. The estimate follows since a branch in $LF(n_0, n-1)$ has time bounded by $c_{n-1}^{-a^4/2} c_{n-2}^{-3b^4}$ (using LF1 and LF2) and $v_{n-1} < c_{n-2}^{-b^4}$ (using (6.31). □

Lemma 7.6. *With total probability, for all n_0 big enough and for all $n \geq n_0$, the following holds.*

Let $j \in VG(n_0, n+1)$, as usual let $R_n(I_{n+1}^j) \subset C_n^{\underline{d}}$ and $\underline{d} = (j_1, ..., j_m)$. Let m_k be biggest possible with

$$v_n + \sum_{j=1}^{m_k} r_n(j_i) \leq k \tag{7.11}$$

(the amount of full returns to level n before time k) and let

$$\beta_k = \sum_{\substack{1 \leq i \leq m_k, \\ j_i \in VG(n_0, n)}} r_n(j_i). \tag{7.12}$$

(the total time spent in full returns to level n which are very good before time k). Then $1 - \frac{\beta_k}{k} < c_{n-1}^{a^8/3}$ if $k > c_n^{-2/n}$.

Proof. Let us estimate first the time i_k which is not spent on non-critical full returns:

$$i_k = k - \sum_{j=1}^{m_k} r_n(j_i). \tag{7.13}$$

This corresponds exactly to v_n plus some incomplete part of the return $j_{m_{k+1}}$. This part can be bounded by $c_{n-1}^{-b^4} + c_{n-1}^{-3b^4}$ (use (6.31) to estimate v_n and LS2 to estimate the incomplete part).

Using LS2 we conclude now that

$$m_k > (k - c_{n-1}^{-b^4}) c_{n-1}^{3b^4} > c_n^{-1/n} \tag{7.14}$$

so m_k is not too small.

Let us now estimate the contribution h_k from bad full returns j_i. The number of such returns must be less than $c_{n-1}^{n/2} m_k$ by LE2 and the estimate on m_k. By LS2 their total time is at most $c_{n-1}^{(n/2)-3b^4} m_k < m_k$.

The non very good full returns on the other hand can be estimated by LE1 (using (7.14)), they are at most $c_{n-1}^{2a^8/3} m_k$. So we can estimate the total time l_k of non very good or bad full returns (with time less then $c_{n-1}^{-a^4/2} c_{n-2}^{-4b^4}$ by Lemma 7.5) by

$$c_{n-1}^{2a^8/3} c_{n-1}^{-a^4/2} c_{n-2}^{-4b^4} m_k, \tag{7.15}$$

while β_k can be estimated from below by

$$(1 - c_{n-1}^{2a^8/3}) c_{n-1}^{-a^4/2} m_k. \tag{7.16}$$

It is easy to see then that $i_k/\beta_k \ll c_{n-1}^{a^4/5}$, $h_k/\beta_k \ll c_{n-1}^{a^4/5}$. We also have $\frac{l_k}{\beta_k} < 2c_{n-1}^{a^8/2}$. So $\frac{i_k+h_k+l_k}{\beta_k}$ is less then $c_{n-1}^{a^8/3}$. Since $i_k + h_k + l_k + \beta_k = k$ we have $1 - \frac{\beta_k}{k} < \frac{i_k+h_k+l_k}{\beta_k}$. □

7.3. **Cool landings.** Let us define the set of cool landings $LC(n_0,n) \subset LE(n_0,n)$, $n_0, n \in \mathbb{N}$, $n \geq n_0$ as the set of all excellent landings $\underline{d} = (j_1, ..., j_m)$ satisfying

LC1: $j_i \in VG(n_0,n)$, $1 \leq i \leq c_{n-1}^{-a^8/2}$.

LC2: For all $c_{n-1}^{-a^8/2} < k \leq m$, $\#\{1 \leq i \leq k,\ j_i \notin VG(n_0,n)\} < (6 \cdot 2^n) c_{n-1}^{a^8/3} k$,

LC3: For $c_{n-1}^{-n/3} \leq k \leq m$, $\#\{1 \leq i \leq k,\ j_i \in B(n_0,n)\} < (6 \cdot 2^n) c_{n-1}^{n/6} k$,

LC4: $j_i \notin B(n_0,n)$, $1 \leq i \leq c_{n-1}^{-n/2}$.

As usual we obtain:

Lemma 7.7. *With total probability, for all n_0 sufficiently big and all $n \geq n_0$,*

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LC(n_0,n) | x \in I_n) < c_{n-1}^{a^8/3} \tag{7.17}$$

and for all n big enough

$$p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LC(n_0,n) | x \in I_n^{\tau_n}) < c_{n-1}^{a^8/3}. \tag{7.18}$$

Proof. We follow the ideas of the proof of Lemma 7.1. Let us start with the first estimate. Notice that by Lemma 7.2 we can estimate the $\tilde{\gamma}_n$-capacity of the complement of excellent landings by $c_n^{2a^4/5}$. The computations below indicate what is lost going from excellent to cool due to each item of the definition:

(LC1) This is a direct estimate analogous to LS2. By Lemma 7.2, the γ_n-capacity of the complement of very good branches is bounded by $c_{n-1}^{a^8}$, so an upper bound for the $\tilde{\gamma}_n$-capacity of the set of landings which do not start with $c_{n-1}^{-a^8/2}$ very good branches is given by $2^n c_{n-1}^{a^8} c_{n-1}^{-a^8/2} \ll c_{n-1}^{a^8/3}$.

(LC2) In order to estimate the set of landings violating LS3, we use the ideas of §5.2. The relevant estimate is (5.17): setting $q = c_{n-1}^{a^8/3}$ and using Lemma 7.2, we see that the $\tilde{\gamma}_n$-capacity of the set of landings violating LC2 for a specific value of $k > c_{n-1}^{-a^8/2}$ is bounded by $2^{-(6 \cdot 2^n) qk}$, and summing up on k we get the upper bound $\sum_{k > c_{n-1}^{-a^8/2}} 2^{-(6 \cdot 2^n) qk} \ll c_{n-1}^{a^8/3}$.

(LC3) The same argument of LC3 (this time setting $q = c_{n-1}^{n/6}$) gives the upper bound $\sum_{k > c_{n-1}^{-n/3}} 2^{-(6 \cdot 2^n) qk} \ll c_{n-1}^{n^2}$,

(LC4) An argument analogous to LC1 gives the upper bound $2^n c_{n-1}^{-n/2} c_{n-1}^{2n} \ll c_{n-1}^{n}$.

Those imply the first estimate. To get the second estimate we argue in the same way: we only need to use Lemma 7.3 to guarantee that $\tau_n \in VG(n_0,n)$ (this avoid problems with LC1 and LC4). □

Using PhPa1' we get

Lemma 7.8. *With total probability, for all n_0 big enough, for all n big enough we have $R_n(0) \in LC(n_0,n)$.*

Lemma 7.9. *With total probability, for all n_0 big enough, for all $n \geq n_0$, $\underline{d} = (j_1, ..., j_m) \in LC(n_0,n)$, and $1 \leq s \leq m$ we have*

$$\sum_{\substack{1 \leq i \leq s,\\ j_i \in VG(n_0,n)}} r_n(j_i) \geq (1 - 2^{-2n}) \sum_{i=1}^{s} r_n(j_i). \tag{7.19}$$

Proof. From LC1, LC2 and Lemma 7.5 we have

$$\sum_{\substack{1 \leq i \leq s,\\ j_i \notin VG(n_0,n) \cup B(n_0,n)}} r_n(j_i) \leq (6 \cdot 2^n) c_{n-1}^{a^8/3} s c_{n-1}^{-a^4/2} c_{n-2}^{-4b^4} \leq c_{n-1}^{a^9} c_{n-1}^{-a^4/2} s, \tag{7.20}$$

and from LC3, LC4 and LS2 we have

$$\sum_{\substack{1 \leq i \leq s,\\ j_i \in B(n_0,n)}} r_n(j_i) \leq (6 \cdot 2^n) c_{n-1}^{n/6} s c_{n-1}^{-3b^4} \leq s, \tag{7.21}$$

while from LC1, LC2 and Lemma 7.4 we have

$$\sum_{\substack{1\leq i\leq s,\\ j_i\in VG(n_0,n)}} r_n(j_i) \geq (1-(6\cdot 2^n)c_{n-1}^{a^8/3})sc_{n-1}^{-a^4/2} \leq \frac{1}{2}c_{n-1}^{-a^4/2}s, \tag{7.22}$$

and the result follows from (7.20), (7.21) and (7.22). □

8. Proof of Theorem B

We must obtain, with total probability, upper and lower (polynomial) bounds for the recurrence of the critical orbit. It will be easier to first study the recurrence with respect to iterates of return branches, and then estimate the total time of those iterates.

Lemma 8.1. *With total probability, for n big enough and for $1 < i \leq s_n$,*

$$\frac{\ln|R_n^i(0)|}{\ln(c_{n-1})} < b^8\left(1+\frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right). \tag{8.1}$$

Proof. From Lemma 4.4, we have

$$\frac{\ln|R_n(0)|}{\ln c_{n-1}} < \frac{\ln(2^{-n}|I_n|)}{\ln c_{n-1}} < 2 \tag{8.2}$$

and the result follows for $i=1$. Let $X\subset I_n$ be a $c_{n-1}^{b^8}$ neighborhood of 0. For n big, we can estimate $\frac{|X|}{|I_n|} < c_{n-1}^{b^8-2}$. Let us show that $R_n^i(0)\notin X$ for $2\leq i\leq c_{n-1}^{-2}$. This requirement can be translated on $R_n(0)$ not belonging to a certain set $Y\subset I_n$ such that

$$Y = \bigcup_{1\leq|\underline{d}|<c_{n-1}^{-2}} (R_n^{\underline{d}})^{-1}(X). \tag{8.3}$$

It is clear that

$$p_\gamma(Y|I_n^{\tau_n}) \leq c_{n-1}^{-2}c_{n-1}^{(b^8-2)/b^4} < c_{n-1}^{b^4-3}. \tag{8.4}$$

Applying PhPa1', the probability that for $2\leq i\leq c_{n-1}^{-2}$ we have $|R_n^i(0)| < c_{n-1}^{b^8}$ is at most $c_{n-1}^{b^4-3}$, which is summable. This implies the result in the range $2\leq i\leq c_{n-1}^{-2}$.

For $j\geq 0$, let $X_j\subset I_n$ be a $c_{n-1}^{b^{4(j+2)})}$ neighborhood of 0. Let K be maximal with $X_K\supset I_{n+1}$. Let $Y_j\subset I_n$ be such that

$$Y_j = \bigcup_{c_{n-1}^{-b^{4j}}\leq|\underline{d}|<c_{n-1}^{-b^{4(j+1)}}} (R_n^{\underline{d}})^{-1}(X_j). \tag{8.5}$$

By Lemma 4.4, it is clear that no X_j intersects $I_n^{\tau_n}$. Thus we can estimate

$$p_\gamma(Y_j|I_n^{\tau_n}) \leq c_{n-1}^{-b^{4(j+1)}}c_{n-1}^{b^{4(j+2)}-2b} < c_{n-1}^{b^{4(j+1)}} \tag{8.6}$$

and

$$p_\gamma(\bigcup_{j=0}^{K} Y_j|I_n^{\tau_n}) < \sum_{j=0}^{\infty} c_{n-1}^{-b^{4j}} < 2c_{n-1}. \tag{8.7}$$

Applying PhPa1', with total probability, the critical point does not belong to $\cup_{j=0}^K Y_j$. This means that for $0\leq j\leq K$ and for $c_{n-1}^{-b^{4j}} < i\leq c_{n-1}^{-b^{4(j+1)}}$, $R_n^i(0)\notin X_j$, which implies $|R_n^i(0)| > c_{n-1}^{b^{4(j+2)}}$. This concludes the proof of the result in the range $c_{n-1}^{-2} < i\leq c_{n-1}^{-b^{4(K+1)}}$.

To conclude the result in the remaining case $c_{n-1}^{-b^{4(K+1)}} < i \leq s_n$, we notice that $|R_n^i(0)| > |I_n|/2 > |X_{K+1}|/2$, so

$$\frac{\ln |R_n^i(0)|}{\ln(c_{n-1})} < \frac{\ln |X_{K+1}|/2}{\ln c_{n-1}} \leq b^{4(K+3)} \leq b^8 \frac{\ln(i)}{\ln(c_{n-1}^{-1})} \tag{8.8}$$

which gives the required estimate. □

For $1 \leq i \leq s_n$, let k_i such that $R_n^i|_{I_{n+2}} = f^{k_i}$.

Lemma 8.2. *With total probability, for n big enough and for $1 \leq i \leq s_n$,*

$$\frac{\ln(k_i)}{\ln(c_{n-1}^{-1})} > \frac{a^4}{3}\left(1 + \frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right). \tag{8.9}$$

Proof. Let us first assume that $c_{n-1}^{-1} \leq i \leq s_n$. By Lemma 7.8, $R_n(0)$ belongs to a cool landing, and by LC2 we get

$$\frac{k_i}{i-1} > c_{n-1}^{-a^4/3}, \tag{8.10}$$

which clearly implies the required estimate.

Using (6.31), we see that $k_i \geq v_n \geq c_{n-1}^{-a}$. Thus for $1 \leq i < c_{n-1}^{-1}$ we have

$$\frac{\ln k_i}{\ln c_{n-1}^{-1}} \geq a > \frac{a^4}{3}\left(1 + \frac{\ln i}{\ln c_{n-1}^{-1}}\right), \tag{8.11}$$

which gives the result. □

Considering $|R_n(0)| = |f^{v_n}(0)| < c_{n-1}$ and using $v_n < c_{n-1}^{-b}$ we get

$$\liminf_{n\to\infty} \frac{-\ln |f^n(0)|}{\ln(n)} \geq a. \tag{8.12}$$

Let now $v_n \leq k < v_{n+1}$. If $|f^k(0)| < k^{-3b^{12}}$ we have $f^k(0) \in I_n$ and so $k = k_i$ for some i. It follows from Lemmas 8.1 and 8.2 that $|f^k(0)| > k^{-3b^{12}}$. Thus

$$\limsup_{n\to\infty} \frac{-\ln |f^n(0)|}{\ln(n)} \leq 3b^{12}. \tag{8.13}$$

9. Hyperbolicity

For $j \neq 0$, we define

$$\lambda_n(j) = \inf_{x\in I_n^j} \frac{\ln(|R_n'(x)|)}{r_n(j)}, \quad \lambda_n = \inf_{j\neq 0} \lambda_n(j). \tag{9.1}$$

Lemma 9.1. *With total probability, for all n sufficiently big, $\lambda_n > 0$.*

Proof. By Lemma 2.3, there exists a constant $\tilde{\lambda}_n > 0$ such that each peridic orbit p of f whose orbit is entirely contained in the complement of I_{n+1} must satisfy $\ln |Df^m(p)| > \tilde{\lambda} m$, where m is the period of p. On the other hand, each non-central branch $R_n|_{I_n^j}$ has a fixed point. By Lemma 4.6, $\mathrm{dist}(R_n|_{I_n^j}) \leq 2^n$ and of course $\lim_{n\to\infty} r_n(j) = \infty$, so we have $\liminf_{n\to\infty} \lambda_n(j) \geq \tilde{\lambda}_n$. On the other hand, for any $j \neq 0$, $\lambda_n(j) > 0$ by Lemma 4.6, so $\lambda_n > 0$. □

Lemma 9.2. *With total probability, for n_0 big enough, we have:*

(9.2) *If $n \geq n_0$ and $j \in VG(n_0, n)$ then* $\lambda_n(j) \geq \lambda_{n_0} \dfrac{1 + 2^{n_0-n}}{2}$,

(9.3) *If $n > n_0$ and $j \in VG(n_0, n)$ then for every $c_{n-1}^{-3/(n-1)} \leq k \leq r_n(j)$ we have* $\inf_{I_n^j} \dfrac{\ln(|Df^k|)}{k} \geq \lambda_{n_0} \dfrac{1 + 2^{n_0-n+\frac{1}{2}}}{2} - c_{n-1}^{2/(n-1)}$.

Proof. Let us prove that if (9.2) holds for a certain value of $n \geq n_0$ then (9.3) and (9.2) hold for $n+1$. This implies the result by induction, since the definition of λ_{n_0} implies that (9.2) holds for n_0. Fix $j \in VG(n_0, n+1)$ and define

$$a_k = \inf_{x \in I^j_{n+1}} \frac{\ln |Df^k(x)|}{k}, \tag{9.4}$$

and let us consider values of k in the range $c_n^{-3/n} \leq k \leq r_{n+1}(j)$ (notice that $k = r_{n+1}(j)$ belongs to this range by Lemma 7.4).

We let $R_n(I^j_{n+1}) \subset C^{\underline{d}}_n$, $\underline{d} = (j_1, ..., j_m)$. Notice that by (6.31), $v_n < c_{n-1}^{-b^4} < k$. Let us say that j_i was completed before k if $v_n + r_n(j_1) + ... + r_n(j_i) \leq k$. Define

$$q_k = \inf_{x \in C^{\underline{d}}_n} \ln |Df^{k-r} \circ f^r(x)| \tag{9.5}$$

where $r = v_n + r_n(j_1) + ... + r_n(j_{m_k})$ with j_{m_k} the last complete return. By Lemma 4.7 we have

$$\frac{-q_k}{k} \leq \frac{-\ln c_n c^5_{n-1}}{c_n^{-3/n}} \ll c_n^{2/n}. \tag{9.6}$$

Let us show that $|DR_n(x)| > c_n^{n^2}$ if $x \in I^j_{n+1}$. Indeed, by Lemma 4.3, $DR_n|_{I_{n+1}} = \phi \circ f$, where ϕ has small distortion, so by Lemma 4.4,

$$|D\phi(x)| > \frac{|R_n(I_{n+1})|}{2|f(I_{n+1})|} > \frac{2^{-n}|I_n|}{|I_{n+1}|^2}, \tag{9.7}$$

while by VG, $|Df(x)| = |2x| > c_n^{n^2} |I_{n+1}|$, so $|DR_n(x)| > c_n^{n^2}$.

Notice also that using Lemma 4.6, for any $m_0 \leq m$, the derivative of $R_n^{m_0}$ in $C^{\underline{d}}_n$ is at least 2^{m_0}. So for $m_0 = c_{n-1}^{-2b^4}$ we have that the derivative of $R_n^{m_0+1}$ in I^j_{n+1} is at least 1. Moreover, still by Lemma 4.6 any complete return (even if not very good) brings in some expansion.

Notice that from LS2

$$k_0 = \sum_{i=1}^{m_0} r_n(j_i) < c_{n-1}^{-2b^4} c_{n-1}^{-3b^4} \ll k, \tag{9.8}$$

so we can use Lemma 7.6 and get

$$a_k > \frac{\beta_k - k_0}{k} \frac{\lambda_{n_0}(1 + 2^{n_0 - n})}{2} - \frac{-q_k}{k} \geq \frac{\lambda_{n_0}(1 + 2^{n_0 - n - 1/2})}{2} - \frac{-q_k}{k}. \tag{9.9}$$

This and (9.6) give (9.3) for $n+1$. If $k = r_{n+1}(j)$, $q_k = 0$ which gives (9.2) for $n+1$. ☐

10. Proof of Theorem A

We must show that with total probability, f is Collet-Eckmann. The argument given here is slightly different from the one in [AM1] and the one sketched in [AM2] (here we use Theorem B to get some estimates, which makes the argument slightly shorter). Let

$$a_k = \frac{\ln |Df^k(f(0))|}{k}. \tag{10.1}$$

Let $\underline{d}^{(n)}(R_n(0)) = (j_1, ..., j_{s_n})$. By Lemma 9.2, each very good return has a definite hyperbolicity by (9.2), while by Lemma 4.6, each return which is not very good brings some (possibly weak) expansion. Thus, for $1 \leq s \leq s_n$ and for n large, Lemma 7.9 implies

$$\frac{\ln |DR_n^s(R_n(0))|}{\sum_{i=1}^s r_n(j_i)} \geq (1 - 2^{-2n}) \lambda_{n_0} \frac{1 + 2^{n_0 - n}}{2} \geq \lambda_{n_0} \frac{1 + 2^{n_0 - n - 1}}{2}. \tag{10.2}$$

In particular,

$$a_{v_{n+1}-1} \geq a_{v_n-1} \frac{v_n - 1}{v_{n+1} - 1} + \lambda_{n_0} \frac{1 + 2^{n_0-n-1}}{2} \frac{v_{n+1} - v_n}{v_{n+1} - 1}. \tag{10.3}$$

Iterating (10.3) implies that for n large we have

$$a_{v_n-1} \geq \lambda_{n_0} \frac{1 + 2^{n_0-n-1}}{2}. \tag{10.4}$$

If $v_n - 1 \leq k < v_{n+1} - 1$, let $0 \leq m_k < s_n$ be maximal such that

$$t_k = v_n - 1 + \sum_{i=1}^{m_k} r_n(j_i) \leq k. \tag{10.5}$$

By (10.2) and (10.4) we have

$$\frac{|Df^{t_k}(f(0))|}{t_k} = \frac{v_n - 1}{t_k} a_{v_n-1} + \frac{|DR_n^{m_k}(R_n(0))|}{\sum_{i=1}^{m_k} r_n(j_i)} \frac{\sum_{i=1}^{m_k} r_n(j_i)}{t_k} \geq \lambda_{n_0} \frac{1 + 2^{n_0-n-1}}{2}. \tag{10.6}$$

Notice that if $\frac{k-t_k}{k} \geq 2^{-2n}$ then $k - t_k \geq c_{n-1}^{-a/2}$ (since $k \geq v_n - 1 > c_{n-1}^{-a} - 1$), so $r_n(j_{m_k+1}) \geq k - t_k > c_{n-1}^{-a^4} c_{n-2}^{-4b^4}$, and we have $j_{m_k+1} \in VG(n_0, n) \cup B(n_0, n)$ by Lemma 7.5. This in turn implies that $j \in VG(n_0, n)$: since $\frac{k-t_k}{k} \leq r_n(j_{m_k+1}) \leq c_{n-1}^{-3b^4}$ (by LS2), if $j_{m_k+1} \in B(n_0, n)$ then by LC4, $k \geq m_k + 1 > c_{n-1}^{-n/2}$, so $\frac{r_n(j_{m_k+1})}{k} \leq c_{n-1}^{n/3} < 2^{-2n}$.

Define $q_k = \ln |Df^{k-t_k}(f^{t_k+1}(0))|$. By Lemma 4.7 and by Theorem B, we have

$$\frac{-q_k}{k} \leq \frac{-\ln(|f^{t_k+1}(0)| c_{n-1}^3)}{k} \leq c_{n-1}^{a^2}, \tag{10.7}$$

since $|f^{t_k+1}(0)| > k^{-C}$ for some $C > 0$. Thus, if $\frac{k-t_k}{k} \leq 2^{-2n}$ we have by (10.6)

$$a_k \geq \frac{t_k}{k} \lambda_{n_0} \frac{1 + 2^{n_0-n-2}}{2} - \frac{-q_k}{k} \geq \frac{\lambda_{n_0}}{2}. \tag{10.8}$$

If $\frac{k-t_k}{k} \geq 2^{-2n}$, we have $j_{m_k+1} \in VG(n_0, n)$, so we can apply (9.3) and conclude that

$$\frac{q_k}{k - t_k} \geq \lambda_{n_0} \frac{1 + 2^{n_0-n+\frac{1}{2}}}{2} - c_{n-1}^{2/(n-1)} \geq \frac{\lambda_{n_0}}{2}, \tag{10.9}$$

which implies, using (10.6) again,

$$a_k \geq \frac{t_k}{k} \lambda_{n_0} \frac{1 + 2^{n_0-n-1}}{2} + \frac{k - t_k}{k} \frac{q_k}{k - t_k} \geq \frac{\lambda_{n_0}}{2}. \tag{10.10}$$

Thus $a_k \geq \frac{\lambda_{n_0}}{2}$ for $v_n - 1 \leq k < v_{n+1} - 1$ for all n sufficiently large, which implies that f is Collet-Eckmann.

References

[ALM] A. Avila, M. Lyubich and W. de Melo. Regular or stochastic dynamics in real analytic families of unimodal maps. To appear in Invent. Math.

[AM1] A. Avila and C. G. Moreira. Statistical properties of unimodal maps: the quadratic family. To appear in Annals of Math.

[AM2] A. Avila and C. G. Moreira. Statistical properties of unimodal maps: smooth families with negative Schwarzian derivative. To appear in Astérisque.

[AM3] A. Avila and C. G. Moreira. Bifurcations of unimodal maps. Preprint (www.arXiv.org).

[AM4] A. Avila and C. G. Moreira. Phase-Parameter relation and sharp statistical properties of unimodal maps. Preprint (www.arXiv.org).

[BBM] V. Baladi, M. Benedicks and V. Maume. Almost sure rates of mixing for i.i.d. unimodal maps. Ann. scient. Éc. Norm. Sup.

[BV] V. Baladi and M. Viana. Strong stochastic stability and rate of mixing for unimodal maps. Ann. scient. Éc. Norm. Sup., v. 29 (1996), 483-517.

[KN] G. Keller and T. Nowicki. Spectral theory, zeta functions and the distribution of periodic points for Collet-Eckmann maps. Comm. Math. Phys., 149 (1992), 31-69.

[L1] M. Lyubich. Combinatorics, geometry and attractors of quasi-quadratic maps. Ann. Math, **140** (1994), 347-404.

[L2] M. Lyubich. Dynamics of quadratic polynomials, III. Parapuzzle and SBR measure. Astérisque, v. 261 (2000), 173 - 200.

[L3] M. Lyubich. Almost every real quadratic map is either regular or stochastic. Ann. of Math. (2) 156 (2002), no. 1, 1-78.

[MvS] W. de Melo and S. van Strien. One-dimensional dynamics. Springer, 1993.

[Pa] J. Palis. A global view of dynamics and a conjecture of the denseness of finitude of attractors. Astérisque, v. 261 (2000), 335 - 347.

[Y] L.-S. Young. Decay of correlations for certain quadratic maps. Comm. Math. Phys., 146 (1992), 123-138.

Collège de France, 3 rue d'Ulm, 75005–Paris, France.
E-mail address: avila@impa.br

IMPA – Estr. D. Castorina 110, 22460-320 Rio de Janeiro – Brazil.
E-mail address: gugu@impa.br