# REDUCIBILITY OR NON-UNIFORM HYPERBOLICITY FOR QUASIPERIODIC SCHRÖDINGER COCYCLES

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Abstract. We show that for almost every frequency $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, for every $C^\omega$ potential $v : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$, and for almost every energy $E$ the corresponding quasiperiodic Schrödinger cocycle is either reducible or non-uniformly hyperbolic (similar results are valid in the smooth category). We describe several applications for the quasiperiodic Schrödinger operator, including persistence of absolutely continuous spectrum under perturbations of the potential. Such results also allow us to complete the proof of the Aubry-André conjecture on the measure of the spectrum of the Almost Mathieu Operator.

## 1. Introduction

A *one-dimensional quasiperiodic $C^r$-cocycle in* $\mathrm{SL}(2,\mathbb{R})$ (briefly, a $C^r$-cocycle) is a pair $(\alpha, A)$ where $\alpha \in \mathbb{R}$ and $A \in C^r(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$. A cocycle should be viewed as a *skew-product*:

$$(\alpha, A) : \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2 \qquad (1.1)$$
$$(x, w) \mapsto (x + \alpha, A(x) \cdot w).$$

The Lyapunov exponent of $(\alpha, A)$ is defined as

$$L(\alpha, A) = \lim \frac{1}{n} \int_{\mathbb{R}/\mathbb{Z}} \ln \|A_n(x)\| dx \geq 0, \qquad (1.2)$$

where $A_n(x) = \prod_{j=n-1}^{0} A(x + j\alpha) = A(x + (n-1)\alpha) \cdots A(x)$ (we will keep the dependence on $\alpha$ implicit).

We say that $(\alpha, A)$ is *uniformly hyperbolic* if there exists a continuous splitting $E_s(x) \oplus E_u(x) = \mathbb{R}^2$, and $C > 0$, $\lambda < 1$ such that

$$\|A_n(x) \cdot w\| \leq C\lambda^n \|w\|, \quad w \in E_s(x), \qquad (1.3)$$
$$\|A_n(x - n\alpha)^{-1} \cdot w\| \leq C\lambda^n \|w\|, \quad w \in E_u(x).$$

Such splitting is automatically unique and thus invariant, that is $A(x)E_s(x) = E_s(x + \alpha)$ and $A(x)E_u(x) = E_u(x + \alpha)$. The set of uniformly hyperbolic cocycles is open in the $C^0$-topology (one allows perturbations both in $\alpha$ and in $A$).

Uniformly hyperbolic cocycles have a positive Lyapunov exponent. If $(\alpha, A)$ has positive Lyapunov exponent but is not uniformly hyperbolic then it will be called *non-uniformly hyperbolic*.

We say that a $C^r$-cocycle $(\alpha, A)$ is $C^r$-reducible if there exists $B \in C^r(\mathbb{R}/2\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ and $A_0 \in \mathrm{SL}(2,\mathbb{R})$ such that

$$B(x + \alpha)A(x)B(x)^{-1} = A_0, \quad x \in \mathbb{R}. \qquad (1.4)$$

We say that $(\alpha, A)$ is $C^r$-reducible modulo $\mathbb{Z}$ if one can take $B \in C^r(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$.[1]

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[1] Obviously, reducibility modulo $\mathbb{Z}$ is a stronger notion than plain reducibility, but in some situations one can show that both definitions are equivalent (see Remark 1.6). The advantage of defining reducibility "modulo $2\mathbb{Z}$" is to include some special situations (notably certain uniformly hyperbolic cocycles).

We say that $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ satisfies a Diophantine condition $DC(\kappa, \tau)$, $\kappa > 0$, $\tau > 0$ if

$$|q\alpha - p| > \kappa |q|^{-\tau}, \quad (p,q) \in \mathbb{Z}^2 \setminus \{(0,0)\}. \tag{1.5}$$

We let $DC = \cup_{\kappa>0,\tau>0} DC(\kappa,\tau)$. Notice that $\cup_{\kappa>0} DC(\kappa,\tau)$ has full Lebesgue measure provided $\tau > 1$.

We say that $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ satisfies a recurrent Diophantine condition $RDC(\kappa,\tau)$ if there are infinitely many $n > 0$ such that $G^n(\{\alpha\}) \in DC(\kappa,\tau)$, where $\{\alpha\}$ is the fractionary part of $\alpha$ and $G$ is the Gauss map $G(x) = \{x^{-1}\}$. We let $RDC = \cup_{\kappa>0,\tau>0} RDC(\kappa,\tau)$. Notice that $RDC(\kappa,\tau)$ has full Lebesgue measure as long as $DC(\kappa,\tau)$ has positive Lebesgue measure, and $\mathbb{R} \setminus RDC$ has Hausdorff dimension $1/2$.

Given $v \in C^k(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, let us consider the Schrödinger cocycle

$$S_{v,E}(x) = \begin{pmatrix} v(x) - E & -1 \\ 1 & 0 \end{pmatrix} \in C^k(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R})) \tag{1.6}$$

($v$ is called the potential and $E$ is called the energy).

There is a fairly good comprehension about the dynamics of Schrödinger cocycles in the case of either small or large potentials.

**Proposition 1.1.** *Let $v \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ be a non-constant potential, and let $\alpha \in DC$. Then*

1. *(Sorets-Spencer* [SS]*) There exists $\lambda_0 = \lambda_0(v) > 0$ such that if $\lambda > \lambda_0$, then for every $E \in \mathbb{R}$, the Lyapunov exponent $L(\alpha, S_{\lambda v,E})$ is positive,*
2. *(Eliasson* [E][2]*) There exists $\lambda_0 = \lambda_0(v,\alpha) > 0$ such that if $0 < \lambda < \lambda_0$, then for almost every $E \in \mathbb{R}$, the cocycle $(\alpha, S_{\lambda v,E})$ is $C^\omega$-reducible.*

*Remark* 1.1. Sorets-Spencer's result is *non-perturbative*: the "largeness" condition $\lambda_0$ does not depend on $\alpha \in DC$ (and one can even do away with the DC altogether). On the other hand, the proof of Eliasson's result is *perturbative*: the "smallness" condition $\lambda_0$ depends in principle on $\alpha \in DC$. We will come back to this issue.

*Remark* 1.2. In general, one can not replace "almost every" by "every" in Eliasson's result above. Indeed, in [E] it is also shown that the set of energies for which $(\alpha, S_{\lambda v,E})$ is not (even $C^0$) reducible is non-empty for generic $v$.

*Remark* 1.3. Let $\alpha \in DC$ and $A \in C^r(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, $r = \infty, \omega$. In this case, $(\alpha, A)$ is uniformly hyperbolic if and only if it is $C^r$-reducible and has a positive Lyapunov exponent. Thus, there are lots of "simple cocycles" for which one has positive Lyapunov exponent/reducibility (and indeed both at the same time), this is the case in particular for $|E|$ large in the Schrödinger case. Those examples are also stable (here we fix $\alpha \in DC$ and stability is with respect to perturbations of $A$).

However, cocycles with a positive Lyapunov exponent/reducible but which are *not* uniformly hyperbolic do happen for a *positive measure set of energies* for many choices of the potential, and in particular in the situations described by the results of Sorets-Spencer (this follows from [B], Theorem 12.14) and Eliasson.

Our main result for Schrödinger cocycles aims to close the gap and describe the situation (for almost every energy) without largeness/smallness assumption on the potential:

**Theorem A.** *Let $\alpha \in RDC$ and let $v : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be a $C^r$ potential, $r = \omega, \infty$. Then, for almost every $E$, the cocycle $(\alpha, S_{v,E})$ is either non-uniformly hyperbolic or $C^r$-reducible.*

[2] This result was originally stated for the continuous time case, but the proof also works for the discrete time case.

For $\theta \in \mathbb{R}$, let

$$R_\theta(x) = \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}. \tag{1.7}$$

Given a $C^r$-cocycle $(\alpha, A)$, we associate a canonical one-parameter family of $C^r$-cocycles $\theta \mapsto (\alpha, R_\theta A)$. Our proof of Theorem A goes through for the more general context of cocycles homotopic to the identity, with the role of the energy parameter replaced by the $\theta$ parameter.

**Theorem A'.** *Let $\alpha \in RDC$, and let $A : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ be $C^r$, $r = \omega, \infty$, and homotopic to the identity*[3]*. Then for almost every $\theta \in \mathbb{R}/\mathbb{Z}$, the cocycle $(\alpha, R_\theta A)$ is either non-uniformly hyperbolic or $C^r$-reducible.*

*Remark* 1.4. Those results are still valid in Gevrey classes of differentiability. It is also possible to obtain results with finite (sufficiently large) differentiability: in this case, the reducibility statements involve loss of derivatives.

*Remark* 1.5. Theorems A and A' generalize to the continuous time case with essentially the same proof.

*Remark* 1.6. One can distinguish two distinct behaviors among the reducible cocycles $(\alpha, A)$ given by Theorems A and A'. The first is uniformly hyperbolic behavior, see Remark 1.3. The second is *totally elliptic* behavior, corresponding to an irrational rotation of $\mathbb{R}^2/\mathbb{Z}^2$. More precisely, we call a cocycle totally elliptic if it is $C^r$-reducible and the constant matrix $A_0$ in (1.4) can be chosen to be a rotation $R_\rho$, where $(1, \alpha, \rho)$ are linearly independent over $\mathbb{Q}$. In this case it is easy to see that the cocycle $(\alpha, A)$ is automatically $C^r$-reducible modulo $\mathbb{Z}$ (possibly replacing $\rho$ by $\rho + \frac{\alpha}{2}$). (To see that almost every reducible cocycle is either uniformly hyperbolic or totally elliptic, it is enough to use Theorems 2.3 and 2.4 which are due to Moser.)

Theorems A and A' give a nice global picture for the theory of quasiperiodic cocycles. They fit with the Palis conjecture for general dynamical systems [Pa], and have a strong analogy with the work of Lyubich in the quadratic family [Ly] (and generalizations such as [ALM]).

This picture improves even more since there are several results which describe reducible and non-uniformly hyperbolic systems. From those, we would like to mention two results about the stability of those properties.

Let us say that a cocycle $(\alpha, A)$ is almost $C^r$-reducible if there exists a sequence $B^{(n)} : \mathbb{R}/2\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ and $A_0 \in \mathrm{SL}(2,\mathbb{R})$ such that $B^{(n)}(x+\alpha)A(x)B^{(n)}(x)^{-1}$ (considered as a function $\mathbb{R}/2\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$) converges to $A_0$ in the $C^r$ topology. It is easy to see that an almost (even only $C^0$) reducible, but non-reducible cocycle has zero Lyapunov exponent. Using Theorems A and A', we conclude that typical $C^\infty$ or $C^\omega$ cocycles which are almost (even only $C^0$) reducible are indeed $C^\infty$ or $C^\omega$-reducible.

The following result can be proved using the ideas of [E], see [AK3] for details:

**Proposition 1.2.** *Let $\kappa > 0$, $\tau > 0$. Almost $C^\infty$-reducible cocycles form an open set of $DC(\kappa,\tau) \times C^\infty(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$.*

So, while reducibility is not an open condition, it is "open modulo zero" (when working inside $RDC \cap DC(\kappa,\tau)$ and with $C^\infty$ or $C^\omega$ potentials).

**Proposition 1.3** (Bourgain-Jitomirskaya [BJ1])**.** *The Lyapunov exponent is a continuous function of $(\alpha, A) \in \mathbb{R} \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ at each $\alpha \in \mathbb{R} \setminus \mathbb{Q}$*[4]*. In particular, the set of cocycles with a positive Lyapunov exponent is open in $(\mathbb{R} \setminus \mathbb{Q}) \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$.*

[3]For the case of cocycles non-homotopic to the identity, see [AK1].

[4]This result was stated in [BJ1] for Schrödinger cocycles, but the proof applies in general.

*Remark* 1.7. Here and in what follows, the space $C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ of real analytic functions $\mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ is supplied with the following topology. A sequence $A^{(n)}$ converges to $A$ if there exists a neighborhood $V$ of $\mathbb{R}/Z$ in $\mathbb{C}/\mathbb{Z}$ such that $A$ and all the $A^{(n)}$ admit holomorphic extensions $V \to \mathrm{SL}(2,\mathbb{C})$ and the (the extensions of) $A^{(n)}$ converge to (the extension of) $A$ uniformly on compacts of $V$. A subset of $C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ is defined to be closed if it is sequentially closed. (The topologies of spaces $C^\omega(\mathbb{R}, \mathrm{SL}(2,\mathbb{R}))$, $C^\omega(\mathbb{R},\mathbb{R})$,... are defined analogously.)

In other words, $C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ is the *inductive limit* $\lim_{a\to 0} C^\omega_a(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$, where for $a > 0$, $C^\omega_a(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ denotes the Banach space of functions $\mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ which admit bounded holomorphic extensions $\{z \in \mathbb{C}/\mathbb{Z}, |\Im(z)| < a\} \to \mathrm{SL}(2,\mathbb{C})$.

Unfortunately, the result of Bourgain-Jitomirskaya is not available in the smooth setting.

Notice that almost $C^\infty$-reducible cocycles are contained (as remarked above) and have "full measure" (by Theorem A') in the space of cocycles which are not non-uniformly hyperbolic. It is thus natural to pose the following question regarding the structure of spaces of quasiperiodic cocycles:

*Problem* 1.1. Is the set of almost $C^\infty$-reducible cocycles equal to the complement of the closure of the set of non-uniformly hyperbolic cocycles in $DC \times C^\omega_a$, $a > 0$?

Obviously this problem can be considered in several different spaces of cocycles ($C^\infty$, $C^\omega$), and with several restrictions on the frequency (particularly interesting is to work with the Diophantine frequency $\alpha$ fixed, or varying in $DC(\kappa,\tau)$). This particular formulation was chosen because it would have immediate interesting applications, some of which are mentioned below.

### 1.1. Application to Schrödinger operators.

We now discuss the application of the previous results to the quasiperiodic Schrödinger operator

$$H_{v,\alpha,x}u(n) = u(n+1) + v(x+\alpha n)u(n) + u(n-1), \quad u \in l^2(\mathbb{Z}), \tag{1.8}$$

where $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, $x \in \mathbb{R}$ and $v : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is at least $C^\infty$. It is well known that the properties of $H_{v,\alpha,x}$ are closely connected to the properties of the family of cocycles $(\alpha, S_{v,E})$, $E \in \mathbb{R}$. Notice for instance that if $(u_n)_{n\in\mathbb{Z}}$ is a solution of $H_{v,\alpha,x}u = Eu$ then

$$\begin{pmatrix} v(x+n\alpha) - E & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} u(n) \\ -u(n-1) \end{pmatrix} = \begin{pmatrix} -u(n+1) \\ u(n) \end{pmatrix}. \tag{1.9}$$

In order to explore this connection, we will use the results of Eliasson to control a neighborhood of reducible cocycles (this can be done in the smooth and the real analytic setting), and the results of Bourgain, Goldstein, Jitomirskaya to control a neighborhood of non-uniformly hyperbolic cocycles (in the real analytic setting).

Let $\Sigma$ be the spectrum of $H(v,\alpha,x)$. It is well known that

$$\Sigma = \{E \in \mathbb{R},\ (\alpha, S_{v,E} \text{ is } \textit{not} \text{ uniformly hyperbolic}\}, \tag{1.10}$$

so $\Sigma = \Sigma(v,\alpha)$ does not depend on $x$.

Let $\Sigma_{sc} = \Sigma_{sc}(\alpha, v, x)$ (respectively, $\Sigma_{ac}$, $\Sigma_{pp}$) be the singular continuous (respectively, absolutely continuous, pure point) part of the spectrum of $H(v,\alpha,x)$.

It has been shown by Last-Simon ([LS], Theorem 1.5) that $\Sigma_{ac}$ does not depend on $x$ for $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ (there are no hypothesis on the smoothness of $v$ beyond continuity). It is known that $\Sigma_{sc}$ and $\Sigma_{pp}$ do depend on $x$ in general.

We will also introduce some decompositions of $\Sigma$ that only depend on the cocycle, and hence are independent of $x$.

We split $\Sigma = \Sigma_0 \cup \Sigma_+$ in the parts corresponding to zero Lyapunov exponent and positive Lyapunov exponent for the cocycle $(\alpha, S_{v,E})$. By Proposition 1.3, $\Sigma_0$ is closed in the real analytic case.

Let $\Sigma_r$ be the set of $E \in \Sigma$ such that $(\alpha, S_{v,E})$ is $C^\infty$-reducible. Let $\Sigma_{ar}$ be the set of $E \in \Sigma$ such that $(\alpha, S_{v,E})$ is almost $C^\infty$-reducible. Using Proposition 1.2, we see that $\Sigma \setminus \Sigma_{ar}$ is closed. It is easy to see that $\Sigma_{ar} \subset \Sigma_0$.

Notice that by the Ishii-Pastur Theorem, we have $\Sigma_{ac} \subset \overline{\Sigma}_0' \subset \overline{\Sigma}_0$, where $\Sigma_0'$ is the set of density points of $\Sigma_0$ (more precisely, $\Sigma_0$ is an "essential support" for the absolutely continuous spectrum).

By Theorem A, $\Sigma_0 \setminus \Sigma_r$ has zero Lebesgue measure if $\alpha \in RDC$ and $v \in C^\infty$. One way to interpret $|\Sigma_0 \setminus \Sigma_r| = 0$ (using the Ishii-Pastur Theorem) is that generalized eigenfunctions in the essential support of the absolutely continuous spectrum are (very regular) Bloch waves. This already gives (in the particular cases under consideration) strong versions of some conjectures in the literature (see for instance the discussion after Theorem 7.1 in [DeS]). (Analogous statements hold in the continuous time case.) In what follows we will discuss some deeper consequences of Theorem A.

1.1.1. *Almost Mathieu.* Certainly the most studied family of potentials in the literature is $v(\theta) = \lambda \cos \theta$, $\lambda > 0$. In this case, $H_{v,\alpha,\theta}$ is called the Almost Mathieu Operator. Before discussing applications for general potentials, it is worth to exemplify some consequences in this context.

The Aubry-André conjecture on the measure of the spectrum of the Almost Mathieu Operator states that the measure of the spectrum of $H_{\lambda \cos \theta,\alpha,x}$ is $|4 - 2\lambda|$ for every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $x \in \mathbb{R}$. This has been already proved for every $\lambda \neq 2$ by [JK], and for every $\alpha$ not of constant type[5] [L]. Using the previous result, we can deal with the last case (which is Problem 5 of [Si]).

**Theorem 1.4.** *The spectrum of $H(\lambda \cos \theta, \alpha, x)$ has Lebesgue measure $|4 - 2\lambda|$ for every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.*

*Proof.* As stated above, it is enough to consider $\lambda = 2$ and $\alpha$ of constant type, in particular $\alpha \in RDC$. Let $\Sigma$ be the spectrum of $H(2 \cos \theta, \alpha, x)$. By Corollary 2 of [BJ1], $\Sigma_+ = \emptyset$. By Theorem A, for almost every $E \in \Sigma_0$, $(\alpha, S_{2\cos\theta,E})$ is $C^\omega$-reducible. Thus, it is enough to show that $(\alpha, S_{2\cos\theta,E})$ is not $C^\omega$-reducible for every $E \in \Sigma$.

Assume this is not the case, that is, $(\alpha, S_{2\cos\theta,E})$ is reducible for some $E \in \Sigma$. To reach a contradiction, we will approximate the potential $2 \cos \theta$ by $\lambda \cos \theta$ with $\lambda > 2$ close to 2. Then, by [E], if $(\lambda, E')$ is sufficiently close to $(2, E)$, either $(\alpha, S_{\lambda\cos\theta,E'})$ is uniformly hyperbolic or $L(\alpha, S_{\lambda\cos\theta,E'}) = 0$. In particular (since the spectrum depends continuously on the potential), there exists $E' \in \mathbb{R}$ such that $L(\alpha, S_{\lambda\cos\theta,E'}) = 0$. But it is well known, see [H], that the Lyapunov exponent of $S_{\lambda\cos\theta,E'}$ is bounded from below by $\max\{\ln \frac{\lambda}{2}, 0\} > 0$ and the result follows[6]. □

*Remark* 1.8. Barry Simon has pointed out to us an alternative argument based on duality that shows that if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and if $E \in \Sigma = \Sigma(2\cos\theta, \alpha)$ then the cocycle $(\alpha, S_{2\cos\theta,E})$ is not $C^\omega$-reducible. Indeed, if $(\alpha, S_{v,E})$ is $C^\omega$-reducible and $E \in \Sigma$, then (by duality) there exists $x \in \mathbb{R}$ such that $E$ is an eigenvalue for $H_{2\cos\theta,\alpha,x}$, and the corresponding eigenvector decays exponentially, hence $L(\alpha, S_{v,E}) > 0$ which gives a contradiction. (This argument actually can be used to show that $(\alpha, S_{v,E})$ is not $C^1$-reducible.)

*Remark* 1.9. Most results about the measure of the spectrum of the Almost Mathieu Operator were based on periodic approximations. On the other hand, our result is based on approximation of the potential or approximation by Diophantine frequencies. This approach also yields new proofs of other

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[5]A number $\alpha \in \mathbb{R}$ is said to be of *constant type* if the coefficients of its continued fraction expansion are bounded. It follows that $\alpha$ is of constant type if and only if $\alpha \in \cup_{\kappa>0} DC(\kappa, 1)$ if and only if $\alpha \in \cup_{\kappa>0} RDC(\kappa, 1)$.

[6]Alternatively, we could have used [E] to conclude that there exists some absolutely continuous spectrum for $H(\lambda \cos \theta, \alpha, x)$, which contradicts positivity of the Lyapunov exponent (Ishii-Pastur). We could also argue by keeping the potential constant and varying $\alpha$. Indeed, any $\alpha$ of constant type can be approximated by $\alpha' \in DC(\kappa, \tau)$ of non-constant type (with $\kappa$, $\tau$ fixed). For such $\alpha'$, absence of absolutely continuous spectrum follows from the zero measure of the spectrum proved in [L], and this gives a contradiction with [E] as before.

cases of Theorem 1.4. For instance, $\lambda \neq 2$ and $\alpha$ of constant type (Theorem 2 of [JK]) can be concluded in this way[7].

By [GJLS], we get:

**Corollary 1.5.** *The spectrum of* $H(2\cos\theta, \alpha, x)$ *is purely singular continuous for every* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, *and for almost every* $x \in \mathbb{R}/\mathbb{Z}$.

For $\alpha \in RDC$ we can be even more precise:

**Theorem 1.6.** *The spectrum of* $H(2\cos\theta, \alpha, x)$ *is purely singular continuous for every* $\alpha \in RDC$, *and for all* $x \in \mathbb{R}/\mathbb{Z}$.

This is a simple application of duality (in its simples form) together with the methods of this paper, and one does not need to use the deeper spectral arguments of [GJLS]. Let us sketch the proof.

*Proof (sketch).* The existence of some absolutely continuous spectrum implies that $|\Sigma| > 0$ and contradicts Theorem 1.4, so one only has to rule out existence of point spectrum. But the existence of an $l^2$ eigenvector for $H(2\cos\theta, \alpha, x)$ associated to some energy $E \in \Sigma$ implies (by duality) that the cocycle $(\alpha, S_{2\cos\theta,E})$ is $L^2$-conjugated to a constant. The results of this paper (see Theorem 5.3 and Remark 5.1) then imply that for $\lambda > 2$ close to 2 and $E'$ close to $E$, the cocycle $(\alpha, S_{\lambda\cos\theta,E'})$ is either uniformly hyperbolic or has zero Lyapunov exponent. As in Theorem 1.4, this implies that $L(\alpha, S_{\lambda\cos\theta,E'}) = 0$ for some $E' \in \mathbb{R}$ contradicting the positivity of the Lyapunov exponent.[8] □

We can also state a result for $\lambda < 2$.

**Theorem 1.7.** *Let* $\lambda < 2$, $\alpha \in RDC$. *For almost every* $E \in \mathbb{R}$, $(\alpha, S_{\lambda\cos\theta,E})$ *is reducible.*

*Proof.* By Corollary 2 of [BJ1], the Lyapunov exponent is zero on the spectrum (that $|\Sigma_+| = 0$ also follows from the work of Jitomirskaya [J] and is enough for our purposes). The result is now a consequence of Theorem A. □

*Remark* 1.10. Let $\lambda > 2$, $\alpha \in RDC$, and let $\Sigma = \Sigma(\lambda\cos\theta, \alpha)$. The preceding theorem yields, by duality, that for almost every $E \in \Sigma$, there exists some $x \in \mathbb{R}$ for which $E$ is an eigenvalue of $H(\lambda\cos\theta, \alpha, x)$ and the corresponding eigenfunction decays exponentially.

## 1.2. More on general potentials.

We now go back to the setting of general potentials and describe some interesting properties of the decompositions we introduced. The following result is related to Proposition 1.2 and can be proved using ideas of Eliasson [E], see [AK3] for details.

**Proposition 1.8.** *Let* $\alpha \in DC$, $v \in C^\infty$. *Then the spectrum of* $H_{v,\alpha,x}$ *is purely absolutely continuous in* $\Sigma_{ar}$, *for every* $x$. *In particular,* $\Sigma_{ar}$ *is either empty or has positive Lebesgue measure.*

Since $\Sigma_0 \setminus \Sigma_{ar}$ has zero Lebesgue measure, we conclude:

**Theorem 1.9.** *Let* $\alpha \in RDC$, $v \in C^\infty$. *Then for every* $x \in \mathbb{R}$, $\Sigma_0 \cap \Sigma_{sc}$ *has zero Lebesgue measure.*

In order to give a more complete picture, we will use the following result:

**Proposition 1.10** (Bourgain-Goldstein [BG]). *Let* $v \in C^\omega$ *and* $x \in \mathbb{R}$. *For almost every* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, *the spectrum of* $H$ *is pure point in* $\Sigma_+$ *with exponentially decaying eigenfunctions.*

Thus we conclude:

**Theorem 1.11.** *Let* $v \in C^\omega$, $x \in \mathbb{R}$. *For almost every* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\Sigma_{sc}$ *has zero Lebesgue measure.*

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[7] For $|\lambda| < 2$ one uses Proposition 1.14 below (which is KAM theoretical) and [L], and the case of $|\lambda| > 2$ follows by duality. Notice that this is the opposite of the approach of [JK], which works with $|\lambda| > 2$ and then applies duality.

[8] We can also use Remark 5.2 to conclude that $(\alpha, S_{2\cos\theta,E})$ is almost reducible, which by Proposition 1.8 implies the existence of some absolutely continuous spectrum and contradicts Theorem 1.4.

We may now summarize the previous results in a nice topological/measure-theoretical description of the spectrum.

**Corollary 1.12** (Separation of the spectrum). *Let $v \in C^\omega$, $x \in \mathbb{R}$. The partition $\Sigma = \Sigma_{ar} \cup \Sigma_+ \cup (\Sigma_0 \setminus \Sigma_{ar})$ has the following properties:*
*Topological*

(T1) *If $\alpha \in DC$, $\Sigma_{ar}$ is open and $\Sigma_0 \setminus \Sigma_{ar}$ is closed,*
(T2) *If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\Sigma_+$ is open.*

*Measure*

(M1) *If $\alpha \in DC$, $\Sigma_{ar}$ is either empty or has positive Lebesgue measure*[9],
(M2) *If $\alpha \in RDC$, $\Sigma_0 \setminus \Sigma_{ar}$ has zero Lebesgue measure.*

*Spectral*

(S1) *If $\alpha \in RDC$, $\Sigma_{ar}$ supports the whole absolutely continuous spectrum and no other,*
(S2) *For almost every $\alpha$, $\Sigma_+$ supports only point spectrum.*

(The contributions of this work are items (M2) and (S1), the others have been stated for completeness.)

One can interpret (S1) (together with (T1)) as saying that the absolutely continuous spectrum is topologically "separated" from the other types of spectrum. It would be interesting to know if $\Sigma_+$ supports the whole point spectrum (for every $x$ and almost every $\alpha$).

1.2.1. *Continuity properties.* Some continuity properties of $\Sigma_{ar}$ and $\Sigma_+$ can be easily obtained using Propositions 1.2 and 1.3: the map $(v, \alpha) \mapsto \Sigma_{ar}$ (respectively $(v, \alpha) \mapsto \Sigma_+$) is lower-semicontinuous in the Hausdorff sense, with respect to $(v, \alpha) \in C^\infty \times DC(\kappa, \tau)$ (respectively $C^\omega \times \mathbb{R} \setminus \mathbb{Q}$). Since $\Sigma_{ac} = \overline{\Sigma}_{ar}$ for $\alpha \in RDC$, $v \in C^\infty$ (by Ishii-Pastur and Theorem A), we conclude:

**Corollary 1.13.** *The map*

$$(v, \alpha) \mapsto \Sigma_{ac} \tag{1.11}$$

*is lower semicontinuous in $C^\infty \times DC(\kappa, \tau)$ at any $\alpha \in RDC$, in the Hausdorff sense.*

Using standard KAM techniques, one easily gets results about continuity of the measure of the reducible part of the spectrum (see [AK2] for details):

**Proposition 1.14.** *Let $\kappa > 0$, $\tau > 0$. The map*

$$(v, \alpha) \mapsto |\Sigma_r| \tag{1.12}$$

*is lower semi-continuous in $C^\infty \times DC(\kappa, \tau)$ at any $\alpha \in RDC$.*

In [AK2] it is also shown how the work of [JK] and Theorem 12.22 of [B] can be used to establish an analogous result regarding the measure of $\Sigma_+$ (for real-analytic potentials):

**Proposition 1.15.** *Let $\kappa > 0$, $\tau > 0$. The map*

$$(v, \alpha) \mapsto |\Sigma_+| \tag{1.13}$$

*is lower semi-continuous in $C^\omega \times DC(\kappa, \tau)$ at any $\alpha \in RDC$.*

Together with Theorem A, those propositions yield (using the simple fact that $|\Sigma|$ is upper semi-continuous):

[9]We believe the argument of Proposition 12.14 of [B] can be adapted to show that this is also the case for $\Sigma_+$.

**Theorem 1.16.** *Let $\kappa > 0$, $\tau > 0$. The map*

$$(v,\alpha) \mapsto |\Sigma| \tag{1.14}$$

*is continuous in $C^\omega \times DC(\kappa,\tau)$ at any $\alpha \in RDC$. The same conclusion holds for $|\Sigma_0|$, $|\Sigma_{ac}|$, $|\Sigma_{ar}|$, $|\Sigma_r|$, and $|\Sigma_+|$.*

1.3. **A non-perturbative version of Eliasson's Theorem.** The following result was proved in [BJ2] using non-perturbative methods:

**Proposition 1.17** (Bourgain-Jitomirskaya)**.** *Let $\alpha \in DC$, $v \in C^\omega$. There exists $\lambda_0 > 0$ (only depending on the bounds of $v$) such that if $0 < \lambda < \lambda_0$, then the spectrum of $H(\lambda v, \alpha, E, x)$ is purely absolutely continuous for almost every $x$*[10].

We can now show that Eliasson's result stated in Proposition 1.1 is indeed a non-perturbative one:

**Theorem 1.18.** *Let $\alpha \in RDC$, $v \in C^\omega$. There exists $\lambda_0 > 0$ (only depending on the bounds of $v$, may be taken the same as in the previous proposition) such that if $0 < \lambda < \lambda_0$, then $(\alpha, S_{\lambda v,E})$ is reducible for almost every $E$.*

*Proof.* By the previous proposition, $\Sigma_{ac} = \Sigma$, so $\Sigma_+ = \emptyset$. □

*Remark* 1.11. Theorem 1.18 also holds in the continuous time case (by reduction to the discrete time case), see [AK3]. Notice that we do not know if the analogous of Proposition 1.17 holds in the continuous time case. One can show that this would follow from a positive answer to Problem 1.1.

1.4. **Integrated density of states, singular continuous spectrum, and an open question.** An immediate consequence of Theorem 1.9 is the following relation between the integrated density of states (i.d.s.) (see [AS] for the definition), the Lyapunov exponent and absolutely continuous spectrum.

**Theorem 1.19.** *Let $\alpha \in RDC$, $v \in C^\omega$. The following are equivalent*

(1) *The i.d.s. is absolutely continuous in $\Sigma_0$,*
(2) *$\Sigma_{sc} \cap \Sigma_0 = \emptyset$ for almost every $x \in \mathbb{R}$.*

*Remark* 1.12. We do not know whether the equivalent statements in the previous theorem also equivalent to

(2') $\Sigma_{sc} \cap \Sigma_0 = \emptyset$ for every $x \in \mathbb{R}$.

We know that the i.d.s. is absolutely continuous in "most" of $\Sigma_0$ (namely, in $\Sigma_{ar}$), and in the case of the Almost Mathieu Operator ($v = \lambda \cos\theta$), the i.d.s. is not absolutely continuous in $\Sigma_0$ only for $\lambda = 2$ (assuming $\alpha$ RDC).

*Problem* 1.2. Is it true that for "typical"[11] potentials $v \in C^\omega$, the i.d.s. is absolutely continuous in $\Sigma_0$ (for almost every $\alpha$)?

(The same question can be posed for potentials given by trigonometric polynomials. In this case, typical means "in a full Lebesgue measure set on the space of trigonometric polynomials of degree $d$, for every $d$".)

We believe this question is of key importance for the following reason: if the answer is yes, then for typical potentials and for almost every $\alpha, x \in \mathbb{R}$, we would have a very clear spectral picture: no singular continuous spectrum and all the point spectrum corresponding to exponentially decaying

[10] Notice that a positive answer to Problem 1.1 would establish this for all $x$.

[11] This should be understood (generalizing the Almost Mathieu case) as a full measure set of parameters in non-trivial analytic families of potentials. Although it is not clear what "non-trivial" should mean in this context, we hope that some definition can be found which would bear analogy to the notion considered in [ALM] for unimodal maps.

eigenfunctions (all the absolutely continuous spectrum corresponds to very regular Bloch waves as discussed before). One could also hope for $\Sigma_+$ to be closed (this implies, for real analytic potentials, that the Lyapunov exponent restricted to $\Sigma_+$ is bounded from below and would give some uniformity properties for the point spectrum), but this seems more likely to be an open and dense property rather than "full measure".

*Remark* 1.13. It would be natural to also ask about absolute continuity of the i.d.s. in $\Sigma_+$ (although we are not aware of any consequence of significance): this is the case at least for the Almost Mathieu operator by duality. However, pushing the analogy with the theory of unimodal maps, one might expect the opposite, that the i.d.s. is not absolutely continuous in $\Sigma_+$ for typical potentials.[12]

1.5. **Outline of the proof of Theorem A.** The proof has some distinct steps, and is based on a renormalization scheme. This point of view, which has already been used in the study of reducibility properties of quasiperiodic cocycles with values in $SU(2)$ and $SL(2,\mathbb{R})$, has proved to be very useful in the non-perturbative case (see [K1], [K2]). However, the scheme we present in this paper is somehow simpler and fits better (at least in the $SL(2,\mathbb{R})$ case) with the general renormalization philosophy (see [S] for a very nice description of this point of view on renormalization):

(1) The starting point is the theory of Kotani[13]. For almost every energy $E$, if the Lyapunov exponent of $(\alpha, S_{v,E})$ is zero, then the cocycle is $L^2$-conjugate to a cocycle in $\mathrm{SO}(2,\mathbb{R})$. Moreover, the *fibered rotation number* of the cocycle (which is closely related to the i.d.s.) is Diophantine with respect to $\alpha$. (The set $\Delta$ of those energies will be precisely the set of energies for which we will be able to conclude reducibility.)
(2) We now consider a smooth cocycle $(\alpha, A)$ which is $L^2$-conjugate to rotations. An explicit estimate allows us to control the derivatives of iterates of the cocycle restricted to certain small intervals.
(3) After introducing the notion of renormalization of cocycles, we interpret item (2) as "a priori bounds" (or precompactness) for a sequence of renormalizations $(\alpha_{n_k}, A^{(n_k)})$.
(4) The recurrent Diophantine condition for $\alpha$ allows us to take $\alpha_{n_k}$ uniformly Diophantine, so the limits of renormalization are cocycles $(\hat{\alpha}, \hat{A})$ where $\hat{\alpha}$ satisfies a Diophantine condition. Those limits are essentially (that is, modulo a constant) conjugate to cocycles in $\mathrm{SO}(2,\mathbb{R})$, and are trivial to analyze: they are always reducible.
(5) Since $\lim(\alpha_{n_k}, A^{(n_k)})$ is reducible, Eliasson's Theorem [E][14] allows us to conclude that some renormalization $(\alpha_{n_k}, A^{(n_k)})$ must be reducible, provided the fibered rotation number of $(\alpha_{n_k}, A^{(n_k)})$ is Diophantine with respect to $\alpha_{n_k}$.
(6) This last condition is actually equivalent to the fibered rotation number of $(\alpha, A)$ being Diophantine with respect to $\alpha$. It is easy to see that reducibility is invariant under renormalization, so $(\alpha, A)$ is itself reducible.

We conclude that for almost every $E \in \mathbb{R}$ such that $L(\alpha, S_{v,E}) = 0$, the cocycle $(\alpha, S_{v,E})$ is reducible, which is equivalent to Theorem A by Remark 1.3.

The above strategy uses $\alpha \in RDC$ in order to take good limits of renormalization. It would be interesting to try to obtain results under the weaker condition $\alpha \in DC$ by working directly with deep renormalizations (without considering limits).

[12]The i.d.s. can be seen as a holonomy map of some combinatorially defined codimension-one lamination in the space of cocycles (determined by the fibered rotation number). The analogous lamination in the unimodal situation [ALM] is not absolutely continuous in the non-uniformly hyperbolic regime [AM].

[13]This step holds in much bigger generality, namely for cocycles over ergodic transformation.

[14]It is probably possible to use simpler KAM schemes, such as the work of Dinaburg-Sinai [DiS], at this stage.

## 2. $L^2$-ESTIMATES

We say that $(\alpha, A)$ is $L^2$-conjugated to a cocycle of rotations if there exists $B : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ such that $\|B\| \in L^2$ and

$$B(x+\alpha)A(x)B(x)^{-1} \in SO(2,\mathbb{R}). \tag{2.1}$$

**Theorem 2.1.** *Let $v : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be continuous. Then for almost every $E$, either $L(\alpha, S_{v,E}) > 0$ or $S_{v,E}$ is $L^2$-conjugated to a cocycle of rotations.*

This is (an immediate corollary of) a result due to Kotani and Deift-Simon (see Theorem 7.1 of [DeS])[15], and is based on beautiful explicit computations.

It turns out that this result generalizes to the setting of Theorem A'.

**Theorem 2.2.** *Let $A : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ be continuous. Then for almost every $\theta \in \mathbb{R}$, either $L(\alpha, R_\theta A) > 0$ or $A$ is $L^2$-conjugated to a cocycle of rotations.*

The proof of this generalization is essentially the same as in the Schrödinger case.

*Remark* 2.1. Both theorems above are valid in a much more general setting, namely for cocycles over transformations preserving a probability measure. The requirement on the cocycle is the minimal to speak of Lyapunov exponents (and Oseledets theory), namely integrability of the logarithm of the norm.

### 2.1. Fibered rotation number.

Besides the Lyapunov exponent, there is one important invariant associated to continuous cocycles which are homotopic to the identity. This invariant, called the *fibered rotation number* will be denoted by $\rho(\alpha, A) \in \mathbb{R}/\mathbb{Z}$, and was introduced in [H], [JM]. The fibered rotation number is a continuous function of $(\alpha, A)$, where $(\alpha, A)$ varies in the space of continuous cocycles which are homotopic to the identity. Another important elementary fact is that both $E \mapsto \rho(\alpha, S_{v,E})$ and $\theta \mapsto \rho(\alpha, R_\theta A)$ have non-decreasing lifts $\mathbb{R} \to \mathbb{R}$, and in particular, those functions have non-negative derivatives almost everywhere. The following result was proved in [M] (see also [DeS] for an optimal estimate).

**Theorem 2.3.** *Let $v \in C^0(\mathbb{R}/\mathbb{Z}, \mathbb{R})$. Then for almost every $E$ such that $L(\alpha, S_{v,E}) = 0$, we have $\frac{d}{dE}\rho(\alpha, S_{v,E}) > 0$.*

This result (and proof) also generalize to the setting of Theorem A':

**Theorem 2.4.** *Let $A \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ be continuous and homotopic to the identity. Then for almost every $E$ such that $L(\alpha, R_\theta A) = 0$, we have $\frac{d}{d\theta}\rho(\alpha, R_\theta A) > 0$.*

*Remark* 2.2. In the Schrödinger case, it is possible to show that the fibered rotation number is a surjective function (of $E$) onto $[0, 1/2]$. In [AS] it is also shown that $N(E) = 2\rho(\alpha, S_{v,E})$ can be interpreted as the integrated density of states.

The arithmetic properties of the fibered rotation number are also important for the analysis of cocycles $(\alpha, A)$. Fix $\alpha \in \mathbb{R}$. Let us say that $\beta \in \mathbb{R}/\mathbb{Z}$ is *$(\kappa, \tau)$-Diophantine with respect to* $\alpha$ if there exists $\kappa > 0$ such that

$$\forall (k,l) \in \mathbb{Z}^2 - \{(0,0)\}, \ |2\beta - k\alpha - l| \geq \frac{\kappa}{(|k|+|l|)^\tau}. \tag{2.2}$$

Notice that for every $\alpha \in DC$, the set of $\beta \in \mathbb{R}$ which are Diophantine with respect to $\alpha$ has full Lebesgue measure. By Lemmas 2.3 and 2.4 we conclude:

**Corollary 2.5.** *Let $\alpha \in DC$, $v \in C^0(\mathbb{R}/\mathbb{Z}, \mathbb{R})$. Then for almost every $E \in \mathbb{R}$ such that $L(\alpha, S_{v,E}) = 0$, we have that $\rho(\alpha, S_{v,E})$ is Diophantine with respect to $\alpha$.*

[15] This was pointed out to us by Hakan Eliasson.

**Corollary 2.6.** *Let $\alpha \in DC$, $A \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$. Then for almost every $\theta \in \mathbb{R}$ such that $L(\alpha, R_\theta A) = 0$, we have that $\rho(\alpha, R_\theta A)$ is Diophantine with respect to $\alpha$.*

## 3. Estimates for derivatives

In this section, we will assume that $(\alpha, A)$ is $L^2$-conjugated to a cocycle of rotations: there exist measurable $B : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ and $R : \mathbb{R}/\mathbb{Z} \to \mathrm{SO}(2,\mathbb{R})$ such that

$$\forall x \in \mathbb{R}/\mathbb{Z} \quad A(x) = B(x+\alpha)R(x)B(x)^{-1} \quad \text{and} \quad \int_{\mathbb{R}/\mathbb{Z}} \phi(x)dx < \infty \tag{3.1}$$

where we set $\phi(x) = \|B(x)\|^2 = \|B(x)^{-1}\|^2$.

We introduce the *maximal function* $S(\cdot)$ of $\phi$:

$$S(x) = \sup_{n \geq 0} \frac{1}{n} \sum_{k=0}^{n-1} \phi(x + k\alpha). \tag{3.2}$$

Since the dynamics of $x \mapsto x + \alpha$ is ergodic on $\mathbb{R}/\mathbb{Z}$ endowed with Haar measure, the Maximal Ergodic Theorem gives us the weak-type inequality

$$\forall M \geq 0, \quad \mathrm{Haar}(\{x \in \mathbb{R}/\mathbb{Z}, S(x) > M\}) \leq \frac{1}{M} \int_{\mathbb{R}/\mathbb{Z}} \phi(x)dx, \tag{3.3}$$

and for a.e $x_0 \in \mathbb{R}/\mathbb{Z}$ the quantity $S(x_0)$ is finite.

If $X \in \mathrm{GL}(2,\mathbb{R})$, we let $\mathrm{Ad}(X)$ be the linear operator in the space of real $2 \times 2$ matrices which is given by $\mathrm{Ad}(X) \cdot Y = X \cdot Y \cdot X^{-1}$. Notice that the operator norm of $\mathrm{Ad}(X)$ satisfies the bound $\|\mathrm{Ad}(X)\| \leq \|X\| \cdot \|X^{-1}\|$.

**Lemma 3.1.** *Assume that $A$ is Lipschitz (with constant $\mathrm{Lip}(A)$). Then for almost every $x_0 \in \mathbb{R}/\mathbb{Z}$, for every $x \in \mathbb{R}/\mathbb{Z}$, we have*

$$\|A_n(x)\| \leq e^{n|x-x_0|\|A\|_{C^0}\mathrm{Lip}(A)S(x_0)\phi(x_0)} \Big(\phi(x_0)\phi(x_0+n\alpha)\Big)^{1/2}. \tag{3.4}$$

*Proof.* We compute $I_n(x_0, x) := A_n(x_0)^{-1}(A_n(x) - A_n(x_0))$:

$$\begin{aligned} I_n(x_0,x) &= A_n(x_0)^{-1}\Big(\prod_{k=n-1}^{0}\big(A(x_0+k\alpha) + (A(x+k\alpha) - A(x_0+k\alpha))\big) - A_n(x_0)\Big) \\ &= \sum_{r=1}^{n} \sum_{0 \leq i_1 < \ldots < i_r \leq n-1} \prod_{j=1}^{r} (\mathrm{Ad}(A_{i_j}(x_0)^{-1}) \cdot H_{i_j}(x_0,x)) \end{aligned} \tag{3.5}$$

where we have set

$$H_i(x_0,x) = A(x_0+i\alpha)^{-1} \cdot (A(x+i\alpha) - A(x_0+i\alpha)), \tag{3.6}$$

so that

$$\|H_i(x_0,x)\| \leq \|A\|_{C^0}\mathrm{Lip}(A)|x - x_0|. \tag{3.7}$$

The assumptions we made give

$$\|A_i(x_0)\| = \|A_i(x_0)^{-1}\| \leq \|B(x_0+i\alpha)^{-1}\| \cdot \|B(x_0)\|, \tag{3.8}$$

that is

$$\|\mathrm{Ad}(A_i(x_0)^{-1})\| \leq (\|B(x_0+i\alpha)^{-1}\| \cdot \|B(x_0)\|)^2 = \phi(x_0)\phi(x_0+i\alpha). \tag{3.9}$$

Thus we have

$$\|I_n(x_0,x)\| \leq \sum_{r=1}^{n} \sum_{0\leq i_1<\ldots<i_r\leq n-1} \prod_{j=1}^{r} \Big( \|A\|_{C^0}\mathrm{Lip}(A)|x-x_0|\phi(x_0)\phi(x_0+i_j\alpha) \Big) \tag{3.10}$$
$$= -1 + \prod_{k=0}^{n-1} \Big( 1 + \|A\|_{C^0}\mathrm{Lip}(A)|x-x_0|\phi(x_0)\phi(x_0+k\alpha) \Big)$$
$$\leq -1 + \exp\Big( \sum_{k=0}^{n-1} \|A\|_{C^0}\mathrm{Lip}(A)|x-x_0|\phi(x_0)\phi(x_0+k\alpha) \Big),$$

hence for every $x \in \mathbb{R}/\mathbb{Z}$,

$$\|A_n(x_0)^{-1}(A_n(x)-A_n(x_0))\| \leq e^{n|x-x_0|\|A\|_{C^0}\mathrm{Lip}(A)\phi(x_0)S(x_0)} - 1, \tag{3.11}$$

which implies

$$\|A_n(x)\| \leq e^{n|x-x_0|\|A\|_{C^0}\mathrm{Lip}(A)\phi(x_0)S(x_0)} \|A_n(x_0)\| \tag{3.12}$$
$$\leq e^{n|x-x_0|\|A\|_{C^0}\mathrm{Lip}(A)\phi(x_0)S(x_0)} \Big( \phi(x_0)\phi(x_0+n\alpha) \Big)^{1/2},$$

which is the conclusion of the lemma. □

We now give estimates for the derivatives.

**Lemma 3.2.** *Assume that $A : \mathbb{R}/\mathbb{Z} \to SL(2,\mathbb{R})$ is of class $C^k$ ($1 \leq k \leq \infty$). Then for every $0 \leq r \leq k$ and any $x \in \mathbb{R}/\mathbb{Z}$*

$$\|(\partial^r A_n)(x)\| \leq C^r n^r \phi(x_0+n\alpha)^{1/2} \Big( c_1(x_0) e^{nc_2(x_0)|x-x_0|} \|\partial^r A\|_{C^0} \Big)^{r+\frac{1}{2}} \|\partial^r A\|_{C^0} \tag{3.13}$$

*where $C$ is an absolute constant and*

$$c_1(x_0) = \phi(x_0)S(x_0)\|A\|_{C^0}^2, \tag{3.14}$$
$$c_2(x_0) = 2S(x_0)\phi(x_0)\|A\|_{C^0}\|\partial A\|_{C^0}.$$

*Proof.* We compute

$$\partial^r A_n(x) = \partial^r \left( \prod_{k=n-1}^{0} A(\cdot + k\alpha) \right)(x) \tag{3.15}$$

which by Leibniz formula is a sum of $n^r$ terms of the form ($s \leq r$)

$$I_{(i^*)}(x) = \left( \prod_{l=n-1}^{i_1+1} A(x+l\alpha) \right) \cdot \partial^{m_1} A(x+i_1\alpha) \cdot \left( \prod_{l=i_1-1}^{i_2+1} A(x+l\alpha) \right) \cdot \tag{3.16}$$
$$\partial^{m_2} A(x+i_2\alpha) \cdot \left( \prod_{l=i_2-1}^{i_3+1} A(x+l\alpha) \right) \cdots$$
$$\partial^{m_s} A(x+i_s\alpha) \cdot \left( \prod_{l=i_s-1}^{0} A(x+l\alpha) \right)$$

where $i^*$ runs through $\mathcal{I} = \{0, \dots, n-1\}^{\{1,\dots,r\}}$ and where $\{i_1, \dots, i_s\} = i^*(\{0, \dots, n-1\})$ satisfy $n-1 \geq i_1 > i_2 > \cdots i_s \geq 0$ and $m_l = \#(i^*)^{-1}(i_l)$ (notice that $m_1 + \ldots + m_s = r$). Each term $I_{(i^*)}$ can be written

(3.17)

$$I_{(i^*)}(x) = \left(\prod_{l=n-1}^{0} A(x+l\alpha)\right) \cdot \mathrm{Ad}\left(\left(\prod_{l=i_1-1}^{0} A(x+l\alpha)\right)^{-1}\right) \cdot \left(A(x+i_1\alpha)^{-1}\partial^{m_1}A(x+i_1\alpha)\right) \cdot$$
$$\mathrm{Ad}\left(\left(\prod_{l=i_2-1}^{0} A(x+l\alpha)\right)^{-1}\right) \cdot \left(A(x+i_2\alpha)^{-1}\partial^{m_2}A(x+i_2\alpha)\right) \cdots$$
$$\mathrm{Ad}\left(\left(\prod_{l=i_s-1}^{0} A(x+l\alpha)\right)^{-1}\right) \cdot \left(A(x+i_s\alpha)^{-1}\partial^{m_s}A(x+i_s\alpha)\right).$$

From the previous lemma,

$$\left\| \prod_{l=i_p-1}^{0} A(x+l\alpha) \right\| \leq \left(K\phi(x_0)\phi(x_0+i_p\alpha)\right)^{1/2}, \tag{3.18}$$

$$\left\| \mathrm{Ad}\left(\prod_{l=i_p-1}^{0} A(x+l\alpha)\right) \right\| \leq \left\| \prod_{l=i_p-1}^{0} A(x+l\alpha) \right\|^2 \leq K\phi(x_0)\phi(x_0+i_p\alpha) \tag{3.19}$$

where

$$K = e^{2n|x-x_0|\phi(x_0)S(x_0)\|A\|_{C^0}\|\partial A\|_{C^0}}, \tag{3.20}$$

and hence we get the following bound

$$\|I_{(i^*)}(x)\| \leq \left(K\phi(x_0)\phi(x_0+n\alpha)\right)^{1/2} \prod_{p=1}^{s} \left(K\phi(x_0)\phi(x_0+i_p\alpha)\|A\|_{C^0}\|\partial^{m_p}A\|_{C^0}\right). \tag{3.21}$$

From this and the convexity (Hadamard-Kolmogorov) inequalities [Ko]

$$\|\partial^m A\|_{C^0} \leq C\|A\|_0^{1-(m/r)}\|\partial^r A\|_{C^0}^{\frac{m}{r}}, \quad 0 \leq m \leq r, \tag{3.22}$$

we deduce (using $\sum_{p=1}^{s} m_p = r$)

$$\begin{aligned} \|I_{(i^*)}(x)\| &\leq \left(K\phi(x_0)\phi(x_0+n\alpha)\right)^{1/2} K^s\phi(x_0)^s\|A\|_{C^0}^s \prod_{p=1}^{s}\left(C\|A\|_{C^0}^{1-\frac{m_p}{r}}\|\partial^r A\|_{C^0}^{\frac{m_p}{r}}\phi(x_0+i_p\alpha)\right) \\ &\leq C^s K^{s+\frac{1}{2}}\phi(x_0)^{s+\frac{1}{2}}\phi(x_0+n\alpha)^{1/2}\|A\|_{C^0}^{2s-1}\|\partial^r A\|_{C^0}\prod_{p=1}^{s}\phi(x_0+i_p\alpha) \\ &\leq C^r\left(K\|A\|_{C^0}^2\phi(x_0)\right)^{r+\frac{1}{2}}\phi(x_0+n\alpha)^{1/2}\|\partial^r A\|_{C^0}\prod_{p=1}^{s}\phi(x_0+i_p\alpha), \end{aligned} \tag{3.23}$$

so that

$$\begin{aligned} \|\partial^r A_n(x)\| &\leq \sum_{i^*\in\mathcal{I}} \|I_{(i^*)}(x)\| \\ &\leq C^r\left(K\|A\|_{C^0}^2\phi(x_0)\right)^{r+\frac{1}{2}}\phi(x_0+n\alpha)^{1/2}\|\partial^r A\|_{C^0}\sum_{i^*\in\mathcal{I}}\phi(x_0+i_1\alpha)\cdots\phi(x_0+i_s\alpha). \end{aligned} \tag{3.24}$$

But the last sum in this estimate satisfies the inequality

$$\sum_{i^*\in\mathcal{I}}\phi(x_0+i_1\alpha)\cdots\phi(x_0+i_s\alpha)=\Big(\phi(x_0)+\ldots+\phi(x_0+(n-1)\alpha)\Big)^r\le n^rS(x_0)^r \tag{3.25}$$

which implies the result. □

We can now conclude easily:

**Lemma 3.3.** *Assume that $A:\mathbb{R}/\mathbb{Z}\to\mathrm{SL}(2,\mathbb{R})$ is $C^k$ $(1\le k\le\infty)$. For almost every $x_*\in\mathbb{R}/\mathbb{Z}$, there exists $K>0$, such that for every $d>0$[16] such that for every $n>n_0(d)$, if $\|\alpha n\|_{\mathbb{R}/\mathbb{Z}}\le\frac{d}{n}$, then*

$$\|\partial^rA_n(x)\|\le K^{r+1}n^r\|A\|_{C^r},\quad |x-x_*|\le\frac{d}{n}. \tag{3.26}$$

*Proof.* Let $X\subset\mathbb{R}/\mathbb{Z}$ be the set of all $x$ such that $S(x)<\infty$, $\phi(x)<\infty$ and which are measurable continuity points of $S$ and $\phi$. This means that for every $\epsilon>0$, $x$ is a density point of

$$Y(x,\epsilon)=S^{-1}(S(x)-\epsilon,S(x)+\epsilon)\cap\phi^{-1}(\phi(x)-\epsilon,\phi(x)+\epsilon). \tag{3.27}$$

The Lebesgue Density Point Theorem implies that $X$ has full Lebesgue Measure.

Fix $x_*\in X$, $d>0$ and $\epsilon>0$. If $n$ is sufficiently big then

$$\left|Y(x_*,\epsilon)\cap\left[x_*-\frac{2d}{n},x_*+\frac{2d}{n}\right]\right|\ge\frac{(4-\epsilon)d}{n}. \tag{3.28}$$

If $\|\alpha n\|_{\mathbb{R}/\mathbb{Z}}<\frac{d}{n}$, this implies

$$\left|(Y(x_*,\epsilon)-\alpha n)\cap Y(x_*,\epsilon)\cap\left[x_*-\frac{d}{n},x_*+\frac{d}{n}\right]\right|\ge\frac{(2-2\epsilon)d}{n}, \tag{3.29}$$

and in particular, each point $x\in\left[x_*-\frac{d}{n},x_*+\frac{d}{n}\right]$ is at distance at most $\frac{2\epsilon d}{n}$ of a point $x_0$ such that $x_0\in Y(x_*,\epsilon)$ and $x_0+\alpha n\in Y(x_*,\epsilon)$. In particular, for every $\delta>0$, if $\epsilon>0$ is sufficiently small then $c_1(x_0)\le c_1(x_*)+\delta$, $c_2(x_0)\le c_2(x_*)+\delta$ where $c_1$ and $c_2$ are as in the previous lemma. The previous lemma implies that

$$\begin{aligned}\|(\partial^rA_n)(x)\|&\le C^rn^r\phi(x_0+n\alpha)^{1/2}\left(c_1(x_0)e^{c_2(x_0)n|x-x_0|}\right)^{r+\frac12}\|\partial^rA\|_{C^0}\\&\le C^rn^r(\phi(x_*)+\epsilon)^{1/2}\left(c_1(x_*)+\delta)e^{2\epsilon d(c_2(x_*)+\delta)}\right)^{r+\frac12}\|\partial^rA\|_{C^0}.\end{aligned} \tag{3.30}$$

It immediately follows that for every $\epsilon>0$, for every $n$ sufficiently big such that $\|\alpha n\|_{\mathbb{R}/\mathbb{Z}}<\frac{d}{n}$, we have

$$\|\partial^rA_n(x)\|\le n^r\Big((C+\epsilon)c_1(x_*)\Big)^{r+1}\|A\|_{C^r},\quad |x-x_*|<\frac{d}{n}. \tag{3.31}$$

□

**Lemma 3.4.** *Assume that $A:\mathbb{R}/\mathbb{Z}\to\mathrm{SL}(2,\mathbb{R})$ is Lipschitz. For almost every $x_*\in\mathbb{R}/\mathbb{Z}$, for every $d>0$, for every $\epsilon>0$, if $n>n_0(d,\epsilon)$ and $\|\alpha n\|_{\mathbb{R}/\mathbb{Z}}\le\frac{d}{n}$, then the matrix $B(x_*)A_n(x)B(x_*)^{-1}$ is $\epsilon$ close to $\mathrm{SO}(2,\mathbb{R})$ provided that $|x-x_*|\le\frac{d}{n}$.*

[16]In what follows, the fact that $K$ is independent of $d$ will not be actually used.

*Proof.* Let $x_*$ be a measurable continuity point of $S$ and $B$. By the same argument of the previous lemma, for $n$ big enough, if $\|\alpha n\|_{\mathbb{R}/\mathbb{Z}} < \frac{d}{n}$, then every $x$ such that $|x - x_*| < \frac{d}{n}$ is at distance at most $\frac{\epsilon}{n}$ from some $x_0$ such that $|S(x_0) - S(x_*)| < \epsilon$, $\|B(x_0) - B(x_*)\| < \epsilon$ and $\|B(x_0 + n\alpha) - B(x_*)\| < \epsilon$. By (3.11), we have

$$\|A_n(x_0)^{-1}(A_n(x) - A_n(x_0))\| \leq e^{n|x-x_0|\|A\|_{C^0}\mathrm{Lip}(A)\phi(x_0)S(x_0)} - 1 \leq K\epsilon \tag{3.32}$$

so it is enough to show that $B(x_*)A_n(x_0)B(x_*)^{-1}$ is close to $\mathrm{SO}(2,\mathbb{R})$. But this is clear since $B(x_0 + \alpha n)A_n(x_0)B(x_0)^{-1} \in \mathrm{SO}(2,\mathbb{R})$ and $B(x_0)$, $B(x_0 + n\alpha)$ are close to $B(x_*)$. □

## 4. Renormalization

Let $\Omega^r = \mathbb{R} \times C^r(\mathbb{R}, \mathrm{SL}(2,\mathbb{R}))$. We will view $\Omega^r$ as a subset of $C^r(\mathbb{R} \times \mathbb{R}^2, \mathbb{R} \times \mathbb{R}^2)$:

$$(\alpha, A) \cdot (x, w) = (x + \alpha, A(x) \cdot w). \tag{4.1}$$

A $C^r$ fibered $\mathbb{Z}^2$-action is a function $\Phi : \mathbb{Z}^2 \to \Omega^r$, such that $\Phi(n,m) \circ \Phi(n', m') = \Phi(n + n', m + m')$. We let $\Lambda^r$ denote the space of $C^r$ fibered $\mathbb{Z}^2$-actions. We endow $\Lambda^r$ with the pointwise topology. This topology is induced from the embedding $\Lambda^r \to \Omega^r \times \Omega^r$, $\Phi \mapsto (\Phi(1,0), \Phi(0,1))$.

Let $\Pi_1 : \mathbb{R} \times C^r(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \to \mathbb{R}$, $\Pi_2 : \mathbb{R} \times C^r(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \to C^r(\mathbb{R}, \mathrm{SL}(2,\mathbb{R}))$ be the coordinate projections. Let also $\gamma^\Phi_{n,m} = \Pi_1 \circ \Phi(n,m) \in \mathbb{R}$ and $A^\Phi_{n,m} = \Pi_2 \circ \Phi(n,m) \in C^r(\mathbb{R}, \mathrm{SL}(2,\mathbb{R}))$.

The action $\Phi$ will be called non-degenerate if $\Pi_1 \circ \Phi : \mathbb{Z}^2 \to \mathbb{R}$ is injective. Let $\Gamma^r$ be the set of non-degenerate actions.

We let $\Lambda^r_0$ be the set of $\Phi \in \Lambda^r$ such that $\gamma^\Phi_{1,0} = 1$ and $\gamma^\Phi_{0,1} \in [0,1]$. For $\Phi \in \Lambda^r_0$, we let $\alpha^\Phi = \gamma^\Phi_{0,1}$. We let $\Gamma^r_0 = \Gamma^r \cap \Lambda^r_0 = \{\Phi \in \Lambda^r_0,\ \alpha^\Phi \in \mathbb{R} \setminus \mathbb{Q}\}$.

### 4.1. Some operations.

Let $\lambda \neq 0$. Define $M_\lambda : \Lambda^r \to \Lambda^r$ by

$$M_\lambda(\Phi)(n,m) = (\lambda^{-1}\gamma^\Phi_{n,m}, x \mapsto A^\Phi_{n,m}(\lambda x)). \tag{4.2}$$

Let $x_* \in \mathbb{R}$. Define $T_{x_*} : \Lambda^r \to \Lambda^r$ by

$$T_{x_*}(\Phi)(n,m) = (\gamma^\Phi_{n,m}, x \mapsto A^\Phi_{n,m}(x + x_*)). \tag{4.3}$$

Let $U \in \mathrm{GL}(2,\mathbb{Z})$. Define $N_U : \Lambda^r \to \Lambda^r$ by

$$N_U(\Phi)(n,m) = \Phi(n', m'), \quad \begin{pmatrix} n' \\ m' \end{pmatrix} = U^{-1} \cdot \begin{pmatrix} n \\ m \end{pmatrix}. \tag{4.4}$$

The operations $M$, $T$, and $N$ will be called rescaling, translation, and base change.

Notice that $M_\lambda M_{\lambda'} = M_{\lambda\lambda'}$, $T_{x_*}T_{x'_*} = T_{x_* + x'_*}$, and $N_U N_{U'} = N_{UU'}$ (that is, $M$, $T$, and $N$ are left actions of $\mathbb{R}^*$, $\mathbb{R}$ and $\mathrm{GL}(2,\mathbb{Z})$ on $\Lambda^r$). Moreover, base changes commute with translations and rescalings.

Notice that $C^r(\mathbb{R}, \mathrm{SL}(2,\mathbb{R}))$ acts in $\Omega^r$ by $\mathrm{Ad}_B(\alpha, A(\cdot)) = (\alpha, B(\cdot + \alpha)^{-1}A(\cdot)B(\cdot))$. This action extends to an action (still denoted $\mathrm{Ad}_B$) on $\Lambda^r$. We will say that $\Phi$ and $\mathrm{Ad}_B(\Phi)$ are $C^r$-conjugate via $B$.

### 4.2. Continued fraction expansion.

Let $0 < \alpha < 1$ be irrational. We will discuss some elementary facts and fix notation regarding the continued fraction expansion

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}. \tag{4.5}$$

Define $\alpha_n = G^n(\alpha)$ where $G$ is the Gauss map $G(x) = \{x^{-1}\}$ ($\{\cdot\}$ denotes fractionary part). The coefficients $a_n$ in (4.5) are given by $a_n = [\alpha_{n-1}^{-1}]$, where $[\cdot]$ denotes integer part. We also set $a_0 = 0$ for convenience. Then

$$\alpha_n = \cfrac{1}{a_{n+1} + \cfrac{1}{a_{n+2} + \cdots}}. \tag{4.6}$$

Let $\beta_n = \prod_{j=0}^{n} \alpha_j$. Define

$$Q_0 = \begin{pmatrix} q_0 & p_0 \\ q_{-1} & p_{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.7}$$

$$Q_n = \begin{pmatrix} q_n & p_n \\ q_{n-1} & p_{n-1} \end{pmatrix} = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_{n-1} & p_{n-1} \\ q_{n-2} & p_{n-2} \end{pmatrix} \tag{4.8}$$

that is,

$$Q_n = U(\alpha_n) \cdots U(\alpha_1), \tag{4.9}$$

where

$$U(x) = \begin{pmatrix} [x^{-1}] & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.10}$$

Then we have

$$\beta_n = (-1)^n (q_n \alpha - p_n) = \frac{1}{q_{n+1} + \alpha_{n+1} q_n}, \tag{4.11}$$

$$\frac{1}{q_{n+1} + q_n} < \beta_n < \frac{1}{q_{n+1}}. \tag{4.12}$$

4.3. **Renormalization.** We define the renormalization operator around 0, $R \equiv R_0 : \Gamma_0^r \to \Gamma_0^r$, by $R(\Phi) = M_{\alpha^{-1}}(N_{-U(\alpha)}(\Phi))$ where $\alpha = \alpha^\Phi$ and $U(\cdot)$ is given by (4.10).

The renormalization operator around $x_* \in \mathbb{R}$, $R_{x_*} : \Gamma_0^r \to \Gamma_0^r$ is defined by $R_{x_*} = T_{x_*}^{-1} \circ R \circ T_{x_*}$.

Notice that if $\Phi \in \Gamma_0^r$ and $\alpha^\Phi = \alpha$ then $R^n(\Phi) = \prod_{j=n-1}^{0} M_{\alpha_j} \circ N_{-U(\alpha_j)} = M_{\beta_{n-1}} \circ N_{(-1)^n Q_n}$.

4.4. **Normalized actions, relation to cocycles.** An action $\Phi \in \Lambda_0^r$ will be called normalized if $\Phi(1,0) = (1, \mathrm{id})$. If $\Phi$ is normalized then $\Phi(0,1) = (\alpha, A)$ can be viewed as a $C^r$-cocycle, since $A$ is automatically defined modulo $\mathbb{Z}$. Inversely, given a $C^r$-cocycle $(\alpha, A)$, $\alpha \in [0,1]$, we associate a normalized action $\Phi_{\alpha,A}$ by setting

$$\Phi_{\alpha,A}(1,0) = (1, \mathrm{id}), \quad \Phi_{\alpha,A}(0,1) = (\alpha, A). \tag{4.13}$$

**Lemma 4.1.** *Any $\Phi \in \Gamma_0^r$ is $C^r$-conjugate to a normalized action. Moreover, if $\Phi_n(1,0) \in \Gamma_0^r$ converges to $(1, \mathrm{id})$ in $\Gamma_0^r$ then one can choose a sequence of conjugacies converging to* id *in the $C^r$ topology*[17].

*Proof.* We first assume that $r \neq \omega$. Let $\Phi(1,0) = (1, A)$. Let $B \in C^r([0, 3/2], \mathrm{SL}(2,\mathbb{R}))$ be such that $B(x) = \mathrm{id}$, $x \in [0, 1/2]$, $B(x) = A(x-1)$, $x \in [1, 3/2]$. Let us extend $B$ to $\mathbb{R}$ forcing $\mathrm{Ad}_B A = \mathrm{id}$ ($B$ is still smooth after the modification). If $A$ is $C^r$ close to id, we can select $B : [0, 3/2] \to \mathrm{SL}(2,\mathbb{R})$ to be $C^r$ close to id, and in this case $B : \mathbb{R} \to \mathrm{SL}(2,\mathbb{R})$ is also $C^r$ close to id.

Let us now assume that $r = \omega$. Let us first deal with the case where (the holomorphic extension of) $A$ is close to the identity in a definite neighborhood of $\mathbb{R}$. Extend $A$ to a real-symmetric $C^\infty$

[17]The reason we refer to sequences instead of speaking of closeness is because the $C^\omega$ topology is not separable. Notice that we will only use the second part of this lemma which is easier to prove.

function $A : \mathbb{C} \to \mathrm{SL}(2, \mathbb{C})$ which is $C^\infty$ close to the identity and which is holomorphic on a definite neighborhood $V$ of $\mathbb{R}$. We will assume that $V$ satisfies (after shrinking)

$$z \in V \implies z+1 \in V, \quad \Re z \leq 0, \tag{4.14}$$

$$z \in V \implies z-1 \in V, \quad \Re z \geq 1, \tag{4.15}$$

$$[0,1] \times [-\epsilon, \epsilon] \subset V. \tag{4.16}$$

Let $B \in C^\infty(\mathbb{C}, \mathrm{SL}(2,\mathbb{C}))$ be $C^\infty$ close to the identity, real-symmetric, and satisfying $B(z+1)^{-1}A(z)B(z)$, $z \in \mathbb{C}$ ($B$ is obtained as in the previous case). Notice that $\overline{\partial} B(z+1) = \overline{\partial} A(z) B(z) + A(z) \overline{\partial} B(z)$, so for $z \in V$ we have $B(z+1)^{-1} \overline{\partial} B(z+1) = B(z+1)^{-1} A(z) \overline{\partial} B(z) = B(z)^{-1} \overline{\partial} B(z)$. Moreover,

$$\|B(z)^{-1} \overline{\partial} B(z)\| < \delta, \quad z \in [0,1] \times [-\epsilon, \epsilon] \tag{4.17}$$

for some small $\delta$.

Given $C : \mathbb{C}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{C})$, we let $D = BC^{-1}$ and we obviously have $D(z+1)A(z)D(z)^{-1} = \mathrm{id}$. We want to choose $C$ so that

$$\overline{\partial} C^{-1}(z) C(z) = -B^{-1}(z) \overline{\partial} B(z), \quad z \in [0,1] \times [-\epsilon, \epsilon], \tag{4.18}$$

for this will assure us that

$$B^{-1}(z) \overline{\partial} D(z) C(z) = B^{-1}(z) \overline{\partial} B(z) + \overline{\partial} C^{-1}(z) C(z) \tag{4.19}$$

vanishes for $z \in [0,1] \times [-\epsilon, \epsilon]$ and also in $V \cap \mathbb{R} \times [-\epsilon, \epsilon]$, and we also want to impose that $C$ is $C^0$ close to the identity. Here the smoothness requirement on $C$ is for it to be of class $W^{1,1}$, that is, it should continuous and have locally integrable distributional derivatives.

Equation (4.18) is equivalent to

$$C^{-1}(z) \overline{\partial} C(z) = B^{-1}(z) \overline{\partial} B(z). \tag{4.20}$$

To conclude, we use the following proposition:

**Proposition 4.2.** *There exists $\kappa > 0$ with the following property. Let $\eta \in L^\infty(\mathbb{R}/\mathbb{Z} \times [-1,1], \mathrm{sl}(2,\mathbb{R}))$ and assume that $\|\eta\|_{L^\infty} < \kappa$. Then there exists $C : \mathbb{R}/\mathbb{Z} \times [-1,1] \to \mathrm{SL}(2,\mathbb{R})$ of class $W^{1,1}$ such that $C(z)^{-1} \overline{\partial} C(z) = \eta$ and $\|C - \mathrm{id}\|_0 \leq \kappa^{-1} \|\eta\|_{L^\infty}$ close to the identity for $z \in \mathbb{R}/\mathbb{Z} \times [-1,1]$. Moreover, $C$ is real-symmetric provided $\eta$ is real-symmetric.*

*Proof.* Let $W^{1,1}(\mathbb{R}/\mathbb{Z} \times [-1,1], \mathrm{sl}(2,\mathbb{R}))$ be the space of continuous maps $a : \mathbb{R}/\mathbb{Z} \times [-1,1] \to \mathrm{sl}(2,\mathbb{R})$ with integrable distributional derivatives, endowed with the natural norm. We can obtain a bounded linear map $P : L^\infty(\mathbb{R}/\mathbb{Z} \times [-1,1], \mathrm{sl}(2,\mathbb{C})) \to W^{1,1}(\mathbb{R}/\mathbb{Z} \times [-1,1], \mathrm{sl}(2,\mathbb{C}))$ which is real-symmetric and solves $\overline{\partial} \circ P = \mathrm{id}$. Indeed $P$ can be given explicitly in terms of the Cauchy transform

$$(P\alpha)(z) = \frac{-1}{\pi} \int_{\mathbb{R} \times [-1,1]} \frac{\alpha(\zeta)}{z - \zeta} d\zeta \wedge d\overline{\zeta} = \lim_{t \to \infty} \frac{-1}{\pi} \int_{[-t,t] \times [-1,1]} \frac{\alpha(\zeta)}{z - \zeta} d\zeta \wedge d\overline{\zeta}. \tag{4.21}$$

Define an analytic map $T : L^\infty(\mathbb{R}/\mathbb{Z} \times [-1,1]) \to L^\infty(\mathbb{R}/\mathbb{Z} \times [-1,1])$ by $T(\cdot) = e^{-P(\cdot)} \overline{\partial} e^{P(\cdot)}$. Then $T(0) = 0$, $DT(0) = \mathrm{id}$. It follows that $T$ is a diffeomorphism in a neighborhood of $\eta = 0$, so we may solve $e^{-P\alpha} \overline{\partial} e^{P\alpha} = \eta$ with $\|\alpha\|_\infty \leq K \|\eta\|_{L^\infty}$ provided $\eta$ is close to 0. It follows that $C = e^{P\alpha}$ satisfies the conclusion of the proposition. □

We may now obtain $C$ with the required properties by taking $\eta = B^{-1} \cdot \overline{\partial} B$ in $[0,1] \times [-\epsilon, \epsilon]$ and $\eta = 0$ otherwise and applying the previous proposition. This concludes the second part of the lemma in the case $r = \omega$.

This argument also works if we only assume that $A$ is close to the identity in the $C^\infty$ topology (indeed the $C^1$ topology is enough, as this is all that we need to get (4.17)), and gives the first part of the lemma also in this case (but we obviously do not get that the holomorphic extension of the normalizing matrix is close to the identity). In order to treat the global case, we first consider $B \in C^\infty(\mathbb{R}, \mathrm{SL}(2,\mathbb{R}))$ with $B(x+1)^{-1}A(x)B(x) = \mathrm{id}$, and then approximate $B$ (in the $C^\infty$ topology) by $B' \in C^\omega(\mathbb{R}, \mathrm{SL}(2,\mathbb{R}))$. Then $B'(x+1)^{-1}A(x)B'(x)$ is $C^\infty$ close to the identity and we can apply the previous case. □

4.5. **Degree and rotation number.** Let $\Phi$ be a $\mathbb{Z}^2$-action. If $w$ is a point of the usual euclidean circle $\mathbb{S}^1 \subset \mathbb{R}^2$ we set

$$f^\Phi_{n,m}(x,v) = \frac{A^\Phi_{n,m}(x)\cdot w}{\|A^\Phi_{n,m}(x)\cdot w\|}, \tag{4.22}$$

and we define

$$\begin{aligned} F^\Phi_{n,m} &: \mathbb{R}\times\mathbb{S}^1 \to \mathbb{R}\times\mathbb{S}^1 \\ &(x,v) \mapsto (x+\gamma^\Phi_{n,m}, f^\Phi_{n,m}(x,v)) \end{aligned} \tag{4.23}$$

If $\pi : \mathbb{R} \to \mathbb{S}^1$ is the projection $\pi(y) = \exp(2\pi i y)$ we can find a continuous lift $d^\Phi_{n,m} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ of $f^\Phi_{n,m}(x,w)w^{-1}$, that is

$$\pi(y + d^\Phi_{n,m}(x,y)) = f^\Phi_{n,m}(x,\pi(y)). \tag{4.24}$$

Observe that such a lift is not uniquely defined, every other lift being of the form $d^\Phi_{n,m}(x,y) + k_{n,m}$, where $k_{n,m}$ is a constant integer. Also, for any $x, y \in \mathbb{R}\times\mathbb{R}$ we have $d^\Phi_{n,m}(x,y+1) = d^\Phi_{n,m}(x,y)$ and thus $d^\Phi_{n,m}(x,w)$ can be defined for any $x\in\mathbb{R}$, $w \in \mathbb{S}^1$.

Let $(e_1,e_2)$ be a directed basis of the $\mathbb{Z}$-module $\mathbb{Z}^2$ (that is if $e_1 = (n_1,m_1)$, $e_2 = (n_2,m_2)$ then we assume that $n_1m_2 - n_2m_1 = 1$). Then it is easy to see that the quantity

$$(d^\Phi_{e_1} \circ F^\Phi_{e_2}(x,w) + d^\Phi_{e_2}(x,w)) - (d^\Phi_{e_2} \circ F^\Phi_{e_1}(x,w) + d^\Phi_{e_1}(x,w)) \tag{4.25}$$

is independent of the choices made for the lifts, does not depend on $(x,w)$ and is a constant integer. Moreover it is shown in [K2] that this integer does not depend on the chosen directed basis $(e_1,e_2)$. This is what we call the *degree* of the action $\Phi$ and denote it by $\deg\Phi$. Also, this integer is invariant by the operation of rescaling, translation and conjugacies that is $\deg(M_\lambda(\Phi)) = \deg(T_{x_*}(\Phi)) = \deg(\mathrm{Ad}_B(\Phi)) = \deg(\Phi)$, and is equal, when the action is normalized, to the usual degree of the map $A : \mathbb{R}/\mathbb{Z} \to SL(2,\mathbb{R})$ defined by $\Phi(0,1) = (\alpha, A(\cdot))$.

Assume now that the action $\Phi$ has *degree zero*. Let us denote by $\mathcal{M}$ the set of measures on $\mathbb{R}\times\mathbb{S}^1$ that project on the first factor to Lebesgue measure on $\mathbb{R}$. It is not difficult to see that one can find a measure $\mu$ in $\mathcal{M}$ that is invariant by $F^\Phi_{n,m}$ for any $(n,m)\in\mathbb{Z}^2$. Take as before $(e_1,e_2)$ a directed basis of $\mathbb{Z}^2$ and define the quantity:

$$(II) = I(0,\gamma^\Phi_{e_2}; d^\Phi_{e_1}) - I(0,\gamma^\Phi_{e_1}; d^\Phi_{e_2}), \tag{4.26}$$

where we have defined for any function $h : \mathbb{R}\times\mathbb{S}^1 \to \mathbb{R}$ and $(a,b)\in\mathbb{R}^2$ the quantity

$$I(a,b;h) = \mathrm{sgn}(b-a)\int_{[a,b]\times\mathbb{S}^1} h(x,v)\,d\mu(x,v). \tag{4.27}$$

If we make other choices for the lifts of $F^\Phi$, the number we obtain just differ by the addition of an element of the *module of frequency of* $\Phi$, that is the $\mathbb{Z}$-module $\Gamma_\Phi$ generated by $\gamma^\Phi_{e_1}$ and $\gamma^\Phi_{e_2}$ where $(e_1,e_2)$ is any directed basis of $\mathbb{Z}^2$ (the module of frequency of $\Phi$ is independent of this basis). Moreover the class of $(II)$ modulo $\Gamma_\Phi$ is invariant by conjugacy and does not depend on $\mu$ (see [K2]). We shall call the element of $\mathbb{R}/\Gamma_\Phi$ thus obtained the *fibered rotation number* of the action $\Phi$ and denote it by $\mathrm{rot}(\Phi)$.

If $\sigma_\lambda : \mathbb{R}/\Gamma_\Phi \to \mathbb{R}/\Gamma_{M_\lambda(\Phi)}$ is the isomorphism of modules induced by $x \mapsto \lambda^{-1}x$ ($\lambda \neq 0$) we also have $\mathrm{rot}(M_\lambda(\Phi)) = \sigma_\lambda(\mathrm{rot}(\Phi))$.

We shall say that an element in $\mathbb{R}/\Gamma_\Phi$ is *$\tau$-Diophantine* ($\tau > 1$) if for some representative $\beta$ and some $\kappa > 0$ one has

$$\forall (k,l) \in \mathbb{Z}^2 - \{(0,0)\},\ |2\beta - k\gamma_{e_1} - l\gamma_{e_2}| \geq \frac{\kappa}{(|k|+|l|)^\tau}. \tag{4.28}$$

This definition is clearly independent of the choice of the representative and of the chosen basis ($\kappa$ then has to be changed). Finally, we say that the action $\Phi$ is $\tau$-Diophantine if $\mathrm{rot}(\Phi)$ is $\tau$-Diophantine. This notion is stable under conjugation, dilatation and translation.

The following result follows immediately from the definition of the fibered rotation number of a cocycle in [H], [JM].

**Lemma 4.3.** *If $\Phi$ is a normalized action of degree 0 which is associated to the cocycle $(\alpha, A)$ then the fibered rotation number $\rho(\alpha, A)$ is a representative of $\mathrm{rot}(\Phi)$. In particular, $\Phi$ is Diophantine if and only if $\rho(\alpha, A)$ is Diophantine with respect to $\alpha$.*

4.6. **Reducibility.** We will say that an action $\Phi \in \Lambda_0^r$ is $C^r$-reducible if it is $C^r$-conjugate to a constant action. It immediately follows that reducibility is invariant under conjugation, translation, rescaling and base change. Thus reducibility is also invariant under renormalization: an action $\Phi \in \Gamma_0^r$ is $C^r$-reducible if and only if its renormalization $R(\Phi)$ is $C^r$-reducible. Moreover, reducibility of a *normalized action* $\Phi_{\alpha,A}$ is equivalent to reducibility modulo $\mathbb{Z}$ of the associated cocycle $(\alpha, A)$.

The following reducibility result is well known:

**Lemma 4.4.** *Let $\Phi \in \Gamma_0^r$, $r = \omega, \infty$ be $C^r$-conjugate to a $\mathrm{SO}(2,\mathbb{R})$ action of degree 0. If $\alpha^\Phi \in DC$ then $\Phi$ is $C^r$-reducible.*

*Proof.* We may assume that $\Phi$ is normalized, since we can always conjugate $\Phi(1,0)$ to $(1, \mathrm{id})$ in $C^r(\mathbb{R}, \mathrm{SO}(2,\mathbb{R}))$: this can be done in the same way as in Lemma 4.1 (it is indeed easier to proceed for the $\mathrm{SO}(2,\mathbb{R})$ case).

Let $(\alpha, A) = \Phi(0,1)$, and let $\phi : \mathbb{R} \to \mathbb{R}$ satisfy $A(x) = R_{\phi(x)}$. Since $\Phi$ is normalized, $A$ is defined modulo $\mathbb{Z}$, and since $\Phi$ is of degree 0, this implies that $\phi$ is defined modulo $\mathbb{Z}$ as well.

Consider the Fourier series

$$\phi(\theta) = \sum_{k \in \mathbb{Z}} \hat{\phi}(k) e^{2k\pi i\theta}, \tag{4.29}$$

and let

$$\psi(\theta) = \sum_{k \in \mathbb{Z}\setminus\{0\}} \hat{\psi}(k) e^{2k\pi i\theta}, \tag{4.30}$$

where

$$\hat{\psi}(k) = \frac{\hat{\phi}(k)}{e^{2k\pi i\alpha} - 1}, \quad k \neq 0 \tag{4.31}$$

so that

$$\phi(x) - \phi(0) = \psi(x+\alpha) - \psi(x). \tag{4.32}$$

The fact that $\alpha \in DC$ implies that $e^{2k\pi i\alpha} - 1 > \kappa k^{-\tau}$ for some $\kappa > 0$, $\tau > 0$. In particular $\psi \in C^r(\mathbb{R}/\mathbb{Z}, \mathbb{R})$.

Let $B(x) = R_{\psi(x)}$. Then $B \in C^r(\mathbb{R}/\mathbb{Z}, \mathrm{SO}(2,\mathbb{R}))$, and we have $B(x+\alpha)^{-1}A(x)B(x) = R_{\phi(0)}$. This implies that $\mathrm{Ad}_B\Phi$ is a constant action. □

The following is a restatement of a result of Eliasson on reducibility of cocycles close to constant ones [E] (see [AK3] for the $C^\infty$ case) in the language of actions.

**Lemma 4.5.** *Let $\Phi \in \Lambda_0^r$ be $C^r$-reducible, $r = \omega, \infty$, and let $\kappa > 0$, $\tau > 0$ be fixed. Let $\Psi_n$ be a sequence of Diophantine actions converging to $\Phi$ in $\Lambda_0^r$ and satisfying $\alpha^{\Psi_n} \in DC(\kappa, \tau)$. Then $\Psi_n$ is $C^r$-reducible for $n$ large enough.*

*Proof.* After performing a conjugation, we may assume that $\Phi(1,0) = (1, \mathrm{id})$ and $\Phi(0,1) = (\alpha_0, A_0)$ where $A_0 \in \mathrm{SL}(2,\mathbb{R})$ is a constant. By Lemma 4.1, there exists a sequence $B^{(n)} \in C^r(\mathbb{R}, \mathrm{SL}(2,\mathbb{R}))$ converging to id which conjugates $\Psi$ to a normalized cocycle $\Psi_n' = \mathrm{Ad}_{B^{(n)}} \Psi_n$. It follows that $\Psi_n'(0,1) = (\alpha_n, A^{(n)})$ converges to $(\alpha_0, A_0)$ in the $C^r$-topology, so Eliasson's result for cocycles applies and $(\alpha_n, A^{(n)})$ is $C^r$-reducible modulo $\mathbb{Z}$ for $n$ large enough. This implies that $\Psi_n'$ and $\Psi_n$ are $C^r$-reducible as well. □

## 5. A priori bounds and limits of renormalization

The language of renormalization allows us to restate Lemma 3.3 as a precompactness result:

**Theorem 5.1** (A priori bounds). *Let $\Phi \in \Gamma_0^r$, $r \geq 1$, be a normalized action, and assume that the cocycle $(\alpha, A) = \Phi(0,1)$ is $L^2$-conjugated to a cocycle of rotations. Then for almost every $x_* \in \mathbb{R}$, there exists $K > 0$ such that for every $d > 0$ and for every $n > n_0(d)$,*

$$\left\|\partial^k A_{1,0}^{R_{x_*}^n \Phi}(x)\right\| \leq K^{k+1} \|A\|_k, \quad 0 \leq k \leq r, \quad |x - x_*| < d. \tag{5.1}$$

$$\left\|\partial^k A_{0,1}^{R_{x_*}^n \Phi}(x)\right\| \leq K^{k+1} \|A\|_k, \quad 0 \leq k \leq r, \quad |x - x_*| < d. \tag{5.2}$$

*In particular, if $r = \omega, \infty$ then $\{R_{x_*}^n(\Phi)\}_n$ is precompact in $\Gamma_0^r$*[18].

*Proof.* Let $x_*$ be as in Lemma 3.3. Then

$$A_{1,0}^{R_{x_*}^n \Phi}(x) = A_{q_{n-1}}(x_* + \beta_{n-1}(x - x_*)), \tag{5.3}$$

$$A_{0,1}^{R_{x_*}^n \Phi}(x) = A_{q_n}(x_* + \beta_{n-1}(x - x_*)). \tag{5.4}$$

Fix $d$ (we may assume $d > 1$). Since $\beta_{n-1} < \frac{1}{q_n} < \frac{1}{q_{n-1}}$, we can apply Lemma 3.3 to conclude for $0 \leq k \leq r$ and for $|x - x_*| < d$,

$$\left\|\partial^k A_{1,0}^{R_{x_*}^n \Phi}(x)\right\| \leq \beta_{n-1}^k \|(\partial^k A_{q_{n-1}})(x_* + \beta_{n-1}(x - x_*))\| \leq (\beta_{n-1} q_{n-1})^k K^{k+1} \|A\|_k \leq K^{k+1} \|A\|_k, \tag{5.5}$$

$$\left\|\partial^k A_{0,1}^{R_{x_*}^n \Phi}(x)\right\| \leq \beta_{n-1}^k \|(\partial^k A_{q_{n-1}})(x_* + \beta_{n-1}(x - x_*))\| \leq (\beta_{n-1} q_n)^k K^{k+1} \|A\|_k \leq K^{k+1} \|A\|_k. \tag{5.6}$$

The precompactness statement is then obvious[19]. □

This result allows us to consider limits of renormalization. Those are easy to analyze due to the following simple corollary of Lemma 3.4:

**Theorem 5.2** (Limits). *Let $\Phi \in \Gamma_0^{\mathrm{Lip}}$ be a normalized action, and assume that the cocycle $(\alpha, A) = \Phi(0,1)$ is $L^2$-conjugated to a cocycle of rotations. Then for almost every $x_* \in \mathbb{R}$, any limit of $R_{x_*}^n(\Phi)$ is conjugate to an action of rotations, via a constant $B \in \mathrm{SL}(2,\mathbb{R})$.*

We can now prove the following rigidity result.

**Theorem 5.3** (Rigidity). *Let $\alpha \in RDC$, and let $A : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ be $C^r$, $r = \omega, \infty$, and homotopic to the identity. If $(\alpha, A)$ is $L^2$-conjugated to a cocycle of rotations, and the fibered rotation number of $(\alpha, A)$ is Diophantine with respect to $\alpha$, then $(\alpha, A)$ is $C^r$-reducible.*

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[18] For $r < \infty$, one obtains precompactness in $\Gamma_0^{r-1+\mathrm{Lip}}$.

[19] Notice that we do not need to use that $K$ does not depend on $d$.

*Proof.* Let $\alpha \in RDC(\kappa,\tau)$ and let $n_k \to \infty$ be such that $\alpha_{n_k} \in DC(\kappa,\tau)$. Let $\Phi \equiv \Phi_{\alpha,A}$.

Consider the renormalizations $\Psi_k = R^{n_k}_{x_*}(\Phi_{\alpha,A})$, where $x_*$ is as in Theorems 3.3 and 5.2. Notice that for every $k$, $\alpha^{\Psi_k} \in DC(\kappa,\tau)$ and the fibered rotation number of $\Psi_k$ is Diophantine with respect to $\alpha^{\Psi_k}$.

Passing to a subsequence, we may assume that $\Psi_k \to \Psi$ in the $C^r$ topology. Since $DC(\kappa,\tau)$ is compact, $\alpha^\Psi = \lim \alpha_{n_k} \in DC(\kappa,\tau)$. By Theorem 5.2, $\Psi$ is $C^r$-conjugate to a $\mathrm{SO}(2,\mathbb{R})$ action, so by Lemma 4.4, $\Psi$ is $C^r$-reducible. Thus Lemma 4.5 applies and we conclude that $\Psi_k$ is $C^r$-reducible for $k$ large enough. It follows that $\Phi$ is reducible, so $(\alpha, A)$ is reducible as well. □

*Proof of Theorems A and A'.* We can now easily prove Theorem A. Let $\alpha \in RDC$, $v \in C^r(\mathbb{R}/\mathbb{Z},\mathbb{R})$, and let $\Delta$ be the set of $E \in \mathbb{R}$ such that $(\alpha, S_{v,E})$ is $L^2$-conjugated to a cocycle of rotations and the fibered rotation number of $(\alpha, S_{v,E})$ is diophantine with respect to $\alpha$. By Theorems 2.1 and 2.5, $\Delta \cup \{E \in \mathbb{R},\ L(\alpha, S_{v,E}) > 0\}$ has full Lebesgue measure in $\mathbb{R}$, and Theorem 5.3 implies that $(\alpha, S_{v,E})$ is $C^r$-reducible for all $E \in \Delta$. This shows that $(\alpha, S_{v,E})$ is $C^r$-reducible for almost every $E \in \mathbb{R}$ such that $L(\alpha, S_{v,E}) = 0$. By Remark 1.3, if $E \in \mathbb{R}$ is such that $L(\alpha, S_{v,E}) > 0$ then $(\alpha, S_{v,E})$ is either non-uniformly hyperbolic or $C^r$-reducible, and the result follows.

This argument also works for Theorem A', using Theorems 2.2 and 2.6 instead of Theorems 2.1 and 2.5. □

*Remark* 5.1. Let $(\alpha, A) \in RDC \times C^\omega$ be $L^2$-conjugated to a cocycle of rotations. Even if we do not make hypothesis on the fibered rotation number of $(\alpha, A)$, this analysis still gives some interesting information. For instance, if $(\alpha_k, A^{(k)}) \in DC(\kappa,\tau) \times C^\omega$ is a sequence of cocycles converging to $(\alpha, A)$, then, for every $k$ sufficiently big, the cocycle $(\alpha_k, A^{(k)})$ is either uniformly hyperbolic or has zero Lyapunov exponent[20]. To see this, it is enough to apply the results of [E] to the limits of subsequences $R^{m_k}_{x_*}(\Phi_{\alpha_k,A^{(k)}})$ for appropriate choices of $m_k \to \infty$.

*Remark* 5.2. More generally, even if one does not make the hypothesis that the fibered rotation number is diophantine in Theorem 5.3, one still concludes that $(\alpha, A)$ is almost $C^\infty$-reducible by applying the results of [AK3].

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