Smoothness of solenoidal attractors

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Abstract

We consider dynamical systems generated by skew products of affine contractions on the real line over angle-multiplying maps on the circle S^1:

$$T : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}, \qquad T(x, y) = (\ell x, \lambda y + f(x))$$

where $\ell \geq 2$, $0 < \lambda < 1$ and f is a C^r function on S^1. We show that, if $\lambda^{1+2s}\ell > 1$ for some $0 \leq s < r - 2$, the density of the SBR measure for T is contained in the Sobolev space $W^s(S^1 \times \mathbb{R})$ for almost all (C^r generic, at least) f.

1 Introduction

In this paper, we study dynamical systems generated by skew products of affine contractions on the real line over angle-multiplying maps on the circle $S^1 = \mathbb{R}/\mathbb{Z}$:

$$T : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}, \qquad T(x, y) = (\ell x, \lambda y + f(x)) \tag{1}$$

where $\ell \geq 2$ is an integer, $0 < \lambda < 1$ is a real number and f is a C^r function on S^1 (for some integer $r \geq 3$). It admits a forward invariant closed subset A such that $\omega(\mathbf{x}) = A$ for Lebesgue almost every point $\mathbf{x} \in S^1 \times \mathbb{R}$. Further, there exists an ergodic invariant probability measure μ on A for which Lebesgue almost every point on $S^1 \times \mathbb{R}$ is generic. The measure μ is called the *SBR measure* for T. If T is locally area contracting, *i.e.*, $\det DT \equiv \lambda\ell < 1$, the subset A is a Lebesgue null subset and hence the SBR measure is totally singular with respect to the Lebesgue measure. In [7], the third named author studied the case where T is locally area expanding, *i.e.*, $\lambda\ell > 1$, and proved that the SBR measure is absolutely continuous with respect to the Lebesgue measure for C^r generic f.

In the present paper, we study the smoothness of the density of the SBR measure in more detail, and the mixing properties of T.

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Theorem 1. *If ℓ and λ satisfy $\lambda^{1+2s}\ell > 1$ for some $0 \le s < r-2$, the density of the SBR measure μ with respect to the Lebesgue measure is contained in the Sobolev space $W^s(S^1 \times \mathbb{R})$ for any f in an open dense subset of $C^r(S^1, \mathbb{R})$.*

Since the elements of $W^s(S^1 \times \mathbb{R})$ for $s > 1$ are continuous up to modification on Lebesgue null subsets from Sobolev's embedding theorem, it follows

Corollary 2. *If $\lambda^3 \ell > 1$ and $r \ge 4$, the attractor A has non-empty interior for f in an open dense subset of $C^r(S^1, \mathbb{R})$.*

Remark. Recently, Bamón, Kiwi and Rivera-Letelier announced the following result: for an open dense subset of $C^{1+\epsilon}$ hyperbolic endomorphisms of the annulus, $\log d + \chi > 0$ implies that the attractor has non-empty interior, where d is the degree of the induced map in homology and χ is the negative Lyapunov exponent of the SBR measure. (See also [2].)

Remark. When $s > 1$, we also obtain that the density of the SBR measure is essentially bounded. Together with the results of Rams in [6], it gives examples of solenoids in higher dimensions for which the invariant measure is equivalent to the Hausdorff measure.

The Perron-Frobenius operator $P : L^1(S^1 \times \mathbb{R}) \to L^1(S^1 \times \mathbb{R})$ is defined by

$$Ph(\mathbf{x}) = \frac{1}{\lambda \ell} \sum_{\mathbf{y} \in T^{-1}(\mathbf{x})} h(\mathbf{y}),$$

and characterized by the property that

$$\frac{dT_* \nu}{d\mathrm{Leb}} = P\left(\frac{d\nu}{d\mathrm{Leb}}\right) \tag{2}$$

for any finite measure ν which is absolutely continuous with respect to the Lebesgue measure Leb on $S^1 \times \mathbb{R}$.

When $s > 1/2$, we obtain a precise spectral description of P, which strengthens considerably Theorem 1.

Theorem 3. *Assume that ℓ and λ satisfy $\lambda^{1+2s}\ell > 1$ for some $1/2 < s < r-2$. Take $\gamma \in \left((\lambda^{1+2s}\ell)^{-1/2}, 1\right)$. For any f in an open dense subset of $C^r(S^1, \mathbb{R})$, there exists a Banach space $\mathcal{B}$ contained in $W^s(S^1 \times \mathbb{R})$ on which the transfer operator P acts continuously with an essential spectral radius at most γ (in particular, P admits a spectral gap, and the correlations of T decay exponentially fast). Moreover, $\mathcal{B}$ can be chosen to contain all functions in $C^{r-1}(S^1 \times \mathbb{R})$ supported in some given (fixed) compact subset of $S^1 \times \mathbb{R}$.*

Since T is uniformly hyperbolic, the exponential decay of correlations was already known. The novel feature of our theorem is that, when the contraction coefficient λ tends to 1, our estimates do *not* degenerate. In fact, the inequality $\lambda < 1$ is used only to ensure that a compact subset of $S^1 \times \mathbb{R}$ is invariant, to get an SBR measure. Hence, our method may probably be generalized to settings with a neutral (or slightly positive) exponent on a compact space.

Fix $\ell \geq 2$ and let $\mathcal{D}_{r,s} \subset (0,1) \times C^r(S^1, \mathbb{R})$ be the set of pairs (λ, f) such that the conclusions of Theorems 1 and 3 hold. Let $\mathcal{D}^\circ_{r,s}$ be the interior of $\mathcal{D}_{r,s}$. The following result shows that Theorems 1 and 3 hold for "almost all" T, in a precise sense:

Theorem 4. *If ℓ and λ satisfy $\lambda^{1+2s}\ell > 1$ for some $0 \leq s < r-2$, there exists a finite collection of C^∞ functions $\varphi_i : S^1 \to \mathbb{R}$, $1 \leq i \leq m$, such that, for any $g \in C^r(S^1, \mathbb{R})$, the subset*

$$\left\{ (t_1, t_2, \cdots, t_m) \in \mathbb{R}^m \;\middle|\; \left(\lambda,\ g(x) + \sum_{i=1}^{m} t_i \varphi_i(x) \right) \notin \mathcal{D}^\circ_{r,s} \right\}$$

is a null subset with respect to the Lebesgue measure on $\mathbb{R}^m$.

We proceed as follows. In the next section, we introduce some definitions related to a transversality condition on the mapping T, which is similar to (but slightly different from) that used in [7]. This transversality condition is proved to be a generic one in the last section. In Section 3, we introduce some norms on the space of C^r functions on $S^1 \times \mathbb{R}$ and prove a Lasota-Yorke type inequality for them, imitating the argument in the recent paper [3] of C. Liverani and the second named author with slight modification. Section 4 is the core of this paper, where we prove a Lasota-Yorke inequality involving the W^s norm and the norm introduced in Section 3. Finally, in Section 5, we show how these Lasota-Yorke inequalities imply the main results of the paper.

2 Some definitions

From here to the end of this paper, we fix an integer $\ell \geq 2$, real numbers $0 < \lambda < 1$ and $0 \leq s < r-2$ satisfying $\lambda^{1+2s}\ell > 1$. We also fix a positive number κ and consider the mapping T for a function f in

$$\mathcal{U} = \mathcal{U}_\kappa = \left\{ f \in C^r(S^1, \mathbb{R}) \; ; \; \|f\|_{C^r} := \max_{0 \leq k \leq r} \sup_{x \in S^1} \left| \frac{d^k}{dx^k} f(x) \right| \leq \kappa \right\}.$$

Fix $\alpha_0 = \kappa/(1-\lambda)$ and let $D = S^1 \times [-\alpha_0, \alpha_0]$. Then we have $T(D) \subset D$. Let $\mathcal{P}$ be the partition of S^1 into the intervals $\mathcal{P}(k) = [(k-1)/\ell, k/\ell)$ for $1 \leq k \leq \ell$. Let $\tau : S^1 \to S^1$ be the map defined by $\tau(x) = \ell \cdot x$. Then the partition $\mathcal{P}^n := \bigvee_{i=0}^{n-1} \tau^{-i}(\mathcal{P})$ for $n \geq 1$ consists of the intervals

$$\mathcal{P}(\mathbf{a}) = \bigcap_{i=0}^{n-1} \tau^{-i}\left(\mathcal{P}(a_{n-i})\right), \qquad \mathbf{a} = (a_i)_{i=1}^n \in \mathcal{A}^n$$

where $\mathcal{A}^n$ denotes the space of words of length n on the set $\mathcal{A} = \{1, 2, \cdots, \ell\}$.

Remark. Notice that $\mathbf{a}$ is the *reverse* of the itinerary of points in $\mathcal{P}(\mathbf{a})$.

For $x \in S^1$ and $\mathbf{a} \in \mathcal{A}^n$, there is a unique point $y \in \mathcal{P}(\mathbf{a})$ such that $\tau^n(y) = x$, which is denoted by $\mathbf{a}(x)$. For $\mathbf{a} = (a_i)_{i=1}^n \in \mathcal{A}^n$, the image of the segment $\mathcal{P}(\mathbf{a}) \times \{0\} \subset S^1 \times \mathbb{R}$ under the iterate T^n is the graph of the function $S(\cdot, \mathbf{a})$ defined by

$$S(x, \mathbf{a}) := \sum_{i=1}^{n} \lambda^{i-1} f(\tau^{n-i}(\mathbf{a}(x))) = \sum_{i=1}^{n} \lambda^{i-1} f([\mathbf{a}]_i(x))$$

where $[\mathbf{a}]_q = (a_i)_{i=1}^q$. For a word $\mathbf{a} = (a_i)_{i=1}^\infty \in \mathcal{A}^\infty$ of infinite length, we define

$$S(x, \mathbf{a}) = \lim_{i \to \infty} S(x, [\mathbf{a}]_i) = \sum_{i=1}^{\infty} \lambda^{i-1} f([\mathbf{a}]_i(x)).$$

For a word $\mathbf{c}$ of length m, let $\mathcal{P}_*(\mathbf{c})$ be the union of the interval $\mathcal{P}(\mathbf{c})$ and the two intervals in $\mathcal{P}^m$ adjacent to it. The function $S(\cdot, \mathbf{a})$ for a word $\mathbf{a} \in \mathcal{A}^n$ with $1 \leq n \leq \infty$ may not be continuous on $\mathcal{P}_*(\mathbf{c})$ when $\mathcal{P}(\mathbf{c})$ has $0 \in S^1$ as its end. Nevertheless the restriction of $S(\cdot, \mathbf{a})$ to $\mathcal{P}(\mathbf{c})$ can be naturally extended to $\mathcal{P}_*(\mathbf{c})$ as a C^r function. Indeed, letting $\tau_{\mathbf{c},\mathbf{a}}^{-i} : \mathcal{P}_*(\mathbf{c}) \to S^1$ be the branch of the inverse of τ^i satisfying $\tau_{\mathbf{c},\mathbf{a}}^{-i}(\mathcal{P}(\mathbf{c})) \subset \mathcal{P}([\mathbf{a}]_i)$, the extension is given by

$$S_{\mathbf{c}}(\cdot, \mathbf{a}) : \mathcal{P}_*(\mathbf{c}) \to \mathbb{R}, \quad S_{\mathbf{c}}(x, \mathbf{a}) := \sum_{i=1}^{n} \lambda^{i-1} f(\tau_{\mathbf{c},\mathbf{a}}^{-i}(x)). \tag{3}$$

For any word $\mathbf{a}$ of finite or infinite length, we have

$$\sup_{x \in \mathcal{P}_*(\mathbf{c})} \max_{0 \leq \nu \leq r} \ell^\nu \left| \frac{d^\nu}{dx^\nu} S_{\mathbf{c}}(x, \mathbf{a}) \right| \leq \alpha_0. \tag{4}$$

For $\mathbf{a}, \mathbf{b} \in \mathcal{A}^q$ and $\mathbf{c} \in \mathcal{A}^p$, we say that $\mathbf{a}$ and $\mathbf{b}$ are *transversal* on $\mathbf{c}$ and write $\mathbf{a} \pitchfork_{\mathbf{c}} \mathbf{b}$ if

$$\left| \frac{d}{dx} S_{\mathbf{c}}(x, \mathbf{a}) - \frac{d}{dx} S_{\mathbf{c}}(y, \mathbf{b}) \right| > 2\lambda^q \ell^{-q} \alpha_0$$

at all points x, y in the closure of $\mathcal{P}_*(\mathbf{c})$. We put

$$\mathbf{e}(q, p) = \max_{\mathbf{c} \in \mathcal{A}^p} \max_{\mathbf{a} \in \mathcal{A}^q} \#\{\mathbf{b} \in \mathcal{A}^q \mid \mathbf{a} \not\pitchfork_{\mathbf{c}} \mathbf{b}\} \qquad \text{and} \qquad \mathbf{e}(q) = \lim_{p \to \infty} \mathbf{e}(q, p).$$

The main argument of the proof will be to construct norms which will satisfy a Lasota-Yorke inequality if $\mathbf{e}(q)$ is not too big for some q. This will readily imply the two main theorems if the norms have sufficiently good properties. To conclude, a transversality argument (similar to the arguments in [7]) will show that, for almost all functions f (in the sense of Theorem 4), $\mathbf{e}(q)$ is not too big for some q.

Henceforth, and until the end of Section 4, we fix a large integer q. By definition, there exists $p_0 \geq 1$ such that $\mathbf{e}(q, p) = \mathbf{e}(q)$ for $p \geq p_0$. We also fix an integer $p \geq p_0$.

3 Perron-Frobenius operator and the norm $\|\cdot\|_\rho^\dagger$

Let $C^r(D)$ be the set of C^r functions on $S^1 \times \mathbb{R}$ whose supports are contained in D. In this section, we define preliminary norms on the space $C^r(D)$ and show Lasota-Yorke type inequalities for them. For the definition of the norms, we prepare a class Ω of C^r curves on $S^1 \times \mathbb{R}$. Let $\gamma : \mathcal{D}(\gamma) \to S^1 \times \mathbb{R}$ be a continuous curve on $S^1 \times \mathbb{R}$ whose domain of definition $\mathcal{D}(\gamma)$ is a compact interval. For $n \geq 0$, there are ℓ^n curves $\tilde{\gamma}_i : \mathcal{D}(\gamma) \to S^1 \times \mathbb{R}$, $1 \leq i \leq \ell^n$, such that $T^n \circ \tilde{\gamma}_i = \gamma$, each of which is called a backward image of γ by T^n. From the hyperbolic properties of T, we can choose positive constants c_i, $1 \leq i \leq r$, so that the following holds: Let Ω be the set of C^r curves $\gamma : \mathcal{D}(\gamma) \to S^1 \times \mathbb{R}$ such that

- the domain of definition $\mathcal{D}(\gamma)$ is a compact interval,
- γ is written in the form $\gamma(t) = (\pi \circ \gamma(t), t)$ and
- $|d^i(\pi \circ \gamma)/dt^i(s)| \leq c_i$ for $1 \leq i \leq r$ and $s \in \mathcal{D}(\gamma)$

where $\pi : S^1 \times \mathbb{R} \to S^1$ is the projection to the first component. Then each backward image $\tilde{\gamma}$ of any $\gamma \in \Omega$ by T^n with $n \geq 1$ is the composition $\hat{\gamma} \circ g$ of a curve $\hat{\gamma} \in \Omega$ and a C^r diffeomorphism $g : \mathcal{D}(\gamma) \to \mathcal{D}(\hat{\gamma})$. Further, we can take a positive constant c so that the diffeomorphism g always satisfies

$$\left| \frac{d^\nu}{ds^\nu}(g^{-1})(s) \right| < c\lambda^n \quad \text{for } s \in \mathcal{D}(\hat{\gamma}) \text{ and } 1 \leq \nu \leq r. \tag{5}$$

We henceforth fix such c, c_i, $1 \leq i \leq r$, and Ω as above. Moreover, the cone

$$\mathbf{C} = \{(u, v) \mid |u| \leq \alpha_0^{-1}|v|\} \tag{6}$$

is invariant under DT^{-1}, whence we can take $c_1 = \alpha_0^{-1}$. Finally, increasing the constants $c_2, \ldots, c_r$ if necessary, we can assume that, whenever I is a segment in $S^1 \times \mathbb{R}$ and J is a component of $T^{-q}(I)$ such that its tangent vectors are all contained in $\mathbf{C}$, then J is the image of an element of Ω (recall that q is fixed once and for all until the end of Section 4).

For a function $h \in C^r(D)$ and an integer $0 \leq \rho \leq r-1$, we define

$$\|h\|_\rho^\dagger := \max_{\alpha+\beta \leq \rho} \sup_{\gamma \in \Omega} \sup_{\varphi \in \mathcal{C}^{\alpha+\beta}(\gamma)} \int \varphi(t) \cdot \partial_x^\alpha \partial_y^\beta h(\gamma(t)) dt$$

where $\max_{\alpha+\beta \leq \rho}$ denotes the maximum over pairs (α, β) of non-negative integers such that $\alpha + \beta \leq \rho$ and $\mathcal{C}^s(\gamma)$ denotes the space of C^s functions φ on $\mathbb{R}$ such that $\operatorname{supp} \varphi \subset \operatorname{Int}(\mathcal{D}(\gamma))$ and $\|\varphi\|_{C^s} \leq 1$. This is a norm on $C^r(D)$. It satisfies

$$\|h\|_{L^1} \leq C\,\|h\|_0^\dagger \leq C\,\|h\|_\rho^\dagger. \tag{7}$$

The following lemma is the main ingredient of this section.

Lemma 5. *There exists a constant A_0 such that*

$$\|P^n h\|_\rho^\dagger \le A_0 \ell^{-\rho n}\|h\|_\rho^\dagger + C(n)\|h\|_{\rho-1}^\dagger \quad \text{for } 1 \le \rho \le r-1, \tag{8}$$

and

$$\|P^n h\|_0^\dagger \le A_0 \|h\|_0^\dagger \tag{9}$$

for $n \ge 0$ and $h \in C^r(D)$, where $C(n)$ may depend on n but not on h.

Proof. Note that the iterate T^n for $n \ge 0$ is locally written in the form

$$T^n(x,y) = (\ell^n x, \lambda^n y + S(\ell^n x)) \tag{10}$$

where S is a C^r function whose derivatives up to order r are bounded by α_0. Consider non-negative integers ρ, α, β satisfying $1 \le \rho \le r-1$ and $\alpha + \beta = \rho$. Differentiating both sides of

$$P^n h(x,y) = \frac{1}{\lambda^n \ell^n} \sum_{(x',y') \in T^{-n}(x,y)} h(x',y')$$

by using (10), we see that the differential $\partial_x^\alpha \partial_y^\beta P^n h(x,y)$ can be written as the sum of

$$\Phi(x,y) = \sum_{(x',y') \in T^{-n}(x,y)} \sum_{k=0}^{\alpha} Q_k(x) \frac{\partial_x^{\alpha-k} \partial_y^{\beta+k} h(x',y')}{\lambda^{(1+\beta+k)n} \ell^{(1+\alpha-k)n}}$$

and

$$\Psi(x,y) = \sum_{(x',y') \in T^{-n}(x,y)} \sum_{a+b \le \rho-1} Q_{a,b}(x) \frac{\partial_x^a \partial_y^b h(x',y')}{\lambda^{(1+b)n} \ell^{(1+a)n}}$$

where $Q_k(\cdot)$ and $Q_{a,b}(\cdot)$ are functions of class C^ρ and C^{a+b} respectively.[1] It is easy to check that the C^ρ norm of $Q_k(\cdot)$ and C^{a+b} norm of $Q_{a,b}(\cdot)$ are bounded by some constant.

For $\gamma \in \Omega$ and $\varphi \in \mathcal{C}^\rho(\gamma)$, we estimate

$$\int \varphi(t) \partial_x^\alpha \partial_y^\beta P^n h(\gamma(t)) dt = \int \varphi(t) \Phi(\gamma(t)) dt + \int \varphi(t) \Psi(\gamma(t)) dt. \tag{11}$$

Let γ_i, $1 \le i \le \ell^n$, be the backward images of the curve γ by T^n and write them as the composition $\hat{\gamma}_i \circ g_i$ of $\hat{\gamma}_i \in \Omega$ and a C^r diffeomorphism g_i. Then we have

$$\int \varphi(t) \Psi(\gamma(t)) dt = \sum_{1 \le i \le \ell^n} \sum_{a+b \le \rho-1} \int \varphi(t) \frac{Q_{a,b}(\pi \circ \gamma(t)) \cdot \partial_x^a \partial_y^b h(\gamma_i(t))}{\lambda^{(1+b)n} \ell^{(1+a)n}} dt$$

$$= \sum_{1 \le i \le \ell^n} \sum_{a+b \le \rho-1} \int \frac{\varphi(g_i^{-1}(s)) \cdot Q_{a,b}(\pi \circ \gamma \circ g_i^{-1}(s)) \cdot (g_i^{-1})'(s) \cdot \partial_x^a \partial_y^b h(\hat{\gamma}_i(s))}{\lambda^{(1+b)n} \ell^{(1+a)n}} ds.$$

[1] Strictly speaking, the functions Q_k and $Q_{a,b}$ are only defined on the ℓ^n-fold covering of S^1, since their definition involves the choice of an inverse branch, but we will keep the dependence on the choice of the inverse branch implicit in the notation.

Since the C^{a+b} norm of the function $s \mapsto \varphi(g_i^{-1}(s)) \cdot Q_{a,b}(\pi \circ \gamma \circ g_i^{-1}(s)) \cdot (g_i^{-1})'(s)$ is bounded by some constant (depending on n) from (5), we have

$$\left| \int \varphi(t)\Psi(\gamma(t))dt \right| \le C(n)\|h\|^{\dagger}_{\rho-1} \tag{12}$$

where $C(n)$ may depend on n but not on h.

The first integral on the right hand side of (11) is written as

$$\begin{aligned}\int \varphi(t)\Phi(\gamma(t))dt &= \sum_{1\le i\le \ell^n} \sum_{k=0}^{\alpha} \int \varphi(t) \frac{Q_k(\pi\circ\gamma_i(t)) \cdot \partial_x^{\alpha-k}\partial_y^{\beta+k} h(\gamma_i(t))}{\lambda^{(1+\beta+k)n}\ell^{(1+\alpha-k)n}} dt \\ &= \sum_{1\le i\le \ell^n} \sum_{k=0}^{\alpha} \int \frac{\varphi(g_i^{-1}(s)) \cdot Q_k(\pi\circ\gamma\circ g_i^{-1}(s)) \cdot (g_i^{-1})'(s) \cdot \partial_x^{\alpha-k}\partial_y^{\beta+k} h(\hat{\gamma}_i(s))}{\lambda^{(1+\beta+k)n}\ell^{(1+\alpha-k)n}} ds.\end{aligned}$$

For a while, we fix $1 \le i \le \ell^n$. Since

$$\begin{aligned}&\frac{d}{ds}(\partial_x^{\alpha-k}\partial_y^{\beta+k-1} h(\hat{\gamma}_i(s))) \\ &\quad = (\pi\circ\hat{\gamma}_i)'(s) \cdot \partial_x^{\alpha-k+1}\partial_y^{\beta+k-1} h(\hat{\gamma}_i(s)) + \partial_x^{\alpha-k}\partial_y^{\beta+k} h(\hat{\gamma}_i(s)),\end{aligned}$$

integration by part yields, for any $\psi \in C^\rho(\mathcal{D}(\hat{\gamma}_i))$,

$$\begin{aligned}\int \frac{d\psi}{ds}(s) \cdot \frac{\partial_x^{\alpha-k}\partial_y^{\beta+k-1} h(\hat{\gamma}_i(s))}{\lambda^{(1+\beta+k)n}\ell^{(1+\alpha-k)n}} ds = &- \int \tilde{\psi}(s) \frac{\partial_x^{\alpha-k+1}\partial_y^{\beta+k-1} h(\hat{\gamma}_i(s))}{\lambda^{(1+\beta+k-1)n}\ell^{(1+\alpha-k+1)n}} ds \\ &- \int \psi(s) \frac{\partial_x^{\alpha-k}\partial_y^{\beta+k} h(\hat{\gamma}_i(s))}{\lambda^{(1+\beta+k)n}\ell^{(1+\alpha-k)n}} ds\end{aligned}$$

where $\tilde{\psi}(s) = \lambda^{-n}\ell^n(\pi\circ\hat{\gamma}_i)'(s)\psi(s)$. This implies

$$\begin{aligned}\left| \int \psi(s) \frac{\partial_x^{\alpha-k}\partial_y^{\beta+k} h(\hat{\gamma}_i(s))}{\lambda^{(1+\beta+k)n}\ell^{(1+\alpha-k)n}} ds \right| \le &\left| \int \tilde{\psi}(s) \frac{\partial_x^{\alpha-k+1}\partial_y^{\beta+k-1} h(\hat{\gamma}_i(s))}{\lambda^{(1+\beta+k-1)n}\ell^{(1+\alpha-k+1)n}} ds \right| \\ &+ C(n)\|\psi\|_{C^\rho}\|h\|^{\dagger}_{\rho-1}\end{aligned} \tag{13}$$

where $C(n)$ may depend on n but not on h nor ψ. Put

$$\psi_0(s) = \varphi(g_i^{-1}(s)) \cdot Q_k(\pi\circ\gamma\circ g_i^{-1}(s)) \cdot (g_i^{-1})'(s).$$

By using the last inequality repeatedly, we obtain

$$\begin{aligned}\left| \int \frac{\psi_0(s)\partial_x^{\alpha-k}\partial_y^{\beta+k} h(\hat{\gamma}_i(s))}{\lambda^{(1+\beta+k)n}\ell^{(1+\alpha-k)n}} ds \right| \le &\left| \int \frac{\psi_{\beta+k}(s)\partial_x^{\rho} h(\hat{\gamma}_i(s))}{\lambda^n \ell^{(1+\rho)n}} ds \right| \\ &+ \sum_{j=0}^{\beta+k-1} C(n)\,\|\psi_j\|_{C^\rho}\,\|h\|^{\dagger}_{\rho-1}\end{aligned}$$

where $\psi_j(s) = \lambda^{-nj}\ell^{nj}((\pi\circ\hat{\gamma}_i)'(s))^j\psi_0(s) = \lambda^{-nj}((\pi\circ\gamma\circ g_i^{-1})'(s))^j\psi_0(s)$. Since $\|\psi_j\|_{C^\rho} < C_0\lambda^n$ for $0\le j\le\beta+k$ for some constant C_0 from (5), we get

$$\left|\int \psi_0(s)\frac{\partial_x^{\alpha-k}\partial_y^{\beta+k}h(\hat{\gamma}_i(s))}{\lambda^{(1+\beta+k)n}\ell^{(1+\alpha-k)n}}dt\right| \le C_0\ell^{-(1+\rho)n}\|h\|_\rho^\dagger + C(n)\|h\|_{\rho-1}^\dagger.$$

Summing up this inequality for γ_i, $1\le i\le \ell^n$, we obtain

$$\left|\int \varphi(t)\Phi(\gamma(t))dt\right| \le C_0\ell^{-\rho n}\|h\|_\rho^\dagger + C(n)\|h\|_{\rho-1}^\dagger$$

for some constant C_0. This and (12) give (8). The proof of (9) is obtained in a similar but much simpler manner. □

4 Main Lasota-Yorke inequality

In this section, we prove the following proposition.

Proposition 6. *There exists a constant B_0 independent of q and a constant $C(q)$ such that, for all $\varphi\in C^r(D)$, for all integer ρ_0 with $s+1<\rho_0\le r-1$,*

$$\|P^q\varphi\|_{W^s}^2 \le \frac{B_0\mathbf{e}(q)}{(\lambda^{1+2s}\ell)^q}\|\varphi\|_{W^s}^2 + C(q)\|\varphi\|_{W^s}\|\varphi\|_{\rho_0}^\dagger.$$

First of all, we introduce some notation and prove some elementary facts concerning the Sobolev norm $\|\cdot\|_{W^s}$. The Fourier transform $\mathcal{F}\varphi$ of $\varphi\in C^r(D)$ is a function on $\mathbb{Z}\times\mathbb{R}$ defined by

$$\mathcal{F}\varphi(\xi,\eta) = \frac{1}{\sqrt{2\pi}}\int_{S^1\times\mathbb{R}}\varphi(x,y)\exp\left(-\mathbf{i}(2\pi\xi x+\eta y)\right)dxdy.$$

For $s\ge 0$ and for $\varphi_1,\varphi_2\in C^r(D)$, we define

$$(\varphi_1,\varphi_2)_{W^s} := (\varphi_1,\varphi_2)_{W^s}^* + (\varphi_1,\varphi_2)_{L^2}$$

where

$$(\varphi_1,\varphi_2)_{W^s}^* := \sum_{\xi=-\infty}^{\infty}\int_{\mathbb{R}}\mathcal{F}\varphi_1(\xi,\eta)\cdot\overline{\mathcal{F}\varphi_2(\xi,\eta)}\cdot((2\pi\xi)^2+\eta^2)^s d\eta.$$

The Sobolev norm is defined by $\|\varphi\|_{W^s} = \sqrt{(\varphi,\varphi)_{W^s}}$. Note that we have

$$(\varphi_1,\varphi_2)_{W^s}^* = \sum_{\alpha+\beta=[s]} b_{\alpha\beta}(\partial_x^\alpha\partial_y^\beta\varphi_1, \partial_x^\alpha\partial_y^\beta\varphi_2)_{W^{s-[s]}}^* \tag{14}$$

where $b_{\alpha\beta}$ are positive integers satisfying $(X^2+Y^2)^{[s]} = \sum_{\alpha,\beta}b_{\alpha\beta}X^{2\alpha}Y^{2\beta}$. Especially, if s is an integer, we have

$$(\varphi_1,\varphi_2)_{W^s}^* = \sum_{\alpha+\beta=s} b_{\alpha\beta}\int_{S^1\times\mathbb{R}}\partial_x^\alpha\partial_y^\beta\varphi_1(x,y)\cdot\overline{\partial_x^\alpha\partial_y^\beta\varphi_2(x,y)}dxdy. \tag{15}$$

In case s is not an integer, we shall use the following formula ([4, pp 240]): there exists a constant $B > 0$ that depends only on $0 < \sigma < 1$ such that

$$(\varphi_1, \varphi_2)^*_{W^\sigma} = \tag{16}$$
$$B \int_{S^1\times\mathbb{R}} dxdy \int_{\mathbb{R}^2} \frac{(\varphi_1(x+u, y+v) - \varphi_1(x,y))\overline{(\varphi_2(x+u, y+v) - \varphi_2(x,y))}}{(u^2+v^2)^{1+\sigma}} dudv.$$

Lemma 7. (1) *For $0 \le t < s \le r$ and $\epsilon > 0$, there is a constant $C(\epsilon, t, s)$ such that*

$$\|\varphi\|^2_{W^t} \le \epsilon \|\varphi\|^2_{W^s} + C(\epsilon, t, s)\|\varphi\|^2_{L^1} \quad \text{for } \varphi \in C^r(D).$$

(2) *For $\epsilon > 0$, there exists a constant $C(\epsilon, s)$ with the following property: if the supports of functions $\varphi_1, \varphi_2 \in C^r(D)$ are disjoint and the distance between them is greater than ϵ, it holds*

$$\left|(\varphi_1, \varphi_2)_{W^s}\right| \le C(\epsilon, s)\|\varphi_1\|_{L^1}\|\varphi_2\|_{L^1}.$$

Proof. (1) follows from the definition of the norm and the fact $\|\mathcal{F}\varphi\|_{L^\infty} \le \|\varphi\|_{L^1}$. If s is an integer, (2) is trivial since $(\varphi_1, \varphi_2)_s = 0$ by (15). Suppose that s is not an integer. Using (14) and (16) with the assumption on the disjointness of the supports and changing variables, we can rewrite $(\varphi_1, \varphi_2)^*_{W^s}$ as

$$-2B \sum_{\alpha+\beta=[s]} \int_{S^1\times\mathbb{R}} dxdy \int_{\mathbb{R}^2} \frac{b_{\alpha\beta} \cdot \partial_x^\alpha \partial_y^\beta \varphi_1(x+u, y+v) \cdot \overline{\partial_x^\alpha \partial_y^\beta \varphi_2(x,y)}}{(u^2+v^2)^{1+\sigma}} dudv$$

where $\sigma = s - [s]$. Integrating $[s]$ times by part on (u, v), then changing variables and integrating again $[s]$ times by part, we obtain

$$(\varphi_1, \varphi_2)^*_{W^s} = \int_{S^1\times\mathbb{R}} dxdy \int_{\mathbb{R}^2} \frac{\varphi_1(x+u, y+v)\overline{\varphi_2(x,y)}\tilde{B}(u,v)}{(u^2+v^2)^{1+\sigma+2[s]}} dudv$$

where $\tilde{B}(u, v)$ is a polynomial of u and v of order $2[s]$. With this and the assumption, we can conclude the inequality in (2). $\square$

The norm $\|\cdot\|^\dagger$ will be used through the following lemma. Let $\mathbf{C}^*$ be the cone in $\mathbb{R}^2$ defined by

$$\mathbf{C}^* = \{(\xi, \eta) \in \mathbb{R}^2 \mid |\eta| \le {\alpha_0}^{-1}|\xi|\},$$

so that $DT^*_{\mathbf{x}}(\mathbf{C}^*) \subset \mathbf{C}^*$ for $\mathbf{x} \in S^1 \times \mathbb{R}$.

Lemma 8. *Let ρ_0 be an integer with $s + 1 < \rho_0 \le r - 1$. Let $\mathbf{a}$ and $\mathbf{c}$ elements of $\mathcal{A}^q$ and $\mathcal{A}^p$ respectively, and $\chi : S^1 \times \mathbb{R} \to \mathbb{R}$ a C^∞ function supported on $\mathcal{P}_*(\mathbf{ca}) \times \mathbb{R}$. Take $(\xi, \eta) \in \mathbb{Z} \times \mathbb{R}\backslash\{(0,0)\}$ such that, for any $\mathbf{x} \in \mathcal{P}_*(\mathbf{ca}) \times \mathbb{R}$, $(DT^q_{\mathbf{x}})^*(\xi, \eta) \in \mathbf{C}^*$. Then, for any $\varphi \in C^r$,*

$$\left|(\xi^2+\eta^2)^{\rho_0/2}\mathcal{F}(P^q(\chi\cdot\varphi))(\xi,\eta)\right| \le C(q,\chi)\,\|\varphi\|^\dagger_{\rho_0}, \tag{17}$$

where $C(q, \chi)$ may depend on q and χ.

Proof. Let (ξ, η) be a vector satisfying the assumption. Let Γ be the set of line segments on $S^1 \times \mathbb{R}$ that are the intersection of a line normal to (ξ, η) with the region $\mathcal{P}_*(\mathbf{c}) \times \mathbb{R}$. We parametrize the segments in Γ by length. Since the support of $P^q(\chi \cdot \varphi)$ is contained in $D \cap (\mathcal{P}_*(\mathbf{c}) \times \mathbb{R})$, the left hand side of (17) is bounded by some constant multiple of

$$\sup_{\gamma \in \Gamma} \int_\gamma \partial^{\rho_0} P^q(\chi \cdot \varphi) dt \tag{18}$$

where ∂ is partial derivative with respect to x if $|\xi| > |\eta|$ and that with respect to y otherwise. For each $\gamma \in \Gamma$, there exists a unique backward image $\tilde{\gamma}$ of T^q that is contained in $\mathcal{P}_*(\mathbf{ca}) \times \mathbb{R}$. If $\mathbf{x} \in \tilde{\gamma}$ and u is tangent to γ at $T^q(\mathbf{x})$, then

$$0 = \langle u, (\xi, \eta) \rangle = \langle (DT^q_{\mathbf{x}})^{-1} u, (DT^q_{\mathbf{x}})^*(\xi, \eta) \rangle.$$

By assumption, $(DT^q_{\mathbf{x}})^*(\xi, \eta) \in \mathbf{C}^*$, whence $(DT^q_{\mathbf{x}})^{-1} u \in \mathbf{C}$ (by definition (6) of $\mathbf{C}$). Hence, $\tilde{\gamma}$ is the composition $\hat{\gamma} \circ \psi$ of an element $\hat{\gamma}$ of Ω and a C^r diffeomorphism ψ. By obvious estimates on the distortion of T^m for $0 \le m \le q$ and by the definition of the norm $\| \cdot \|^\dagger_{\rho_0}$, we get that (18) is bounded by $C \|\varphi\|^\dagger_{\rho_0}$. $\square$

Let $\{\chi_{\mathbf{c}} : S^1 \to \mathbb{R}\}_{\mathbf{c} \in \mathcal{A}^p}$ be a C^∞ partition of unity subordinate to the covering $\{\operatorname{Int} \mathcal{P}_*(\mathbf{c})\}_{\mathbf{c} \in \mathcal{A}^p}$, whence $\operatorname{supp}(\chi_{\mathbf{c}}) \subset \operatorname{Int} \mathcal{P}_*(\mathbf{c})$. Define a function $\chi_{\mathbf{ca}}$ by $\chi_{\mathbf{ca}}(\tau^{-q}_{\mathbf{c},\mathbf{a}} x) = \chi_{\mathbf{c}}(x)$ if $x \in \mathcal{P}_*(\mathbf{c})$, and extend it by 0 elsewhere. Then the functions $\chi_{\mathbf{ca}}$ for $(\mathbf{a}, \mathbf{c}) \in \mathcal{A}^q \times \mathcal{A}^p$ are again a C^∞ partition of unity. To keep the notation simple, we will still use $\chi_{\mathbf{c}}$ and $\chi_{\mathbf{ca}}$ to denote $\chi_{\mathbf{c}} \circ \pi$ and $\chi_{\mathbf{ca}} \circ \pi$.

Lemma 9. *There is a constant $C > 0$ such that, for any $\varphi \in C^r(D)$, it holds*

$$\sum_{(\mathbf{a},\mathbf{c}) \in \mathcal{A}^q \times \mathcal{A}^p} \|\chi_{\mathbf{ca}} \varphi\|^2_{W^s} \le 2\|\varphi\|^2_{W^s} + C\|\varphi\|^2_{L^1} \tag{19}$$

and

$$\|\varphi\|^2_{W^s} \le 7 \sum_{\mathbf{c} \in \mathcal{A}^p} \|\chi_{\mathbf{c}} \varphi\|^2_{W^s} + C\|\varphi\|^2_{L^1}. \tag{20}$$

Proof. Since the claims are obvious when $s = 0$, we assume $s > 0$. Let t be the largest integer that is (strictly) less than s. Then for every $\epsilon > 0$ we have

$$\sum_{(\mathbf{a},\mathbf{c}) \in \mathcal{A}^q \times \mathcal{A}^p} \|\chi_{\mathbf{ca}} \varphi\|^2_{W^s} \le (1 + \epsilon)\|\varphi\|^2_{W^s} + C(\epsilon)\|\varphi\|^2_{W^t}.$$

Indeed, we can check this by using (15) if s is an integer and by using (14) and (16) instead of (15) otherwise. Hence (19) follows from lemma 7(1).

From lemma 7(2), we have $(\chi_{\mathbf{c}}\varphi, \chi_{\mathbf{c}'}\varphi)_{W^s} \le C\|\chi_{\mathbf{c}}\varphi\|_{L^1}\|\chi_{\mathbf{c}'}\varphi\|_{L^1} \le C\|\varphi\|^2_{L^1}$ for some constant $C > 0$ if the closures of $\mathcal{P}_*(\mathbf{c})$ and $\mathcal{P}_*(\mathbf{c}')$ do not intersect. Also we have $(\chi_{\mathbf{c}}\varphi, \chi_{\mathbf{c}'}\varphi)_{W^s} \le (\|\chi_{\mathbf{c}}\varphi\|^2_{W^s} + \|\chi_{\mathbf{c}'}\varphi\|^2_{W^s})/2$ in general. Applying these to $\|\varphi\|^2_{W^s} = \sum_{(\mathbf{c},\mathbf{c}') \in \mathcal{A}^p \times \mathcal{A}^p} (\chi_{\mathbf{c}}\varphi, \chi_{\mathbf{c}'}\varphi)_{W^s}$, we obtain (20). $\square$

We start the proof of Proposition 6. From (20), we have

$$\begin{aligned}\|P^q(\varphi)\|_{W^s}^2 &\le 7\sum_{\mathbf{c}\in\mathcal{A}^p}\|\chi_{\mathbf{c}}P^q(\varphi)\|_{W^s}^2 + C\|\varphi\|_{L^1}^2\\ &\le 7\sum_{\mathbf{c}\in\mathcal{A}^p}\left\|\sum_{\mathbf{a}\in\mathcal{A}^q}P^q(\chi_{\mathbf{ca}}\varphi)\right\|_{W^s}^2 + C\|\varphi\|_{L^1}^2.\end{aligned}$$

So we will estimate

$$\left\|\sum_{\mathbf{a}\in\mathcal{A}^q}P^q(\chi_{\mathbf{ca}}\varphi)\right\|_{W^s}^2 = \sum_{(\mathbf{a},\mathbf{b})\in\mathcal{A}^q\times\mathcal{A}^q}(P^q(\chi_{\mathbf{ca}}\varphi),P^q(\chi_{\mathbf{cb}}\varphi))_{W^s}$$

for $\mathbf{c}\in\mathcal{A}^p$.

Consider first a pair $(\mathbf{a},\mathbf{b})\in\mathcal{A}^q\times\mathcal{A}^q$ such that $\mathbf{a}\pitchfork_{\mathbf{c}}\mathbf{b}$. For any $(\xi,\eta)\in\mathbb{Z}\times\mathbb{R}\backslash\{(0,0)\}$, this implies that either $(DT^q_{\mathbf{x}})^*(\xi,\eta)\in\mathbf{C}^*$ for all $\mathbf{x}\in\mathcal{P}_*(\mathbf{ca})\times\mathbb{R}$, or $(DT^q_{\mathbf{x}})^*(\xi,\eta)\in\mathbf{C}^*$ for all $\mathbf{x}\in\mathcal{P}_*(\mathbf{cb})\times\mathbb{R}$. Let U be the set of all $(\xi,\eta)\in\mathbb{Z}\times\mathbb{R}$ such that the first possibility holds, and $V=(\mathbb{Z}\times\mathbb{R})\backslash U$. If $(\xi,\eta)\in U$, by Lemma 8, there exists a constant $C>0$ such that $|\mathcal{F}P^q(\chi_{\mathbf{ca}}\varphi)(\xi,\eta)|\le C(\xi^2+\eta^2)^{-\rho_0/2}\|\varphi\|^\dagger_{\rho_0}$. Moreover, $|\mathcal{F}P^q(\chi_{\mathbf{ca}}\varphi)(\xi,\eta)|\le C\|\varphi\|_{L^1}$, which is bounded by $C\|\varphi\|^\dagger_{\rho_0}$ by (7). Hence, $|\mathcal{F}P^q(\chi_{\mathbf{ca}}\varphi)(\xi,\eta)|\le C(1+\xi^2+\eta^2)^{-\rho_0/2}\|\varphi\|^\dagger_{\rho_0}$. So we have, for some constant C,

$$\begin{aligned}&\left|\sum_{\xi=-\infty}^{\infty}\int\mathbf{1}_U(\xi,\eta)\cdot(1+\xi^2+\eta^2)^s\mathcal{F}P^q(\chi_{\mathbf{ca}}\varphi)\cdot\overline{\mathcal{F}P^q(\chi_{\mathbf{cb}}\varphi)}\,d\eta\right|\\ &\le C\left(\sum_{\xi=-\infty}^{\infty}\int\mathbf{1}_U(\xi,\eta)\cdot(1+\xi^2+\eta^2)^s|\mathcal{F}P^q(\chi_{\mathbf{ca}}\varphi)|^2d\eta\right)^{1/2}\|P^q(\chi_{\mathbf{cb}}\varphi)\|_{W^s}\\ &\le C\|\varphi\|^\dagger_{\rho_0}\|\varphi\|_{W^s},\end{aligned}$$

since the function $(1+\xi^2+\eta^2)^{-\rho_0+s}$ is integrable by the assumption $s<\rho_0-1$. The same inequality holds on V, and we obtain

$$|(P^q(\chi_{\mathbf{ca}}\varphi),P^q(\chi_{\mathbf{cb}}\varphi))_{W^s}|\le C\|\varphi\|^\dagger_{\rho_0}\|\varphi\|_{W^s}. \tag{21}$$

For the sum over $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}\not\pitchfork_{\mathbf{c}}\mathbf{b}$, we have

$$\begin{aligned}\sum_{\mathbf{a}\not\pitchfork_{\mathbf{c}}\mathbf{b}}(P^q(\chi_{\mathbf{ca}}\varphi),P^q(\chi_{\mathbf{cb}}\varphi))_{W^s} &\le \sum_{\mathbf{a}\not\pitchfork_{\mathbf{c}}\mathbf{b}}\frac{\|P^q(\chi_{\mathbf{ca}}\varphi)\|_{W^s}^2+\|P^q(\chi_{\mathbf{cb}}\varphi)\|_{W^s}^2}{2}\\ &\le \mathbf{e}(q)\sum_{\mathbf{a}\in\mathcal{A}^q}\|P^q(\chi_{\mathbf{ca}}\varphi)\|_{W^s}^2.\end{aligned} \tag{22}$$

For the terms in the last sum, we have the estimate

$$\|P^q(\chi_{\mathbf{ca}}\varphi)\|_{W^s}^2\le\frac{C_0\|\chi_{\mathbf{ca}}\varphi\|_{W^s}^2}{\lambda^{(1+2s)q}\ell^q}+C\|\varphi\|_{L^1}^2 \tag{23}$$

where C_0 is a constant that depends only on λ, ℓ and κ. Indeed, we can check this by using (15) and (10) if s is an integer and by using (14) and (16) instead of (15) otherwise.

From (21), (22), (23), (19) and (7), we obtain

$$\begin{aligned}\sum_{\mathbf{c}\in\mathcal{A}^p}\left\|\sum_{\mathbf{a}\in\mathcal{A}^q}P^q(\chi_{\mathbf{ca}}\varphi)\right\|_{W^s}^2 &\le \frac{C_0\mathbf{e}(q)}{\lambda^{(1+2s)q}\ell^q}\sum_{(\mathbf{a},\mathbf{c})\in\mathcal{A}^q\times\mathcal{A}^p}\|\chi_{\mathbf{ca}}\varphi\|_{W^s}^2 + C\|\varphi\|_{W^s}\|\varphi\|_{\rho_0}^\dagger \\ &\le \frac{2C_0\mathbf{e}(q)}{\lambda^{(1+2s)q}\ell^q}\|\varphi\|_{W^s}^2 + C\|\varphi\|_{W^s}\|\varphi\|_{\rho_0}^\dagger,\end{aligned}$$

and hence Proposition 6.

5 Proof of the main theorems

We will use Lemma 5 and Proposition 6 to study the properties of P acting on the space $C^r(D)$ equipped with the norms $\|\cdot\|_{\rho_0}^\dagger$ and $\|\cdot\|_{W^s}$.

Lemma 10. *Let $\delta\in(\ell^{-1},1)$. There exists $C>0$ such that, for integer $1\le\rho\le r-1$, for $n\in\mathbb{N}$,*

$$\|P^nh\|_\rho^\dagger \le C\delta^{\rho n}\|h\|_\rho^\dagger + C\|h\|_{\rho-1}^\dagger.$$

Proof. We prove it by induction on ρ. Let $\rho\ge 1$. By Lemma 5, there exists $N\in\mathbb{N}$ and $C>0$ such that

$$\left\|P^Nh\right\|_\rho^\dagger \le \delta^{\rho N}\|h\|_\rho^\dagger + C\|h\|_{\rho-1}^\dagger. \tag{24}$$

By the inductive assumption (and Lemma 5 in the $\rho=1$ case), $\|P^nh\|_{\rho-1}^\dagger \le C\|h\|_{\rho-1}^\dagger$. Hence, iterating (24) gives the conclusion. □

Lemma 11. *Let $\delta\in(\ell^{-1},1)$, and let $0\le\rho_1<\rho_0\le r-1$ be integers. Let $\nu(\rho_0,\rho_1)=\sum_{j=\rho_1+1}^{\rho_0}\frac{1}{j}$. There exists $C>0$ such that, for $n\in\mathbb{N}$,*

$$\|P^nh\|_{\rho_0}^\dagger \le C\delta^{n/\nu(\rho_0,\rho_1)}\|h\|_{\rho_0}^\dagger + C\|h\|_{\rho_1}^\dagger.$$

Proof. Let n be a multiple of $(r-1)!$. Then Lemma 10 implies by induction over $\rho_1+1\le\rho\le\rho_0$ that

$$\left\|P^{(\frac{1}{\rho}+\cdots+\frac{1}{\rho_1+1})n}h\right\|_\rho^\dagger \le C\delta^n\|h\|_\rho^\dagger + C\|h\|_{\rho_1}^\dagger.$$

For $\rho=\rho_0$, we obtain $\left\|P^{\nu(\rho_0,\rho_1)n}h\right\|_{\rho_0}^\dagger \le C\delta^n\|h\|_{\rho_0}^\dagger + C\|h\|_{\rho_1}^\dagger$. □

Theorem 12. *Assume that $\frac{B_0\mathbf{e}(q)}{(\lambda^{1+2s}\ell)^q}<1$. Let $0\le\rho_1<\rho_0\le r-1$ be integers with $s<\rho_0-1$, and let $\nu=\nu(\rho_0,\rho_1)$ be as in the previous lemma. Let*

$$\gamma\in\left(\max\left(\ell^{-1/\nu},\sqrt{\frac{(B_0\mathbf{e}(q))^{1/q}}{\lambda^{1+2s}\ell}}\right),1\right).$$

Let $\|\varphi\| := \|\varphi\|_{W^s} + \|\varphi\|^{\dagger}_{\rho_0}$. *There exists a constant* C *such that, for all* $n \in \mathbb{N}$,

$$\|P^n \varphi\| \leq C\gamma^n \|\varphi\| + C \|\varphi\|^{\dagger}_{\rho_1} .$$

Proof. Since $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ and $\sqrt{ab} \leq \epsilon a + \epsilon^{-1} b$, Proposition 6 implies

$$\|P^q \varphi\|_{W^s} \leq \left(\frac{(B_0 \mathbf{e}(q))^{1/q}}{\lambda^{1+2s}\ell} \right)^{q/2} \|\varphi\|_{W^s} + \epsilon \|\varphi\|_{W^s} + C(\epsilon) \|\varphi\|^{\dagger}_{\rho_0} .$$

Since $\left(\frac{(B_0 \mathbf{e}(q))^{1/q}}{\lambda^{1+2s}\ell} \right)^{q/2} < \gamma^q$, taking ϵ small enough yields

$$\|P^q \varphi\|_{W^s} \leq \gamma^q \|\varphi\|_{W^s} + C \|\varphi\|^{\dagger}_{\rho_0} .$$

Iterating this equation K times gives

$$\left\|P^{Kq} \varphi\right\|_{W^s} \leq \gamma^{Kq} \|\varphi\|_{W^s} + C(K) \|\varphi\|^{\dagger}_{\rho_0} , \tag{25}$$

for some constant $C(K)$. If K is large enough, the choice of γ and Lemma 11 also yield

$$\left\|P^{Kq} \varphi\right\|^{\dagger}_{\rho_0} \leq \frac{\gamma^{Kq}}{2} \|\varphi\|^{\dagger}_{\rho_0} + C'(K) \|\varphi\|^{\dagger}_{\rho_1} . \tag{26}$$

Fix such a K, and define a norm $\|\varphi\|^* := \|\varphi\|_{W^s} + 2C(K)\gamma^{-Kq} \|\varphi\|^{\dagger}_{\rho_0}$. Adding (25) and (26) gives

$$\left\|P^{Kq} \varphi\right\|^* \leq \gamma^{Kq} \|\varphi\|^* + C \|\varphi\|^{\dagger}_{\rho_1} .$$

Iterating this equation (and remembering $\|P^n \varphi\|^{\dagger}_{\rho_1} \leq C \|\varphi\|^{\dagger}_{\rho_1}$ for some constant C independent of n, by Lemma 10), we obtain the conclusion of the theorem for the norm $\|\cdot\|^*$. Since it is equivalent to the original norm $\|\cdot\|$, this concludes the proof. □

Corollary 13. *If* $B_0 \mathbf{e}(q) < (\lambda^{1+2s}\ell)^q$, *the conclusion of Theorem 1 holds for the transformation* T.

Proof. Take $\rho_0 = r - 1$ and $\rho_1 = 0$. They satisfy the assumptions of Theorem 12 since $s < r - 2$.

We fix a non-negative function $\Psi_0 \in C^r(D)$ such that $\int \Psi_0 d\text{Leb} = 1$. Put $\nu_0 = \Psi_0 \cdot \text{Leb}$ and $\Psi_n = P^n \Psi_0$ for $n \geq 1$. From (2), the density of $T^n_* \nu_0$ is Ψ_n. As the sequence $T^n_* \nu_0$ converges to the SBR measure μ for T weakly, we have

$$\lim_{n \to \infty} (\Psi_n, \varphi)_{L^2} = \int \varphi d\mu \tag{27}$$

for any continuous function φ on $S^1 \times \mathbb{R}$ with compact support. By Theorem 12, the sequence Ψ_n for $n \geq 1$ is bounded with respect to the norm $\|\cdot\|$, hence also for the norm $\| \cdot \|_{W^s}$. Then there is a subsequence $n(i) \to \infty$ such that $\Psi_{n(i)}$ converges weakly to some element Ψ_∞ in the Hilbert space $W^s(S^1 \times \mathbb{R})$. This and (27) imply $\int \Psi_\infty \varphi d\text{Leb} = \int \varphi d\mu$ for any continuous function φ on $S^1 \times \mathbb{R}$ with compact support. Thereby the density of the SBR measure μ is $\Psi_\infty \in W^s(S^1 \times \mathbb{R})$. □

Corollary 14. *Let $1/2 < s < r-2$. Assume that $\frac{B_0\mathbf{e}(q)}{(\lambda^{1+2s}\ell)^q} < 1$. If*

$$\gamma \in \left(\sqrt{\frac{(B_0\mathbf{e}(q))^{1/q}}{\lambda^{1+2s}\ell}}, 1 \right),$$

the conclusion of Theorem 3 holds for the transformation T and this γ.

Proof. Let ρ_0 be the smallest integer such that $s < \rho_0 - 1$, and ρ_1 the largest integer such that $\rho_1 < s - 1/2$. They satisfy the assumptions of Theorem 12. Moreover, $\nu(\rho_0, \rho_1) \leq 1 + \frac{1}{2} + \frac{1}{3} < 2$. Hence, $\ell^{-1/\nu} < \frac{1}{\sqrt{\ell}} < \sqrt{\frac{(B_0\mathbf{e}(q))^{1/q}}{\lambda^{1+2s}\ell}}$.

Let $\mathcal{B}$ be the completion of $C^r(D)$ with respect to the norm $\|\cdot\|$. It is a Banach space included in $W^s(D)$ and containing $C^{r-1}(D)$. Theorem 12 gives a Lasota-Yorke inequality between $\mathcal{B}$ and the space $\mathcal{B}'$ obtained by completing $C^r(D)$ for the norm $\|\cdot\|_{\rho_1}^{\dagger}$. Hence, the result is a standard consequence of Hennion's Theorem [5], if we can prove that the unit ball of $\mathcal{B}$ is relatively compact in $\mathcal{B}'$.

The embedding of $\mathcal{B}$ in $W^s(D)$ is continuous. Let $t \in (\rho_1 + 1/2, s)$. The embedding of $W^s(D)$ in $W^t(D)$ is compact by Sobolev's embedding theorem. To conclude, it is sufficient to check that the injection $W^t(D) \to \mathcal{B}'$ is continuous. Since $t > \rho_1 + 1/2$, [1, Theorem 7.58 (iii)] (applied with $p = q = 2$, $k = 1$ and $n = 2$) proves that, for any smooth curve $\mathcal{C} \subset D$, for any $\varphi \in W^t(D)$,

$$\left\|\partial_x^\alpha \partial_y^\beta \varphi\right\|_{L^2(\mathcal{C})} \leq C(\mathcal{C}) \|\varphi\|_{W^t(D)}$$

whenever α and β are non-negative integers satisfying $\alpha + \beta \leq \rho_1$. The constant $C(\mathcal{C})$ can be chosen uniformly over all curves of Ω, and we obtain $\|\varphi\|_{\rho_1}^{\dagger} \leq C\|\varphi\|_{W^t(D)}$. $\square$

For $\beta > 0, \kappa > 0$ and $\lambda \in (0,1)$, let

$$\mathcal{E}(\beta, \kappa, \lambda) = \left\{ f \in \mathcal{U}_\kappa \; ; \; \limsup_{q \to \infty} \frac{1}{q} \log \mathbf{e}(q) > \beta \right\}.$$

Note that this definition depends on κ and λ through $\mathbf{e}(q)$, since $\mathbf{e}(q)$ is defined in terms of $\alpha_0 = \kappa/(1-\lambda)$.

Since the quantity $\mathbf{e}(q)$ depends on $f \in \mathcal{U}_\kappa$ upper semi-continuously and since we can take arbitrarily large κ in the beginning, Theorems 1, 3 and 4 follow from Corollaries 13 and 14 and the next proposition.

Proposition 15. *For any $\beta > 0$ and $\lambda > 0$, there is a finite collection of C^∞ functions $\varphi_i : S^1 \to \mathbb{R}$, $i = 1, 2, \cdots, m$ and a constant $D_0 > 0$ such that, for any $\kappa > D_0$ and any C^r function $g \in \mathcal{U}_{\kappa - D_0}$, the subset*

$$\left\{ (t_1, t_2, \cdots, t_m) \in [-1,1]^m \;\middle|\; g + \sum_{i=1}^m t_i \varphi_i \in \mathcal{E}(\beta, \kappa, \lambda) \right\}$$

is a Lebesgue null subset on $[-1,1]^m$.

This proposition has essentially been proved in [7]. For completeness, we give a proof of it in the next section.

6 Genericity of the transversality condition

In this section, we give a proof of Proposition 15. For a C^2 function g and C^∞ functions φ_i, $1 \le i \le m$, on S^1, we consider a family of functions

$$f_{\mathbf{t}}(x) = g(x) + \sum_{i=1}^{m} t_i \varphi_i(x) : S^1 \to \mathbb{R} \tag{28}$$

and the corresponding family of maps

$$T_{\mathbf{t}} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}, \qquad T_{\mathbf{t}}(x, y) = (\ell x, \lambda y + f_{\mathbf{t}}(x)) \tag{29}$$

with parameters $\mathbf{t} = (t_1, t_2, \cdots, t_m) \in [-1, 1]^m \subset \mathbb{R}^m$. Put

$$S(x, \mathbf{a}; \mathbf{t}) = \sum_{i=1}^{n} \lambda^{i-1} f_{\mathbf{t}}([\mathbf{a}]_i(x)) \tag{30}$$

for $\mathbf{t} \in [-1, 1]^m$ and a word $\mathbf{a} \in \mathcal{A}^n$ of length $1 \le n \le \infty$. For a point $x \in S^1$ and a sequence $\sigma = (\mathbf{a}_0, \mathbf{a}_1, \cdots, \mathbf{a}_k)$ of elements in $\mathcal{A}^\infty$, we consider an affine map $G_{x,\sigma} : \mathbb{R}^m \to \mathbb{R}^k$ defined by

$$G_{x,\sigma}(\mathbf{t}) = \left(\frac{d}{dx} S(x, \mathbf{a}_i; \mathbf{t}) - \frac{d}{dx} S(x, \mathbf{a}_0; \mathbf{t}) \right)_{i=1,2,\cdots,k}. \tag{31}$$

If the affine map $G_{x,\sigma}$ is surjective, we define its Jacobian by

$$\mathbf{Jac}(G_{x,\sigma}) = \frac{\mathrm{Leb}_k([0,1]^k)}{\mathrm{Leb}_k(G_{x,\sigma}^{-1}([0,1]^k) \cap \mathrm{Ker}(G_{x,\sigma})^\perp)}$$

where Leb_k is the k-dimensional Hausdorff measure and $\mathrm{Ker}(G_{x,\sigma})^\perp$ is the orthogonal complement of the kernel of the linear part of $G_{x,\sigma}$, whence

$$\mathrm{Leb}(G_{x,\sigma}^{-1}(Y) \cap [-1, 1]^m) \le C_0 \frac{\mathrm{Leb}(Y)}{\mathbf{Jac}(L)} \quad \text{for any Borel subset } Y \subset \mathbb{R}^k \tag{32}$$

where C_0 is a constant that depends only on the dimensions m and k.

For $0 < \gamma \le 1$, $\delta > 0$ and $n \ge 1$, we say that the family $T_{\mathbf{t}}^n$ is (γ, δ)-*generic* if the following property holds: for any finite sequence $\{\mathbf{a}_i\}_{i=0}^d$ in $\mathcal{A}^\infty$ such that $[\mathbf{a}_i]_n$ are mutually distinct, for any $x \in S^1$ and for any integer $0 < k < \gamma d$, we can choose a subsequence $\sigma = (\mathbf{b}_0, \mathbf{b}_1, \cdots, \mathbf{b}_k)$ of length k among $\{\mathbf{a}_i\}_{i=0}^d$ so that $G_{x,\sigma}$ is surjective and satisfies $\mathbf{Jac}(G_{x,\sigma}) > \delta$. It is proved in [7] that

Proposition 16 ([7], Proposition 15). *For given $0 < \lambda < 1$, $\ell \ge 2$ and $n \ge 1$, there exists a finite collection of C^∞ functions φ_i, $1 \le i \le m$, such that the corresponding family $T_{\mathbf{t}}^n$ is $(1/(n+1), 1/2)$-generic, regardless of the C^2 function g.*

Recall that we are considering fixed λ and ℓ. Let $\beta > 0$ be the positive number in the statement of Proposition 15. We can and do take integers $N_0 \ge 2$, $d_0 \ge 2$ and $n_0 \ge 1$ such that

$$\lambda^{N_0-1}\ell^2 < 1, \quad d_0/(n_0+1) > N_0+1 \quad \text{and} \quad (d_0+1)\exp(-\beta n_0/2) < 1/2. \tag{33}$$

Let φ_i, $1 \le i \le m$, be the C^∞ functions in the conclusion of Proposition 16 for these λ, ℓ and $n = n_0$. Let $D_0 = \sum_{i=1}^m \|\varphi_i\|_{C^r}$. Hence, if $g \in \mathcal{U}_{\kappa - D_0}$ and $(t_1, \ldots, t_m) \in [-1,1]^m$, then $g + \sum t_i\varphi_i \in \mathcal{U}_\kappa$. In order to prove the conclusion of Proposition 15, we pick arbitrary $g \in \mathcal{U}_{\kappa-D_0}$ and consider the family $T_{\mathbf{t}}$ defined by (28) and (29).

For an integer q, we put $p(q) = [q\log(\ell/\lambda)/\log\ell] + 1$. For a word $\mathbf{c}$ of finite length, let $x_{\mathbf{c}}$ be the left end of $\mathcal{P}(\mathbf{c})$. We fix a word $\mathbf{a}_\infty \in \mathcal{A}^\infty$ arbitrarily and, for any word $\mathbf{a}$ of finite length, we put $\bar{\mathbf{a}} = \mathbf{a}\mathbf{a}_\infty$.

Lemma 17. *If $f_{\mathbf{t}} \in \mathcal{E}(\beta,\kappa)$, we can take arbitrarily large integer q such that there exist $1 + d_0$ words $\mathbf{a}_i$, $0 \le i \le d_0$, in $\mathcal{A}^q$ and a word $\mathbf{c} \in \mathcal{A}^{p(q)}$ satisfying*

(E1) $|\frac{d}{dx}S(x_{\mathbf{c}}, \bar{\mathbf{a}}_i; \mathbf{t}) - \frac{d}{dx}S(x_{\mathbf{c}}, \bar{\mathbf{a}}_j; \mathbf{t})| \le 8\lambda^q\ell^{-q}\alpha_0$ *for any $1 \le i, j \le d_0$, and*

(E2) $[\mathbf{a}_i]_{n_0} \ne [\mathbf{a}_j]_{n_0}$ *if $i \ne j$.*

Proof. By assumption, we can take an arbitrarily large $\tilde{q}$ such that there exist a point $x \in S^1$ and subset $E \subset \mathcal{A}^{\tilde{q}}$ such that $\#E \ge \exp(\beta\tilde{q})$ and

$$\left|\frac{d}{dx}S(x, \mathbf{a}; \mathbf{t}) - \frac{d}{dx}S(x, \mathbf{b}; \mathbf{t})\right| \le 4\lambda^{\tilde{q}}\ell^{-\tilde{q}}\alpha_0 \quad \text{for } \mathbf{a} \text{ and } \mathbf{b} \text{ in } E. \tag{34}$$

For each $0 \le j \le [\tilde{q}/n_0]$, we introduce an equivalence relation $\sim_j$ on E such that $\mathbf{a} \sim_j \mathbf{b}$ if and only if $[\mathbf{a}]_{jn_0} = [\mathbf{b}]_{jn_0}$, and let

$$\nu(j) = \max_{\mathbf{a}\in E} \#\{\mathbf{b} \in E \mid \mathbf{b} \sim_j \mathbf{a}\}.$$

Since $\nu(0) = \#E \ge \exp(\beta\tilde{q})$ while $\nu(j) \le \ell^{\tilde{q}-jn_0}$ obviously, there exists $0 \le j \le [\tilde{q}/n_0]$ such that $\nu(j+1) < \exp(-\beta n_0/2)\nu(j)$. Let j_* be the minimum of such integers j and put $q = \tilde{q} - n_0 j_*$. Then we have $\nu(j_*) \ge \exp(\beta q)$ and $q \ge \beta\tilde{q}/(2\log\ell)$. The equivalence class H w.r.t. $\sim_{j_*}$ of maximum cardinality contains at least (d_0+1) non-empty equivalence classes w.r.t. $\sim_{j_*+1}$, because

$$\nu(j_*) - (d_0+1)\nu(j_*+1) > \nu(j_*) - (d_0+1)\exp(-\beta n_0/2)\nu(j_*) > 0$$

by (33). So we can take $\mathbf{b} \in \mathcal{A}^{\tilde{q}-q}$ and $\mathbf{a}_i \in \mathcal{A}^q$, $0 \le i \le d$, such that $\mathbf{b}\mathbf{a}_i \in H$ for $0 \le i \le d_0$ and that (E2) holds. Put $x' = \mathbf{b}(x)$. It follows from (34) that

$$\left|\frac{d}{dx}S(x', \mathbf{a}_i; \mathbf{t}) - \frac{d}{dx}S(x', \mathbf{a}_j; \mathbf{t})\right| \le 4\lambda^q\ell^{-q}\alpha_0 \quad \text{for } 0 \le i, j \le d_0. \tag{35}$$

Take $\mathbf{c} \in \mathcal{A}^{p(q)}$ such that $x' \in \mathcal{P}(\mathbf{c})$. Since the distance between $x_{\mathbf{c}}$ and x' is bounded by $\ell^{-p(q)} \le \lambda^q/\ell^q$, the condition (E1) follows from (35) and (4). $\square$

Let $\mathcal{B}^q$ be the set of pairs $(\sigma, \mathbf{c})$ of a sequence $\sigma = (\mathbf{b}_i)_{i=0}^{N_0}$ in $\mathcal{A}^q$ and $\mathbf{c} \in \mathcal{A}^{p(q)}$ such that $\mathbf{Jac}(G_{x_{\mathbf{c}},\bar{\sigma}}) > 1/2$, where $\bar{\sigma} = (\bar{\mathbf{b}}_i)_{i=0}^{N_0}$. For $(\sigma, \mathbf{c}) \in \mathcal{B}^q$ with $\sigma = (\mathbf{b}_i)_{i=0}^{N_0}$, we put

$$Y(\sigma, \mathbf{c}) = G_{x_{\mathbf{c}},\bar{\sigma}}^{-1}([-8(\lambda/\ell)^q\alpha_0, 8(\lambda/\ell)^q\alpha_0]^{N_0})$$

and $Y(q) := \bigcup_{(\sigma,\mathbf{c})\in\mathcal{B}^q} Y(\sigma, \mathbf{c})$. Since the family $T_{\mathbf{t}}^{n_0}$ is $(1/(n_0+1), 1/2)$-generic, the conclusion of Lemma 17 and the second condition in (33) imply that, if $f_{\mathbf{t}} \in \mathcal{E}(\beta, \kappa, \lambda)$, the parameter $\mathbf{t}$ is contained in $Y(q)$ for infinitely many q. Using (32) and the simple estimate $\#\mathcal{B}^q \le \ell^{q(N_0+1)+p(q)}$, we get

$$\mathrm{Leb}\,(Y(q)) \le C\ell^{q(N_0+1)+p(q)}(\lambda/\ell)^{qN_0}$$

for some constant $C > 0$. By the first condition in (33), the left hand side converges to 0 exponentially fast as $q \to \infty$. Therefore we obtain the conclusion of Proposition 15 by Borel-Cantelli lemma.

References

[1] Adams, R.A. Sobolev spaces. Pure and Applied Mathematics, Vol. 65. Academic Press, 1975.

[2] Bamón, R., Kiwi, J., Rivera-Letelier, J. and Urzúa, R., *On the topology of solenoidal attractors of the cylinder*, preprint, Math. ArXiv math.DS/0403523.

[3] Gouëzel, S. and Liverani, C., Banach spaces adapted to Anosov systems, preprint, Math. ArXiv math.DS/0405278

[4] Hörmander, L. The analysis of linear partial differential operators. I. 2nd edition, Springer-Verlag, Berlin, 1990

[5] Hennion, H., *Sur un théorème spectral et son application aux noyaux lipschitziens.* Proc. Amer. Math. Soc. **118**, 627–634 (1993)

[6] Rams, M., *Absolute continuity for the SBR measure for non-linear fat baker maps.* Nonlinearity, **16**, 1649–1655 (2003)

[7] Tsujii, M., *Fat solenoidal attractors.* Nonlinearity, **14**, 1011–1027 (2001)