EQUATIONS FOR FORMAL TORIC DEGENERATIONS

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Abstract. Let (R, m) be a complete equicharacteristic noetherian local domain and ν a valuation of its field of fractions whose valuation ring R_ν dominates R with trivial residue field extension $k \simeq k_\nu$. In order to extend to all rational valuations the torific method (se below) of local uniformization which works for rational valuations with finitely generated semigroup, new methods are needed. One such method is to approximate the given valuation by semivaluations (valuations on quotients, also called pseudo-valuations) with finitely generated semigroups. We produce equations in a generalized power series ring for the algebra encoding the degeneration. We use this to represent ν as the limit of a sequence of semivaluations of R with finitely generated semigroups.

Contents

Definitions. For us a valuation of a noetherian local domain (R, m) is an inclusion of R in a valuation ring (R_ν, m_ν). Usually R_ν will be contained in

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the field of fractions K of R. A valuation ring is a commutative domain in which given two non-zero elements, one divides the other. This implies that it is local. We refer to [3, Chapter VI, §1, §3, §4] for the definitions and facts stated in the next paragraphs.

The valuation is the quotient morphism of groups $\nu\colon K^* \to K^*/U$ where U is the multiplicative group of units of R_ν. This quotient, which we will denote by Φ, is totally ordered by the relation $x \leq y$ if $\frac{y}{x} \in R_\nu$.

The rational rat.rk.ν of the valuation is the rational rank $\dim_{\mathbf{Q}}\Phi \otimes_{\mathbf{Z}} \mathbf{Q}$ of the abelian group Φ. The rank (or height) of the valuation is the length of the maximal sequence of convex subgroups of Φ. It is also the Krull dimension of the ring R_ν which is not noetherian in general.

If the valuation is centered at m, which means that $m_\nu \cap R = m$, the dimension of the valuation is defined to be the transcendence degree $\mathrm{tr}.k_\nu/k$ of $k_\nu = R_\nu/m_\nu$ over $k = R/m$ and one has Abhyankar's inequality

$$\mathrm{tr}.k_\nu/k + \mathrm{rat.rk}.\nu \leq \mathrm{dim}R.$$

If equality holds one says that ν is an Abhyankar valuation of R.

1. Introduction

We continue here the program, presented in [29], [30] and [32], to prove the existence of torific embeddings for valuations centered in equicharacteristic excellent local domains (R, m) of dimension $d \geq 1$ with an algebraically closed residue field. Here is its structure:

(1) Reduce the general case to the case of *rational* valuations, those with trivial residue field extension: $R_\nu/m_\nu = R/m = k$. They correspond to k-rational points of the Zariski-Riemann manifold of the field of fractions of R.
(2) Reduce the case of rational valuations to the case where R is complete.
(3) Assuming that R is complete, reduce the case of a rational valuation to that of rational Abhyankar valuations (rational rank=dim$R$).
(4) Assuming that R is complete, prove the existence of "torific embeddings"[1] for rational Abhyankar valuations using embedded resolution of affine toric varieties, which is blind to the characteristic, applied to the toric graded ring $\mathrm{gr}_\nu R$ associated to the filtration defined by the valuation.

Step 1) was dealt with in [29, Section 3.6].
Step 2), except in the cases of rank one or Abhyankar valuations, is waiting for the proof of a conjecture about the extension of valuations of R to its completion (see [29, *Proposition 5.19*], [14, Conjecture 1.2] and [32, Problem B]), which is in progress in [15].
We note that in the case where we start from a complete local domain, the strict transform under a birational ν-extension $R \subset R' \subset R_\nu$ is understood to be the completion of the local ring R', which is essentially of finite type over R, at the center of the valuation.

[1] A torific embedding of an algebroid space with a valuation is an embedding into an affine space such that the valuation can be uniformized in an embedded manner by toric birational maps.

Step 4) was made in [30].
The essential remaining hurdle is step 3), and the main result of this paper: is a step in this direction.

The paper is organized as follows:
- The first part revisits the valuative Cohen theorem of [30] which relates the ring R and the graded k-algebra $\mathrm{gr}_\nu R$ associated to the valuation. This graded algebra is a twisted semigroup k-algebra of the semigroup of values Γ of ν on R. Here the main result is is a structure theorem (Theorem 3.42) for the set of (possibly infinitely many) equations which define the total space of the specialization of R to $\mathrm{gr}_\nu R$.

- The second part uses this result to produce (Theorem 4.8) a sequence of Abhyankar semivaluations (or quasi-valuations) $(\nu_{B_\alpha})_{a\in\mathbf{N}}$ of R supported on r-dimensional quotients of R, whose finitely generated semigroups $\Gamma_{B_a}\subset\Gamma$ fill up Γ as a grows. Here r is the rational rank of the valuation ν. This corresponds to a part of a conjecture stated in [32, Part 1].

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2. Preliminaries

A valuation ν with value group Φ on a local domain R determines a filtration of R by the ideals

$$\mathcal{P}_\varphi(R)=\{x\in R/\nu(x)\geq\varphi\} \text{ and } \mathcal{P}^+_\varphi(R)=\{x\in R/\nu(x)>\varphi\} \text{ for } \varphi\in\Phi.$$

We note that this is not in general a filtration in the usual sense because the totally ordered abelian value group Φ of ν may not be well ordered so that an element of Φ may not have a successor. However, the successor ideal $\mathcal{P}^+_\varphi(R)$ is well defined and if φ has a successor φ^+ we have $\mathcal{P}^+_\varphi(R)=\mathcal{P}_{\varphi^+}(R)$.

In this text we shall deal with local subrings R of the valuation ring R_ν which are dominated by R_ν, so that $m_\nu\cap R=m$ and the semigroup $\Gamma=\nu(R\setminus\{0\})$ is contained in $\Phi_{\geq 0}$. By definition $\mathcal{P}_\varphi(R)=\mathcal{P}_\varphi(R_\nu)\cap R$ and $\mathcal{P}_\varphi(R_\nu)=R_\nu$ for $\varphi\leq 0$ so that the graded k-algebra

$$\mathrm{gr}_\nu R=\bigoplus_{\varphi\in\Gamma}\mathcal{P}_\varphi(R)/\mathcal{P}^+_\varphi(R)\subset\bigoplus_{\varphi\in\Phi_{\geq 0}}\mathcal{P}_\varphi(R_\nu)/\mathcal{P}^+_\varphi(R_\nu)=\mathrm{gr}_\nu R_\nu$$

associated to the ν-filtration on R is the graded k-subalgebra of $\mathrm{gr}_\nu R_\nu$ whose non zero homogeneous elements have degree in the sub-semigroup[2] Γ of $\Phi_{\geq 0}$.

In view of the defining property of valuation rings, for all $\varphi\in\Phi$ any two elements of $\mathcal{P}_\varphi(R_\nu)\setminus\mathcal{P}^+_\varphi(R_\nu)$ differ by multiplication by a unit of R_ν so that their images in $\mathcal{P}_\varphi(R)/\mathcal{P}^+_\varphi(R)$ differ by multiplication by a non zero element of $k_\nu=R_\nu/m_\nu$. Thus, the homogeneous components of $\mathrm{gr}_\nu R_\nu$ are all one dimensional vector spaces over k_ν.

[2] I follow a tradition of calling semigroup of values of a valuation what is actually a monoid.

If the valuation ν is rational, the same is true of the non zero homogeneous components of $\mathrm{gr}_\nu R$ as $k = R/m$ vector spaces. Then one has a toric description of $\mathrm{gr}_\nu R$ as follows.
Since R is noetherian the semigroup Γ is well ordered and so has a minimal system of generators $\Gamma = \langle \gamma_1, \ldots, \gamma_i, \ldots \rangle$, where γ_{i+1} is the smallest non zero element of Γ which is not in the semigroup $\langle \gamma_1, \ldots, \gamma_i \rangle$ generated by its predecessors. We emphasize here that the γ_i are indexed by an ordinal $I \leq \omega^h$, where h is the rank (or height) of the valuation.

The graded k-algebra $\mathrm{gr}_\nu R$ is then minimally generated by homogeneous elements $(\overline{\xi}_i)_{i\in I}$ with $\deg \overline{\xi}_i = \gamma_i$ and we have a surjective morphism of graded k-algebras, which we denote by $\mathrm{gr}_w \pi$ for reasons which will appear in Theorem 3.13 below.

$$\mathrm{gr}_w \pi \colon k[(U_i)_{i\in I}] \longrightarrow \mathrm{gr}_\nu R, \quad U_i \mapsto \overline{\xi}_i,$$

where $k[(U_i)_{i\in I}]$ is the polynomial ring in variables indexed by I, graded by giving U_i the degree γ_i.

If the non zero homogeneous components of $\mathrm{gr}_\nu R$ are one dimensional k-vector spaces, the kernel of $\mathrm{gr}\pi$ is generated by isobaric binomials $(U^{m^\ell} - \lambda_\ell U^{n^\ell})_{\ell \in L}$, with $\lambda_\ell \in k^*$, where U^m represents a monomial in the U_i's (see [29, Proposition 4.2]). These binomials correspond to a generating system of relations between the generators γ_i of the semigroup, and for all practical purposes we can think of $\mathrm{gr}_\nu R$ as the semigroup algebra over k of the semigroup Γ. This is asserted in [29, Proposition 4.7] and made more precise in subsection 3.1 below.
We assume that the set $(U^{m^\ell} - \lambda_\ell U^{n^\ell})_{\ell \in L}$ is such that none of the binomials is in the ideal generated by the others. In particular, since the kernel of $\mathrm{gr}\pi$ *is a prime ideal, the binomials $U^{m^\ell} - \lambda_\ell U^{n^\ell}$ are irreducible in $k[(U_i)_{i\in I}]$, which means, if k is algebraically closed, that the vectors $m^\ell - n^\ell$ are primitive.*

Since the semigroup of weights is well ordered and there are only finitely many binomials of a given weight, this minimality property can be achieved by a transfinite cleaning process eliminating at each step the binomials of least weight belonging to the ideal generated by binomials $U^{m^\ell} - \lambda_\ell U^{n^\ell}$ in the kernel of $\mathrm{gr}\pi$, necessarily of smaller weight.

By general results in [29, §2.3] the ring R is a deformation of $\mathrm{gr}_\nu R$ [3].

More precisely, we have the following generalized version of what we have just seen and of [29, Propositions 2.2 and 2.3]. Recall that an order function on a commutative ring R with values in a totally ordered abelian group Φ is a map $o \colon R\setminus\{0\} \to \Phi_{\geq 0}$ satisfying $o(xy) \geq o(x)+o(y)$ and $o(x+y) \geq \min(o(x), o(y))$. An order function defines a filtration of R by ideals

$$\mathcal{F}_\varphi(R) = \{x \in R | o(x) \geq \varphi\} \text{ and } \mathcal{F}^+_\varphi(R) = \{x \in R | o(x) > \varphi\},$$

where we agree that $o(0)$ is larger than any element of Φ. We assune that $\mathcal{F}^+_0 = m$, the maximal ideal of R and remark that $\mathcal{F}_\varphi(R) = R$ for $\varphi \leq 0$.

[3]Nowadays the corresponding specialization of R to $\mathrm{gr}_\nu R$ is deemed to be, when the semigroup is finitely generated, an example of "toric degeneration".

Then we define the associated graded ring of the order function

$$\mathrm{gr}_o R = \bigoplus_{\varphi\in\Phi_{\geq 0}} \mathcal{F}_\varphi(R)/\mathcal{F}^+_\varphi(R).$$

This algebra is not in general a domain, or even reduced. It is reduced if and only if $o(x^n) = no(x)$ and is a domain if and only if the order function is a valuation.

Proposition 2.1. *Let R be a local k-algebra with residue field k and o an order function on R as above.*

- *The $k[v^{\Phi_{\geq 0}}]$-algebra*

$$\mathcal{A}_o(R) = \bigoplus_{\varphi\in\Phi} \mathcal{F}_\varphi(R)v^{-\varphi} \subset R[v^\Phi]$$

is faithfully flat, where the algebra structure comes from the composition of inclusions $k[v^{\Phi_{\geq 0}}] \subset R[v^{\Phi_{\geq 0}}] \subset \mathcal{A}_o(R)$, the last inclusion being that of elements of degree ≤ 0.
- *The natural map $\mathcal{A}_o(R) \to \mathrm{gr}_o R$ defined by $x_\varphi v^{-\varphi} \mapsto x_\varphi \bmod.\mathcal{F}^+_\varphi$ induces an isomorphism of graded rings*

$$\mathcal{A}_o(R)/(v^\varphi, \varphi > 0) \simeq \mathrm{gr}_o R,$$

where $(v^\varphi, \varphi > 0)$ denotes the ideal generated by the v^φ with $\varphi > 0$.
- *The inclusion $R[v^{\Phi_{\geq 0}}] \subset \mathcal{A}_o(R)$ induces, after taking rings of fractions with respect to the multiplicative subset $(v^{\Phi_{>0}})$ an isomorphism*

$$(v^{\Phi_{>0}})^{-1}R[v^{\Phi_{\geq 0}}] \simeq (v^{\Phi_{>0}})^{-1}\mathcal{A}_o(R).$$

- *Given a character $\chi\colon \Phi \to k^*$, the surjection $\mathcal{A}_o(R) \to R$ defined by $\sum x_\varphi v^{-\varphi} \mapsto \sum x_\varphi \chi(-\varphi)$ induces an isomorphism*

$$\mathcal{A}_o(R)/(v^\varphi - \chi(\varphi)_{\varphi\in\Phi})\mathcal{A}_o(R) \simeq R.$$

In the algebraic category, the proposition states that the map

$$\mathrm{Spec}\mathcal{A}_o(R) \to \mathrm{Spec}k[v^{\Phi_{\geq 0}}]$$

is faithfully flat, its special fiber is $\mathrm{Specgr}_o R$ *and its general fiber is* $\mathrm{Spec}R$.

Proof. The proof is exactly the same as that of [29, Propositions 2.2 and 2.3]. □

The construction of the algebra $\mathcal{A}_o(R)$, with another intent and the order function associated to the powers of an ideal, goes back to work of David Rees in the 1950's. See [29, Section 2].

Coming back to filtrations induced by rational valuations, when in addition $\mathrm{gr}_\nu R$ is finitely generated, not only is the faithfully flat specialization of R to $\mathrm{gr}_\nu R$ equidimensional (see Proposition 3.47 below), so that we have equality in Abhyankar's inequality below, but the space associated to this toric degeneration (see section 3.8 below) is "equiresolvable at ν" in the sense that there exist generators $(\xi_i)_{i\in I}$ of the maximal ideal of R whose initial forms $(\overline{\xi}_i)_{i\in I}$ are generators of the k-algebra $\mathrm{gr}_\nu R$ and some birational toric maps which provide, in the associated coordinates $(\overline{\xi}_i)_{i\in I}$, an embedded resolution of singularities of the affine toric variety $\mathrm{Specgr}_\nu R$, and when applied to the generators ξ_i of the maximal ideal of R and localized at the point (or center) picked by ν, provide an embedded local uniformization of the valuation ν.

Of course, in the case where $\mathrm{gr}_\nu R$ is not finitely generated, we must adapt this notion of "ν-equiresolvability" since $\mathrm{Specgr}_\nu R$ does not even have a resolution of singularities, embedded or not.
By a theorem of Piltant (see [29], proposition 3.1) we know however that the Krull dimension $\mathrm{dimgr}_\nu R$ is the rational rank $r(\nu)$ of the valuation even when $\mathrm{gr}_\nu R$ is not finitely generated.
For rational valuations of a noetherian local domain R of dimension d, Abhyankar's inequality $r(\nu) \leq d$ then reads as the following inequality of Krull dimensions

$$(A) \qquad \mathrm{dimgr}_\nu R \leq \mathrm{dim} R.$$

Those for which equality holds, called (zero dimensional) Abhyankar valuations, were the subject of [30], where it was shown that their graded algebra is quasi-finitely generated[4], and that they could be uniformized in an embedded manner in a torific embedding[5].

The proof reduced the excellent equicharacteristic case to the complete equicharacteristic case using the good behavior of Abhyankar valuations with respect to completion shown in [30, Section 7.2]. The case of complete equicharacteristic local domains was then reduced to a combinatorial problem using a valuative version of Cohen's structure theorem to obtain equations describing the specialization of R to $\mathrm{gr}_\nu R$.
The classical Cohen structure theorem presents a complete equicharacteristic noetherian local ring R as a quotient of a power series ring in finitely many variables by an ideal generated by deformations adding terms of higher degree to the homogeneous polynomials defining the associated graded ring $\mathrm{gr}_m R$ as a quotient of a polynomial ring.
The valuative Cohen theorem presents a complete equicharacteristic noetherien local ring endowed with a rational valuation ν as a quotient of a generalized power series ring in such a way that the corresponding morphism of associated graded rings is a presentation of $\mathrm{gr}_\nu R$ as a quotient of a generalized weighted polynomial ring by a binomial ideal. In a second part, it describes the kernel of the surjection to R as topologically generated by overweight deformations of the binomials.
The geometric interpretation of the first part of the valuative Cohen theorem is that after embedding the algebroid germ X corresponding to R in an affine space $\mathbf{A}^{|I|}(k)$ by the generators $(\xi_i)_{i\in I}$, the valuation ν becomes the trace on X of a *monomial* valuation (a.k.a. weight in the sense of [30, Definition 2.1]) on the the ambient possibly infinite dimensional affine space.
The interpretation of the second part is that the image of the embedding in question is defined by a (possibly infinite) set of equations which constitutes an *overweight deformation* (see definition 3.2 below) in a generalized sense

[4]This means that there is a local R-algebra R' essentially of finite type dominated by the valuation ring R_ν (a birational ν-modification) such that $\mathrm{gr}_\nu R'$ is finitely generated. For more on finite and non finite generation of $\mathrm{gr}_\nu R$, see [6].

[5]A proof of local uniformization for Abhyankar valuations of algebraic function fields, with assumptions on the base field, had appeared in [20] and a more general one in [**?**]. See also [**?**].

of a set of binomial equations defining the generalized affine toric variety $\mathrm{Specgr}_\nu R$.
This allows us to produce (relatively) explicit equations for the total space of the degeneration of the algebroid (or formal) germ corresponding to R to $\mathrm{Specgr}_\nu R$; see section 3.8 below. This is of course necessary to show that a (partial) embedded toric resolution of $\mathrm{Specgr}_\nu R$ extends to a uniformization of ν.

Remark 2.2. Just as Kaplansky's embedding theorem (see [18] for valued fields, [26] for regular rings, and [33]) is a valuative generalization of the Newton-Puiseux theorem, the valuative Cohen theorem is a valuative generalization of the Cohen structure theorem. It would be useful to write a proof of the valuative Cohen Theorem in the mixed characteristic case, as suggested in [29, Remark before §3].

Example 2.3. Let R be the algebra of an algebroid branch over an algebraically closed field k. That is, a complete equicharacteristic noetherian one dimensional local domain with residue field k. The integral closure of R is $k[[t]]$ and R has only one valuation ν: the t-adic valuation. Its value semigroup on R is a numerical semigroup and therefore finitely generated (see [30, Corollary 6.3], [12], [24]). Let $I = (\gamma_0, \gamma_1, \dots, \gamma_g)$ be the ordered set of degrees of a minimal set of generators $(\overline{\xi}_i)_{i\in I}$ of the k-algebra $\mathrm{gr}_\nu R \subset k[t]$. We then have

$$\mathrm{gr}_\nu R \simeq k[t^{\gamma_0}, \dots, t^{\gamma_g}],$$

the algebra of a monomial curve with the same semigroup of t-adic values as R.
In characteristic zero and for plane branches, the set I is in order preserving bijection with the set $(\beta_0, \beta_1, \dots, \beta_g)$ of characteristic Puiseux exponents. For positive characteristic see [29, Example 6.2] or [31].
Representatives $\xi_i \in R$ of the $(\overline{\xi}_i)_{i\in I}$ embed the curve in $\mathbf{A}^{g+1}(k)$ as a curve defined by equations that are overweight deformations of the binomials defining the monomial curve $\mathrm{Spec}k[t^\Gamma]$ (read [29, Example 5.69] in the formal power series context). This is the first example of a torific embedding.

In view of the constancy of fiber dimension in a flat degeneration such as that of R to $\mathrm{gr}_\nu R$ in [29, Proposition 2.3] in the case of finite generation, strict inequality in inequality (A) implies that the k-algebra $\mathrm{gr}_\nu R$ is not finitely generated (see Proposition 3.47 below). Then ν-equiresolvability should mean that some *partial* toric resolutions of the singularities of the toroidal scheme $\mathrm{Specgr}_\nu R$, affecting only finitely many of the $(\overline{\xi}_i)_{i\in I}$, when applied to the elements $\xi_i \in R$ instead of the $\overline{\xi}_i \in \mathrm{gr}_\nu R$, should uniformize the valuation ν on R.

The problem is to find the appropriate finite subset of the set L of binomial equations written above. It should generate a prime ideal in a finite set of variables and have the property that some of the embedded toric resolutions of the toric variety corresponding to this prime binomial ideal uniformize the valuation. Theorem 4.8 below provides the appropriate finitely generated approximations to $\mathrm{gr}_\nu R$.

3. Valued complete equicharacteristic noetherian local rings and the valuative Cohen theorem

3.1. **The graded algebra of a rational valuation is a twisted semigroup algebra.** The purpose of this section is to make sure that we can apply to the graded algebras asssociated to rational valuations the results and techniques known for semigroup algebras even when the graded algebra is not finitely generated.

Let ν be a rational valuation centered in a noetherian local domain R with residue field k. denoting by $\overline{k}$ the algebraic closure of k, given a minimal system of generators $(\overline{\xi}_i)_{i\in I}$ of the k-algebra $\mathrm{gr}_\nu R$, a simple algebraic argument based on the fact that the group Φ has a finite rational rank shows that there exist elements $\rho_i\in(\overline{k})^*$ such that the morphism of $\overline{k}$-algebras $\overline{k}\otimes_k \mathrm{gr}_\nu R\to\overline{k}[t^\Gamma]$ determined by $1\otimes_k\overline{\xi}_i\mapsto\rho_i t^{\gamma_i}$ is an isomorphism of graded algebras. By [12, Theorems 17.1 and 21.4] we know that the Krull dimension of $k[t^\Gamma]$ is the rational rank r of ν and by Piltant's Theorem that $\mathrm{gr}_\nu R$ is also an r-dimensional domain. Considering the surjective composed morphism

$$\overline{k}[(U_i)_{i\in I}]\longrightarrow\overline{k}[t^\Gamma],\ \ U_i\mapsto\rho_i t^{\gamma_i},$$

all that is needed is that the $U_i=\rho_i$ are non zero solutions of the system of binomial equations $(U^{m^\ell}-\lambda_\ell U^{n^\ell}=0)_{\ell\in L}$ which generate the kernel. There exists a finite set $F\subset I$ such that the $(\gamma_i)_{i\in F}$ rationally generate the value group Φ of our valuation. The binomial equations involving only the variables $(U_i)_{i\in F}$ are finite in number and have non zero solutions $\rho_i\in(\overline{k})^*$ by the nullstellensatz and quasi-homogeneity. All the other γ_i are rationally dependent on these. The result then follows by a transfinite induction in the well ordered set I: assuming the algebraicity to be true for binomial equations corresponding to the relations between the $(\gamma_j)_{j\in F\cup\{j\in I|j<i\}}$ and choosing non zero algebraic solutions ρ_j, we substitute ρ_j for U_j in the binomials involving the variables $(U_j)_{j\in F\cup\{j\in I|j<i\}}$ and U_i to obtain binomial equations in U_i with coefficients in $k((\rho_j)_{j\in F\cup\{j\in I|j<i\}})$. By rational dependence we know there is at least one such equation, of the form $U_i^{n_i}-m_i(\rho)=0$ where $m_i(\rho)$ is a non zero Laurent term in the $(\rho_j)_{j\in F}$, and in $k((\rho_j)_{j\in F\cup\{j\in I|j<i\}})[U_i,U_i^{-1}]$ the ideal generated by the equations coming from all binomials involving U_i is principal so that up to multiplication by a root of unity we have a well defined solution $\rho_i\in k^*$ and this gives the result. So we have proved the following slight generalization of known results (compare with [29, Proposition 4.7] and, for the case of submonoids of $\mathbf{Z}^r$, [4, Exercise 4.3]):

Proposition 3.1. *Let $\Gamma=\langle(\gamma_i)_{i\in I}\rangle$ be the semigroup of values of a rational valuation ν on a noetherian local domain R with residue field k. Given a minimal set $(\overline{\xi}_i)_{i\in I}$ of homogeneous generators of the k-algebra $\mathrm{gr}_\nu R$, with $\deg\xi_i=\gamma_i$, the map which to an isomorphism ρ of graded $\overline{k}$-algebras*

$$\rho\colon\overline{k}\otimes_k\mathrm{gr}_\nu R\simeq\overline{k}[t^\Gamma]$$

associates the family of elements $(\rho(1\otimes_k\overline{\xi}_i)t^{-\gamma_i})_{i\in I}$ of $(\overline{k})^*$ defines a bijection from the set of such isomorphisms to the set of $\overline{k}$-rational points of $\mathrm{Specgr}_\nu R$

all of whose coordinates in $\mathrm{Spec}\overline{k}[(U_i)_{i\in I}]$ *are non zero, a generalized torus orbit. This set is not empty.*

The graded algebra $\mathrm{gr}_\nu R$ is a *twisted semigroup algebra* in the sense of [17, Definition 10.1]. In particular, if k is algebraically closed, there are many isomorphisms of graded k-algebras from $\mathrm{gr}_\nu R$ to $k[t^\Gamma]$.
For a different viewpoint on the same result, and some interesting special cases, see [2].

3.2. **Overweight deformations of prime binomial ideals.** This section is a reminder of [30, Section 3]. Let $w\colon \mathbf{N}^I \to \Phi_{\geq 0}$ be a semigroup morphism defining a weight on the variables $(u_i)_{i\in I}$ of a polynomial or power series ring over a field k, with values in the positive part $\Phi_{\geq 0}$ of a totally ordered group Φ of finite rational rank. If we are dealing in finitely many variables, the semigroup of values taken by the weight is well-ordered by a theorem of B. Neumann in [22, Theorem 3.4].

Let us consider the power series case, in finitely many variables, and the ring $S = k[[u_1,\ldots,u_N]]$. Consider the filtration of S indexed by $\Phi_{\geq 0}$ and determined by the ideals $\mathcal{Q}_\varphi$ of elements of weight $\geq \varphi$, where the weight of a series is the minimum weight of a monomial appearing in it. Defining similarly $\mathcal{Q}^+_\varphi$ as the ideal of elements of weight $> \varphi$, the graded ring associated to this filtration is the polynomial ring

$$\bigoplus_{\varphi\in\Phi_{\geq 0}} \mathcal{Q}_\varphi/\mathcal{Q}^+_\varphi = k[U_1,\ldots,U_N],$$

with $U_i = \mathrm{in}_w u_i$, graded by $\mathrm{deg}U_i = w(u_i)$.

Definition 3.2. Given a weight w as above, a (finite dimensional) overweight deformation is the datum of a prime binomial ideal $(u^{m^\ell} - \lambda_\ell u^{n^\ell})_{1\leq\ell\leq s},\ \lambda_\ell \in k^*$, of $S = k[[u_1,\ldots,u_N]]$ such that the vectors $m^\ell - n^\ell \in \mathbf{Z}^N$ generate the lattice of relations between the $\gamma_i = w(u_i)$, and of series

$$(OD)\qquad \begin{array}{l} F_1 = u^{m^1} - \lambda_1 u^{n^1} + \Sigma_{w(p)>w(m^1)} c_p^{(1)} u^p \\ F_2 = u^{m^2} - \lambda_2 u^{n^2} + \Sigma_{w(p)>w(m^2)} c_p^{(2)} u^p \\ \ldots\ldots \\ F_\ell = u^{m^\ell} - \lambda_\ell u^{n^\ell} + \Sigma_{w(p)>w(m^\ell)} c_p^{(\ell)} u^p \\ \ldots\ldots \\ F_s = u^{m^s} - \lambda_s u^{n^s} + \Sigma_{w(p)>w(m^s)} c_p^{(s)} u^p \end{array}$$

in $k[[u_1,\ldots,u_N]]$ such that, with respect to the monomial order determined by w, they form a standard basis for the ideal which they generate: their initial forms generate the ideal of initial forms of elements of that ideal.
Here we have written $w(p)$ for $w(u^p)$ and the coefficients $c_p^{(\ell)}$ are in k.

The dimension of the ring $R = k[[u_1,\ldots,u_N]]/(F_1,\ldots,F_s)$ is then equal to the dimension of $k[[u_1,\ldots,u_N]]/(u^{m^1} - \lambda_1 u^{n^1},\ldots,u^{m^s} - \lambda_s u^{n^s})$, which is the rational rank of the group generated by the weights of the u_i.
The structure of overweight deformation endows the ring R with a rational valuation. Its filtration is the image in R of the filtration by weight in S. In other words, the valuation of a nonzero element of R is the maximum weight of its representatives in $k[[u_1,\ldots,u_N]]$. It is a rational Abhyankar valuation.

Remark 3.3. In particular, R is a domain.

3.3. **A reminder and complements about the valuative Cohen theorem.** When one wishes to compare an equicharacteristic noetherian local domain R endowed with a rational valuation ν with its associated graded ring $\mathrm{gr}_\nu R$, the simplest case is when R contains a field of representatives k and the k-algebra $\mathrm{gr}_\nu R$ is finitely generated. After re-embedding the space X corresponding to R in an affine space where $\mathrm{Specgr}_\nu R$ can be embedded one can again, at least if R is complete, write down equations for the faithfully flat degeneration of X to $\mathrm{Specgr}_\nu R$ within that new space (see [29, §2.3]) and use their special form as overweight deformations to prove local uniformization for the valuation ν as explained in [30, §3]. However, without some strong finiteness assumptions on the ring R and the valuation ν, one cannot hope for such a pleasant situation. The valuative Cohen theorem shows that when R is complete one can obtain a similar state of affairs without any other finiteness assumption than the noetherianity of R.

Let ν be a rational valuation on a complete equicharacteristic noetherian local domain R. Since R is noetherian, the semigroup of values $\Gamma = \nu(R\setminus\{0\})$ is countable and well ordered, and has a minimal set of generators $(\gamma_i)_{i\in I}$ which is indexed by an ordinal $I \leq \omega^{h(\nu)}$ according to [36, Appendix 3, Proposition 2].

Introduce variables $(u_i)_{i\in I}$ in bijection with the γ_i and consider the k-vector space of all sums $\Sigma_{e\in E} d_e u^e$ where E is any set of monomials in the u_i and $d_e \in k$. Since the values semigroup Γ is combinatorially finite this vector space is in fact, with the usual multiplication rule, a k-algebra $k\widehat{[(u_i)_{i\in I}]}$, which we endow with a weight by giving u_i the weight γ_i. Combinatorial finiteness means that there are only finitely many monomials with a given weight, and we can enumerate them according to the lexicographic order of exponents. Thus, the set of monomials u^m can be seen as a well ordered subset, for the lexicographic order, of the product $\Gamma \times \mathfrak{N}(\mathbf{N})$, where $\mathfrak{N}(\mathbf{N})$ represents the set of finite sequences of non negative integers, each being ordered lexicographically. Combinatorial finiteness also implies that the initial form of every series with respect to the weight filtration is a polynomial, so that we have:

Proposition 3.4. *The graded algebra of* $k\widehat{[(u_i)_{i\in I}]}$ *with respect to the weight filtration is the polynomial algebra* $k[(U_i)_{i\in I}]$ *with* $U_i = \mathrm{in}_w u_i$*, graded by giving* U_i *the weight* γ_i*.* □

The k-algebra $k\widehat{[(u_i)_{i\in I}]}$ is endowed with a monomial valuation given by the weight $w(\Sigma_{e\in E} d_e u^e) = \min_{d_e\neq 0} w(u^e)$. This valuation is rational since all the γ_i are > 0. Note that 0 is the only element with value ∞ because here ∞ is an element larger than any element of Γ. With respect to the "ultrametric" given by $u(x,y)=w(y-x)$, the algebra $k[\widehat{(u_i)_{i\in I}}]$ is spherically complete[6] by [30, Theorem 4.2].

Lemma 3.5. *Let A,B be two sets of (not necessarily distinct) monomials. If a monomial u^c appears infinitely many times among the monomials u^{a+b} with $u^a\in A,\ u^b\in B$, then at least one decomposition $c=a+b$ must appear infinitely many times. Thus, some u^a or u^b with $c=a+b$ must appear infinitely many times in A or B respectively.*

Proof. The combinatorial finiteness of the semigroup Γ implies that the number of posssible distinct decompositions $c=a+b$ is finite. Thus, at least one of them must occur infinitely many times. □

Definition 3.6. Let J be an ideal in $k[\widehat{(u_i)_{i\in I}}]$. The closure of J with respect to the ultrametric w is the set of elements of $k[\widehat{(u_i)_{i\in I}}]$ which are a transfinite sum of elements of J. That is, the set of elements which can be written $y=\sum_{\tau\in T}(y_{\tau+1}-y_\tau)$ where T is a well ordered set, $(y_\tau)_{\tau\in T}$ is a pseudo-convergent sequence and for each $\tau\in T$ we have $y_{\tau+1}-y_\tau\in J$.

Proposition 3.7. *The closure of an ideal of $k[\widehat{(u_i)_{i\in I}}]$ is an ideal.*

Proof. This follows from the fact that a transfinite series exists in $k[\widehat{(u_i)_{i\in I}}]$ if any given monomial appears only finitely many times. Since that must be the case for the sum y of the definition, it is also the case for the sum or difference of two such elements, and for the product with an element of the ring. All the terms of such sums are in J.

□

Proposition 3.8. *For any series $h(u)\in k[\widehat{(u_i)_{i\in I}}]$ which is without constant term, the series $\sum_{a=0}^{\infty}h(u)^a$ converges in $k[\widehat{(u_i)_{i\in I}}]$.*

Proof. Since there is no constant term, we have to prove that any non constant monomial can appear only finitely many times in the series. If the value group is of rank one, the result is clear by the archimedian property. We proceed by induction on the rank. If Ψ_1 is the largest non trivial convex subgroup of the value group Φ, we can write $h(u)=h_1(u)+h_2(u)$ where $h_1(u)$ contains all the terms in $h(u)$ which involve only variables with weight in Ψ_1. Then $h(u)^a=\sum_{i=0}^{a}\binom{a}{i}h_1(u)^ih_2(u)^{a-i}$. If a monomial appears infinitely many times in the $(h(u)^a)_{a\in\mathbf{N}}$ then applying lemma 3.5 to the sets of monomials appearing in the $(h_1(u)^i)_{i\in\mathbf{N}}$ and $(h_2(u)^j)_{j\in\mathbf{N}}$ respectively shows that some monomial must appear infinitely many times in the collection of series $(h_1(u)^i)_{i\in\mathbf{N}}$, which we exclude by the induction hypothesis applied to Ψ_1, or some monomial must appear infinitely many times in the collection of series $(h_2(u)^j)_{j\in\mathbf{N}}$, which is impossible because the semigroup $\Phi_+\setminus\Psi_1$ is archimedian since Φ/Ψ_1 is. □

[6] A *pseudo-convergent,* or *pseudo-Cauchy* sequence of elements of $k[\widehat{(u_i)_{i\in I}}]$ is a sequence $(y_\tau)_{\tau\in T}$ indexed by a well ordered set T without last element, which satisfies the condition that whenever $\tau<\tau'<\tau"$ we have $w(y_{\tau'}-y_\tau)<w(y_{\tau"}-y_{\tau'})$ and an element y is said to be a *pseudo-limit* of this pseudo-convergent sequence if $w(y_{\tau'}-y_\tau)\leq w(y-y_\tau)$ for $\tau,\tau'\in T,\ \tau<\tau'$. Spherically complete means that every pseudo-convergent sequence has a pseudo-limit.

Corollary 3.9. *The ring $k[\widehat{(u_i)_{i\in I}}]$ is local, with maximal ideal $\hat{m}$ consisting of series without constant term and residue field k. The maximal ideal is the closure of the ideal generated by the $(u_i)_{i\in I}$.*

Indeed the proposition shows that the non invertible elements of the ring are exactly those without constant term, which form an ideal. The second assertion follows from the fact that any series without constant term is a transfinite sum of w-isobaric polynomials in the u_i.

Proposition 3.10. *The local ring $k[\widehat{(u_i)_{i\in I}}]$ is henselian.*

Proof. This follows from the fact that it is spherically complete, that the value group of w is of finite rational rank, and the characterization of henselian rings by the convergence of pseudo-convergent sequences of étale type found in [9, Proposition 3.7]. □

Corollary 3.11. *The henselization of the localization $k[(u_i)_{i\in I}]_{\hat{m}\cap k[(u_i)_{i\in I}]}$ of the polynomial ring $k[(u_i)_{i\in I}]$ at the maximal ideal which is the ideal of polynomials without constant term, is contained in $k[\widehat{(u_i)_{i\in I}}]$.*

Proof. By proposition 3.8, any element of $k[(u_i)_{i\in I}]$ which is not in the maximal ideal is invertible in $k[\widehat{(u_i)_{i\in I}}]$, which is henselian. □

Corollary 3.12. *If the series $h(u)$ has no constant term, given any other series $g(u)\in k[\widehat{(u_i)_{i\in I}}]$ and a monomial u^m, writing $g(u)=\sum_{a\in\mathbf{N}} g_a(u)(u^m)^a$, where no term of a $g_a(u)$ is divisible by u^m, the substitution $\sum_{a\in\mathbf{N}} g_a(u)h(u)^a$ gives an element of $k[\widehat{(u_i)_{i\in I}}]$: in this ring, we can substitute a series without constant term for any monomial, and in particular for any variable.*

Proof. Since any monomial appears finitely many times in $g(u)$, by combinatorial finiteness, any monomial can appear only finitely many times among the series $(g_a(u))_{a\in\mathbf{N}}$.
Should a monomial appear infinitely many times among the terms of the sum $\sum_{a\in\mathbf{N}} g_a(u)h(u)^a$, by lemma 3.5 applied to the set of monomials appearing in the $(g_a(u))_{a\in\mathbf{N}}$ and the set of monomials appearing in the $(h(u)^a)_{a\in\mathbf{N}\setminus\{0\}}$, a monomial would have to appear infinitely many times among the $h(u)^a$ and this would contradict proposition 3.8. □

In conclusion, the k-algebra $k[\widehat{(u_i)_{i\in I}}]$ is regular in any reasonable sense and has almost all of the properties of the usual power series rings except of course noetherianity if Γ is not finitely generated. If Γ is finitely generated, $k[\widehat{(u_i)_{i\in I}}]$ is the usual power series ring in finitely many variables, equipped with a weight.

Theorem 3.13. (The valuative Cohen theorem, part 1) *Let R be a complete equicharacteristic noetherian local domain and let ν be a rational valuation of R. We fix a field of representatives $k\subset R$ of the residue field R/m and a minimal system of homogeneous generators $(\overline{\xi}_i)_{i\in I}$ of the graded k-algebra $\mathrm{gr}_\nu R$. There exist choices of representatives $\xi_i\in R$ of the $\overline{\xi}_i$ such that the application $u_i\mapsto \xi_i$ determines a surjective morphism of k-algebras*

$$\pi\colon k[\widehat{(u_i)_{i\in I}}]\longrightarrow R$$

which is continuous with respect to the topologies associated to the filtrations by weight and by valuation respectively. The associated graded morphism with respect to these filtrations is the morphism

$$\mathrm{gr}_w\pi\colon k[(U_i)_{i\in I}] \longrightarrow \mathrm{gr}_\nu R, \quad U_i \mapsto \overline{\xi}_i$$

whose kernel is a prime ideal generated by binomials $(U^{m^\ell} - \lambda_\ell U^{n^\ell})_{\ell\in L},\ \lambda_\ell \in k^*$.
If the semigroup $\Gamma = \nu(R\setminus\{0\})$ *is finitely generated or if the valuation* ν *is of rank one, one may take any system of representatives* $(\xi_i)_{i\in I}$.

This is proved in [30, §4] and we recall that we have:

Proposition 3.14. *Let* $p\colon S \to R$ *be a surjective morphism of commutative local domains and let* $\mathcal{Q} = (\mathcal{Q}_\varphi)_{\varphi\in\Phi_{\geq 0}}$ *be a filtration of* S *by ideals indexed by the positive part* $\Phi_{\geq 0}$ *of a totally ordered abelian group* Φ *of finite rational rank, and such that* $\bigcap_{\varphi\in\Phi_{\geq 0}} \mathcal{Q}_\varphi = \{0\}$ *and the set of* $\varphi \in \Phi_{\geq 0}$ *such that* $\mathcal{Q}_\varphi/\mathcal{Q}_\varphi^+ \neq 0$ *is well ordered. Assume further that* R *is noetherian and complete. Let* $(\mathcal{F}_\varphi)_{\varphi\in\Phi_{\geq 0}} = (p(\mathcal{Q}_\varphi))_{\varphi\in\Phi_{\geq 0}}$ *be the image filtration of* R. *The kernel of the induced surjective map of graded rings* $\mathrm{gr}_\mathcal{Q} S \to \mathrm{gr}_\mathcal{F} R$ *consists of the* $\mathcal{Q}$*-initial forms of the elements of the kernel* F *of the morphism* p.

Proof. From the definition we have that $x \in R$ is in $\mathcal{F}_\varphi \setminus \mathcal{F}_\varphi^+$ if and only if x has a representative $\tilde{x} \in S$ such that $\tilde{x} \in \mathcal{Q}_\varphi \setminus \mathcal{Q}_\varphi^+$. We can use the proof of [30, Proposition 3.3] to verify that any $x \in R$ has a representative of maximum $\mathcal{Q}$-order. An element $\hat{x} \in S$ whose initial form $\mathrm{in}_\mathcal{Q}\hat{x}$ is mapped to zero in $\mathrm{gr}_\mathcal{F} R$ is not a representative of maximum order of its image $x \in R$. It satifies $\hat{x} - \tilde{x} \in F$, where $\tilde{x}$ is a representative of maximum order. Thus $\mathrm{in}_\mathcal{Q}(\hat{x} - \tilde{x}) = \mathrm{in}_\mathcal{Q}\hat{x} \in \mathrm{in}_\mathcal{Q} F$. Conversely, if $\mathrm{in}_\mathcal{Q} x_1 \in \mathrm{in}_\mathcal{Q} F$ it means precisely that it is the initial form of an element $x_1 - y \in F$ with y of $\mathcal{Q}$-order larger that that of x_1, so that x_1 is not a representative of maximum order of its image in R and the image of $\mathrm{in}_\mathcal{Q} x_1$ in $\mathrm{gr}_\mathcal{F} R$ is zero. □

Corollary 3.15. (See [30, Proposition 3.6]) *The initial form* $\mathrm{in}_w h(u)$ *of an element* $h(u) \in k[\widehat{(u_i)_{i\in I}}]$ *is in the kernel of* $\mathrm{gr}_w\pi$ *if and only if* $w(h(u)) < \nu(\pi(h(u)))$.

Proof. The ν-initial form of an element x of R is of the form $c\overline{\xi}^e$ with $c \in k^*$ and $\overline{\xi}^e$ a monomial in the $\overline{\xi}_i$. The weight of the term $cu^e \in k[\widehat{(u_i)_{i\in I}}]$ is $\nu(x)$ which shows that the ν-adic filtration of R is the image by the morphism π of the w-adic filtration of $k[\widehat{(u_i)_{i\in I}}]$. □

In other words, the filtration of R by the valuation ideals is the image by π of the filtration by weight on $k[\widehat{(u_i)_{i\in I}}]$ and the kernel of the associated graded morphism encodes the difference between the two.

Remarks 3.16. 1) The only case where the morphism π is an isomorphism is the case where the ring R is regular and the valuation ν is monomial.

2) After giving this result its name, I realized that Cohen's structure theorem had roots in valuation theory. It is an analogue for complete noetherian local rings of a structure theorem for complete valued fields due to H. Hasse and F.K. Schmidt. See [25, Section 4.3]. However, this result was itself inspired

by a conjecture of Krull in commutative algebra which directly inspired I.S. Cohen; see [5, Introduction].

3) The sum $\sigma = \sum u^m$ of *all* monomials is an element of $k\widehat{[(u_i)_{i\in I}]}$ and can be deemed to represent the transfinite product $\prod_{i\in I}(\frac{1}{1-u_i})$ since in the expansion of $\frac{1}{1-u_i}$ each power of u_i appears exactly once and with coefficient 1.
If we consider the canonical morphism of topological k-algebras

$$\kappa\colon k\widehat{[(u_i)_{i\in I}]} \to k[[t^\Gamma]], \quad u_i \mapsto t^{\gamma_i}$$

defined in [30, introduction to section 4],[7] by construction a monomial of weight γ in the variables u_i encodes a representation of γ as a combination of the generators γ_i so that the image by κ of the series σ is the generalized generating series $\sum_{\gamma\in\Gamma} p(\gamma)t^\gamma \in k[[t^\Gamma]]$ where $\gamma \mapsto p(\gamma) \in \mathbf{N}$ is a partition function: the number of distinct ways of writing γ as a combination with coefficients in $\mathbf{N}$ of the generators γ_i. Using the morphism κ, we see that it satisfies a generalized version of Euler's identity for the generating series of partitions:

$$\sum_{\gamma\in\Gamma} p(\gamma)t^\gamma = \prod_{i\in I}(\frac{1}{1-t^{\gamma_i}}).$$

We observe that the semigroup $\Phi_{\geq 0}$ is itself well ordered and combinatorially finite if and only if its approximation sequences constructed in [30, Proposition 2.3]

$$\mathbf{N}_0^r \subset \mathbf{N}_1^r \subset \cdots \subset \mathbf{N}_h^r \subset \mathbf{N}_{h+1}^r \subset \cdots \Phi_{\geq 0}$$

by linearly nested free subsemigoups $\mathbf{N}^r$, where r is the rational rank of Φ, are finite. Indeed, if such a sequence is infinite there must be infinite decreasing sequences of elements of $\Phi_{\geq 0}$ because if the linear inclusion $\mathbf{N}_h^r \subset \mathbf{N}_{h+1}^r \subset \Phi_{\geq 0}$, encoded by a matrix with entries in $\mathbf{N}$, is not an equality, the sum of the basis element of $\mathbf{N}_{h+1}^r$ is strictly smaller than the sum of basis elements of $\mathbf{N}_h^r$. Conversely, we know that $\mathbf{N}^r$ is well ordered because it is the semigroup of values of a monomial valuation on the noetherian ring $k[[x_1, \ldots, x_r]]$.

Therefore, if $\Phi_{\geq 0}$ is well ordered, we have $\Phi_{\geq 0} = \mathbf{N}^r$ where r is the rational rank of Φ. The same construction as above can be applied when taking variables u_i indexed by all the elements of $\mathbf{N}^r$ instead of only generators and taking $r = 1$ we recover the usual partition function $p(n)$ and the usual Euler identity.
The series $\sum_{\gamma\in\Gamma} p(\gamma)t^\gamma$ can also be viewed as the generalized Hilbert-Poincaré series of the graded algebra $\mathrm{gr}_w k\widehat{[(u_i)_{i\in I}]}$ since $p(\gamma)$ is the dimension of the k-vector space generated by the different monomials of degree γ. We note that the Hilbert-Poincaré series of $\mathrm{gr}_\nu k[[t^\Gamma]] = k[t^\Gamma]$ is again, up to a change in the meaning of the letter t, the sum $\sum_{\gamma\in\Gamma} t^\gamma$ of all monomials.

[7] which minimally presents $k[[t^\Gamma]]$ as a quotient of a generalized power series ring.

3.4. **The valuative Chevalley Theorem.** In this section we dig a little deeper into the relationship between the valuative Cohen theorem and Chevalley's theorem (see [29], section 5, and [3], Chap. IV, §2, No. 5, Cor.4), which is essential in its proof. Let R be a complete equicharacteristic noetherian local domain and let ν be a rational valuation centered in R. Let us take as set $(\gamma_i)_{i\in I}$ the entire semigroup Γ. If the valuation ν is of rank one, this set is of ordinal ω (see [36, Appendix 3, Proposition 2]) and cofinal in $\Phi_{\geq 0}$ since by [7, Theorem 3.2 and §4] the semigroup Γ has no accumulation point in $\mathbf{R}$ because we assume that R is noetherian (see also [27, Lemma 3.16]). Since we have $\bigcap_{\gamma\in\Gamma}\mathcal{P}_\gamma = (0)$, it is an immediate consequence of Chevalley's theorem that given a sequence of elements of R of strictly increasing valuations, their m-adic orders tend to infinity. This is no longer true for valuations of rank > 1 as evidenced by the following:

Example 3.17. Let $R = k[[x, y]]$ equipped with the monomial rank two valuation μ with value group $(\mathbf{Z}^2)_{lex}$ such that $\mu(x) = (1, 0)$ and $\mu(y) = (0, 1)$. The elements $(x + y^i)_{i\geq 1}$ have strictly increasing values $(1, i)$ but their m-adic order remains equal to 1.

This is the reason why in the valuative Cohen theorem the representatives of generators of the associated graded algebra have to be chosen. In this section we present some consequences of this choice.

Let $(\gamma_i)_{i\in I}$ be a well ordered set contained, with the induced order, in the non negative semigroup of a totally ordered group Φ of finite rational rank. By the theorem of B. Neumann in [22] which we have already quoted, the semigroup Γ generated by these elements is well ordered. Let

$$(0) = \Psi_h \subset \Psi_{h-1} \subset \cdots \subset \Psi_1 \subset \Psi_0 = \Phi$$

be the sequence of convex subgroups of Φ, where h is the rank of Φ and Ψ_k is of rank $h - k$. Let us consider the canonical morphism $\lambda\colon \Phi \to \Phi/\Psi_{h-1}$. With these notations we can state the following inductive definition:

Definition 3.18. A subset B of a well ordered subset $(\gamma_i)_{i\in I}$ of $\Phi_{\geq 0}$ is said to be *initial in* $(\gamma_i)_{i\in I}$ when

- If $h = 1$, then B is of the form $\{\gamma_i | i \leq i_0\}$.
- If $h > 1$, then $\lambda(B)$ is initial in the set $(\lambda(\gamma_i))_{i\in I} \subset \Phi/\Psi_{h-1}$ of the images of the γ_i and for every $\varphi_{h-1} \in \lambda(B)$ we have that the set of differences $B \cap \lambda^{-1}(\varphi_{h-1}) - \tilde{\varphi}_{h-1}$ is initial in the set of differences $\{\gamma_i)_{i\in I}\} \cap \lambda^{-1}(\varphi_{h-1}) - \tilde{\varphi}_{h-1} \subset (\Psi_{h-1})_{\geq 0}$, where $\tilde{\varphi}_{h-1}$ is the smallest γ_i contained in $\lambda^{-1}(\varphi_{h-1})$.

One verifies by induction on the rank that the intersection of initial subsets is initial, so that we have an initial closure of a subset C of $(\gamma_i)_{i\in I}$, the intersection of the initial subsets containing C.

Remark 3.19. If the valuation is of rank > 1, it is not true that if $\gamma_i \in B$ and $\gamma_j < \gamma_i$, then $\gamma_j \in B$.

The useful features of initial sets are what comes now and especially the valuative Chevalley Theorem below which shows that finite initial subsets provide a way of approximating the countable ordinal I indexing the generators of Γ by nested finite subsets which is appropriate for the valuative Cohen Theorem.

To simplify notations, we shall sometimes identify each γ_i with its index i, as in the proof below.

Proposition 3.20. *Let the $(\gamma_i)_{i\in I}$ be as above. The initial closure of a finite subset of $(\gamma_i)_{i\in I}$ is finite.*

Proof. This is a variant of the proof of lemma 5.58 of [29]. Let C be a finite subset of I. If the valuation is of rank one the ordinal of I is at most ω (see [36, Appendix 3, Proposition 2]) and the initial closure of C is the set of elements of I which are less than or equal to the largest element of C, and it is finite. Assume now that the result is true for valuations of rank $\leq h-1$ and let ν be a valuation of rank h. Let $\lambda\colon \Phi \to \Phi_{h-1} = \Phi/\Psi_{h-1}$ be the morphism of groups corresponding to the valuation ν_{h-1} of height $h-1$ with which ν is composed. Set $C_1 = \lambda(C)$ and let I_1 index the distinct $(\lambda(\gamma_i))_{i\in I}$. We have a natural monotone map $I \to I_1$, which we still denote by λ. By the induction hypothesis, we have a finite set $\tilde{C}_1$ containing C_1 and initial in I_1. Define $\tilde{C}$ as follows: it is the union of finite subsets $\tilde{C}_{i_1}$ of the $\lambda^{-1}(i_1)$ for $i_1 \in \tilde{C}_1$, where $\tilde{C}_{i_1}$ is the set of elements of $\lambda^{-1}(i_1)$ which are smaller than or equal to the largest element of $C \cap \lambda^{-1}(i_1)$. By *loc. cit.*, Lemma 4, or Proposition 3.17 of [29], each of these sets if finite and $\tilde{C} = \bigcup_{i_1\in\tilde{C}_1} \tilde{C}_{i_1}$ is the initial closure of C. □

Corollary 3.21. *Given a finite set $B_0 \subset I$, there is a countable sequence $(B_a)_{a\in\mathbf{N}_{>0}}$ of nested finite initial subsets containing B_0 whose union is I:*

$$B_0 \subset B_1 \subset \cdots \subset B_a \subset B_{a+1} \subset \cdots \subset I$$

Proof. Consider the sequence of morphisms

$$\Phi \xrightarrow{\lambda_{h-1}} \Phi/\Psi_{h-1} \xrightarrow{\lambda_{h-2}} \Phi/\Psi_{h-2} \cdots\cdots \xrightarrow{\lambda_2} \Phi/\Psi_2 \xrightarrow{\lambda_1} \Phi/\Psi_1 \longrightarrow 0.$$

Denote by μ_i the compositum $\lambda_i \circ \lambda_{i+1} \circ \cdots \circ \lambda_{h-1}\colon \Phi \to \Phi/\Psi_i$.

Let $B_{0,1} \subset \Phi/\Psi_1$ be the initial closure of the union of the image of B_0 in Φ/Ψ_1 and the smallest element of the image of I in Φ/Ψ_1 which is not in that image. For each $\varphi_1 \in B_{0,1}$ take the initial closure in $\lambda_1^{-1}(\varphi_1)$ of the union of $\lambda_1^{-1}(\varphi_1) \cap \mu_2(B_0)$ and the smallest element of $\lambda_1^{-1}(\varphi_1) \cap \mu_2(I)$ which is not in $\lambda_1^{-1}(\varphi_1) \cap \mu_2(B_0)$. The union of the results of this construction is a finite initial subset $B_{0,2}$ of $\mu_2(I)$ which contains $\mu_2(B_0)$.
We repeat the same construction starting from $B_{0,2} \subset \mu_2(I)$ to build a finite initial subset $B_{0,3}$ of $\mu_3(I)$, and so on until we have created a finite initial subset B_1 of I which strictly contains B_0 unless $B_0 = I$. Then we apply the same construction replacing B_0 by B_1 to obtain B_2, and so on.

Let us now prove that the union of the B_a is I. We keep the notations of Definition 3.18 and Proposition 3.20, where $\lambda_{h-1} = \lambda$. The result is true if Φ is of rank one because a strictly increasing sequence of elements of I is cofinal in I (see [7, Theorem 3.2]). Let us assume that it is true for groups of rank $\leq h-1$ and assume that the result is not true for rank h. Let ι be the smallest element of I which is not in $\bigcup_{t\in\mathbf{N}} B_a$. Using our induction hypothesis, define a_0 as the least a such that $\lambda(\iota) \in \lambda(B_a)$. Let $\tilde{\varphi}_\iota$ be the smallest element of $I \cap \lambda^{-1}(\lambda(\iota))$. By the definition of initial closure, for $a \geq a_0$ the elements of each set of differences $B_a \cap \lambda^{-1}(\lambda(\iota)) - \tilde{\varphi}_\iota$ are initial in $I \cap \lambda^{-1}(\lambda(\iota)) - \tilde{\varphi}_\iota \subset \Psi_{h-1}$. We are reduced to the rank one case and since by construction the $B_a \cap \lambda^{-1}(\lambda(\iota))$ grow with a we obtain that for large enough a we have $\iota \in B_a \cap \lambda^{-1}(\lambda(\iota))$ and a contradiction. □

Corollary 3.22. *The finite initial subsets of I with inclusion maps form a direct system whose limit is I.* □

Theorem 3.23. (The valuative Chevalley theorem) *Let R be a complete equicharacteristic noetherian local domain and let ν be a rational valuation of R. Denote by Γ the semigroup of values of ν and by $(\gamma_i)_{i\in I}$ a minimal set of generators of Γ. Assume that the set I is infinite and let D be an integer. Let B_0 be a finite subset of I. There exist elements $(\xi_i)_{i\in I}$ in R such that $\nu(\xi_i) = \gamma_i$ for all $i \in I$ and a finite initial subset $C(D)$ of $(\gamma_i)_{i\in I}$ containing B_0 and such that whenever $\gamma_i \notin C(D)$, then $\xi_i \in m^{D+1}$.*

Proof. This is essentially a consequence of the valuative Cohen theorem of [30], or rather of its proof. Recall that the choice of the ξ_i in the proof of the valuative Cohen theorem is such that if the image of ξ_i in a quotient $R/\mathfrak{p}_q$ is in a power of the maximal ideal, then ξ_i lies in the same power of the maximal ideal in R, for all primes $\mathfrak{p}_q$ which are the centers of valuations with which ν is composed. So if $\mathfrak{p}_1$ is the center of the valuation ν_1 of height one with which ν is composed, we may assume by induction that the result is true for the ξ_i whose value is in the convex subgroup Ψ_1 of Φ associated to ν_1. By a result of Zariski (see [29, corollary 5.9]), we know that the ν_1-adic topology on R is finer than the m-adic topology, which means that there exists a $\varphi_1 \in \Phi_1 = \Phi/\Psi_1$ such that if $\nu_1(\xi_i) \geq \varphi_1$ then $\xi_i \in m^{D+1}$. So we are interested only in the ξ_i whose ν_1-value is positive and less that φ_1. If $\nu_1(m) > 0$, they are finite in number (see [29, proposition 3.17]) so it suffices to add them to the finite set constructed at the previous inductive stage. If not, since Φ_1 is of rank one, there are only finitely many values of the $\nu_1(\xi_i)$ which are positive and less than φ_1. Let η be one of these. Denote by $\lambda\colon \Phi \to \Phi_1$ the natural projection. Again using the argument of the proof of the valuative Cohen theorem, and in particular Chevalley's theorem applied to the filtration of $\mathcal{P}_\eta/\mathcal{P}_\eta^+$ by the $\mathcal{P}_\varphi/\mathcal{P}_\eta^+$ with $\lambda(\varphi) = \eta$, we have that for each η there is a $\varphi_0(\eta) \in \lambda^{-1}(\eta)$ such that if $\nu(\xi_i) \in \lambda^{-1}(\eta)$ and $\nu(\xi_i) \geq \varphi_0(\eta)$ then $\xi_i \in m^{D+1}$. Again by ([29], proposition 3.17) there are finitely many generators of Γ in $\lambda^{-1}(\eta)$ which are $\leq \varphi_0(\eta)$. It suffices to add to our finite set the union of these finite sets over the finitely many values η that are $\leq \varphi_1$. □

Proposition 3.24. *If the rational valuation ν is of rank one, Theorem 3.23 is equivalent to the existence for each integer $D > 0$ of a finite initial set $C(D)$ in Γ such that for $\gamma \notin C(D)$ we have $\mathcal{P}_\gamma \subset m^{D+1}$.*

Proof. If the set I is infinite, the γ_i are cofinal in $\Phi_{\geq 0}$ since by [7, Theorem 3.2 and §4] the semigroup Γ has no accumulation point in $\mathbf{R}$ because we assume that R is noetherian. Therefore we have $\bigcap_{i\in I} \mathcal{P}_{\gamma_i} = (0)$ where the ordinal of I is ω. By Chevalley's theorem (see [3, Chap. IV, §2, No. 5, Cor. 4]), all but finitely many of the ξ_i are in m^{D+1}. Conversely, assuming the result of Theorem 3.23, recall that $\mathcal{P}_\gamma$ is generated by the monomials ξ^e of value $\geq \gamma$ and let $s \in I$ be the smallest index such that $t \geq s$ implies that $\xi_t \in m^{D+1}$. Let $(\gamma_i)_{i\in J}$ be the finite set of the $\gamma_j < \gamma_s$. For any monomial

ξ^e such that some decomposition of $\nu(\xi^e)$ as sum of γ_i contains a γ_t with $t \geq s$ we have $\xi^e \in m^{D+1}$. Let $C(D)$ be the set of elements of Γ which are $\leq D\gamma_s$. If $\nu(\xi^e) = \gamma > D\gamma_s$, either its decomposition along the γ_i contains a γ_t with $t \geq s$ or it can be written $\gamma = \sum_{i\in J} a_i\gamma_i$ with $a_i \in \mathbf{N}$ and then since $(\sum_{i\in J} a_i)\sup(\gamma_i)_{i\in J} > D\gamma_s$ we must have $(\sum_{i\in J} a_i) \geq D+1$ and thus $\xi^e \in m^{D+1}$ since the ξ_i are in m.
If the set I is finite, taking as initial set $C(D)$ the set of elements of Γ which are $\leq D\gamma_s$, where now γ_s is the largest of the γ_i, the last argument of the proof shows that $\mathcal{P}_\gamma \subset m^{D+1}$ for $\gamma \notin C(D)$. □

Remark 3.25. So we see that in the rank one case Theorem 3.23 is indeed equivalent to the result given by Chevalley's theorem.

Corollary 3.26. *In the situation of the valuative Cohen theorem, given an infinite collection of monomials $(u^{q_\ell})_{\ell\in L}$ such that each monomial appears only finitely many times, for any integer D all but finitely many of the ξ^{q_ℓ} are in m^{D+1}.*

Proof. Set $C = C(D)$ and let us consider the monomials u^{q_ℓ} which involve only the finitely many variables u_i with $i \in C$. If there are finitely many such monomials, we are done. Otherwise, since each monomial appears only finitely many times and we are now dealing with finitely many variables, the monomials which are such that $|q_\ell| \leq D$ are finite in number and we are done again since the ξ_i are in m and by construction all the other monomials are in m^{D+1}. □

Remark 3.27. Since each monomial appears only finitely many times, the sum $\sum_{\ell\in L} u^{q_\ell}$ exists in $k\widehat{[(u_i)_{i\in I}]}$ and therefore its image $\sum_{\ell\in L}\xi^{q_\ell}$ must converge in R, albeit in a transfinite sense. The corollary shows an aspect of this.

3.5. **Finiteness in the valuative Cohen theorem.** Since the binomials $(U^{m^\ell} - \lambda_\ell U^{n^\ell})_{\ell\in L}$ with $\lambda_\ell \in k^*$ generate the w-initial ideal of the kernel F of the morphism $\pi\colon k\widehat{[(u_i)_{i\in I}]} \longrightarrow R$, for each $\ell \in L$ there is at least one element of the form $u^{m^\ell} - \lambda_\ell u^{n^\ell} + \sum_{w(p)>w(u^{m^\ell})} c_p u^p$ which is in F (overweight deformation of its initial binomial). Let us call such elements F_ℓ.

When the rational valuation ν on the noetherian complete local domain R is of rank >1 and the semigroup Γ is not finitely generated, not only does one have to choose the representatives $\xi_i \in R$ of generators of the k-algebra $\mathrm{gr}_\nu R$ carefully (see [30]) in order to avoid adding infinitely many times the same element in sums such as $\sum_{i\in I}\xi_i$, but the equations F_ℓ whose initial forms are the binomials $(u^{m^\ell} - \lambda_\ell u^{n^\ell})_{\ell\in L}$, $\lambda_\ell \in k^*$ also have to be chosen carefully in order to avoid similar accidents when writing elements of the closure in $k\widehat{[(u_i)_{i\in I}]}$ of the ideal which they generate. In this subsection we show how the noetherianity of the local domain R can be used to prove the existence of good choices of the $(F_\ell)_{\ell\in L}$.

Proposition 3.28. *The elements F_ℓ can be chosen, without modifying the initial binomial, in such a way that each one involves only a finite number* *of variables and no monomial u^p appears in infinitely many of them or in infinitely many products $A_\ell F_\ell$, where the A_ℓ are terms such that there are at most finitely many products $A_\ell F_\ell$ of any given weight.*

Proof. According to equation (E) of subsection 3.3, the image in R of a topological generator is of the form $\xi^{m^\ell} - \lambda_\ell \xi^{n^\ell} - \sum_p c_p^{(\ell)} \xi^p$ with $\nu(\xi^p) > \nu(\xi^{m^\ell})$ for all exponents p and of course it is zero in R. Since the ring R is noetherian, the ideal generated by the ξ^p which appear in the series is finitely generated, say by $\xi^{e_1}, \ldots, \xi^{e_s}$. Let us choose finitely many of the ξ_j which generate the maximal ideal of R, call them collectively Ξ and call U the collection of the corresponding variables in $\widehat{k[(u_i)_{i\in I}]}$. Then our series can be rewritten as $\xi^{m^\ell} - \lambda_\ell \xi^{n^\ell} - \sum_{j=1}^s G_j^{(\ell)}(\Xi)\xi^{e_j}$, where $G_j^{(\ell)}(\Xi) \in R$ is a series of terms in the elements of Ξ, possibly with a non zero constant term. Our series is the image of the element $u^{m^\ell} - \lambda_\ell u^{n^\ell} - \sum_{j=1}^s G_j^{(\ell)}(U) u^{e_j}$ of F. This element has the same initial binomial as our original series since $w(u^{e_j}) > w(u^{m^\ell})$ for all j and involves only finitely many variables. It can replace our original series as element of the ideal F with initial form $u^{m^\ell} - \lambda_\ell u^{n^\ell}$.

To prove the second assertion, remember from [30, §4] or section 3.3 above that there is a well-ordering on the set of all monomials obtained by embedding it into $\Gamma \times \mathfrak{N}(\mathbf{N})$ thanks to the finiteness of the fibers of the map $u^e \mapsto w(u^e)$. Fix a choice of $(F_\ell)_{\ell\in L}$ and consider all k-linear combinations of the F_ℓ of the form $F_\ell - \mu F_{\ell'}$ which do not modify the initial binomial. If the set of monomials u^e which appear infinitely many times as terms in all such linear combinations is not empty, it has a smallest element u^{e_0}. Note that we may assume that such a monomial does not appear in the initial binomials because there are only finitely many initial binomials involving a given finite set of variables.
Denote by L_0 the set of indices $\ell \in L$ such that u^{e_0} appears in F_ℓ as above, and by $L_1 \subset L_0$ the finite set of indices $\ell \in L_0$ such that the weight of the initial form of F_ℓ is minimal.
Denote by L_2 the set of indices in $L_0 \setminus L_1$ such that the weight of the initial form of F_ℓ is minimal, and define recursively in the same manner finite subsets L_ι indexed by an ordinal. Now for each $\ell \in L_1$ we can replace F_ℓ by $F_\ell - \mu_{\ell,\ell'} F_{\ell'}$ with an $\ell' \in L_2$ without changing the initial binomial, where the constant $\mu_{\ell,\ell'}$ is chosen to eliminate the monomial u^{e_0} from the difference. In so doing we create a new system of overweight deformations of the collection of binomial ideals and cannot add a monomial of lesser weight than u^{e_0} which might appear infinitely many times since u^{e_0} was the smallest in the original family $(F_\ell)_{\ell\in L}$. We continue this operation with $\ell \in L_\iota, \ell' \in L_{\iota+1}$. In the end we have a new system of overweight deformations obtained by linear combinations and in which u^{e_0} cannot appear. This contradiction shows that we can choose the F_ℓ so that no monomial appears infinitely many times.
It follows from lemma 3.5 that under our assumption, for any collection of isobaric polynomials A_ℓ, the $A_\ell(F_\ell - (u^{m^\ell} - \lambda_\ell u^{n^\ell}))$, cannot contain infinitely many times the same monomial. On the other hand since each A_ℓ is an isobaric polynomial, a monomial can appear only finitely many times in all

the $A_\ell u^{m^\ell}$ or $A_\ell u^{n^\ell}$, so that altogether no monomial can appear infinitely many times in the collection of series $A_\ell F_\ell$. □

Theorem 3.29. (The valuative Cohen theorem, part 2) *The kernel F of π is the closure of the ideal generated by the elements F_ℓ obtained as in Proposition 3.28 as $U^{m^\ell} - \lambda_\ell U^{n^\ell}$ runs through a set of generators of the kernel of $\mathrm{gr}_w\pi$. In a slightly generalized sense, these generators form a standard basis of the kernel of π with respect to the weight w.*

Proof. Let h be a non zero element of F; its weight is finite but its image in R has infinite value. By Corollary 3.15, $\mathrm{in}_w h$ belongs to the ideal F_0 generated by the $(U^{m^\ell} - \lambda_\ell U^{n^\ell})_{\ell\in L}$.
Thus, there exists a finite set of elements $(F_\ell)_{\ell\in L_1}$, with $L_1 \subset L$, and isobaric polynomials $(A_\ell)_{\ell\in L_1}$ in the u_i such that $w(h - \sum_{\ell\in L_1} A_\ell^{(1)} F_\ell) > w(h)$. Since this difference is still in F, we can iterate the process and build a series indexed by the set T of weights of the successive elements, say y_a, with $y_0 = h, y_1 = h - \sum_{\ell\in L_1} A_\ell^{(1)} F_\ell$ and, if $\tau + 1$ is the successor of τ in the index set T, $y_\tau - y_{\tau+1} = \sum_{\ell\in L_\tau} A_\ell^{(\tau)} F_\ell$, such that $w(y_{\tau+1}) > w(y_\tau)$, so that $w(y_\tau) = w(y_{\tau+1} - y_\tau)$. The sequence $(y_\tau)_{\tau\in T}$ is a pseudo-convergent sequence with respect to the ultrametric $u(x, y) = w(y - x)$ which, according to [30, §4] has 0 as a pseudo-limit in the spherically complete ring $k[\widehat{(u_i)_{i\in I}}]$. This is of course not sufficient to prove what we want. However, in view of proposition 3.28, no monomial can appear in infinitely many of the $y_\tau - y_{\tau+1}$. If $h \neq \sum_{\tau<\rho}(y_{\tau+1} - y_\tau)$, the initial form of $h - \sum_{\tau<\rho}(y_{\tau+1} - y_\tau)$ is in F_0 and we can continue the approximation. Thus, the transfinite sum $\sum_{\tau\in T}(y_\tau - y_{\tau+1})$ exists in $k[\widehat{(u_i)_{i\in I}}]$ and, by definition of T, has to be equal to h so that 0 is indeed the limit of the sequence y_τ. This shows that h is in the closure of the ideal generated by the F_ℓ. □

Remark 3.30. The statement about the F_ℓ topologically generating the kernel of π is part a) of the asterisked[8] proposition 5.49 of [29]. Part b) of that proposition is incorrect, as the next section shows.

From now on we shall assume not only that the $(\xi_i)_{i\in I}$ are representatives in R of the $\overline{\xi}_i \in \mathrm{gr}_\nu R$ which make the valuative Cohen theorem valid, but also that each of the equations F_ℓ we consider involves only finitely many variables and the F_ℓ's topologically generate the kernel of π.

In view of their definition the $(\xi_i)_{i\in I}$ generate the maximal ideal of R. Since R is noetherian there is a finite subset $J \subset I$ such that the elements $(\xi_i)_{i\in J}$ minimally generate this maximal ideal. According to ([29], 5.5), for each $\ell \in I \setminus J$ there must be among the topological generators (E) of the ideal F one which contains u_ℓ as a term. We detail the argument here:

The element ξ_ℓ is expressible as a series $h((\xi_i)_{i\in J})$. Therefore the series $u_\ell - h((u_i)_{i\in J})$ must belong to the ideal F. There must be an element H of the closure of the ideal generated by the series $(F_\ell)_{\ell\in L}$ such that

$$w(u_\ell - h((u_i)_{i\in J}) - H) > \gamma_\ell.$$

[8] An asterisked proposition in that text means it is endowed with hope but not with a proof.

Thus, the series H must contain u_ℓ as a term so that at least one of the topological generators of F must contain u_ℓ as a term. An argument given in ([30], proof of proposition 7.9) establishes the following:

Proposition 3.31. *Up to a change of the representatives ξ_i we may assume that each variable u_i with $i \in I\setminus J$ appears as a term in a topological generator of* kerπ *of the form:*

$$u^{m^i} - \lambda_i u^{n^i} - g_i((u_j)_{j<i}) - u_i, \qquad (F_i)$$

where every term of the series $g_i((u_j)_{j<i})$ has weight $> w(u^{m^i})$ and $< w(u_i)$.

Proof. Up to multiplication of variables by a non zero constant and corresponding modification of the λ_ℓ, we may assume that for each $i \in I \setminus J$, the variable u_i appears with coefficient minus one in F_i. Let i_1 be the smallest element of $I \setminus J$ such that there exists an equation F_i as above containing u_i linearly and not having the form described above. We write $g((u_j)_{j<i_1})$ for the series of terms of weight $< i_1$. Such terms use only variables of weight $< w(u_{i_1})$ i.e., of index $< i_1$ so that the condition is stronger than what the notation suggests. The remaining part of F_{i_1} can be written $-u_{i_1} + \sum_{w(p)>\gamma_{i_1}} c_p^{(i_1)} u^p$. If we replace the representative $\xi_{i_1} \in R$ by $\xi_{i_1} - \sum_{w(p)>\gamma_{i_1}} c_p^{(i_1)} \xi^p$ without changing the previous ξ_j, we modify the equation F_{i_1} into the desired form. We note that in view of the definition of i_1 we may assume that no variable of index $< i_1$ appears as a term in (F_{i_1}). We then continue with the smallest index i_2 such that the corresponding equation does not have the desired form, and obtain the result by transfinite induction. □

Proposition 3.32. *Each F_ℓ is irreducible in $k\widehat{[(u_i)_{i\in I}]}$.*

Proof. Should it be reducible, since the kernel F of π is prime, one of its factors should be in F and by Theorem 3.29 and the fact that the kernel of $\mathrm{gr}_w\pi$ is prime, the initial binomial of F_ℓ should be in the ideal generated by the initial forms of the other generators of F, which is impossible in view of the assumption we made on the generating binomials.

□

Remark 3.33. Since the initial forms of the F_ℓ are the binomials defining our toric variety $\mathrm{Specgr}_\nu R$, Theorem 3.29 indeed provides equations for the degeneration to $\mathrm{Specgr}_\nu R$, as we shall see more precisely below in section 3.8. However, since the binomials $(U^{m^\ell} - \lambda_\ell U^{n^\ell})_{\ell\in L}$ come in bulk, only being ordered by their weight, it is difficult to make use of them. The next two subsections address this difficulty.

3.6. **The valuative Cohen theorem for power series rings.** In this section we prove that, in the case where R is a power series ring, the equations provided by the valuative Cohen theorem can then be given a more specific form.

Assume that R is a power series ring over a field k. Since by the valuative Cohen Theorem the ξ_i generate the maximal ideal, there is a subset $J \subset I$ such that the $(\xi_i)_{i\in J}$ form a minimal system of generators of the maximal ideal of R. Note that we state nothing about the rational independence of their valuations. By Proposition 3.31 for each $i \in I \setminus J$ there is a series of the form

$$u^{m^i} - \lambda_i u^{n^i} - g_i((u_j)_{j<i}) - u_i, \qquad (F_i)$$

among the topological generators of kerπ.

Generators of the relations between the values of the $(\xi_i)_{i\in J}$ are encoded by finitely many binomials $(U^{m^\ell} - \lambda_\ell U^{n^\ell})_{\ell\in L_0}$ which according to Theorem 3.29 are the initial forms for the weight w of series $(F_\ell)_{\ell\in L_0}$ which are among the topological generators of kerπ.

Theorem 3.34. *If R is regular, i.e., a power series ring with coefficients in k, the kernel of the morphism $\pi\colon k\widehat{[(u_i)_{i\in I}]} \to R$ asssociated to a rational valuation ν by Theorems 3.13 and 3.29 is the closure of the ideal generated by the $(F_i)_{i\in I\setminus J}$.*

Proof. It suffices to show that all the $(F_\ell)_{\ell\in L}$ of Theorem 3.29 are in the ideal generated by the F_i. It is impossible for any F_ℓ to use only variables with indices in J because if that were the case its image in R by π would be a non trivial relation between the $(\xi_i)_{i\in J}$, contradicting the fact that these are a minimal set of generators of the maximal ideal in a regular local ring. Given F_ℓ with $\ell \in L$, let i be the largest index which is not in J of a variable appearing in F_ℓ. Let us substitute $u_i + F_i = u^{m^i} - \lambda_i u^{n^i} - g_i((u_j)_{j<i})$ in place of u_i in F_ℓ, according to corollary 3.12. Denote by $F_\ell^{[i]}$ the result of this substitution. It involves only finitely many variables, and those whose indices are not in J are all of index $< i$. The difference $F_\ell - F_\ell^{[i]}$ is a multiple of F_i and $F_\ell^{[i]}$ is in kerπ. If $F_\ell^{[i]} = 0$, since F_ℓ is irreducible it should be equal to F_i up to multiplication by a unit, which proves the assertion in this case. Thus we may assume that $F_\ell^{[i]} \neq 0$. Now we can again take the variable of highest index not in J, say i_1, appearing in $F_\ell^{[i]}$, make the substitution of u_{i_1} by $u_{i_1} + F_{i_1}$ and repeat the argument; if the result of the substitution is zero, then F_ℓ belonged to the ideal generated by F_i and F_{i_1}. Otherwise we have a non zero element in kerπ involving only variables of index $< i_1 < i$ and variables with index in J. Since the indices are elements of a well ordered set, after finitely many such steps, either we obtain zero which shows that F_ℓ is in the ideal generated by finitely many F_i's, or we obtain a non zero element of kerπ involving only the variables $(u_i)_{i\in J}$. Again the image of this element by π would be a non trivial relation between the elements of a minimal system of generators of the maximal ideal of R, which is impossible since R is regular. Thus we obtain a contradiction if F_ℓ did not belong to the ideal generated by the F_i. □

Remark 3.35. Theorem 3.34 can be interpreted as a manifestation of the "abyssal phenomenon" of [29, section 5.6]. If the values of the generators $(\xi_i)_{i\in J}$ of the maximal ideal of our power series ring are rationally independent, the semigroup Γ is finitely generated, and we have an Abhyankar, even monomial, valuation. If such is not the case even after some birational ν-modification, after [30, Section 7], the rational rank of the valuation is strictly less than the dimension of R and we are in the non-Abhyankar case. The semigroup Γ is not finitely generated and the binomial relations in $\mathrm{gr}_\nu R$ expressing the rational dependance of values of the $(\xi_i)_{i\in J}$ cannot give rise to relations in R, as they should in view of the faithful flatness of the degeneration of R to $\mathrm{gr}_\nu R$, because R is regular. The role of the equations $(F_i = 0)_{i\in I\setminus J}$ is to prevent their expression in R, which would decrease the dimension and introduce singularities, by sending this expression to infinity. Of course one could point out that something like this is necessary to deform a non-noetherian ring such as $\mathrm{gr}_\nu R$ into the noetherian R, but that leaves out the precise structure of the deformation, which is significant even when $\mathrm{gr}_\nu R$ is noetherian; see Example 2.3.

In order to make full use of the valuative Cohen Theorem, we need a notion of a finite set of generators of the semigroup which is adapted to the generators F_i of Proposition 3.31. Except in rank one, the overweight condition does not imply that the indices of the variables appearing in the series $g_i((u_j)_{j<i})$ above are in the initial set B; see Remark 3.19. This is an obstruction, in the case where we consider finitely many variables, to the process of successive elimination of the variables u_i which we used in the proof of Theorem 3.34. The purpose of the next proposition is to correct this.

Proposition 3.36. *In the situation of the valuative Cohen Theorem, each finite initial set B is contained in a finite set B^{full} such that the indices of the finitely many variables appearing in F_i for $i \in B^{full} \setminus J$ belong to B^{full}.*

Proof. Consider the equations F_i for $i \in B$ and the largest weight $\gamma_{i_{max}}$ in B. In view of the construction of the F_i, the weight of each of the variables u_j with $j \notin B$ appearing in the F_i for $i \in B$ is $< \gamma_{i_{max}}$. Let us add these weights to the set B to obtain a finite set B_1. We now take the equations F_j with $j \in B_1\setminus B$ corresponding to these new variables, with a new maximum weight $< \gamma_{i_{max}}$ and repeat the process, producing a strictly decreasing sequence of weights, which has to stop after finitely many steps since Γ is well ordered. When it stops, it means that no new variables are needed. Denoting by B^{full} the finite set of indices thus created, we see that it has the property of the Proposition. □

Definition 3.37. The finite subset B^{full} associated as above to a finite subset B of I and an appropriate set F of topological generators of the kernel of the morphism π of the valuative Cohen theorem is called the *F-filling* or for short *filling* of B. It depends not only on the semigroup but also on the equations $(F_\ell)_{\ell\in L}$. By construction, filling preserves inclusions.

Remark 3.38. The fillings of finite initial sets have all the properties of finite initial sets with respect to the valuative Chevalley Theorem, which is the important property of initial sets for us.

From now on the notation B shall designate the filling of an initial set B.

By the faithful flatness of the toric degeneration, each binomial $u^{m^\ell} - \lambda_\ell u^{n^\ell}$ is the initial binomial of an equation F_ℓ and thus the F_ℓ are a (generalized) standard basis for the kernel of π (see [29, Proposition 5.53]). It needs not be the case that the F_i, which when R is a power series ring topologically generate the same ideal, are such a standard basis. See example 4.1 for a very special case where this happens.

3.7. **Lifting the valuation to a power series rings.** We can present our complete equicharacteristic local domain R as a quotient of a power series ring $S = k[[x_1, \ldots, x_n]]$ over k by a prime ideal P. Our rational valuation ν then appears as a valuation on S/P which we can compose with the PS_P-adic valuation μ_1 of the regular local ring S_P, with center on S the ideal P and residue field $k_{\mu_1} = S_P/PS_P = \mathrm{Frac}(S/P)$ to obtain a rational valuation μ on S; see [3, Chapter VI, no. 4, Proposition 2]. By definition of the symbolic powers $P^{(e)} = P^e S_P \cap S$, the associated graded ring of S with respect to μ_1 is $\mathrm{gr}_{\mu_1} S = \bigoplus_{e \in \mathbf{N}} \frac{P^{(e)}}{P^{(e+1)}}$.

The graded ring $\mathrm{gr}_\mu S$ is then the graded ring associated to the filtration of $\mathrm{gr}_{\mu_1} S$ induced on each homogeneous component by the valuation ideals $\mathcal{P}_\varphi / P^{(e+1)}$ with $P^{(e+1)} \subseteq \mathcal{P}_\varphi \subseteq P^{(e)}$, where $\varphi \in \{e\} \oplus \Phi$.

Let us pick a minimal system of generators $p_1, \ldots, p_f$ for the ideal P. We note that a regular system of parameters for the regular local ring S_P consists of $n-d$ of the p_q's. We shall assume in the sequel that the dimension of S is minimal, which means that its dimension is the embedding dimension of R.

Proposition 3.39. *1) The valuation μ on S obtained as explained above is rational and its value group is $(\mathbf{Z} \oplus \Phi)_{lex}$.*
2) The generators $p_1, \ldots, p_f$ of the ideal P can be chosen so that $\mu(p_1) < \mu(p_2) < \ldots < \mu(p_f)$ and their μ-initial forms $\mathrm{in}_\mu p_1, \ldots, \mathrm{in}_\mu p_f$ are part of a minimal system of generators of $\mathrm{gr}_\mu S$. All the generators of the semigroup of μ which are of μ_1-value > 0 and are not among the $\mu(p_q)$ are of value $> \mu(p_f)$.
3) With such a choice the ideal PS_P is minimally generated by $p_1, \ldots, p_{n-d}$, which constitute a regular sequence in S. In particular the μ_1-initial forms $\mathrm{in}_{\mu_1} p_1, \ldots, \mathrm{in}_{\mu_1} p_{n-d}$ of $p_1, \ldots, p_{n-d}$ are non-zero in $P/P^{(2)}$.

Proof. Let $S_\mu \subset S_{\mu_1}$ be the inclusion of valuation rings corresponding to the fact that μ is composed with μ_1 and let m_{μ_1} be the intersection with S_μ of the maximal ideal of S_{μ_1}. The valuation ring R_ν is equal to the quotient S_μ / m_{μ_1} and has the same residue field as S_μ. By ([34], §4), the value group of μ is an ordered extension of $\mathbf{Z}$ by Φ, which can only be $(\mathbf{Z} \oplus \Phi)_{lex}$ (see [34, Remarque before section 5]).

If $\mu(p_1) = \mu(p_2)$ for example, their initial forms in $\mathrm{gr}_\mu S$ are proportional since μ is rational, and we may replace p_2 by $p_2 - \lambda p_1$ whose value is $> \mu(p_1)$ if λ is chosen appropriately in k, and so on. In view of the example given on page 200 of [11], some of the series p_q may belong to $P^{(2)}$. However, that is not possible for those which constitute a regular system of parameters for PS_P. [9]

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[9]When the field k is of characteristic zero, it is likely that none of the p_q may belong to $P^{(2)}$. We do not know whether $P^{(2)} \subset (x_1, \ldots, x_n)P$ (see [11]) which would give the result immediately by Nakayama's Lemma, but we can obtain a partial result:
If $p_q \in P^{(2)}$, there is an $h \notin P$ such that $hp_q \in P^2$. In characteristic zero this implies that $\frac{\partial p_q}{\partial x_i} \in P$ for $i = 1, \ldots, n$. By [28, Chap. 0, 0.5], p_q is then integral over mP so that if we denote by $P_1 \subset P$ the ideal generated by the generators of P which are not in $P^{(2)}$, we have $P_1 + \overline{mP} = P$. Since the ideal P is integrally closed, by the integral Nakayama Lemma of [28, lemme 2.4], with $\mathfrak{n}_1 = 0$, we have $\overline{P_1} = P$.

To prove the second assertion of 2), let $\zeta \in S$ be a representative of the smallest element of the semigroup of μ on S whose μ_1-value is > 0. This element has to be part of a minimal system of generators of $\mathrm{gr}_\mu S$. By construction we have $\mu(\zeta) \leq \mu(p_1)$. We also have $\mathrm{in}_{\mu_1}\zeta \in P/P^{(2)}$, which is generated as S/P-module by the images of the p_q, so that $\mu(\zeta) \geq \min(\mu(p_q)) = \mu(p_1)$ and finally $\mu(\zeta) = \mu(p_1)$.

We now proceed by induction and assume that the μ-initial forms of $p_1, \ldots, p_j$ are the first j generators of $\mathrm{gr}_\mu S$ whose μ_1-value is > 0. Since S is complete, it is complete for the valuation μ (see [29, Proposition 5.10]) and we may assume that $\mathrm{in}_\mu p_{j+1}$ is not the initial form $\mathrm{in}_\mu(\sum_{q=1}^{j} a_q p_q)$ of an element of the ideal of S generated by $p_1, \ldots, p_j$; if it were, we might replace p_{j+1} by $p_{j+1} - \sum_{q=1}^{j} a_q p_q$ without affecting its role as generator of P while increasing the μ-value, and continue until we either reach a contradiction with the minimality of the system of generators or obtain a generator with the desired property. This is essentially the same as the argument in the proof of lemma 4.6 of [30].

Since μ is a rational valuation and $\mathrm{in}_\mu p_{j+1}$ is not the initial form of an element of the ideal generated by $p_1, \ldots, p_j$, it cannot be a monomial in the $\mathrm{in}_\mu u_i$ and the $\mathrm{in}_\mu p_1, \ldots, \mathrm{in}_\mu p_j$ so that $\mu(p_{j+1})$ is not in the semigroup Δ_j generated by the μ-values of the elements of $S \setminus P$ and $\mu(p_1), \ldots, \mu(p_j)$. Let $\mu(\zeta)$ be a generator of the semigroup Δ of μ on S which is not in Δ_j and is $\leq \mu(p_{j+1})$.

We can write $\zeta = \sum_{q=1}^{j} a_q p_q + \sum_{q=j+1}^{f} a_q p_q$ and if $\mu(\zeta) < \mu(p_{j+1})$ the μ-initial form of ζ must come from the first sum and thus be the initial form of an element of the ideal of S generated by $p_1, \ldots, p_j$. Now $\mathrm{in}_\mu(p_{j+1})$ is a monomial necessarily involving some of these $\mathrm{in}_\mu(\zeta)$ so either $\mu(p_{j+1})$ is the smallest of the $\mu(\zeta)$ and we have what we want, or $\mathrm{in}_\mu(p_{j+1})$ is the initial form of an element of the ideal of S generated by $p_1, \ldots, p_j$ and we have a contradiction.
Assertion 2) now follows by induction on j.
To prove 3) we use the fact that by construction each p_j does not belong to the ideal generated by $p_1, \ldots, p_{j-1}$ for $2 \leq j \leq n-d$, so that $p_1, \ldots, p_{n-d}$ minimally generate the ideal PS_P. □

Corollary 3.40. *If B is a finite initial set in the minimal set of generators $(\gamma_i)_{i \in I}$ of the value semigroup of the valuation ν, then $B \bigcup \{\mu(p_1), \ldots, \mu(p_{n-d})\}$ and $B \bigcup \{\mu(p_1), \ldots, \mu(p_f)\}$ are finite initial sets in the minimal set of generators of the value semigroup of the valuation μ.*
The filling of $B \bigcup \{\mu(p_1), \ldots, \mu(p_f)\}$ adds only variables u_i.

Proof. This follows from part 2) of the Proposition and the definition of initial sets. □

Given the rational valuation μ of rational rank $r+1 < n$ on the power series ring $S = k[[x_1, \ldots, x_n]]$ which we have just built, the generators γ_i of the semigroup Γ are part of the minimal system of generators of the semigroup Δ of μ. Other elements of the minimal system of generators of Δ are the $\mu(p_1), \ldots, \mu(p_f) \in \mathbf{Z} \oplus \Phi$ by proposition 3.39, and finally the other members of this minimal system of generators of Δ which are $> \mu(p_f)$ and which we denote by δ_a.

We therefore have a surjection of graded k-algebras

$k[(U_i)_{i\in I}, V_1, \dots, V_f, (W_a)_{a\in A}] \to \mathrm{gr}_\mu S,\ U_i \mapsto \mathrm{in}_\mu \eta_i, V_q \mapsto \mathrm{in}_\mu p_q, W_a \mapsto \mathrm{in}_\mu \zeta_a,$

where $\zeta_a \in S$ with $\mu(\zeta_a) = \delta_a$.

By the valuative Cohen theorem we can lift the $\xi_i \in R$ to elements $\eta_i \in S$, the $(\xi_i)_{i\in J}$ which minimally generate the maximal ideal of R lifting to $(\eta_i)_{i\in J}$ with the same property for S since we assumed that the dimension of S is the embedding dimension of R. We can choose representatives ζ_a in S for the elements of $\mathrm{gr}_\mu S$ of degree δ_a in such a way that the theorem applies, the p_q being finite in number can be kept, and so we obtain a continuous surjective morphism of topological k-algebra

$$\Pi\colon k[(u_i)_{i\in I}, \widehat{v_1, \dots,} v_f, (w_a)_{a\in A}] \to S,\ u_i \mapsto \eta_i,\ v_q \mapsto p_q,\ w_a \mapsto \zeta_a,$$

to which we can apply Theorem 3.34 and obtain the following

Lemma 3.41. *Let $\theta \in S$ be an element whose initial form in $\mathrm{gr}_\mu S$ is part of a minimal system of generators of this k-algebra and let $z \in \{(u_i)_{i\in I}, (v_q)_{1\le q\le s}, (w_a)_{a\in A}\}$ be the corresponding variable. A topological generator F_z of the kernel of Π which contains $-z$ as a term according to Theorem 3.34 can be chosen so that the following holds: If θ is one of the η_i or one of the p_q, the initial binomial of F_z involves only the variables $(u_i)_{i\in I}$. If θ is one of the ζ_a, each term of the initial binomial of F_z involves some of the v_j.*

Proof. If θ is one of the η_i the result is clear, since the variables in the initial binomial must be of value less than that of η_i. If θ is one of the p_q, say p_ℓ, there cannot even exist a F_z whose initial binomial is not in $k[(u_i)_{i\in I}]$ because if that was the case, both terms of the initial binomial would contain a monomial in the p_q and then p_ℓ would be in the ideal generated by the p_q of smaller value, which contradicts the minimality of the system of generators. Now if w_a corresponds to $\zeta_a \in P$ which is not one of the p_q, there exist $a_q \in k[(u_i)_{i\in I}, \widehat{v_1, \dots,} v_f, (w_a)_{a\in A}]$ such that $w_a - \sum_{q=1}^f a_q v_q$ is in $\ker\Pi$ and is not zero. The process of expressing this difference as a series

$$w_a - \sum_{q=1}^{f} a_q v_q = \sum A_\ell F_\ell$$

in the topological generators of $\ker\Pi$ begins with expressing the initial form for the weight of that difference as a combination of binomials which are part of a generating system for $\ker\mathrm{gr}\Pi$. In a minimal such expression there can be no term which is a monomial in the u_i because there is no such term in the initial form of the difference. Since the values can only increase from there and in view of the fact that the valuation μ is composed with the PS_P-adic valuation, the sum $\sum A_\ell F_\ell$ can contain no term using only the u_i. On the other hand, since w_a is a term in the difference it must appear as a term in that sum, and we know that one of the F_ℓ must contain w_a (or $-w_a$) as a term. It follow that one of the F_ℓ appearing in the sum, which by construction has an initial binomial which uses some of the v_q, w_b, contains w_a (or $-w_a$) as a term. ∎

Keeping the notations of this section, let us now denote by $\mathcal{P}$ the closure of the ideal of $k[\widehat{(u_i)_{i\in I}, v_1, \dots, v_f}, (w_a)_{a\in A}]$ generated by the $(v_q)_{q=1,\dots,f}$ and the $(w_a)_{a\in A}$. We have a commutative diagram:

$$(*)\qquad \begin{array}{ccc} k[\widehat{(u_i)_{i\in I}, v_1, \dots, v_f}, (w_a)_{a\in A}] & \xrightarrow{\ \Pi\ } & S \\ \downarrow & & \downarrow \\ k[\widehat{(u_i)_{i\in I}}] & \xrightarrow{\ \pi\ } & R \end{array}$$

where the left vertical arrow is the quotient by the ideal $\mathcal{P}$. It follows from Lemma 3.41 that we can now prove:

Theorem 3.42. (Structure of rational valuations on complete equicharacteristic noetherian local domains)
Let R be a complete equicharacteristic noetherian local domain and let $k \subset R$ be a field of representatives of the residue field of R. Let ν be a rational valuation of R and let $(\xi_i)_{i\in I}$ be representatives in R of a minimal set of generators $(\overline{\xi}_i)_{i\in I}$ of the k-algebra $\mathrm{gr}_\nu R$ for which the valuative Cohen theorem holds. Let $J \subset I$ be a minimal set such that the $(\xi_i)_{i\in J}$ generate the maximal ideal of R and let $(U^{m^\ell} - \lambda_\ell U^{n^\ell})_{\ell\in L}$ be a minimal set of generators of the kernel of the surjective morphism of k-algebras $k[(U_i)_{i\in I}] \to \mathrm{gr}_\nu R,\quad U_i \mapsto \overline{\xi}_i$.

Then a set of topological generators for the kernel of the continuous surjective morphism $\pi\colon k[\widehat{(u_i)_{i\in I}}] \to R$ given by the valuative Cohen theorem is structured as follows:

- *For each $i \in I \setminus J$, a generator of the form*
$$F_i = u^{m^i} - \lambda_i u^{n^i} - g_i((u_j)_{j<i}) - u_i,$$
where the initial binomial is one of the $(u^{m^\ell} - \lambda_\ell u^{n^\ell})_{\ell\in L}$, the series $g_i((u_j)_{j<i}) \in k[\widehat{(u_i)_{i\in I}}]$ depends only on variables of index $< i$ and the weight of each of its terms is $> w(u^{m^i})$ and $< w(u_i)$.
- *A finite subset of the complement in L of the set of binomials used above is in bijection with generators of the form*
$$F_q = u^{m^q} - \lambda_q u^{n^q} - g_q(u),$$
with $g_q(u) \in k[\widehat{(u_i)_{i\in I}}]$ and the weight of each of its terms is $> w(u^{m^q})$.
- *The series $(F_\ell)_{\ell\in L}$ constitute a (generalized) standard basis with respect to the weight w, in the sense of Definition 3.2, for the ideal of $k[\widehat{(u_i)_{i\in I}}]$ topologically generated by the $(F_i)_{i\in I\setminus J}, (F_q)_{q=1,\dots,f}$.*

Moreover, each of the $(F_\ell)_{\ell\in L}$ uses only finitely many of the variables u_i.

Proof. From the diagram above we see that we have the equality $\Pi^{-1}(P) = \ker\Pi + \mathcal{P}$ and that $\ker\pi$ is the image of this ideal in $k[\widehat{(u_i)_{i\in I}}]$. So we obtain topological generators of $\ker\pi$ by reducing the topological generators of $\ker\Pi$ modulo $\mathcal{P}$. By Theorem 3.34 the ideal $\ker\Pi$ is generated by the $(F_i)_{i\in I\setminus J}, (F_{v_q})_{q=1,\dots,f}, (F_{w_a})_{a\in A}$. By Lemma 3.41, modulo $\mathcal{P}$ those which involve a $-w_a$ as a term vanish entirely, those which involve a $-v_q$ as a last term do not vanish since their initial binomial involves only the u_i and

become the F_q of the theorem. Those which involve a $-u_i$ as a last term pass to the quotient without modification and become the $F_i \in \ker\pi$.
Finally, by construction the initial forms of the F_ℓ generate the kernel of the morphism $\mathrm{gr}_w\pi$, which is the initial ideal of the kernel of π with respect to the weight w. □

Remarks 3.43. 1) We know that, since the $(\tilde{\eta}_j)_{j\in J}$ generate the maximal ideal of S, we can write each p_q as a series in the $(\tilde{\eta}_j)_{j\in J}$. In fact the morphism Π induces an isomorphism from the sub k-algebra $k[[(u_j)_{j\in J}]]$ of $k[\widehat{(u_i)_{i\in I}, v_1, \ldots, v_f}, (w_a)_{a\in A}]$ to S and the $p_q((u_j)_{j\in J}) - v_q$ belong to the kernel of Π. *The point is to obtain each $p_q(\tilde{\eta}_j)_{j\in J})$, as we do using the topological generator of* $\ker\Pi$ *which contains $-v_q$ as a term, as the result of successive eliminations through equations whose initial forms are binomials belonging to our set of generators of the kernel of* $\mathrm{gr}_w\Pi$. *This is what will allow us to obtain a quotient of R as an overweight deformation of a finite number of these binomials.*
2) The theorem shows the persistence of the abyssal phenomenon of Remark 3.35 even when R is not regular.
3) An interpretation of the theorem is that it presents the valuation ν on R as the image of the *monomial* valuation on $k[\widehat{(u_i)_{i\in I}}]$ given by the weights, in the sense that the value of an element of R is the maximum weight of its representatives in $k[\widehat{(u_i)_{i\in I}}]$ (see [30, Proposition 3.3]). In other words, the ν-adic filtration on R is the image by the morphism π of the w-adic filtration on $k[\widehat{(u_i)_{i\in I}}]$.
4) The semigroup Δ of values of the valuation μ on S requires more generators than the $(0, \gamma_i)$, the $\mu(p_j)$ and some of the (n, γ), $n \in \mathbf{N} \setminus \{0\}, \gamma \in \Gamma$, and their description would be interesting. We do not need it here thanks to Lemma 3.41.

3.8. **Equations for the toric degeneration.** We can now write the equations for the toric degeneration of R to $\mathrm{gr}_\nu R$ in the natural parameters, the v^φ, where φ runs through a system of generators for the semigroup $\Phi_{\geq 0}$, although there is not a minimal one in general. See [29, §4].

Recall from Proposition 2.1 or [29, Section 2.3] that the algebra encoding the toric degeneration of the ring R to its associated graded ring $\mathrm{gr}_\nu R$ is the valuation algebra

$$\mathcal{A}_\nu(R) = \bigoplus_{\varphi\in\Phi} \mathcal{P}_\varphi(R) v^{-\varphi} \subset R[v^\Phi].$$

Having fixed a field of representatives $k \subset R$ of the residue field, let us consider the $k[v^{\Phi_{\geq 0}}]$-algebra $k[v^{\Phi_{\geq 0}}][\widehat{(\hat{u}_i)_{i\in I}}] = k[v^{\Phi_{\geq 0}}] \otimes_k k[\widehat{(\hat{u}_i)_{i\in I}}]$, which is the same construction as $k[\widehat{(u_i)_{i\in I}}]$ in section 3.3 but where the series have coefficients in $k[v^{\Phi_{\geq 0}}]$. Each variable $\hat{u}_i$ still has weight γ_i and we continue to write $w(p)$ for $w(\hat{u}^p)$.

In [29, Section 2.4] we defined on the algebra $\mathcal{A}_\nu(R)$ a valuation $\nu_\mathcal{A}$ by

$$\nu_\mathcal{A}(\sum x_\varphi v^{-\varphi}) = \min(\nu(x_\varphi)), \text{ with } \mathrm{gr}_{\nu_\mathcal{A}} \mathcal{A}_\nu(R) \simeq k[v^{\Phi_{\geq 0}}] \otimes_k \mathrm{gr}_\nu R,$$

an isomorphism as (bi-)graded $k[v^{\Phi_{\geq 0}}]$-algebras. See *loc.cit.* for details.

Proposition 3.44. *With the notations of Theorem 3.13, the* $(\xi_i v^{-\gamma_i})_{i\in I}$ *constitute a minimal system of homogeneous topological generators of the* $k[v^{\Phi_{\geq 0}}]$*-algebra* $\mathcal{A}_\nu(R)$ *in the sense that the morphism of* $k[v^{\Phi_{\geq 0}}]$*-algebras*

$$\hat\pi\colon k[v^{\Phi_{\geq 0}}][\widehat{(\hat u_i)_{i\in I}}] \longrightarrow \mathcal{A}_\nu(R),\quad \hat u_i \mapsto \xi_i v^{-\gamma_i}$$

is surjective and continuous for the topologies defined by the weight and the valuation $\nu_{\mathcal{A}}$ *respectively.*

Proof. By the valuative Cohen theorem (Theorem 3.13), if $x \in \mathcal{P}_\varphi(R)$ we can write $x = \sum_e d_e \xi^e$ where the $d_e \in k^*$ and the ξ^e are monomials in the ξ_i of increasing valuations, with the minimum value being $\geq \varphi$. Let us write $-\varphi = -\nu(\xi^e) + \eta_e$ with $\eta_e \geq 0$. Note that the η_e corresponding to the smallest index in the sum is zero if $\nu(x) = \varphi$ and all the others are > 0. Then we can write $xv^{-\varphi} = \sum_e d_e v^{\eta_e}\xi^e v^{-\nu(\xi^e)}$ and since the $\xi^e v^{-\nu(\xi^e)}$ are monomials in the $\xi_i v^{-\gamma_i}$ and the minimality is clear as well as the continuity, this proves the result. $\square$

We now think of the $\hat u_i = \xi_i v^{-\gamma_i}$ as defining an embedding of the total space of the toric degeneration into the space corresponding to the algebra $k[v^{\Phi_{\geq 0}}][\widehat{(\hat u_i)_{i\in I}}]$. We are seeking equations for the image of this embedding, that is, generators for the kernel of the morphism $\hat\pi$ of Proposition 3.44 above.
If a series $H((\hat u_i))_{i\in I})$ is in the kernel, we see that by definition of the morphism, we have $H((v^{\gamma_i}\hat u_i)_{i\in I}) = 0$ in R, and conversely. so that the change of variables $u_i \mapsto v^{\gamma_i}\hat u_i$, which is well defined over $\mathrm{Spec}(v^{\Phi_+})^{-1}k[v^{\Phi_{\geq 0}}]$, carries the kernel of the morphism π of Theorem 3.13 to the kernel of $\hat\pi$. It realizes the isomorphism, valid over $\mathrm{Spec}(v^{\Phi_+})^{-1}k[v^{\Phi_{\geq 0}}]$ of the fiber of $\mathrm{Spec}\mathcal{A}_\nu(R) \to \mathrm{Spec}k[v^{\Phi_{\geq 0}}]$ with the space corresponding to R.
Now for each topological generator $F_\ell = u^{m^\ell} - \lambda_\ell u^{n^\ell} + \cdots$ of the kernel of π we see that $F_\ell(v^\gamma \hat u)$, with the obvious contracted notation, is divisible, as an element of $k[v^{\Phi_{\geq 0}}][\widehat{(\hat u_i)_{i\in I}}]$, by $v^{w(m^\ell)} = v^{w(n^\ell)}$. The ideal of $k[v^{\Phi_{\geq 0}}][\widehat{(\hat u)_i)_{i\in I}}]$ topologically generated by the

$$\hat F_\ell = v^{-w(m^\ell)}F_\ell(v^\gamma\hat u) = \hat u^{m^\ell} - \lambda_\ell \hat u^{n^\ell} + \sum_{w(p)>w(m^\ell)} c_p v^{w(p)-w(m^\ell)}\hat u^p$$

is contained in the kernel of $\hat\pi$, is equal to it and also isomorphic to the kernel of π over $\mathrm{Spec}(v^{\Phi_+})^{-1}k[v^{\Phi_{\geq 0}}]$, while the space it defines modulo the ideal $(v^{\Phi_{>0}})$ is the same as the space defined by $\mathcal{A}_\nu(R)$ modulo that ideal, namely $\mathrm{Specgr}_\nu R$. By flatness of $\mathcal{A}_\nu(R)$ over $k[v^{\Phi_{\geq 0}}]$, they have to coincide.

Combining this with Theorem 3.42 we have obtained the following:

Theorem 3.45. *With the notations of Theorem 3.42, a set of topological generators in* $k[v^{\Phi_{\geq 0}}][\widehat{(\hat u_i)_{i\in I}}]$ *for the ideal* $\ker\hat\pi$ *defining the total space of the degeneration of* R *to* $\mathrm{gr}_\nu R$ *is as follows:*

- *For each* $i \in I \setminus J$*, a generator of the form*

$$\hat F_i = \hat u^{m^i} - \lambda_i \hat u^{n^i} - \hat g_i(v, (\hat u_j)_{j<i}) - v^{\gamma_i - w(\hat u^{m^i})}\hat u_i,$$

where the initial binomial is one of the $(U^{m^\ell} - \lambda_\ell U^{n^\ell})_{\ell\in L}$, *the series* $\hat{g}_i(v, (\hat{u}_j)_{j<i}) \in k[v^{\Phi_{\geq 0}}][\widehat{(\hat{u}_i)_{i\in I}}]$ *depends only on variables* u_i *of index* $< i$ *and the weight of each of its terms is* $> w(\hat{u}^{m^i})$ *and* $< w(\hat{u}_i)$.

- *A finite subset of the complement in* L *of the set of binomials* $(\hat{u}^{m^\ell} - \lambda_\ell \hat{u}^{n^\ell})_{\ell\in L}$ *used by the* $\hat{F}_i$ *is in bijection with generators of the form*

$$\hat{F}_q = \hat{u}^{m^q} - \lambda_q \hat{u}^{n^q} - \hat{g}_q(v, \hat{u}),$$

with $\hat{g}_q(v, \hat{u}) \in k[v^{\Phi_{\geq 0}}][\widehat{(\hat{u}_i)_{i\in I}}]$ *and the weight of each of its terms is* $> w(\hat{u}^{m^q})$.

Moreover, we have:

- *The series* $(\hat{F}_\ell)_{\ell\in L}$ *constitute a standard basis for the ideal* $\mathrm{ker}\hat{\pi}$ *with respect to the weight filtration.*

Each series uses finitely many of the variables $\hat{u}_i$ *and each term appearing in the series* $\hat{F}_i$ *or* $\hat{F}_q$ *is of the form* $c_e v^{w(\hat{u}^e)-w(\hat{u}^{m^i})}\hat{u}^e$ *or* $c_e v^{w(\hat{u}^e)-w(\hat{u}^{m^q})}\hat{u}^e$ *with* $c_e \in k^*$. □

Remark 3.46. In view of [30, Proposition 2.3], the ring $k[v^{\Phi_{\geq 0}}]$ is the union of a nested sequence of polynomial rings in r variables, where r is the rational rank of Φ:

$$k[x_1^{(0)}, \dots, x_r^{(0)}] \subset \dots \subset k[x_1^{(h)}, \dots, x_r^{(h)}] \subset k[x_1^{(h+1)}, \dots, x_r^{(h+1)}] \subset \dots \subset k[v^{\Phi_{\geq 0}}],$$

where each variable $x_i^{(h)}$ is mapped to a monomial in the $x_j^{(h+1)}$, $j = 1, \dots, r$. Therefore, whenever we deal with a finite set of series F_ℓ, for example the $F_\ell|B$ of section 4 below, since we may assume that they use only finitely many of the variables u_i, the parameter space of the toric degeneration can be taken to be an r-dimensional affine space $\mathbf{A}^r(k)$.

Proposition 3.47. *If the semigroup* Γ *is finitely generated, the valuation* ν *is Abhyankar.*

Proof. If Γ is finitely generated, there are finitely many variables and finitely many equations involved to encode the specialization of R to $\mathrm{gr}_\nu R$, which means, in view of Remark 3.46, that it corresponds to a morphism

$$k[x_1^{(h)}, \dots, x_r^{(h)}] \to \hat{\mathcal{A}}_\nu^{(h)}(R)$$

for sufficiently large h, where $\hat{\mathcal{A}}_\nu^{(h)}(R)$ is the algebra corresponding to the degeneration as in Theorem 3.42, with the v^φ replaced by elements of $k[x_1^{(h)}, \dots, x_r^{(h)}]$. This morphism is still faithfully flat while the rings are now noetherian and we can apply [3, AC VIII, §4, no.4, Corollaire 1] to prove the equidimensionality of the fibers, that is, $\mathrm{dimgr}_\nu R = \mathrm{dim} R$. □

Remark 3.48. This is one direction of [30, Theorem 7.21].

Example 3.49. Consider the overweight deformation, as in section 3.2:

$$y^2 - x^3 - u, \; u^2 - x^{s-1}y$$

of the binomial ideal $y^2 - x^3, u^2 - x^{s-1}y$, where x, y, u have weights $4, 6, 2s+1$ respectively, with $s \geq 6$. Set $\hat{x} = v^{-4}x, \hat{y} = v^{-6}y, \hat{u} = v^{-(2s+1)}u$.
Substituting $x = v^4\hat{x}, y = v^6\hat{y}, u = v^{2s+1}\hat{u}$ in the equations, we obtain $v^{12}\hat{y}^2 - v^{12}\hat{x}^3 - v^{2s+1}\hat{u} = 0, v^{4s+2}\hat{u}^2 - v^{4s+2}\hat{x}^{s-1}\hat{y} = 0$.

After dividing each equation by the highest possible power of v, we obtain the equations for the total space of the toric degeneration over $\mathrm{Spec} k[v]$ of the formal curve in $\mathbf{A}^3(k)$ with ring $R = k[[x,y,u]]/(y^2 - x^3 - u, u^2 - x^{s-1}y)$ to the monomial (toric) curve $\hat{x} = t^4, \hat{y} = t^6, \hat{u} = t^{2s+1}$:

$$\hat{y}^2 - \hat{x}^3 - v^{2(s-6)+1}\hat{u} = 0,\ \hat{u}^2 - \hat{x}^{s-1}\hat{y} = 0.$$

Example 3.50. Let us go back to Example 2.3 and let $R \subset k[[t]]$ be the algebras of an algebroid plane branch and its normalization, endowed with the t-adic valuation. We have an inclusion of valuation algebras $\mathcal{A}_\nu(R) \subset \mathcal{A}_\nu(k[[t]]) \simeq k[v][[t]][v^{-1}t]$. As above, we introduce $\hat{t} = v^{-1}t$ to embed the space corresponding to $\mathcal{A}_\nu(k[[t]])$ in the affine space $\mathbf{A}^1(k[v])$ over the affine line $\mathbf{A}^1(k)$. The $\xi_i \in R$ of Example 2.3, viewed as elements $\xi_i(t) \in k[[t]]$, parametrize the embedding of our curve in $\mathbf{A}^{g+1}(k)$. They can be written

$$\xi_i(t) = t^{\gamma_i} + \sum_{e>\gamma_i} c_e^{(i)} t^e, \quad \text{with } c_e^{(i)} \in k.$$

The generators $v^{-\gamma_i}\xi_i$ of $\mathcal{A}_\nu(R)$ then become elements $v^{-\gamma_i}\xi_i(v\hat{t}) \in k[v][[\hat{t}]]$ which realize in a parametric way the degeneration of our curve to the monomial curve with the same semigroup, as in [**?**, Section 3]:

$$v^{-\gamma_i}\xi_i(v\hat{t}) = \hat{\xi}_i(v,\hat{t}) = \hat{t}^{\gamma_i} + \sum_{e>\gamma_i} c_e^{(i)} v^{e-\gamma_i}\hat{t}^e, \quad \text{with } c_e^{(i)} \in k.$$

A version of this example appears in [29, Section 5.4].

One can compare the degenerations, by equations and by parametrization, on the following example. It is the case $s = 6$ of Example 3.49, where the equation of the plane branch has been changed a little in order to simplify the parametrization, and we assume characteristic $\neq 2$.
The equations and the parametrization correspond for $v = 1$ to the torific embedding in $\mathbf{A}^3(k)$ of the plane branch with equation

$$(y^2 - x^3)^2 - x^5 y - \frac{1}{16}x^7 = 0.$$

They are

$$\hat{y}^2 - \hat{x}^3 - v\hat{u} = 0, \quad \hat{u}^2 - \hat{x}^5\hat{y} - \frac{1}{16}v^2\hat{x}^7 = 0$$

$$\hat{x} = \hat{t}^4, \hat{y} = \hat{t}^6 + \frac{1}{2}v\hat{t}^7, \hat{u} = \hat{t}^{13} + \frac{1}{4}v\hat{t}^{14}.$$

Remarks 3.51. 1) The assumption that the characteristic is $\neq 2$ is geometrically meaningful: the curve given parametrically by $x = t^4, y = t^6 + t^7$ which in characteristic $\neq 2$ has the same semigroup $\langle 4, 6, 13\rangle$ as this example, has in characteristic 2 the semigroup $\langle 4, 6, 15\rangle$.
In characteristic $\neq 0$, given a parametric presentation $x(t), y(t)$ in $k[[t]]$ of a formal plane branch, the relation between the two series, the semigroup of the branch, and the characteristic of k is very mysterious to us. See, however, [1]. In characteristic zero the question was settled by Zariski in [35, Chap. 1].
2) Since $\mathcal{A}_\nu(R)$ is a Φ-graded algebra, the component of degree φ being $\mathcal{P}_\varphi v^{-\varphi}$, it is natural that equations and parametrizations for the toric degeneration should be homogeneous (or rather isobaric) when v is given weight

-1. Thus, one can write directly equations for the total space of the degeneration from the overweight deformation.

4. Approximating a valuation by Abhyankar semivaluations

4.1. **The basic example of approximation of a valuation by Abhyankar semivaluations.** This example comes from the interpretation found in [29, Example 4.20] of an example of Zariski.

Let k be an algebraically closed field and let ν be a rational valuation of rational rank one on the ring $R = k[[x, y]]$. It cannot be Abhyankar, and therefore the semigroup $\Gamma = \nu(R \setminus \{0\})$ is not finitely generated (see [30, beginning of §7]). Up to a change of variables we may assume that $\nu(x) < \nu(y)$ and that $\nu(y)$ is the smallest non-zero element of Γ which is not in $\Gamma_0 = \mathbf{N}\nu(x)$. By [8, Theorem 1.1 and Lemma 2.1], the semigroup Γ is generated by positive rational numbers $\nu(x), \nu(y), (\gamma_i)_{i\geq 2}$ and we can normalize the valuation by setting $\gamma_0 = \nu(x) = 1$. Here, as in Example 2.3, we denote the smallest element of Γ by γ_0 instead of γ_1 to underscore the special role played historically by the coordinate x.
Moreover, denoting by Φ_i (resp. Γ_i) the subgroup (resp. subsemigroup) of $\mathbf{Q}$ generated by $1, \nu(y), \nu(u_2), \ldots, \nu(u_i)$, with $\Phi_0 = \mathbf{Z}$, Φ_1 (resp. Γ_1) generated by $\nu(x), \nu(y)$, and denoting by n_i the index $[\Phi_i : \Phi_{i-1}]$, we have that $\gamma_{i+1} > n_i\gamma_i$ and as a consequence $n_i\gamma_i \in \Gamma_{i-1}$ and the relations between the γ_i are generated by the expressions of this last result as

$$n_i\gamma_i = t_0^{(i)}\gamma_0 + t_1^{(i)}\gamma_1 + \sum_{j=2}^{i-1} t_j^{(i)}\gamma_j \quad \text{with } t_j^{(i)} \in \mathbf{N}, \quad 0 \leq j \leq i-1,$$

where we may assume that $t_j^{(i)} < n_j$ for $j \geq 1$.
According to Theorem 3.34, by a good choice of the representatives $\xi_i \in k[[x, y]]$ of the generators γ_i the morphism

$$k[\widehat{x, y, (u_i)_{i\geq 2}}] \to k[[x, y]], \ x \mapsto x, y \mapsto y, u_i \mapsto \xi_i,$$

is well defined and surjective, and has a kernel generated up to closure by

$$\begin{array}{l} y^{n_1} - x^{t_0^{(1)}} - g_1(x, y) - u_2, \\ \vdots \\ u_i^{n_i} - x^{t_0^{(i)}} y^{t_1^{(i)}} \prod_{2\leq q\leq i-1} u_q^{t_q^{(i)}} - g_i(x, y, (u_j)_{2\leq j\leq i}) - u_{i+1} \\ u_{i+1}^{n_{i+1}} - x^{t_0^{(i+1)}} y^{t_1^{(i+1)}} \prod_{2\leq j\leq i} u_j^{t_j^{(i+1)}} - g_{i+1}(x, y, (u_j)_{2\leq j\leq i+1}) - u_{i+2} \\ \vdots \\ \vdots \end{array}$$

with the series $g_i(x, y, (u_j)_{2\leq j\leq i}) \in k[[(x, y, (u_j)_{2\leq j\leq i}]]$ satisfying the overweight condition and having each term of weight $< \gamma_{i+1}$.

Now if we keep only the first i equations, and set $u_j = 0$ for $j \geq i+1$, we obtain by elimination of the variables u_j , $2 \leq j \leq i$ the equation $p_i(x, y) = 0$ of a plane branch C_i in the formal affine plane. It has a unique valuation, which is Abhyankar and has a semigroup Γ_i generated by $1, \nu(y), \nu(u_j)_{2\leq j\leq i}$. The unique valuation on the branch C_i is an Abhyankar semivaluation on

$k[[x,y]]$ and the corresponding semigroups $\Gamma_i = \langle 1, \gamma_1, \gamma_2, \ldots, \gamma_i \rangle$ fill the semigroup Γ as i grows. The intersection numbers with C_i at the origin of the elements $\xi_j \in k[[x,y]]$ are the γ_j, for $2 \leq j \leq i$.

We note that by construction the valuation of $p_i(x,y)$ is that of u_{i+1}, that is γ_{i+1}.
Finally, by Chevalley's theorem, there exists a function $\beta\colon \mathbf{N} \to \mathbf{N}$ with $\beta(i)$ tending to infinity with i, such that $p_i(x,y) \in (x,y)^{\beta(i)}$.

To the branch C_i is associated a rank two valuation ν_i on $k[[x,y]]$ defined as follows: each element $h \in k[[x,y]]$ can be written uniquely as $h = p_i^n \tilde{h}$ with $\tilde{h}$ not divisible by p_i and then the restriction of $\tilde{h}$ to C_i has a value $\gamma \in \Gamma_i$.
The map $h \mapsto (n, \gamma) \in \mathbf{Z} \times \Phi_i$ is the rank two valuation ν_i, which is a rational Abhyankar valuation dominating $k[[x,y]]$. The valuations ν_i approximate ν in the sense that given any $h \in k[[x,y]]$, for sufficiently large i the element $p_i(x,y) \in (x,y)^{\beta(i)}$ does not divide h so that $\nu_i(h) \in \{0\} \times \Gamma_i$ is also $\nu(h)$ by what we have seen above. So we can indeed approximate the non-Abhyankar valuation ν of rank one and rational rank one by Abhyankar valuations of $k[[x,y]]$ of rank two and rational rank two, but the fundamental fact is the approximation of ν by Abhyankar semivaluations.

Remark 4.1. If the algebraically closed field k is of characteristic zero, each equation $p_i(x,y) = 0$ has for roots the conjugates of a Puiseux expansion $y_i(x)$ with $i+1$ characteristic exponents $\beta_0, \ldots, \beta_i$ which coincide, up to renormalization, with the Puiseux expansion up to and excluding the $(i+1)$-th Puiseux exponents of the roots of the equations $p_j = 0$ for $j > i$ (see [23]). As i tends to infinity these Puiseux series $y_i(x)$ converge to a series $y_\infty(x) \in k[[x^{\mathbf{Q}_{\geq 0}}]]$ which is not a root of any polynomial or power series in x, y because the denominators of the exponents are not bounded. Thus, given $h(x,y) \in k[[x,y]] \setminus \{0\}$, the series $h(x, y_\infty(x))$ is not zero, and the map which to a series $h(x,y)$ associates the order in x of $h(x, y_\infty(x))$ defines a valuation on $k[[x,y]]$, which is the valuation ν. For more details in a more general situation, see [21].
The relationship between these Puiseux exponents β_j and the generators of the semigroup is quite simple (see [35, Chap.1]): it is $\gamma_{j+1} - n_j\gamma_j = \beta_{j+1} - \beta_j$. For a wider perspective one can consult [13] and [10].

These beautiful facts are unfortunately missing when one tries to understand valuations over fields of positive characteristic. Even for branches the behaviour of solutions in generalized power series $y(x) \in k[[x^{\mathbf{Q}_{\geq 0}}]]$ of algebraic equations is much more complicated when k is of positive characteristic (see [19]) and as far as we know the relationship of these solutions with the semigroup of values has not been clarified (see [31, Section 1]).

The purpose of this section is to show that the situation described above is in fact quite general. We start from Theorem 3.42.

4.2. **Definition of the ideals K_B.** Let us choose a finite initial subset B_0 of I which contains:

- The set J of indices of the elements $(\xi_i)_{i \in J}$ minimally generating the maximal ideal of R;

- A set of $r = \text{rat.rk.}\Phi$ indices $i_1, \dots, i_r$ such that the $(\gamma_{i_t})_{t=1,\dots,r}$ rationally generate the group Φ;
- The indices of the finite set of variables u_i appearing in the equations $(F_q)_{1\leq q\leq f}$ of Theorem 3.42.

Let B be (the filling of) a finite initial subset of I containing B_0. Denote by ι_B the injection $k[[(u_i)_{i\in B}]] \subset k[[(u_i)_{i\in I}]]$ and consider in R the ideal K_B of R generated by the images by the morphism of k-algebras

$$\pi \circ \iota_B \colon k[[(u_i)_{i\in B}]] \longrightarrow R\ ;\quad u_i \mapsto \xi_i$$

of the F_ℓ whose initial binomial involves only variables whose index is in B and in which all the u_i with $i \notin B$ have been set equal to 0.
These series are denoted by $F_\ell|B$. Note that the F_ℓ which use only variables in B are mapped to zero in this operation and the images by $\pi \circ \iota_B$ of the other $F_\ell|B$ are contained in the ideal generated by the $(\xi_i)_{i\notin B}$. .
The ideal K_B is the image by $\pi\circ\iota_B$ of the ideal $\mathcal{K}_B$ of $k[[(u_i)_{i\in B}]]$ generated by the $F_\ell|B$.

Consider the following commutative diagram, which results from our constructions.

(**)
$$\begin{array}{ccccc} k[[(u_i)_{i\in B}]] & \overset{\iota_B}{\hookrightarrow} & \widehat{k[(u_i)_{i\in I}]} & \overset{\pi}{\longrightarrow} & R \\ & & \searrow^{\pi_B} & & \downarrow \\ & & & & R/K_B \end{array}$$

Lemma 4.2. *Assuming that B is full, the kernel of the surjective morphism $\pi\circ\iota_B$ is generated by the $(F_i)_{i\in B\setminus J}, (F_q)_{1\leq q\leq f}$ and the kernel of π_B is generated by the $F_\ell|B$ for the F_ℓ whose initial binomial is in $k[[(u_i)_{i\in B}]]$, which include the generators of the kernel of the morphism $\pi\circ\iota_B$.*

Proof. The morphism $\pi\circ\iota_B$ is surjective because B contains the indices of a set of generators of the maximal ideal of R. To prove the first statement, we note that the $(F_i)_{i\in B\setminus J}$ and $(F_q)_{1\leq q\leq f}$ in question belong to the kernel of $\pi\circ\iota_B$. Let us go back to section 3.7 and let again $(\xi_i)_{i\in J}$ be elements of our set of representatives $(\xi_i)_{i\in I}$ minimally generating the maximal ideal of R. In view of point 2) of Remark 3.43, the intersection of the kernel of $\pi\circ\iota_B$ with the subalgebra $k[[(u_i)_{i\in J}]]$ of $k[[(u_i)_{i\in B}]]$ is generated by the series $p_q((u_i)_{i\in J}),\ q = 1,\dots,f$.
Remembering of course the fact that the finite set B is full we can use the F_i to successively eliminate the variables $u_i,\ i\notin J$ from the F_q as we did in the proof of Theorem 3.34. This shows that modulo the ideal generated by the $(F_i)_{i\in B\setminus J}$, the series F_q belongs to the ideal of $k[[(u_i)_{i\in B}]]$ generated by the $p_q((u_i)_{i\in J}),\ q = 1,\dots,f$. On the other hand, by the same process, any element of the kernel of $\pi\circ\iota_B$ is congruent modulo the ideal generated by the F_i to an element of the restriction to $k[[(u_i)_{i\in J}]]$ of that kernel and so, modulo the ideal generated by the F_i, belongs to the ideal generated by the $p_q((u_i)_{i\in J}),\ q = 1,\dots,f$. This proves the result. The second statement follows from the first and the definition of K_B.

□

Proposition 4.3. *Let B be (the filling of) a finite initial set of the set of generators of the semigroup Γ.*

(1) The quotient R/K_B is an overweight deformation of the prime binomial ideal generated by the binomials $u^{m^\ell} - \lambda_\ell u^{n^\ell}$ which are contained in $k[(u_i)_{i\in B}]$. In particular, K_B is a prime ideal.

(2) Given an integer D there exists (the filling of) a finite initial set $B(D)$ such that for any (filling of a) finite initial set B containing $B(D)$ we have $K_B \subset m^{D+1}$.

(3) For each set B as above, we have:

$$\nu(K_B) = \min_{i\notin B}\gamma_i.$$

In particular, $\xi_i \notin K_B$ for $i \in B$.

Proof. The ring R/K_B is the quotient of $k[[(u_i)_{i\in B}]]$ by the ideal generated by the $F_\ell|B$ and the kernel of $\pi\circ\iota_B$. Let us denote by $\mathcal{P}_B$ the filtration of R/K_B which is the image of the filtration by weight in $k[[(u_i)_{i\in B}]]$. In the language of Proposition 2.1 this means that we consider the order function on R/K_B which associates to an element the maximum weight of its representatives in $k[[(u_i)_{i\in B}]]$. The induced morphism

$$\mathrm{gr}_w(\pi_B)\colon \mathrm{gr}_w k[[(u_i)_{i\in B}]] = k[(U_i)_{i\in B}] \to \mathrm{gr}_{\mathcal{P}_B}(R/K_B)$$

is surjective by construction. By Proposition 3.14, its kernel is the w-initial ideal of the kernel of π_B, which by Lemma 4.2 is generated by the binomials $U^{m^\ell} - \lambda_\ell U^{n^\ell}$ which are in $k[(U_i)_{i\in B}]$. These are exactly the binomials corresponding to the relations between the $(\overline{\xi}_i)_{i\in B}$.
Thus, the k-algebra $\mathrm{gr}_{\mathcal{P}_B}(R/K_B)$ is, by Proposition 3.1, a twisted semigroup algebra for the semigroup which is the image in Φ of $\mathbf{N}^b = \bigoplus_{i=1}^b \mathbf{N}e_i$, with $b = |B|$, by the map $e_i \mapsto \gamma_i$. This semigroup is $\Gamma_B = \langle(\gamma_i)_{i\in B}\rangle$. Since B contains B_0, the rational rank of the group it generates is r so that $\dim \mathrm{gr}_{\mathcal{P}_B}(R/K_B) = r$. By the flatness of the degeneration of R/K_B to $\mathrm{gr}_{\mathcal{P}_B}(R/K_B)$, which is finitely generated (see Proposition 3.47), this implies that R/K_B has dimension r. We now have to show that it is an overweight deformation of $\mathrm{gr}_{\mathcal{P}_B}(R/K_B)$. This follows from the next Lemma.

In the following lemma, an overweight unfolding *is a deformations of equations which adds only terms of higher weight, without any condition on the initial forms of the elements of the ideal generated by the deformed equations.*

Lemma 4.4. *Let $\mathbf{F}_0 = (u^{m^\ell} - \lambda_\ell u^{n^\ell})_{\ell\in L}$ be a prime binomial ideal in $k[[u_1,\dots,u_N]]$ corresponding to a minimal system of generators of the relations between the generators of a finitely generated semigroup $\Gamma = \langle\gamma_1,\dots,\gamma_N\rangle$ generating a totally ordered abelian group of rational rank r. Let w be the weight on $k[[u_1,\dots,u_N]]$ defined by giving u_i the weight γ_i.*

Let $(F_\ell = u^{m^\ell} - \lambda_\ell u^{n^\ell} + \sum_{w(u^p)>w(u^{m^\ell})} c_p^{(\ell)} u^p)_{\ell\in L}$ be overweight unfoldings of the binomials. Let $\mathbf{F}$ be the ideal of $k[[u_1,\dots,u_N]]$ generated by the $(F_\ell)_{\ell\in L}$ and $R = k[[u_1,\dots,u_N]]/\mathbf{F}$.
If $\dim R = r$, then the unfolding is an overweight deformation of the binomial ideal.

Exactly as in the proof of Theorem 3.3 of [30], we can define an order function with values in Γ on R by associating to a non-zero element of R the highest weight of its preimages in $k[[u_1, \ldots, u_N]]$. Then we have, by Proposition 2.1, a faithfully flat specialization of R to $k[[u_1, \ldots, u_N]]/\mathrm{in}_w\mathbf{F}$, where $\mathrm{in}_w\mathbf{F}$ is the w-initial ideal of the ideal $\mathbf{F}$, and $\dim(k[[u_1, \ldots, u_N]]/\mathrm{in}_w\mathbf{F}) = \dim R$. On the other hand we have the exact sequence

$$(0) \to \mathrm{in}_w\mathbf{F}/\mathbf{F}_0 \to k[[u_1, \ldots, u_N]]/\mathbf{F}_0 \to k[[u_1, \ldots, u_N]]/\mathrm{in}_w\mathbf{F} \to (0),$$

and since $\mathbf{F}_0$ is prime and the domain $k[[u_1, \ldots, u_N]]/\mathbf{F}_0$ is of dimension r (see [29, Proposition 3.7]), the equality $\dim R = r$ implies the equality $\mathrm{in}_w\mathbf{F} = \mathbf{F}_0$ which means that we have an overweight deformation. This proves the first assertion of the proposition.

The second part of Proposition 4.3 follows from the fact that by Theorem 3.23 we can choose $B(D)$ so that the $(\xi_i)_{i \notin B(D)}$ are all in m^{D+1} and by construction the image in R of $F_\ell|B \in k[[(u_i)_{i \in B(D)}]]$ differs from zero by a series all of whose terms are in the ideal generated by the $(\xi_i)_{i \notin B(D)}$. This remains true for any finite set B containing $B(D)$.

The ideal K_B is contained in the ideal generated by the $(\xi_i)_{i \notin B}$, so that $\nu(K_B) \geq \min_{i \notin B}\gamma_i$. Since B is full, if u_j is the variable of weight $\min_{i \notin B}\gamma_i$, all the other variables appearing in the series F_j belong to B. Setting $u_j = 0$ in the series F_j and taking the image of the result in R creates an easily written element of value $\min_{i \notin B}\gamma_i$ in K_B. $\square$

To summarize, the ring R/K_B is indeed an overweight deformation of the prime binomial generated by the $u^{m^\ell} - \lambda_\ell u^{n^\ell}$ which are in $k[[(u_i)_{i \in B}]]$ in the sense of subsection 3.2 and [30, Section 3], and the induced valuation, which we shall denote by ν_B, has to be Abhyankar, of rational rank $r = \mathrm{rat.rk.}\nu = \dim R/K_B$. The value semigroup of this Abhyankar valuation is the subsemigroup Γ_B of Γ generated by the $(\gamma_i)_{i \in B}$. The filtration of R/K_B associated to ν_B is the filtration $\mathcal{P}_B$ introduced above.

Remarks 4.5. 1) The reader can verify that in example 4.1, if we take $i \geq 3$, forget the equations beginning with $u_j^{n_j}$ for $j \geq i + 1$ and set $u_j = 0$ for $j \geq i$ we obtain by successive elimination of the u_i with $i \geq 2$ the equation of a plane branch $p_i(x, y) = 0$ with semigroup $\langle 1, \nu(y), \nu(u_2), \ldots, \nu(u_i)\rangle$ for its only valuation, which is Abhyankar. Of course in this case $R = k[[x, y]]$ is regular. The fact that the valuation of $p_i(x, y)$ is γ_i is generalized in the third statement of the proposition.
2) The ideal K_B is zero exactly when all the F_ℓ whose initial binomial is in $k[(u_i)_{i \in B}]$ are in $k[[(u_i)_{i \in B}]]$ and then R is an overweight deformation of the prime binomial ideal generated by the initial binomials of the F_ℓ. This implies that $\dim R = r$ and ν is an Abhyankar valuation. See Remark 4.9 below.

4.3. **Properties of the ideals K_B.** We now study some properties of the collection of the ideals K_B following in particular from Proposition 4.3.

Notice that given an inclusion $B \subset B'$, the set of generators of the ideal $K_{B'}$ contains more F_ℓ's because there are more initial binomials, but a series $F_\ell|B$ in $\mathcal{K}_B$ must become in $\mathcal{K}_{B'}$ a series $F_\ell|B'$ which is in general different. Thus, an inclusion $B \subset B'$ does not imply an inclusion between K_B and

$K_{B'}$. However, as we just saw, the m-adic order of the ideals K_B grows with B and tends to infinity.
By construction, the w-initial ideal of $\mathcal{K}_B$ is contained in the w-initial ideal of $\mathcal{K}_{B'}$.

Corollary 4.6. (of Proposition 4.3) *With the notations of Corollary 3.21, we have* $\bigcap_{B\supset B_0} K_B = \bigcap_{a\in\mathbf{N}} K_{B_a} = (0)$.

Proof. This follows from the fact that R is noetherian and the second part of the proposition. □

Corollary 4.7. *Let $\mathcal{J} \subset R$ be the jacobian ideal, defining the singular locus of the formal germ associated to R. There exists a finite initial set $B_{\mathcal{J}}$ containing B_0 and such that if an initial set B contains $B_{\mathcal{J}}$ we have that $\mathcal{J} \not\subset K_B$ so that the localization R_{K_B} is a regular ring. In particular, in the sequence of Corollary 3.21, we have that $J \not\subset K_{B_a}$ and $R_{K_{B_a}}$ is a regular ring for large enough a.*

Proof. Since R is a domain and a quotient of a power series ring, we know that the ideal $\mathcal{J}$ is not zero. Since R is noetherian, we know that $\bigcap_{j=1}^{\infty} m^j = (0)$. Let D be the largest integer E such that $J \subset m^E$ in R. If we take $B_{\mathcal{J}} = B(D)$ as in the proposition, we have that $K_{B_{\mathcal{J}}} \subset m^{D+1}$. Thus, if $B_{\mathcal{J}} \subset B$ it is impossible for $\mathcal{J}$ to be contained in K_B. □

Theorem 4.8. *The Abhyankar semivaluations ν_B of R are better and better approximations of the valuation ν as the finite initial set B grows in the sense that:*

(1) The union of the finitely generated semigroups $\Gamma_B = \langle(\gamma_i)_{i\in B}\rangle$ is equal to Γ, and in particular the nested union of the Γ_{B_a} corresponding to the sequence of Corollary 3.21 is equal to Γ.
(2) Given an inclusion $B \subset B'$, if $x \in R$ is such that $x \notin K_B \cup K_{B'}$, we have the inequality $\nu_B(x) \leq \nu_{B'}(x)$.
(3) For any $x \in R \setminus K_B$ we have the inequality $\nu_B(x) \leq \nu(x)$ and for any $x \in R \setminus \{0\}$ there exists a finite set B such that for any $B' \supset B$ we have $\nu_{B'}(x) = \nu(x)$. In particular, for any $x \in R \setminus \{0\}$ there is an index $a_0(x)$ in the sequence of Corollary 3.21 such that for $a \geq a_0(x)$ we have $\nu_{B_a}(x) = \nu(x)$.

Proof. The first statement follows from Corollary 3.21. By definition (see the line just before Remark 4.5 and Proposition 3.14) of the Abhyankar valuation on R/K_B, the ν_B-value of the image $\overline{x} \in R/K_B$ of an element $x \in R$ is the maximum weight of its representatives in $k[[(u_i)_{i\in B}]]$. Let $h^B(\overline{x})$ be such a representative. Since $h^B(\overline{x})$ is also a representative of $\overline{x}$ in $k[\widehat{[(u_i)_{i\in I}}]$, we have $\nu_B(\overline{x}) = w(h^B(\overline{x})) \leq \nu(x)$.
Given an inclusion $B \subset B'$ and $x \in R \setminus (K_B \cup K_{B'})$, we can choose representatives of maximum weight $h^B, h^{B'}$ in $k[[(u_i)_{i\in B'}]]$ of the images of x in R/K_B and $R/K_{B'}$ respectively. We must then have $h^B - h^{B'} \in \mathcal{K}_B + \mathcal{K}_{B'}$ so that $\mathrm{in}_w(h^B - h^{B'}) \in \mathrm{in}_w(\mathcal{K}_B + \mathcal{K}_{B'}) = \mathrm{in}_w\mathcal{K}_{B'}$. If $w(h^{B'}) < w(h^B)$ we get $\mathrm{in}_w h^{B'} \in \mathrm{in}_w\mathcal{K}_{B'}$ which contradicts the maximality of the weight. This proves the inequality $\nu_B(x) \leq \nu_{B'}(x)$.

Given $x \in R \setminus \{0\}$ there exists a representative of x in $k\widehat{[(u_i)_{i\in I}]}$ which is of weight $\nu(x)$. By the argument of the proof of Proposition 3.28 we can choose a representative h of x which has the same weight and involves only finitely many of the variables u_i. Let B be the filling of a finite initial set containing B_0 and these variables. We now have $h \in k[[(u_i)_{i\in B}]]$ and $\mathrm{in}_w h \notin \mathrm{in}_w \mathcal{K}_B$ so that $h \notin \mathcal{K}_B + \ker(\pi \circ \iota_B)$. Thus we have $x \notin K_B$ and $\nu_B(x) = w(h) = \nu(x)$. The second statement of part 2) of the theorem follows from the inequalities $\nu_B(x) \leq \nu_{B'}(x) \leq \nu(x)$ we have seen above. □

Remarks 4.9. 1) We have concentrated on the case where ν is not an Abhyankar valuation because that is the unsolved case. The case of Abhyankar valuations with finitely generated semigroups is solved from the viewpoint of this paper in [30] but there are Abhyankar valuations with infinitely generated semigroup (see [6]). Let ν be such a valuation, so that in our construction we have $r = d$. For a given finite subset B as above, the ideal K_B is zero, so that all the series $F_\ell|B$ belong to the ideal generated by the $(F_i)_{i\in B\setminus J}, (F_q)_{1\leq q\leq f}$. This implies that the ring R is presented as an overweight deformation of the binomial ideal generated by the initial forms of the $F_\ell|B$ and as such is endowed with a valuation ν_B with a finitely generated semigroup. In this case, the valuation ν is approximated by valuations with finitely generated semigroups.
2) Just as in Example 4.1, using Corollary 4.7, we can, for large enough B, compose the valuation ν_B with the $K_B R_{K_B}$-adic valuation of the regular local ring R_{K_B}, to obtain a valuation $\tilde{\nu}_B$ on R of rational rank $r+1$. These valuations are Abhyankar if $r + 1 = \dim R$ and approximate ν in the sense that given $x \in R$, for large enough B we have $\tilde{\nu}_B(x) = \nu(x)$ since $x \notin K_B$.

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