

About the Jacobi-Perron algorithm

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When trying to understand the geometry of a singular varieties by comparing them with non singular singular ones, a classical approach is to project the singular space X to a non singular one of the same dimension and carefully describe the accidents which occur. Of this, Łojasiewicz was a grand master.

This lecture, however, is about another approach, which is to dominate the algebraic variety X , say over a field k , by non singular spaces by proper algebraic morphisms $\pi: X' \rightarrow X$ where π is an isomorphism over the non singular locus of X . This implies that the field $k(X')$ of rational functions on X' with coefficients in k is the same as $k(X)$; the morphism π is birational.

This led Zariski to define the Zariski-Riemann manifold $\mathcal{X} = \varprojlim X'$, the projective limit of all proper birational maps over X .

It is the generalisation of the normalisation of curves, which produces Riemann surfaces. In dimensions ≥ 2 , it is much more complicated, not even a scheme.

So the question arises to factor the natural map $\Pi: \mathcal{X} \rightarrow X$ through a birational map $\Pi: \mathcal{X} \rightarrow X' \rightarrow X$, where X' is non singular. In other words, approximate $\mathcal{X}$ by non singular algebraic varieties X' over X .

In characteristic zero this, and much more, was proved by Hironaka. In positive characteristic and dimension > 3 it is not known.

Even though it is not a variety, the space $\mathcal{X}$ has local rings which are valuation rings.

A valuation ring is a commutative ring without zero divisors R_ν such that for any $x, y \in R_\nu$, both $\neq 0$, either x divides y or y divides x . It is equivalent to saying that any non zero finitely generated ideal $(x_1, \dots, x_n) \subset R_\nu$ is also generated by one of the a_i .

This implies that R_ν is a local ring: the non invertible elements form an ideal m_ν , which is the only maximal ideal.

If R_ν is one of them, corresponding to a point $\nu \in \mathcal{X}$, the case where ν is a *rational* point of $\mathcal{X}$, i.e., $R_\nu/m_\nu = k$ is already interesting. It will be assumed from now on.

A valuation ring is noetherian only if it is a one dimensional regular local ring. The valuation is the order with respect to the maximal ideal.

The quotient $(R_\nu \setminus \{0\})/U_\nu$ of the multiplicative semigroup of R_ν by the group of units of R_ν , is an abelian semigroup $\Phi_{\geq 0}$ totally ordered by the relation $\bar{x} \leq \bar{y}$ if $y/x \in R_\nu$. The natural morphism

$$\nu: R_\nu \setminus \{0\} \rightarrow (R_\nu \setminus \{0\})/U_\nu = \Phi_{\geq 0},$$

often completed by mapping $0 \in R_\nu$ to an element ∞ larger than any element of $\Phi_{\geq 0}$, is called the valuation associated to R_ν .

It satisfies

$$\nu(xy) = \nu(x) + \nu(y)$$

$$\nu(x + y) \geq \min(\nu(x), \nu(y))$$

$$\nu(x) = \infty \leftrightarrow x = 0.$$

This extends to the field of fractions K of R_ν by $\nu(x/y) = \nu(x) - \nu(y)$.

The totally ordered abelian group Φ of which $\Phi_{\geq 0}$ is the non negative part is called the value group of the valuation and consists of the values taken by ν on K^* , where K is the field of fractions of R_ν .

Thinking of the field of fraction K of R as a field of algebraic functions on an algebraic variety one may think of ν as a generalized order of vanishing, or of pole, with values in Φ , of the rational function x along "something". In dimension one it is the order of vanishing, or of the pole, of a rational function at a point of the curve.

In higher dimension it can be, for example in the case where $K = k(x, y)$, the field of rational functions in two variables, the "order of vanishing" of a rational function $h(x, y)$ along a curve parametrised by $x = t, y = t^\tau$, where τ is a positive irrational number" : the order in t of $h(t, t^\tau)$.

It can also be the order of vanishing along a "curve"

$$x(t) = \sum_{s \in A} a_s t^s, \quad y(t) = \sum_{s \in B} b_s t^s,$$

where A, B are well ordered subsets of $\mathbf{Q}_{\geq 0}$ whose denominators are unbounded, with say $x(0) = y(0) = 0$. This ensures that the series $h(x(t), y(t)) \neq 0$ and it has a least exponent, which is the valuation of h .

The possible value groups of valuations of the field of rational functions can be very diverse: Any subgroup of $\mathbf{Q}$, with the induced order, $(\mathbf{Z}^r)_{lex}$, $\mathbf{Z}_{\tau_1} \oplus \mathbf{Z}_{\tau_2} \oplus \cdots \oplus \mathbf{Z}_{\tau_r}$, where the $\tau_i \in \mathbf{R}_{>0}$ are rationally independent.

For valuations of a field of algebraic functions in n variables over k , or of an n -dimensional local domain R if we assume that the valuation corresponds to a rational point of $\mathcal{X}$, we have that the rational rank $r(\nu) = \dim_{\mathbf{Q}} \Phi \otimes_{\mathbf{Z}} \mathbf{Q}$ is $\leq n$ and the archimedian rank, the least integer $h(\nu)$ such that Φ is an ordered subgroup of $(\mathbf{R}^h)_{lex}$, is $\leq r(\nu)$. If $r(\nu) = \dim R$, we have an Abhyankar valuation of R and the value group is $\mathbf{Z}^r$.

A totally ordered abelian group has two main invariants: Its rational rank, $r(\Phi) = \dim_{\mathbf{Q}} \Phi \otimes_{\mathbf{Z}} \mathbf{Q}$, and its archimedean rank which we just saw. Archimedean rank one means that Φ can be embedded in $\mathbf{R}$ as an ordered group.

An aside:

There is a topology on the set of monomial orderings of $\mathbf{Z}^r$, where a basis of open sets is indexed by pairs of distinct points $a, b \in \mathbf{Z}^r$ and elements of the basis are $U_{a,b} = \{< / a < b\}$. For $r \geq 2$ this topological space is homeomorphic to the Cantor set. This is due to Sikora.

A rational point $\nu \in \mathcal{X}$ has an image in X and an image x' in each birational modification X' of X . At each of these points the variety has a local ring $R_{X',x'}$ so that we have an inductive system of local rings $R_{X',x'} \subset R_\nu$.

One issue is to show that there are regular local rings in this inductive system, a.k.a. the problem of local uniformization, and another issue, on which I concentrate today, is to find how to produce nested sequences of noetherian local rings growing from $R_{X,x}$ to R_ν .

A simplified form of a local ring R endowed with a valuation ν is the associated graded ring. Let us define the ideals:

$$\mathcal{P}_\phi(R) = \{x \in R \mid \nu(x) \geq \phi\}, \quad \mathcal{P}_\phi^+(R) = \{x \in R \mid \nu(x) > \phi\}, \quad \text{and}$$

$$\mathrm{gr}_\nu R = \bigoplus_{\phi \in \Phi_{\geq 0}} \mathcal{P}_\phi(R) / \mathcal{P}_\phi^+(R).$$

In the sequel we always assume that R , our $R_{X,x}$, satisfies $R \subset R_\nu$, so that ν takes only positive values on R , and that $m_\nu \cap R$ is the maximal ideal $m = m_{X,x}$ of R , with $R/m = R_\nu/m_\nu = k$ (rational valuation). Thus, $\text{gr}_\nu R$ is a graded k -algebra and so is $\text{gr}_\nu R_\nu$.

For example, if k is algebraically closed, $\text{gr}_\nu R_\nu \simeq k[t^{\Phi_{\geq 0}}]$ the semigroup algebra of the semigroup group $\Phi_{\geq 0}$ with coefficients in k .

More generally, if k is algebraically closed, denoting by Γ the semigroup $\nu(R \setminus \{0\}) \subset \Phi_{\geq 0}$, the k -algebra $\operatorname{gr}_{\nu} R$ is isomorphic to $k[t^{\Gamma}]$. It is just the subalgebra of $\operatorname{gr}_{\nu} R_{\nu}$ whose homogeneous components have degree in Γ .

If $R \subset \mathbf{C}\{t\}$ is the ring of a complex analytic branch, with the t -adic valuation ν_t , then

$$\mathrm{gr}_\nu R \simeq k[t^{\gamma_1}, \dots, t^{\gamma_g}],$$

where the $\gamma_i \in \mathbf{N}$ minimally generate the numerical semigroup of values $\nu_t(R \setminus \{0\})$.

The k -algebra $\operatorname{gr}_\nu R$ is not in general finitely generated, but its Krull dimension is the rational rank r . For rational valuations, Abhyankar's inequality is

$$\dim \operatorname{gr}_\nu R \leq \dim R.$$

The philosophy is that for rational valuations, with k algebraically closed, the semigroup of values contains all the information necessary to produce local uniformization.

The Jacobi-Perron algorithm is a generalisation to higher dimensions of the continued fraction expansion of a real number viewed geometrically. It has a role in the approximation of valuation rings by local rings of non singular algebraic varieties and I will try to explain the connection.

If

$$\tau = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots a_k + \frac{1}{\dots}}}}$$

shortened into $\tau = [a_0, a_1, \dots, a_k, \dots]$ is the continued fraction expansion of a positive real number τ , and the $\frac{p_k}{q_k} = [a_0, a_1, \dots, a_k]$ are the *reduced fractions*.

The construction of the integers a_k is as follows: $a_0 = \lfloor \tau \rfloor$, the integral part of τ , $a_1 = \lfloor \frac{1}{\tau - a_0} \rfloor$.
Set $\tau^{(1)} = \frac{1}{\tau - a_0}$ and $a_1 = \lfloor \tau^{(1)} \rfloor$, then $\tau^{(2)} = \frac{1}{\tau^{(1)} - a_1}$, and so on.

A little experimentation gives: $p_0 = a_0, q_0 = 1$; $p_1 = a_0 a_1 + 1, q_1 = a_1$, $p_2 = a_0 + a_2(a_0 a_1 + 1), q_2 = 1 + a_2 a_1$ so that we are led to *conjecture* the general rule of formation for the numerators and denominators of the reduced fractions

$$p_{k+1} = p_{k-1} + a_{k+1} p_k, \quad q_{k+1} = q_{k-1} + a_{k+1} q_k,$$

with the initial conditions $p_0 = a_0, q_0 = 1$.

It follows by induction that

$$p_k q_{k+1} - p_{k+1} q_k = (-1)^k.$$

It is time for a geometric interpretation, which goes back to H.J. Stephen Smith.

Let $\mathbf{N}^2$ be the lattice of integral points in the first quadrant of $\mathbf{R}^2$ with coordinates i, j . Consider the line $j = \tau i$, with $\tau > 1$ and the boundaries B_1 and B_2 of the convex hulls in $\mathbf{R}_{\geq 0}^2$ of the points of $\mathbf{N}^2$ which are below and above the line.

Apart from the two half coordinate axis, they are polygons, with infinitely many vertices if τ is irrational. The vertices below the line have coordinates (p_k, q_k) for k even, starting from $(p_0, q_0) = (1, a_0 = \lfloor \tau \rfloor)$ and those above the line have coordinates (p_k, q_k) for k odd, starting with $(p_{-1}, q_{-1}) = (0, 1)$.

We see that the vectors $A_k = (p_k, q_k)$ and $A_{k+1} = (p_{k+1}, q_{k+1})$ generate in the first quadrant a *regular* cone σ_k , which means that they form a basis of the integral lattice, which contains the vector $\mathbf{w} = (1, \tau)$.

Since the a_k are all ≥ 1 , for every k and $i \geq 1$, the vectors A_{k+1+i} are strictly inside the cone σ_k .

Let now $\tau_1, \dots, \tau_m$ be positive real numbers. Let us first assume that they are rationally independent. We are going to approximate the vector $\mathbf{w} = (\tau_1, \dots, \tau_m) \in \mathbf{R}_{\geq 0}^m$ by cones generated by integral vectors forming a basis of the integral lattice.

Define $\tau_1^{(1)}, \dots, \tau_m^{(1)}$ as follows:

$$\tau_1 = \tau_m^{(1)}, \tau_2 = \tau_1^{(1)} + a_2^{(0)} \tau_m^{(1)}, \dots, \tau_m = \tau_{m-1}^{(1)} + a_m^{(0)} \tau_m^{(1)},$$

with $a_j^{(0)} = \lfloor \frac{\tau_1}{\tau_j} \rfloor$.

Now repeat starting from $\tau_1^{(1)}, \dots, \tau_m^{(1)}$ and iterate.

After h steps, one has written

$$\tau_i = A_i^{(h)} \tau_1^{(h)} + \dots + A_m^{(h)} \tau_m^{(h)},$$

with $\tau_j^{(h)} \geq 0$.

In other words,

$$\mathbf{w} = \tau_1^{(h)} \mathbf{A}^{(h)} + \tau_2^{(h)} \mathbf{A}^{(h+1)} + \dots + \tau_m^{(h)} \mathbf{A}^{(h+m-1)},$$

The coefficients $\tau_i^{(j)}$ are non negative integers, and the matrix of the vectors

$$\mathbf{A}^{(h)}, \mathbf{A}^{(h+1)}, \dots, \mathbf{A}^{(h+m-1)}$$

has determinant $(-1)^{h(m-1)}$.

The $\tau_i^{(h)}$ are the coordinate of $\mathbf{w}$ in the coordinate system $\mathbf{A}^{(h)}, \mathbf{A}^{(h+1)}, \dots, \mathbf{A}^{(h+m-1)}$.

If we apply this to $(1, \tau) \in \mathbf{R}_{\geq 0}^2$. i.e, $m = 2$ we see that it produces the (p_k, q_k) as vectors $A^{(h)}$.

Moreover, as h grows the directions in $\mathbf{P}^{m-1}(\mathbf{R})$ of the vectors $A^{(h)}$ tend to the direction of w . So we have a sequence of vectors $A^{(h)}$ with positive integral coordinates whose directions in $\mathbf{P}^{m-1}(\mathbf{R})$ spiral to the direction of w and such that any consecutive m of them as above form a basis of the integral lattice such that w is contained in the convex cone $\sigma^{(h)} = \langle A^{(h)}, A^{(h+1)}, \dots, A^{(h+m-1)} \rangle$ which they generate.

The convex dual of a cone $\sigma \subset \mathbf{R}^m$ is the set of $(a_1, \dots, a_m) \in \check{\mathbf{R}}^m$ such that

$$\sum_1^m a_i x_i \geq 0 \quad \forall x \in \sigma$$

It is a cone, generated by the linear forms which vanish on the faces of σ .

Note that by construction we have for each $h \geq 1$ the equality

$$A^{(h)} - A^{(h+m)} + a_2^{(h)} A^{(h+1)} + \dots + a_m^{(h)} A^{(h+m-1)} = 0,$$

which shows, since the $a_j^{(h)} = [\tau_j^{(h)} / \tau_1^{(h)}]$ are non-negative, that we have

$$A^{(h+m)} \in \sigma^{(h)} = \langle A^{(h)}, \dots, A^{(h+m-1)} \rangle$$

and therefore $\sigma^{(h+1)} \subset \sigma^{(h)}$, that is $\check{\sigma}^{(h)} \subset \check{\sigma}^{(h+1)}$.

Geometrically, starting from the first quadrant, or indeed any regular rational cone $\sigma^{(0)}$ in the first quadrant containing the vector $\mathbf{w}$, the Jacobi-Perron algorithm builds a nested sequence

$$\sigma^{(0)} \supset \sigma^{(1)} \supset \sigma^{(2)} \supset \dots \supset \sigma^{(h)} \supset \dots \ni \mathbf{w}$$

of regular rational cones

$$\sigma^{(h)} = \langle \mathbf{A}^{(h)}, \mathbf{A}^{(h+1)}, \dots, \mathbf{A}^{(h+m-1)} \rangle,$$

converging to $\mathbf{w}$.

Let us now consider the subgroup $\Phi \subset \mathbf{R}$ generated by $\tau_1, \dots, \tau_m$, with the induced order.

We shall be interested in the totally ordered semigroup $\Phi_{\geq 0}$ of non negative elements.

It coincides with the m -uples of integers $a_1, \dots, a_m$ such that

$$\sum_{i=1}^m a_i \tau_i \geq 0.$$

The convex dual $\check{\sigma}^{(h)}$ of $\sigma^{(h)}$ is contained in the half space $\check{\mathbf{w}} = \sum_{i=1}^m a_i \tau_i \geq 0$, the integral points of which form the semigroup $\Phi_{\geq 0}$. Since $\sigma^{(h)}$ is a regular cone, so is its convex dual.

So we have a nested sequence of regular rational cones

$$\check{\sigma}^{(0)} \subset \check{\sigma}^{(1)} \subset \dots \subset \check{\sigma}^{(h)} \subset \dots \subset \check{\mathbf{w}}$$

and a nested sequence of free semigroups isomorphic to $\mathbf{N}^m$

$$\check{\sigma}^{(0)} \cap \mathbf{Z}^m \subset \check{\sigma}^{(1)} \cap \mathbf{Z}^m \subset \dots \subset \check{\sigma}^{(h)} \cap \mathbf{Z}^m \subset \dots \subset \check{\mathbf{w}} \cap \mathbf{Z}^m = \Phi_{\geq 0}$$

with $\mathbf{Z}$ -linear injections.

The algebra of the free semigroup $\check{\sigma}^{(h)} \cap \mathbf{Z}^m$ is a polynomial algebra $k[x_1^{(h)}, \dots, x_m^{(h)}]$ contained in $k[t^{\Phi_{\geq 0}}]$, and since by assumption there are no integral points on the hyperplane $\sum_{i=1}^m a_i \tau_i = 0$ except the origin, the semigroup $\Phi_{\geq 0}$ is the union of the $\check{\sigma}^{(h)} \cap \mathbf{Z}^m$ as $h \rightarrow \infty$.

This proves that $k[t^{\Phi_{\geq 0}}]$ is the union, or direct limit, of nested polynomial subalgebras.

The morphisms between these polynomial algebras correspond by duality to the expressions of the $A^{(h+m)}$ as linear combinations with non negative integral coefficients of $(A^{(h)}, \dots, A^{(h+m-1)})$ and are therefore monomial as announced.

If we now consider a group with one more generator $\tau_{m+1} > 0$ which is rationally dependent on $\tau_1, \dots, \tau_m$, the new weight vector $\mathbf{w} = (\tau_1, \dots, \tau_m, \tau_{m+1}) \in \mathbf{R}^{m+1}$ is contained in a rational simplicial cone $\sigma \subset \mathbf{R}^{m+1}$ generated by m integral vectors $v_1, \dots, v_m$ of the first quadrant forming part of a basis of the integral lattice.

In the end, we can prove again that for any totally ordered abelian group of rank one and rational rank r , one has the same approximation of $\Phi_{\geq 0}$ by free subsemigroups of rank r . For example:

$$\mathbf{N} = \mathbf{N}_{(1)} \rightarrow \mathbf{N}_{(2)} \cdots \rightarrow \mathbf{N}_{(h)} \rightarrow \mathbf{N}_{(h+1)} \rightarrow \cdots \rightarrow \mathbf{Q}_{\geq 0}.$$

The map $\mathbf{N}_{(h)} \rightarrow \mathbf{N}_{(h+1)}$ is multiplication by $h + 1$.

By induction on the archimedean rank (see §4 of [T1], one can extend this result to any totally ordered group of finite rational rank, in an algorithmic form.

The idea of the induction is based on the approximation of $\mathbf{Z}_{lex, \geq 0}^2$ by free subsemigroups $\mathbf{N}_{(h)}^2 = \{a(0, 1) + b(1, -h)\}$ with $(a, b) \in \mathbf{N}^2$.

From now on, we change the notation m coming from Jacobi-Perron to the notation r of rational rank of valuations.

Let R_ν be the valuation ring of a valuation with value group Φ , of finite rational rank r . Choose r homogeneous elements $x_1^{(0)}, \dots, x_r^{(0)} \in \text{gr}_\nu R_\nu$ whose valuations are rationally independent. The graded algebra $\text{gr}_\nu R_\nu$ is the union of a nested family of polynomial algebras in r variables over $k_\nu = R_\nu/m_\nu$ with maps between them sending each variable to a term:

$$k_\nu[x_1^{(0)}, \dots, x_r^{(0)}] \subset \dots \subset k_\nu[x_1^{(h)}, \dots, x_r^{(h)}] \subset k_\nu[x_1^{(h+1)}, \dots, x_r^{(h+1)}] \subset \dots \subset \text{gr}_\nu R_\nu.$$

This can be viewed as implying a graded version of local uniformization: a finitely generated graded k_ν -subalgebra of $\text{gr}_\nu R_\nu$ is contained in a polynomial (i.e., graded regular) subalgebra.

The main result

Let $R \subset R_\nu$ be a rational Abhyankar valuation of an excellent equicharacteristic local domain R of dimension r with algebraically closed residue field k . Assume that R contains a field of representatives of k . Choose r homogeneous elements $x_1^{(0)}, \dots, x_r^{(0)} \in \text{gr}_\nu R$ whose valuations are rationally independent. The ring R_ν is the union of a nested family indexed by $\mathbf{N}$ of local domains $R^{(h)}$ essentially of finite type over R , which are regular for $h \geq 1$:

$$R = R^{(0)} \subset R^{(1)} \subset \dots \subset R^{(h)} \subset R^{(h+1)} \subset \dots \subset R_\nu,$$

where each inclusion $R^{(h)} \subset R^{(h+1)}$, including the first one $R^{(0)} \subset R^{(1)}$, is obtained by localizing at the point picked by the valuation a birational map of finite type, which for $h \geq 1$ is monomial with respect to suitable minimal sets of generators of the maximal ideals of $R^{(h)}$ and $R^{(h+1)}$.

The graded version

The proof of the main result relies on local uniformization of Abhyankar valuations to prove the existence of a regular $R^{(1)}$ with coordinates having rationally independent values, and then follows from what we have seen:

The associated graded rings $\mathrm{gr}_\nu R^{(h)}$ for $h \geq 1$ form a nested system of polynomial rings in r variables over k containing $k[x_1^{(0)}, \dots, x_r^{(0)}]$, where the inclusion maps

$$k[x_1^{(0)}, \dots, x_r^{(0)}] \subset \dots \subset k[x_1^{(h)}, \dots, x_r^{(h)}] \subset k[x_1^{(h+1)}, \dots, x_r^{(h+1)}] \subset \dots \subset \mathrm{gr}_\nu R_\nu$$

send each variable to a term, and the inclusions are birational, with the possible exception of the first one, which is composed:

$$k[x_1^{(0)}, \dots, x_r^{(0)}] \subset \mathrm{gr}_\nu R \subset k[x_1^{(1)}, \dots, x_r^{(1)}].$$

The proof of the main result is essentially to lift the monomial relations of the nested sequence of polynomial rings to create new monomials to add to the rings $R^{(h)}$ at each step. For details see the proof of Theorem 7.27 of reference [T2] below.

If you want to know more, 1

There are more details in

- §4.3 of:

[T1] B. Teissier, *Valuations, deformations, and toric geometry*, in *Valuation Theory and its applications, Vol. II*, Fields Inst. Commun. 33, AMS., Providence, RI., 2003, 361-459. Available at <http://webusers.imj-prg.fr/~bernard.teissier/>

See in particular page 395.

-[T2] B. Teissier, *Overweight deformations of affine toric varieties and local uniformization*, in *Valuation theory in interaction*, Proceedings of the second international conference on valuation theory, Segovia–El Escorial, 2011. Edited by A. Campillo, F-V. Kuhlmann and B. Teissier. European Math. Soc. Publishing House, Congress Reports Series, Sept. 2014, 474-565.

There is certainly a geometric relation with Zariski's Perron transform, but it is not clear to me at this point; my viewpoint is different. See

-[EH] Samar ElHitti, *Perron transforms*, Communications in Algebra, 42 (2014) 2003-2045.

If you want to know more, 2

A good reference for the Jacobi-Perron algorithm and its use for local uniformisation is:

-[H-P] W.V.D. Hodge and D. Pedoe, *Methods of Algebraic Geometry, Volume 3*, Cambridge Mathematical Library, Cambridge University Press, 1954.