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On Random Surfaces

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Summary

This thesis is about random surfaces and their applications. There are many different notions of random surfaces, but in this text they will (mainly) be surfaces obtained from randomly gluing together an even number of triangles along their sides. By choosing a metric on the underlying triangle, such surfaces can be given various metric structures. In this text we shall mainly study the geometry of these surfaces. This will rely heavily on the connections between random surfaces, random cubic graphs and random pairs of permutations in the symmetric group.

Concretely, we study the distribution of short closed curves on random surfaces and its dependence on the topology of the surfaces. We will consider this distribution for (compact and punctured) random surfaces with hyperbolic metrics and random surfaces with more general Riemannian metrics. Besides curves on random surfaces, we will also consider probility distributions associated to subsurfaces of random surfaces and curves on random cubic graphs.

For the final chapter, which contains joint work with Hugo Parlier, we study a different subject: curve, pants and flip graphs. These are graphs parameterizing various types of topological data on a given surface. We will study the topological complexity, in the form of the genus, of these graphs and their quotients by the mapping class group.

All the results in this text can be either found in or derived from the articles [**Pet13**], [**Pet14**] and [**PP14**].

Résumé

Cette thèse étudie les surfaces aléatoires et leurs applications. Il existe une grande variété de notions de surfaces aléatoires. Dans ce texte une surface aléatoire sera (principalement) une surface obtenue par un recollement aléatoire d'un nombre pair de triangles le long de leurs côtés. Si on definit une métrique sur le triangle, on obtient une métrique sur chaque surface aléatoire. Dans ce texte, nous étudierons principalement la géometrie de ces surfaces. On s'appuiera sur les liens entre les surfaces aléatoires, les graphes cubiques aléatoires et les paires aléatoires de permutations dans le groupe symétrique.

On étudiera en particulier la distribution des courbes fermées courtes sur des surfaces aléatoires et sa dépendance en la topologie des surfaces. On considérera cette distribution pour des surfaces aléatoires (compactes et à pointes) avec des métriques hyperboliques et des métriques riemanniennes plus générales. Outre les courbes sur les surfaces aléatoires on étudiera aussi les distributions de probabilité associées aux sous-surfaces et aux courbes sur des graphes cubiques aléatoires.

Pour le dernier chapitre, qui contient des travaux en collaboration avec Hugo Parlier, nous étudierons un sujet différent: les graphes des courbes, pantalons et flips. Ce sont des graphes qui paramètrent différents types de données topologiques d'une surface fixée. Nous étudierons la complexité topologique (le genre) de ces graphes et de leurs quotients sous l'action du groupe des difféotopies.

Tous les résultats dans ce texte peuvent être trouvé dans ou déduit des articles [**Pet13**], [**Pet14**] et [**PP14**]

Samenvatting

Dit proefschrift gaat over stochastische oppervlakken en hun toepassingen. Er bestaan verschillende begrippen van stochastische oppervlakken, maar in deze tekst zullen we het (voornamelijk) hebben over oppervlakken die verkregen worden door het willekeurig aan elkaar lijmen van driehoeken. Deze oppervlakken kunnen van een metriek worden voorzien door een metriek op de onderliggende driehoek te definiëren. In deze tekst zullen we hoofdzakelijk de meetkunde van zulke oppervlakken onderzoeken. Dit zal sterk leunen op de verbanden tussen stochastische oppervlakken, stochastische kubische grafen en stochastische paren van permutaties in de symmetrische groep.

We zullen voornamelijk de verdeling van korte gesloten curves in stochastische oppervlakken bestuderen. We beschouwen deze verdeling voor stochastische oppervlakken met hyperbolische metrieken (met en zonder cuspen) en meer algemene Riemannsche metrieken. Naast curves in oppervlakken, zullen we ook verdelingen van deeloppervlakken en curves in stochastische kubische grafen bestuderen.

Voor het laatste hoofdstuk, wat gezamelijk werk met Hugo Parlier bevat, veranderen we enigzins het onderwerp en bestuderen we curve-, broek- en flipgrafen. Dit zijn grafen die verschillende types topologische data van een gegeven oppervlak parameteriseren. We onderzoeken de topologische complexiteit, in de vorm van de genus, van deze grafen en hun quotiënten door de afbeeldingsklassegroep.

Al de resultaten in deze tekst kunnen of gevonden ofwel afgeleid worden van de artikelen [**Pet13**], [**Pet14**] en [**PP14**].

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During the final year of my doctoral studies I have spent a semester at Brown University. I am very grateful for the hospitality of the Mathematics Department there and in particular to Jeff Brock for being my host. I have enjoyed my stay in Providence very much and it would not have been half as nice without Alex, Brian, Dale, Francesco, Kenny, Mamikon, Peter, Ren and Sara.

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Introduction

The subject of this thesis is best described as 'the use of combinatorial constructions, often related to graph theory, in the study of surfaces'. In particular, the largest part of this text is concerned with the study of random surfaces.

Random surfaces, and random manifolds in general, can be seen as manifold analogues to random graphs and random simplicial complexes. As is the case for random graphs and random simplicial complexes, there exist different notions of random surfaces that find applications in various areas of mathematics and physics. Besides these applications, random surfaces can also be used to understand the geometry of a 'typical' surface and in some cases even to show the existence of of surfaces with specific properties.

One way to obtain a notion of a random surface comes from measures of finite volume on the moduli space of hyperbolic surfaces of a fixed genus, like the Weil-Petersson measure. Given such a measure, one obtains a natural notion of a random point in this moduli space. This model for random surfaces has been studied by Guth, Parlier and Young in [**GPY11**] and Mirzakhani in [**Mir13**] and we will briefly touch upon it in this text (see Section 2.7). In this setting, studying random surfaces comes down to computing volumes of (subsets of) moduli spaces, which in general is a highly non-trivial problem.

The main model we will consider, is however given by randomly gluing together an even number of triangles along their sides. The idea to use this model to study the geometry and topology of a typical hyperbolic surface originated in [**BM04**], where Brooks and Makover define a random surface to be a random gluing of ideal hyperbolic triangles. This model has many nice features. For instance, when one compactifies the surfaces after the gluing, the obtained set of surfaces is dense in any moduli space of compact surfaces. Furthermore, this model is deeply connected with a very well studied model for random cubic graphs, which means that results on random cubic graphs can sometimes be translated into results on random hyperbolic surfaces. These random surfaces have also appeared in different contexts. Curiously, the first computations for the behaviour of the Weil-Petersson volume of general moduli spaces of surfaces with punctures were based on a cell decomposition of decorated Teichmüller space with cells labelled by these random surfaces [**Pen92**]. This is one of the many indications that there might be a connection between the two models. Furthermore, these techniques have also been used to study the topology of moduli spaces [**HZ86**] and mapping class groups [**Har86**]. Finally, this model of random surfaces has also been considered by physicists in relation with quantum gravity [**BIZ80**]. The topology and geometry of these random surfaces themselves has for example been investigated in [**BM04**], [**Gam06**] and [**GPY11**] (see Section 2.7). Our main goal will be to understand the distributions of short curves on random surfaces and how these depend (or in fact don't depend) on the topology of these surfaces.

Besides random surfaces we will also consider curve, pants and flip graphs. A lot is known about the geometry of these graphs (cf. [**Aou13**], [**Bow06**], [**Bow14**], [**CRS13**], [**HPW13**] and [**MM99**]) and how this relates to the geometry of Teichmüller spaces (cf. [**Brc03**], [**BF06**], [**BMM10**], [**Ham07**] and [**Raf05**]), mapping class groups and three manifolds (cf. [**BCM12**], [**Min99**] and [**Min10**]). We will study the topological complexity of these graphs and their quotients by the mapping class group. The connection to the previous subject, besides the combinatorial nature of curve pants and flip graphs, is that in the cases of the pants and flip graphs we will be able to use tools from the theory of random cubic graphs and random surfaces.

The set up of this text is as follows. Chapter 1 discusses the necessary preliminaries and also settles some notation. In Chapter 2 we define random surfaces and discuss known results.

Because of the connection between random surfaces and random cubic graphs, Chapter 3 is entirely about random cubic graphs. We recall some known results and we also need to prove some new bounds on distributions of cycles on such graphs. In particular, we prove that short circuits on cubic graphs do in general not separate the graph into two disjoint pieces (Theorem 3.18).

In Chapter 4, we study the probability that we find a fixed labelled subsurface in a random surface. These probabilities are mainly interesting because they help us understand the probability that an unlabelled subsurface appears in a random surface. The main result in this context will be that these probabilities are asymptotically (when the number of triangles grows) independent of the genus in a suitable sense (they do not change when we restrict to surfaces of maximal genus or non-negligible subsets of surfaces, see Theorems 4.7 and 4.11).

In Chapter 5 we will investigate the length spectrum of a random surface. We will prove that, when we let the number of triangles tend to infinity, the random variables that count the numbers of curves of fixed given lengths converge to independent Poisson distributed random variables with means given in terms of combinatorial data (Theorem 5.1). As an application we compute the limits of probability distribution of the systole in the (punctured and compact) hyperbolic case (Corollary 5.2). This will all be independent of the genus in the same sense as before. We also show that the limiting distribution of the genus is independent of the systole (Corollary 5.3). With these probability distributions given, we compute the limit of the expected value of the systole in the (punctured and compact) hyperbolic case (Theorems 5.5 and 5.7). The obstruction to immediately using the given probability distribution is that we need the probabilities to converge to their limits in a controlled way. Proving that this is indeed the case is the main component of the proof of Theorems 5.5 and 5.7.

After this, in Chapter 6, we turn our attention to the Riemannian case. In this setting we obtain genus-independent bounds (in terms of the geometry of the underlying triangle) on the limiting probability distribution of the systole when the number of triangles tends to infinity (Corollary 6.3). Using these, we derive bounds on the limit supremum and limit infimum of the expected value of the systole in this setting (Theorem 6.5).

In the final chapter (Chapter 7), containing joint work with Hugo Parlier, we investigate the genus of curve pants and flip graphs and their quotients by the mapping class group. It turns out that, except in a finite number of cases, the full graphs have infinite genus (Theorems 7.1, 7.3 and 7.9). The genera of their quotients are necessarily finite. We determine the asymptotic behavior (as the genus of the underlying surdace tends to infinity) of these up to given multiplicative constants (Theorems 7.4 and 7.10). In the case of the curve graph this is a direct application of a classical theorem by Ringel and Youngs, in the other two cases we use counting methods coming from random cubic graphs and random surfaces.

CHAPTER 1

Preliminaries

The aim of this chapter will be provide an overview of the background material that we will need later on. The goal of the chapter is not to give a careful treatment of all these preliminaries, but rather to make the text somewhat self contained and also to settle some notation. In every section we will give references to texts that contain more complete expositions of the subject at hand.

1.1. Geometry

First and foremost, this thesis is about geometry. That means that we will be studying manifolds. Mostly, these shall be smooth manifolds. But not always, and when they are not this will be clear from the context. We will be particularly interested in surfaces, i.e. 2-manifolds, and sets of surfaces that sometimes carry natural manifold structures themselves. Our surfaces will often carry hyperbolic metrics, so we will spend some time in this section on recalling facts from hyperbolic geometry. Proofs of these facts can be for example be found in [**Bea83**] and [**Bus92**]. For the preliminaries on manifolds we refer the reader to one of the many books on differential geometry, like for instance [**doC92**].

1.1.1. Hyperbolic geometry. We start with (2-dimensional) hyperbolic geometry. The hyperbolic plane was originally introduced as an example of a space that satisfies all Euclid's axioms for geometry except the famous parallel postulate. Later on, it turned out to be fundamental in the study of low-dimensional manifolds.

In the usual terminology the hyperbolic plane is an abstract geometric space that can be represented by several 'models' (like $\mathbb{R}^n$ is a model for Euclidean geometry). These models are various Riemannian manifolds that are all isometric. There are also multiple generalizations of the hyperbolic plane to higher dimensions: real and complex hyperbolic spaces. The first of these retains the curvature properties and the second one the complex structure.

We will describe two models that share the property of being *conformal*: they sit in $\mathbb{R}^2$ and the Euclidean angle between two tangent vectors is also the corresponding hyperbolic angle. In other words, the angle you see in the pictures is the actual hyperbolic angle. The lengths are however strongly distorted in both models. The first model is the Poincaré upper half plane model:

Definition 1.1. The *Poincaré upper half plane model for the hyperbolic plane* is the manifold:

$$\mathbb{H}^2 = \{z \in \mathbb{C};\ \mathrm{Im}(z) > 0\}$$

equipped with a Riemannian metric given by:

$$ds^2 = \frac{|dz|^2}{\mathrm{Im}(z)^2}$$

When one works out the metric coming from this Riemannian metric, one obtains the following formula:

$$d_{\mathbb{H}^2}(z,w) = \cosh^{-1}\left(1 + \frac{|z-w|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)}\right)$$

The geodesics in the upper half plane model are vertical lines and half circles that have their center on the real line. This implies that between any pair of points there is a unique geodesic.

The second model we will use is the Poincaré disk model:

Definition 1.2. The *Poincaré disk model for the hyperbolic plane* is the manifold:

$$\mathbb{D}^2 = \{z \in \mathbb{C};\ |z| < 1\}$$

equipped with a Riemannian metric given by:

$$ds^2 = 2\frac{|dz|^2}{(1-|z|^2)^2}$$

The metric in this model works out to:

$$d_{\mathbb{D}^2}(z,w) = 2\tanh^{-1}\left(\left|\frac{z-w}{1-z\overline{w}}\right|\right)$$

and geodesics in this case are diameters and circle arcs orthogonal to the unit circle in $\mathbb{C}$.

Other widely models for the hyperbolic plane are the Lorentz model, which is based on a hyperboloid in $\mathbb{R}^3$ and turns out to be particularly useful in the study of higher dimensional real hyperbolic geometry, and the Klein-Beltrami model which can be obtained as a projectivization of the latter.

An important role is played by the group of orientation preserving isometries of the hyperbolic plane. We will state the results in terms of the upper half plane model (because all the models are isometric, the other groups we would study are all isomorphic). The group of orientation preserving isometries of $\mathbb{H}^2$ is isomorphic to:

$$\mathrm{PSL}_2(\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R});\ \det(A) = 1\} \,/\, \sim$$

where $A \sim A'$ if and only if $A = \pm A'$. $\mathrm{PSL}_2(\mathbb{R})$ acts on $\mathbb{H}^2$ by:

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \cdot z = \frac{az+b}{cz+d}$$

Note that this action extends to $\mathbb{H}^2 \cup \mathbb{R} \cup \{\infty\}$. The set $\mathbb{R} \cup \{\infty\}$ is called the *boundary* of $\mathbb{H}^2$ and is denoted $\partial\mathbb{H}^2$. We will sometimes write $\overline{\mathbb{H}^2}$ for $\mathbb{H}^2 \cup \partial\mathbb{H}^2$. In the unit disk model the corresponding set is denoted by $\partial\mathbb{D}^2$ and is the unit circle. Every distinct pair of points on $\partial\mathbb{H}$ determines a unique geodesic that has these points as its 'endpoints'.

Elements of $\mathrm{PSL}_2(\mathbb{R})$ are classified as follows:

DEFINITION 1.3. Let $[A] \in \mathrm{PSL}_2(\mathbb{R})$ and write $t = \mathrm{tr}\,(A)^2$.
- If $t < 4$, then $[A]$ is called *elliptic*.
- If $t = 4$, then $[A]$ is called *parabolic*.
- If $t > 4$, then $[A]$ is called *hyperbolic*

Note that the square in the definition above is necessary to make the classification well-defined.

For a hyperbolic element $[A] \in \mathrm{PSL}_2(\mathbb{R})$ we have the following formula for its *translation length* $\tau_{[A]}$. This is the distance over which the element displaces a point in $\mathbb{H}^2$, minimized over all points:

$$\tau_{[A]} = 2\cosh^{-1}\left(\frac{\mathrm{tr}\,(A)}{2}\right)$$

Hyperbolic triangles will play an important role in this text. A hyperbolic triangle is the convex hull of three distinct points in $\overline{\mathbb{H}^2}$ (i.e. the smallest closed set containing the geodesic segments between the three points and the geodesic segment between any pair of its elements), these points will be called the *corners* of the triangle. We have the following fact about triangles:

PROPOSITION 1.1. *If T and T' are two hyperbolic triangles with the same angles at the corners then they are isometric. If these angles are α, β and γ then the area of T is:*

$$\pi - \alpha - \beta - \gamma$$

A corner of T that lies on $\partial\mathbb{H}^2$ will be called an *ideal* corner. If all the corners of T lie on $\partial\mathbb{H}^2$ then T will be called an ideal triangle. Note that the angle at an ideal corner is necessarily 0.

One of the end goals of this section is to explain how to use the hyperbolic plane to construct metrics on finite type surfaces. Before we do so, we will recall some facts about these finite type surfaces, starting with their definition.

1.1.2. Finite type surfaces. A surface will be said to be of *finite type* if its fundamental group is finitely presented. About such surfaces we have the following classical nineteenth century theorem, which we state for oriented surfaces with boundary components.

THEOREM 1.2. The classification of finite type surfaces: *Let X be an oriented surface of finite type (possibly with boundary). Then X is diffeomorphic to the connected sum of a sphere with a finite number of tori, out of which a finite number of points and open disks have been removed.*

To settle notation for all finite type surfaces we have the following definition:

DEFINITION 1.4. Let $g, b, n \in \mathbb{N}$. The surface $\Sigma_{g,b,n}$ will denote the connected sum of the sphere with g tori out of which b open disks and n points have been removed. g will be called the *genus*, b the number of *boundary components* and n de number of *punctures* of the surface. The triple (g, b, n) will be called the *type* of the surface. The number:

$$\kappa\left(\Sigma_{g,b,n}\right) = 3g + n + b - 3$$

is called the *complexity* of the surface.

Generally we will write: $\Sigma_{g,0,0} = \Sigma_g$ and $\Sigma_{g,0,n} = \Sigma_{g,n}$.

There also exists a classification of infinite type surfaces, based on the 'ends' of the surface. Roughly speaking, these are the parts of the surface where the genus or the number of punctures flies off to infinity. We will however will not need it in this text and refer the the interested reader to [**Ric63**].

We will often build surfaces out of triangles. For reference we record the fact that every surface can be triangulated in the following theorem:

THEOREM 1.3. *$\Sigma_{g,b,n}$ is diffeomorphic to a triangulated surface out of which n corners have been removed for any $g, b, n \in \mathbb{N}$*

In general we will be quite flexible in use of the word triangulation. To us a triangulation of a surface will mean a set of simple arcs on the surface with pairwise disjoint interiors such that the complement of these arcs is a disjoint union of triangles. This means that we allow two sides of the same triangle to be glued together and we also allow gluings of two triangles along more than one side. It is quite standard not to allow these two things. However, for us it will conventient to allow them and the results we use here are still valid in this slightly more general context.

Triangulations also give rise to a topological invariant: the Euler characteristic. Which was, as the name suggests, first studied by Euler in the case of the sphere.

Definition 1.5. Let X be a triangulated surface. The Euler characteristic of X is given by:

$$\chi(X) = V - E + F$$

where V is the number of corners, E the number of arcs and F the number of triangles of the triangulation.

We have the following proposition:

Proposition 1.4. *Let $g, b, n \in \mathbb{N}$. Then:*

$$\chi(\Sigma_{g,b,n}) = 2 - 2g - n - b$$

Note that this proposition also tells us that the Euler characteristic is a diffeomorphism invariant and does not depend on the triangulation. Finally, we remark that all the results of this section also hold in an appropriate sense when we allow triangulations with polygons with a higher number of sides.

1.1.3. Geometric surfaces. Like we said before, we want to do geometry, so we need metrics on our surfaces. We will discuss two ways to describe metrics on surfaces. The first one is a hyperbolic structure:

Definition 1.6. A surface with a *hyperbolic structure* is a surface S with an atlas $\{(U_i, \varphi_i)\}_{i=1}^N$, where U_i are open sets covering S and $\varphi_i : U_i \to \mathbb{R}^2$ homeomorphisms on their images such that:
- $\varphi(U_i) \subset \mathbb{H}^2$.
- for all i, j such that $U_i \cap U_j \neq \emptyset$, the transition map

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

is a restriction of an isometry of $\mathbb{H}^2$.

A surface with such a structure naturally carries a Riemannian metric that is locally isometric to $\mathbb{H}^2$. If a surface is diffeomorphic to a surface with a hyperbolic structure we say it *carries* such a structure. Furthermore, if the induced metric is complete, we call the structure *complete*. Finally, a hyperbolic structure is also naturally a *conformal* (or *complex* or *Riemann surface*) structure: a structure in which the chart images are interpreted as subset of $\mathbb{C}$ and the transition maps are biholomorphic maps.

We have the following theorem, which is a combination of the Gauss-Bonnet theorem and the Koebe Poincaré uniformization theorem:

Theorem 1.5. Uniformization Theorem: *An oriented smooth finite type surface X carries a hyperbolic structure if and only if $\chi(X) < 0$. Furthermore. in every equivalence class of conformal structures on a given surface one can find a unique complete hyperbolic structure.*

When we combine the theorem above with Proposition 1.4, we see that 'most' surfaces are hyperbolic. We also stress that the theorem above only says that a hyperbolic structure is only unique up to conformal transformations. This does not imply that there is a unique such structure on a given diffeomorphism type of surfaces. This is intentional, because the number of different hyperbolic structures is in fact uncountable in most cases. We will see more about this in the next section.

Surfaces with hyperbolic structures can also be described by discrete torsion free subgroups of $\mathrm{Isom}^+(\mathbb{H}^2)$.

PROPOSITION 1.6. *Let* $\Gamma < \mathrm{Isom}^+(\mathbb{H}^2)$ *be a discrete torsion free subgroup. Then:*

$$S = \mathbb{H}^2 / \Gamma$$

carries a complete hyperbolic structure in which the transition maps are restrictions of elements of Γ.

Conversely, let S *be a surface without boundary equipped with a complete hyperbolic structure. Then there exists a discrete and torsion free subgroup* $\Gamma < \mathrm{Isom}^+(\mathbb{H}^2)$ *such that:*

$$S = X / \Gamma$$

We could also replace the statement 'in which the transaction maps are restrictions of elements of Γ' by saying that the projection map $\pi : \mathbb{H}^2 \to S$ is a local isometry.

This description using groups helps to understand the geometry of these surfaces, using the following proposition:

PROPOSITION 1.7. *Let* $\Gamma < \mathrm{Isom}^+(\mathbb{H}^2)$ *be discrete and torsion free.*

- *If* $g \in \Gamma$ *is hyperbolic then* $\mathbb{H}^2/\Gamma$ *contains a closed curve of length* τ_g
- *If* $g \in \Gamma$ *is parabolic then* $\mathbb{H}^2/\Gamma$ *contains an isometric copy of the surface:*

$$\left\{z \in \mathbb{H}^2;\ \mathrm{Im}(z) > h\right\} / z \mapsto z+1$$

for some $h > 0$. *Such a subsurface is called a* cusp. *The image of a horzontal line in such a cusp will be called a* horocycle *around this cusp.*

The method of constructing surfaces with hyperbolic structures that we will actually use most is gluing together smaller surfaces equipped with hyperbolic structures. We have the following theorem (which can be found as Theorem 1.3.5 in [**Bus92**]):

THEOREM 1.8. *Let S_1 and S_2 be two surfaces with hyperbolic structures such that $\partial S_1, \partial S_2 \neq \emptyset$. Given $A \subset \partial S_1$ and $B \subset \partial S_2$ and an isometry $\varphi : A \to B$ such that:*

- *The angles in S_1 at x and at $\varphi(x)$ in S_2 add up to 2π for all x in the interior of A.*
- *The angle in S_1 at y plus that at $\varphi(y)$ in S_2 is less than π for all $y \in \partial A$*

Then:

$$S = (S_1 \sqcup S_2)/\, x \sim \varphi(x)$$

carries a unique hyperbolic structure such that the projection map:

$$\pi : S_1 \sqcup S_2 \to S$$

is a local isometry. Furthermore, if the structures on S_1 and S_2 are complete and for $i = 1, 2$ and any pair of connected components C and D of ∂S_i we have:

$$\inf \{d(x, y);\ x \in C,\ y \in D\} > 0$$

Then the structure on S is complete.

A particular example of such gluings that we will use extensively is formed by gluings of ideal hyperbolic triangles. Note however that these do not satisfy the completeness conditions. It turns out that if one is careful with the gluing (to be explained later) then the resulting structure will be complete.

The final fact about hyperbolic surfaces we need is about non-peripheral[1] homotopically non-trivial curves on them and can for instance be found as Theorem 1.6.6 in [**Bus92**]:

THEOREM 1.9. *Let S be a surface endowed with a hyperbolic metric and let γ be a homotopically non-trivial and non-peripheral curve on S. Then there exists a unique geodesic on S homotopic to γ.*

The second type of metrics we will consider in fact includes the first type. Namely, we will consider metrics coming from gluings of triangles equipped with a Riemannian metric. We will discuss the details of these metrics in Section 2.5.3.

1.1.4. Systolic inequalities. The lengths of closed curves tell a lot about the geometry of a surface, or any type of manifold for that matter. In a sense, the simplest invariant of a metric to consider in this context is the systole of a surface:

DEFINITION 1.7. Let (S, ds^2) be a finite type Riemannian surface that contains at least one homotopically non-trivial, non-peripheral curve. Then the *systole* of (S, g) is the number:

$$\mathrm{sys}(S, ds^2) = \inf \{\ell_{ds^2}(\gamma);\ \gamma \text{ a homotopically non-trivial, non-peripheral curve on } S\}$$

where $\ell_{ds^2}(\gamma)$ denotes the length of γ with respect to ds^2.

[1] non-peripheral means not homotopic to a boundary component or puncture.

The term 'systole' is also often used for a curve that realizes the shortest length, in which case what we call the systole is sometimes called the systolic length. We will however stick to systole, both for the length and for a curve that realizes it.

Gromov proved the following inequality for Riemannian surfaces:

THEOREM 1.10. [**Gro83**] *There exists a constant $C > 0$, independent of g, such that:*

$$\sup\left\{\frac{\mathrm{sys}(\Sigma_g, ds^2)}{\mathrm{area}(\Sigma_g, ds^2)^2};\ ds^2 \text{ a Riemannian metric}\right\} \leq \frac{\log(g)^2}{g}(C + o(1))$$

as $g \to \infty$.

The notation $o(1)$ is Landau notation and means that the error term tends to 0 as $g \to \infty$. In the same article, Gromov also proved systolic inequalities for higher dimensional manifolds satisfying certain topological conditions. If we restrict to hyperbolic surfaces the inequality follows from a simple area argument, first observed by Buser [**Bus92**]. From that argument one obtains $C = \frac{1}{\pi}$ for the hyperbolic case.

1.2. Parameter spaces

Having defined all these surfaces, the next natural question is: what kind of surfaces can one construct? This is a question underlying a lot of research in geometry and also a question that leads to the study of random surfaces. One possible approach to trying to solve this question is to put all the surfaces (or other geometric or topological objects) of interest in a box and study the box. This box is what we will call a parameter space. In this section we will define multiple parameter spaces, parameterizing various types of data that come from surfaces.

1.2.1. Teichmüller space.

The first example we are interested in is the Teichmüller space of a surface. We first define it as a set:

DEFINITION 1.8. Let $g, b, n \in \mathbb{N}$ such that $\chi(\Sigma_{g,b,n}) < 0$. We define the Teichmüller space of $\Sigma_{g,b,n}$ as:

$$\mathcal{T}(\Sigma_{g,b,n}) = \left\{(S, f);\ \begin{array}{c} S \text{ a surface with a hyperbolic structure,} \\ f : \Sigma_{g,b,n} \to S \text{ a homeomorphism} \end{array}\right\} \Big/ \sim$$

where:

$$(S, f) \sim (S', f')$$

if and only if there exists an isometry $m : S \to S'$ such that:

$$(f')^{-1} \circ m \circ f : \Sigma_{g,b,n} \to \Sigma_{g,b,n}$$

is isotopic to the identity. The map f in $[(S, f)] \in \mathcal{T}(\Sigma_{g,b,n})$ is called the *marking* of the point in Teichmüller space.

There are many equivalent ways to put a topology on $\mathcal{T}(\Sigma_{g,b,n})$. We will follow [**Bus92**] and use pants decompositions.

DEFINITION 1.9. A *pants decomposition* of $\Sigma_{g,b,n}$ is a set of distinct isotopy classes of non-trivial simple closed curves $P = \{\gamma_i\}_{i=1}^k$ on $\Sigma_{g,b,n}$ such that:

$$\Sigma_{g,b,n} \setminus \left(\bigcup_{i=1}^{k} \gamma_i \right)$$

is homeomorphic to a disjoint union of copies of $\Sigma_{0,3}$ (*pairs of pants*).

If we are given a pants decomposition of $\Sigma_{g,b,n}$ then, using the markings, we obtain a pants decomposition of every surface in $\mathcal{T}(\Sigma_{g,b,n})$. Because of the hyperbolic structure associated to such a surface, the curves in the pants decomposition all have a length at this point. We can move through Teichmüller space by varying these lengths. Another way to move is to 'twist' at a curve in the pants decomposition: we can cut the surface along the given curve and glue it back together with a twist. If we fix a basepoint in $\mathcal{T}(\Sigma_{g,b,n})$, this moving around gives us a map:

$$\Phi_P : (\mathbb{R}_+ \times \mathbb{R})^{3g+b+n-3} \times \mathbb{R}_+^b \to \mathcal{T}(\Sigma_{g,b,n})$$

where the two dimensions come from counting the number of interior and boundary curves of a pants decomposition of $\Sigma_{g,b,n}$. We have the following theorem (which can be found in [**Bus92**]):

THEOREM 1.11. *Let P be a pants decomposition of $\Sigma_{g,b,n}$. The map:*

$$\Phi_P : (\mathbb{R}_+ \times \mathbb{R})^{3g+b+n-3} \times \mathbb{R}_+^b \to \mathcal{T}(\Sigma_{g,b,n})$$

is a bijection.

Using this map we can topologize $\mathcal{T}(\Sigma_{g,b,n})$. The coordinates given by the map above are called *Fenchel-Nielssen coordinates.* In general we will not mention the map Φ_P or the fixed pants decomposition P and denote a point in Teichmüller space by:

$$(\ell_1, \tau_1, \ell_2, \tau_2 \ldots, \ell_{3g+b+n-3}, \tau_{3g+b+n-3}, \beta_1, \beta_2 \ldots, \beta_b) \in \mathcal{T}(\Sigma_{g,b,n})$$

where the ℓ_i's denote the lengths of the curves in the given pants decomposition and the τ_i's the twists.

1.2.2. The Mapping class group and Moduli space.

The next parameter space we will study is the moduli space of curves, or just moduli space. For us it will be easiest to define it as a quotient of Teichmüller space by the mapping class group, which we shall define first:

Definition 1.10. Let $g, b, n \in \mathbb{N}$. We define the groups:

$$\mathrm{Diffeo}^+(\Sigma_{g,b,n}) = \left\{ f \in \mathrm{Diffeo}(\Sigma_{g,b,n})\,; \begin{array}{c} f \text{ does not permute punctures or} \\ \text{boundary components and} \\ \text{preserves the orientation} \end{array} \right\}$$

and:

$$\mathrm{Diffeo}_0^+(\Sigma_{g,b,n}) = \left\{ f \in \mathrm{Diffeo}^+(\Sigma_{g,b,n})\,;\ f \text{ is isotopic to the identity} \right\}$$

We define the mapping class group of $\Sigma_{g,b,n}$ as:

$$\mathrm{Mod}(\Sigma_{g,b,n}) = \mathrm{Diffeo}^+(\Sigma_{g,b,n}) \big/ \mathrm{Diffeo}_0^+(\Sigma_{g,b,n})$$

The reason we write 'Mod' is that this group is also known as Teichmüller's modular group, in analogy with the case of the torus where the mapping class group is isomorphic to $\mathrm{SL}_2(\mathbb{Z})$. In this text, using 'Mod' is particularly practical, because we will also study an object called the modular curve graph, which also abbreviates to MCG.

A special set of elements in $\mathrm{Mod}(\Sigma_{g,b,n})$ that we will use later on is formed by *Dehn twists* around simple closed curves. Given a non-peripheral homotopically essential simple closed curve $\alpha \subset \Sigma_{g,b,n}$ we define the Dehn twist around α to be the mapping class:

$$D_\alpha \in \mathrm{Mod}(\Sigma_{g,b,n})$$

that can be obtained by cutting $\Sigma_{g,b,n}$ along α and regluing it with a full twist.

$\mathrm{Mod}(\Sigma_{g,b,n})$ acts on $\mathcal{T}(\Sigma_{g,b,n})$ by:

$$g \cdot [(S, f)] = [(S, f \circ g^{-1})]$$

As we've already said, the quotient of this action is called moduli space:

Definition 1.11. Let $g, b, n \in \mathbb{N}$ such that $\chi(\Sigma_{g,b,n}) < 0$. We define the moduli space of $\Sigma_{g,b,n}$ as:

$$\mathcal{M}(\Sigma_{g,b,n}) = \mathcal{T}(\Sigma_{g,b,n}) / \mathrm{Mod}(\Sigma_{g,b,n})$$

So $\mathcal{M}(\Sigma_{g,b,n})$ is automatically a topological space with the quotient topology. It should be noted that the action of $\mathrm{Mod}(\Sigma_{g,b,n})$ has non-trivial stabilizers. This means that, while $\mathcal{T}(\Sigma_{g,b,n})$ is a manifold, $\mathcal{M}(\Sigma_{g,b,n})$ is not. It is however an orbifold.

$\mathcal{M}(\Sigma_{g,b,n})$ can also be described as the space of complete hyperbolic stuctures with geodesic boundary on $\Sigma_{g,b,n}$ up to isometries that do not permute boundary components or punctures. Note that this means that in moduli space as we define it here the punctures and boundary components are still marked.

1.2.3. The Weil-Petersson Metric. The Weil-Petersson metric is a Kähler metric on Teichmüller space that descends to a metric on Moduli space. Its original definition is somewhat lengthy. Because we will not use it in the main part of the text, we will take a shortcut and use a theorem of Wolpert as a definition. This expresses the symplectic form induced by the Weil-Petersson metric in a simple way in terms of Fenchel-Nielssen coordinates:

DEFINITION 1.12. [**Wol81**] Let $g, n \in \mathbb{N}$. The Weil-Petersson symplectic form on $\mathcal{T}(\Sigma_{g,n})$ is given by:

$$\omega_{WP} = \sum_{i=1}^{3g+n-3} d\ell_i \wedge d\tau_i$$

This form induces a volume form:

$$\wedge^{3g+n-3}\omega_{WP} = (3g+n-3)! \cdot d\ell_1 \wedge d\tau_1 \wedge \ldots \wedge d\ell_{3g+n-3} \wedge d\tau_{3g+n-3}$$

Up to the factor in front, this is the restriction of the standard Euclidean volume element to $(\mathbb{R}_+ \times \mathbb{R})^{3g+n-3} \subset \mathbb{R}^{6g+2n-6}$. Hence, Teichmüller space has infinite volume in this metric.

ω_{WP} is invariant under the action of the mapping class group. This means that both ω_{WP} and its associated volume form descend to Moduli space. It is a theorem by Wolpert [**Wol85**] that this form extends smoothly to a compactification $\overline{\mathcal{M}(\Sigma_{g,n})}$ of Moduli space (called the Deligne-Mumford compactification, which is constructed by allowing surfaces with length 0 curves). This implies that the Weil-Petersson volume of Moduli space is finite.

1.2.4. The curve graph. The final six parameter spaces we will consider are all graphs. All these graphs have higher dimensional analogues (simplicial complexes parametrizing the same data), but we will restrict to the graphs. The first one is the curve graph. This graph records curves on a surface and their intersection properties. Its formal definition is the following:

DEFINITION 1.13. Let $g, n \in \mathbb{N}$ such that $\kappa(\Sigma_{g,n}) \geq 2$. The *curve graph* $\mathcal{C}(\Sigma_{g,n})$ is the graph with:

- vertices: isotopy classes of non-trivial and non-peripheral simple closed curves on $\Sigma_{g,n}$.
- edges: vertices α and β share an edge if and only if they can be realized disjointly on $\Sigma_{g,n}$.

There also exists a sensible definition of the curve graph when $\kappa(\Sigma_{g,n}) = 1$. Because this coincides with the pants graph, we postpone it to the next section. Furthermore, note that the curve graph does not 'see' the difference between boundary components and punctures, hence it makes sense to only define it for surfaces with punctures.

Instead of considering the curves up to isotopy we can also consider them up to homeomorphism. This gives rise to the modular curve graph:

DEFINITION 1.14. Let $g, n \in \mathbb{N}$ such that $\kappa(\Sigma_{g,n}) \geq 2$. The *modular curve graph* $\mathcal{MC}(\Sigma_{g,n})$ is the graph with:

vertices: homeomorphism types of non-trivial and non-peripheral simple closed curves on $\Sigma_{g,n}$.

edges: vertices α and β share an edge if and only if they can be realized disjointly on $\Sigma_{g,n}$.

One can obtain $\mathcal{MC}(\Sigma_{g,n})$ as a quotient of $\mathcal{C}(\Sigma_{g,n})$ by the action of the extended mapping class group $\mathrm{Mod}_{\mathrm{ext}}(\Sigma_{g,n})$ defined as follows:

$$\mathrm{Mod}_{\mathrm{ext}}(\Sigma_{g,n}) = \frac{\mathrm{Diffeo}(\Sigma_{g,n})}{\mathrm{Diffeo}_0(\Sigma_{g,n})}$$

where the subscript 0 again denotes isotopy to the identity. The difference between this group and $\mathrm{Mod}(\Sigma_{g,n})$ is that in $\mathrm{Mod}_{\mathrm{ext}}(\Sigma_{g,n})$ we do allow orientation reversing diffeomorphisms and diffeomorphisms that permute the punctures.

It follows from theorems by Ivanov and Korkmaz that in general $\mathrm{Mod}_{\mathrm{ext}}(\Sigma_{g,n})$ is in fact the full simplicial automorphism group of $\mathcal{C}(\Sigma_{g,n})$:

THEOREM 1.12. [**Iva88**], [**Kor99**] *If* $(g, n) \notin \{(1, 2), (2, 0)\}$ *then the simplicial automorphism group of* $\mathcal{C}(\Sigma_{g,n})$ *is* $\mathrm{Mod}_{\mathrm{ext}}(\Sigma_{g,n})$.

This means that in general considering $\mathcal{MC}(\Sigma_{g,n})$ amounts to looking at the curve graph up to simplicial automorphism. What happens in the remaining two cases is also known by work of Korkmaz [**Kor99**] and Luo [**Luo00**].

For a closed surface Σ_g, the graph $\mathcal{MC}(\Sigma_g)$ is particularly simple. There are exactly

$$\left\lfloor \frac{g}{2} \right\rfloor + 1$$

homeomorphism types of simple closed curves on a closed surface of genus g: there is one non-separating type and $\lfloor \frac{g}{2} \rfloor$ separating types depending on how much genus they leave on either side. Figure 1.1 below shows examples in genus 4 and 5:

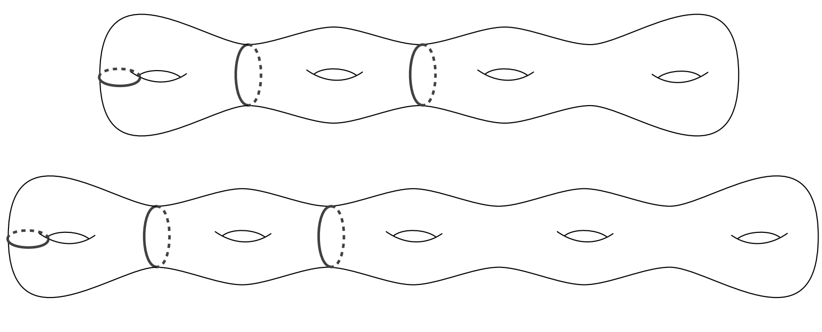


FIGURE 1.1. Topological types of curves on surfaces of genus 4 and 5.

It is not difficult to see that these curves can be realized disjointly, so $\mathcal{MC}(\Sigma_g)$ contains a complete graph on $\lfloor \frac{g}{2} \rfloor + 1$ vertices. In addition there are loops at vertices corresponding to curves that can be realized by two different disjoint isotopy classes of curves simultaneously. For odd g this is every curve and for even genus this is every curve except for the separating curve that separates the surface into two parts of genus $\frac{g}{2}$.

1.2.5. The pants graph. On a surface $\Sigma_{g,n}$ with $\kappa(\Sigma_{g,n}) \geq 1$, the pants graph $\mathcal{P}(\Sigma_{g,n})$ is a graph on isotopy classes of pants decompositions of $\Sigma_{g,n}$. In order to define the edges in $\mathcal{P}(\Sigma_{g,n})$, we need the notion of an *elementary move* on an curve in a pants decomposition:

- If the curve is a boundary curve of two distinct pairs of pants then an elementary move on this curve consists of replacing it with a curve that intersects it twice.
- If the curve is a boundary curve of one pair of pants then an elementary move means replacing it with a curve that intersects it once.

The figures below illustrate the types of elementary moves. Elementary moves 1 and 2 correspond to the first type and elementary move 3 to the second type.

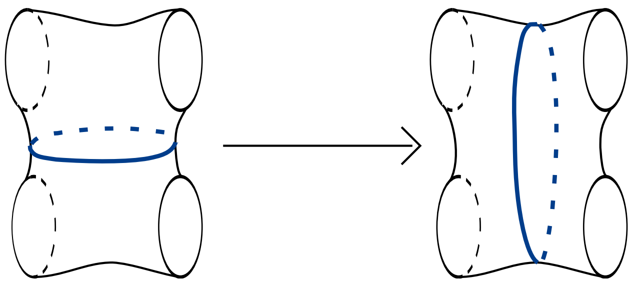


FIGURE 1.2. Elementary move 1. The black closed curves are either part of the pants decomposition or boundary components.

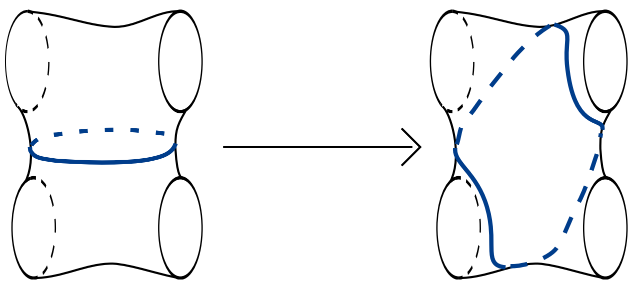


FIGURE 1.3. Elementary move 2. The black closed curves are either part of the pants decomposition or boundary components.

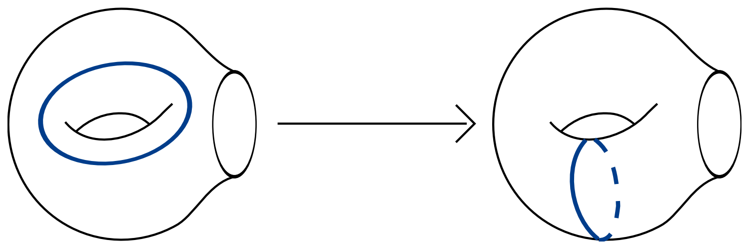


FIGURE 1.4. Elementary move 3. The black closed curve is either part of the pants decomposition or a boundary component.

If we start with a pants decomposition and perform an elementary move on one of its curves, the resulting set of curves will again be a pants decomposition of the surface. Elementary moves will allow us to define edges in the pants graph which we formally define as follows:

DEFINITION 1.15. Let $g, n \in \mathbb{N}$ such that $\kappa(\Sigma_{g,n}) \geq 1$. The *pants graph* $\mathcal{P}(\Sigma_{g,n})$ is the graph with:

vertices: isotopy classes of pants decompositions of $\Sigma_{g,n}$.

edges: vertices P and P' share an edge if P' can be obtained from P by performing an elementary move on one of its curves.

As before, we define the modular pants graph as the graph defined on the same objects up to homeomorphism.

DEFINITION 1.16. The *modular pants graph* $\mathcal{MP}(\Sigma_{g,n})$ is the graph with:

vertices: homeomorphism types of pants decompositions of $\Sigma_{g,n}$.

edges: vertices P and P' share an edge if and only if P' can be obtained from P by performing an elementary move on one of its curves.

We again have an action of $\mathrm{Mod}_{\mathrm{ext}}(\Sigma_{g,n}) \curvearrowright \mathcal{P}(\Sigma_{g,n})$ through the action on curves and:

$$\mathcal{MP}(\Sigma_{g,n}) = \frac{\mathcal{P}(\Sigma_{g,n})}{\mathrm{Mod}_{\mathrm{ext}}(\Sigma_{g,n})}$$

As for the curve graph before, this time by work of Margalit, the extended mapping class group is the automorphism group of the pants graph, except in some low complexity cases:

THEOREM 1.13. [**Mar04**] *If* $(g, n) \notin \{(0, 3), (1, 1), (1, 2), (2, 0)\}$ *then* $\mathrm{Mod}_{\mathrm{ext}}(\Sigma_{g,n})$ *is the full automorphism group of* $\mathcal{P}(\Sigma_{g,n})$

Again it is also known what happens in the remaining cases and again we will not go into this and refer the reader to Margalit's paper [**Mar04**].

Finally, when $\kappa(\Sigma_{g,n}) = 1$ then we will set:

$$\mathcal{C}(\Sigma_{g,n}) = \mathcal{P}(\Sigma_{g,n}) \text{ and } \mathcal{MC}(\Sigma_{g,n}) = \mathcal{MP}(\Sigma_{g,n})$$

1.2.6. The flip graph. The final type of graphs we will be interested in is flip graphs. These are graphs of isotopy classes of triangulations of $\Sigma_{g,n}$ with vertices that lie in the punctures (and thus we suppose $n > 0$). We note again that we include all triangulations and not only the simple ones.

Given such a triangulation, a flip in one of the arcs of the triangulation consists of replacing this edge with its 'opposite diagonal'. Phrased otherwise, a flip is the removal of an arc and its replacement with the only other possible arc that can complete the resulting multi-arc into a triangulation. Note that if an arc only belongs to one triangle, it is impossible to flip it. A flip is illustrated in Figure 1.5.

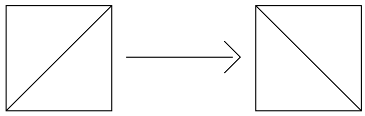


FIGURE 1.5. A flip.

We can now define the flip graph:

DEFINITION 1.17. Let $n > 0$. The *flip graph* $\mathcal{F}(\Sigma_{g,n})$ is the graph with:

vertices: isotopy types of triangulations of $\Sigma_{g,n}$ with vertices in the punctures.
edges: vertices T and T' share an edge if and only if T' can be obtained from T by performing a flip on one of the arcs of T.

The modular version of the flip graph is defined as follows.

DEFINITION 1.18. Let $n > 0$.The *modular flip graph* $\mathcal{MF}(\Sigma_{g,n})$ is the graph with:

vertices: homeomorphism types of triangulations of $\Sigma_{g,n}$.
edges: vertices T and T' share an edge if and only if T' can be obtained from T by performing a flip on one of the edges of T.

Again, it is the quotient of the flip graph by the extended mapping class group and, except for in some low complexity cases, the quotient of the flip graph by its automorphisms, by the following result of Korkmaz and Papadopoulos:

THEOREM 1.14. [**KP12**] *Let* $(g, n) \notin \{(0, 3), (1, 1)\}$ *then* $\mathrm{Mod}_{\mathrm{ext}}(\Sigma_{g,n})$ *is the full simplicial automorphism group of* $\mathcal{F}(\Sigma_{g,n})$.

1.3. The genus of a graph

In the final chapter of this text we will be studying the topological complexity of the graphs defined in the previous section. We are going to measure the topological complexity of a graph by its genus:

Definition 1.19. Let Γ be a graph. The genus of Γ is given by:

$$\gamma(\Gamma) = \min\{g;\ \text{there exists a continuous injection } f : \Gamma \to \Sigma_g\}$$

where the minimum of the empty set is ∞.

The main tool we will use to get bounds on the genera of graphs is the following proposition. The lower bound is due to Beineke and Harary and the upper bound is a well known fact that essentially bounds the genus of a graph in terms of its Betti number (or cycle rank). The *girth* of a graph is the number of edges in the shortest circuit in the graph.

Proposition 1.15. [**BH65.1**] *Let Γ be a connected graph that is not a tree. Furthermore, let Γ have p vertices, q edges and girth h. Then:*

$$1 + \frac{1}{2}\left(1 - \frac{2}{h}\right) q - \frac{1}{2} p \ \leq\ \gamma(\Gamma)\ \leq\ \frac{1}{2} + \frac{1}{2} q - \frac{1}{2} p$$

The proof of the lower bound uses a classical theorem by Youngs (Theorem 4.3 in [**You63**]) which says that every minimal genus embedding $\Gamma \hookrightarrow \Sigma_\gamma$ is such that $\Sigma_\gamma \backslash \Gamma$ is disjoint union of open 2-cells. Because the use of the girth, the lower bound is invalid for trees. The upper bound, that actually does hold for trees, follows from constructing an explicit embedding. Both bounds are sharp. The sharpness of the lower bound can for instance be seen from the n-cube skeleton (cf. [**Rin55**], [**BH65.2**]) and that of the upper bound can be seen from the set of trees.

The main consequence of Proposition 1.15 that we are interested in is that if we have a sequence of graphs $\{\Gamma_n\}_{n \in \mathbb{N}}$ such that:

$$\frac{p(\Gamma_n)}{q(\Gamma_n)} \to 0 \text{ as } n \to \infty$$

and $h(\Gamma_n) \geq h \geq 3$ for all $n \in \mathbb{N}$ then:

$$\left(\frac{1}{2} - \frac{1}{h}\right) q(\Gamma_n) \lesssim \gamma(\Gamma_n) \lesssim \frac{1}{2} q(\Gamma_n) \text{ as } n \to \infty$$

In other words, just by knowing the number of edges and vertices, we can determine the asymptotic behaviour of the genus of our sequence up to a multiplicative constant.

We will also need to know the genus of some specific graphs. We start with a classical theorem of Ringel and Youngs about the genus of a complete graph K_n on n vertices:

Theorem 1.16. [**RY68**] *Let $5 \leq n \in \mathbb{N}$. We have:*

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$$

The other theorem we need is about the bipartite complete graph $K_{m,n}$. This is the graph with vertices $\{x_1, \ldots, x_m, y_1, \ldots, y_n\}$ and edges $\{x_i, y_j\}$ for all $i = 1, \ldots m$, $j = 1, \ldots, n$.

THEOREM 1.17. [**Rin65**] *Let $m, n \in \mathbb{N}$. We have:*

$$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$$

1.4. Probability theory

The use of the word random in the title of this text suggests that we will need some results from probability theory as well. These we will summarize in this section. The basics of probability theory, which we shall not discuss in this section, can for instance be found in [**Fel68**]. Most of the basics we need can also be found in [**Bol85**].

We will generally denote *probability measures* by $\mathbb{P}$ and *probability spaces* by Ω (technically we need to speak of σ-algebras as well, but in all the examples we treat, these are the obvious ones, so we skip over this detail). We will not make a distinction in terminology between probability measures and probability distributions coming from random variables (in any case, a probabiliy measure can be interpreted as the probability distribution of the random variable $\mathrm{Id} : \Omega \to \Omega$). The *conditional probabiliy* of '$A \subset \Omega$', conditioned on $B \subset \Omega$ will be denoted $\mathbb{P}[A|\, B]$ and the *expected value* of a *random variable* $X : \Omega \to \mathbb{R}$ (i.e. a measurable map) will be denoted $\mathbb{E}[X]$, given that it exists, which will also not be an issue in this text.

1.4.1. Some probability distributions.

We will now recall some widely used probability distributions. We shall restrict ourselves to the distributions that will be needed afterwards.

The type of probability space that we will encounter most in this text is actually a probability space where Ω is a finite set and $\mathbb{P} : 2^{\Omega} \to [0,1]$, where 2^{Ω} denotes the power set of Ω, is defined by:

$$\mathbb{P}_N[A] = \frac{|A|}{|\Omega|} \text{ for all } A \subset \Omega$$

We wil sometimes call such a distribution, or a continuous analogue of it, a *uniform distribution*. Sometimes we will denote such a distribution by $\mathbb{U} : 2^X \to [0,1]$.

The normal distribution with average $\mu \in \mathbb{R}$ and standard deviation $\sigma \in (0,\infty)$ will be denoted $\mathcal{N}(\mu, \sigma)$.

The next distribution, which we will encounter often in this text, is the Poisson distribution:

DEFINITION 1.20. Let $X : \Omega \to \mathbb{N}$ be a random variable on a probability space Ω and $\lambda \in (0, \infty)$. We will say that X is *Poisson distributed* with mean λ if:

$$\mathbb{P}_N [X = k] = \frac{\lambda^k e^{-\lambda}}{k!} \text{ for all } k \in \mathbb{N}$$

In this definition '$\mathbb{P}_N [X = k]$' is shorthand for $\mathbb{P}_N [\{\omega \in \Omega;\ X(\omega) = k\}]$. We will very often use this and analogous abbreviations.

The last type of distribution we want to define is the Poisson-Dirichlet distribution. In order to define this distribution we first need to define the following set:

$$\Delta_\infty = \left\{ x \in [0,1]^{\mathbb{N}};\ \sum_{i=0}^{\infty} x_i = 1 \right\}$$

The distribution, which will be defined on Δ_∞ is defined as follows:

DEFINITION 1.21. Let $X_i : [0,1] \to [0,1]$ be uniformly distributed random variables for all $i \in \mathbb{N}$. We define the random variables $Y_i : [0,1] \to [0,1]$ by:

$$Y_0 = X_0, \text{ and } Y_i = X_i \prod_{j=0}^{i-1} (1 - X_j) \text{ for all } i \geq 1$$

We define:

$$Y = (Y_{i_0}, Y_{i_1}, \ldots)$$

where the bijection $j \mapsto i_j$ is chosen such that:

$$Y_{i_j} \geq Y_{i_{j+1}} \text{ and if } Y_{i_j} = Y_{i_{j+1}} \text{ then } i_j \leq i_{j+1}$$

The random variable $Y : [0,1]^{\mathbb{N}} \to \Delta_\infty$ is said to be *Poisson-Dirichlet distributed.*

A nice interpretation of the sequence $(Y_0, Y_1, \ldots)$ is that of breaking of a stick into pieces. First the stick is broken into two pieces uniformly, which gives Y_0: the length of the piece on the right of the breaking point. After that, the piece on the left of the breaking point is broken into two pieces uniformly again, which gives Y_1 and this process is continued ad infinitum. The sequence $(Y_{i_0}, Y_{i_1}, \ldots)$ is then just a reordering of the pieces of stick according to their length.

1.4.2. Inequalities. Sometimes it turns out to be less difficult to determine the expected value of a random variable than its probability distribution. In some of these cases, the expected value helps one estimate tail probabilities through Markov's inequality:

THEOREM 1.18. Markov's inequality: *Let Ω be a probability space and $X : \Omega \to \mathbb{R}$ a non-negative valued random variable such that $\mathbb{E}[X]$ exists. Then:*

$$\mathbb{P}_N [X \geq x] \leq \frac{\mathbb{E}[X]}{x} \textit{ for all } x \in (0, \infty)$$

This implies Chebyshev's inequality:

COROLLARY 1.19. Chebychev's inequality: *Let Ω be a probability space and $X : \Omega \to \mathbb{R}$ a random variable such that $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$ exist. Then:*

$$\mathbb{P}_N\left[|X - \mathbb{E}[X]| \geq x\right] \leq \frac{\mathbb{E}\left[(X - \mathbb{E}[X])^2\right]}{x^2}$$

This inequality implies that the numerator on the right hand side is a measure for how much the given random variable deviates from its average. The following definition gives it a name:

DEFINITION 1.22. Let Ω be a probability space and $X : \Omega \to \mathbb{R}$ a random variable such that $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$ exist. Then the *variance* of X is defined as:

$$\mathbb{V}\mathrm{ar}[X] = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$$

The final estimate we need is the following:

LEMMA 1.20. *Let Ω be a probability space and $X : \Omega \to \mathbb{R}$ a random variable such that $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$ exist. Then:*

$$\mathbb{P}_N[X = 0] \leq \frac{\mathbb{V}\mathrm{ar}[X]}{\mathbb{E}[X]^2 + \mathbb{V}\mathrm{ar}[X]}$$

This can be derived from the Cauchy inequality and can for instance be found as Equation 1.3 in [**Bol85**].

1.4.3. Convergence. We also want to be able to say something about how close two random variables are to each other. There are many different (inequivalent) ways to do this. We will use the following definition:

DEFINITION 1.23. Let E be a measure space with σ-algebra Σ_E. Furthermore, let $X : \Omega_1 \to E$ and $Y : \Omega_2 \to E$ be two random variables on probability spaces Ω_1 and Ω_2 respectively. Then we define the *total variational distance* between X and Y as:

$$d(X, Y) = \sup\left\{\left|\mathbb{P}_1[X \in A] - \mathbb{P}_2[Y \in A]\right|;\ A \in \Sigma_E\right\}$$

This definition allows us to say when a sequence of random variables converges to a fixed random variable:

DEFINITION 1.24. Let E be a measure space. Furthermore, let $\{X_n : \Omega_n \to E\}_{n=1}^{\infty}$ be a sequence of random variables on a sequence of probability space $\{\Omega_n\}_{n=1}^{\infty}$ and $X : \Omega \to E$ is a random variable on a probability space Ω such that:

$$\lim_{n \to \infty} d(X_n, X) = 0$$

Then we say that X_n converges to X *in distribution* as $n \to \infty$ and we write:

$$X_n \xrightarrow{d} X \text{ as } n \to \infty$$

We can also say when two sequences of random variables behave the same way asymptotically:

Definition 1.25. Let E be a measure space. Furthermore, let $\{X_n : \Omega_{1,n} \to E\}_{n=1}^{\infty}$ and $\{Y_n : \Omega_{2,n} \to E\}_{n=1}^{\infty}$ be two sequences of random variables on two sequences of probability spaces $\Omega_{1,n}$ and $\Omega_{2,n}$ respectively, such that:

$$\lim_{n\to\infty} d(X_n, Y_n) = 0$$

Then we write:

$$X_n \overset{d}{\sim} Y_n \text{ as } n \to \infty$$

We also need the following form of convergence of a random variable, which in fact implies convergence in distribution:

Definition 1.26. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of $\mathbb{R}$-valued random variables on a probability space $(\Omega, \Sigma, \mathbb{P})$. Furthermore let $X : \Omega \to \mathbb{R}$ be a random variable such that:

$$\mathbb{P}_N \left[\lim_{n\to\infty} X_n = X\right] = 1$$

Then we say that X_n converges to X *almost surely* for $n \to \infty$.

We note that the expression on the right hand side in this definition needs a non-trivial definition itself, which can be found [**Fel68**].

One of the main tools in this text will be the following result about random variables converging to Poisson variables and is some form of what is sometimes called the *method of moments*. In this theorem we write $(X)_k = X \cdot (X_1 - 1) \cdots (X_1 - k + 1)$ for any random variable X.

Theorem 1.21. Poisson approximation: *Let $k \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_k \in (0, \infty)$. Let $X_{n,i} : \Omega_n \to \mathbb{N}$ be a random variable on the probability space $(\Omega_n, \mathbb{P}_n)$ for all $n \in \mathbb{N}$ and $i = 1, \ldots, k$ such that:*

$$\lim_{n\to\infty} \mathbb{E}\left[(X_{n,1})_{r_1} (X_{n,2})_{r_2} \cdots (X_{n,k})_{r_k}\right] = \lambda_1^{r_1} \lambda_2^{r_2} \cdots \lambda_k^{r_k}$$

for all $r_1, \ldots, r_k \in \mathbb{N}$. Then:

$$X_{n,i} \xrightarrow{d} X_i \text{ for } n \to \infty \text{ and } i = 1, \ldots, k$$

where:

- *X_i is a Poisson distributed random variable with mean λ_i.*
- *All the X_i are independent of all the other X_j for all $1 \leq i \neq j \leq k$.*

The numbers $\mathbb{E}\left[(X_{n,1})_{r_1} (X_{n,2})_{r_2} \cdots (X_{n,k})_{r_k}\right]$ are sometimes called the *joint factorial moments* of the random variables $X_{n,1}, \ldots, X_{n,k}$.

1.4.4. Random permutations. Random surfaces can also be described by specific pairs of random permutations. So, in this section we will describe some results on random elements of finite groups.

DEFINITION 1.27. Let G be a finite group. The *uniform probability measure on G* is the measure $\mathbb{U}_G : 2^G \to [0,1]$ defined by:

$$\mathbb{U}_G(A) = \frac{|A|}{|G|} \text{ for all } A \subset G$$

Two examples that we will be particularly interested in are the uniform probability measures on the symmetric group S_N and the alternating group A_N on N letters. We will state two convergence results. The proofs of these in the classical case of the symmetric group can for example be found in Chapter 3 of [**Pit06**]. The fact that they are also true in the case of the alternating group is due to Gamburd [**Gam06**]. The normalized cycle lengths of a permutation are the lengths of its cycles, in a decomposition of the permutation into disjoint cycles, divided by N.

PROPOSITION 1.22. [**Pit06**] [**Gam06**] *Let $N \in \mathbb{N}$ and let $G = \mathrm{S}_N$ or $G = \mathrm{A}_N$. Furthermore, let*

$$L_N = (L_{N,1}, \ldots, L_{N,N!}) : G \to [0,1]^{N!}$$

denote the normalized cycle lengths of a permutation, ordered by size. Then L_N, as a random variable on $(G, \mathbb{U}_G)$, converges in distribution to the Poisson-Dirichlet distribution on Δ_∞ when $N \to \infty$.

This proposition implies the following:

PROPOSITION 1.23. [**Pit06**] [**Gam06**] *Let $N \in \mathbb{N}$ and let $G = \mathrm{S}_N$ or $G = \mathrm{A}_N$. Furthermore, let $C_N : G \to \mathbb{N}$ count the number of cycles of a permutation. Then:*

$$\frac{C_N}{\log(N)} \to 1 \textit{ almost surely, and } \frac{C_N - \log(N)}{\sqrt{\log N}} \xrightarrow{d} \mathcal{N}(0,1) \textit{ as } N \to \infty$$

as a random variable on $(G, \mathbb{U}_G)$.

Because we understand the properties of a uniformly chosen element of S_N or A_N quite well, we would like to bound the distance of other probability distributions to the uniform distribution. To this end we will use the Diaconis-Shahshahani upper bound lemma. Before we can state this lemma, we need the following definition:

Definition 1.28. Let G be a finite group.

- The set of irreducible unitary representations of G will be denoted $\hat{G}$.
- For $(\rho : G \to \mathrm{GL}(V_\rho)) \in \hat{G}$ and a probability measure $\mathbb{P}$ on G. The *Fourier transform* of $\mathbb{P}$ at ρ is the linear map:

$$\hat{\mathbb{P}}(\rho) = \sum_{g\in G} \mathbb{P}[g]\rho(g) : V_\rho \to V_\rho$$

We now have the following two lemmas by Diaconis and Shahshahani:

Lemma 1.24. [**DS81**] *Let G be a finite group and $\mathbb{P}$ a probability measure on G that is constant on conjugacy classes. Furthermore, let $(\rho : G \to \mathrm{GL}(V_\rho)) \in \hat{G}$. Then:*

$$\hat{\mathbb{P}}(\rho) = \frac{1}{\dim(\rho)} \sum_{K \text{ conjugacy class of } G} \mathbb{P}\left[K\right] |K|\, \zeta^\rho(K) I_{\dim(\rho)}$$

where $\mathbb{P}\left[K\right]$ is the value of $\mathbb{P}$ on a single element in K.

The second lemma is known as the Diaconis-Shahshahani upper bound lemma:

Lemma 1.25. [**DS81**] *Let G be a finite group and let $\mathbb{P}$ be a probability measure on G. Then:*

$$d\left(\mathbb{P}, \mathbb{U}_G\right)^2 \leq \frac{1}{4} \sum_{\substack{\rho\in\hat{G} \\ \rho\neq\mathrm{id}}} \dim(\rho)\mathrm{tr}\left(\hat{\mathbb{P}}(\rho)\overline{\hat{\mathbb{P}}(\rho)}\right)$$

1.5. Combinatorics

In this section we will briefly recall some combinatorics.

1.5.1. Posets and Möbius functions. The first combinatorial fact we will need is the so called Möbius inversion formula. For a detailed treatment of this material we refer the reader to Chapter 3 of [**Sta97**].

We start with the definition of a partialy ordered set (or poset):

Definition 1.29. A *partially ordered set* is a set P together with a relation $\leq_P$ such that:

1. For all $x \in P$ we have: $x \leq_P x$ (reflexivity).
2. If $x, y \in P$ such that $x \leq_P y$ and $y \leq_P x$ then $x = y$ (antisymmetry).
3. If $x, y, z \in P$ such that $x \leq_P y$ and $y \leq_P z$ then $x \leq_P z$ (transitivity).

Furthermore, if for all $x, y \in P$ the set $\{z \in P;\ x \leq_P z \leq_P y\}$ is finite, $(P, \leq_P)$ is called *locally finite.*

Generally we will drop the subscript P in $\leq_P$. Furthermore, if $x, y \in P$ such that $x \leq y$ and $x \neq y$ then we will write $x < y$.

Next up is the definition of the Möbius function of a poset:

DEFINITION 1.30. Let $(P, \leq)$ be a locally finite poset. The *Möbius function* of $(P, \leq)$ is the function $\mu : P \times P \to \mathbb{Z}$ given recursively by:

$$\mu(x,y) = 0 \text{ for all } x > y \in P$$

$$\mu(x,x) = 1 \text{ for all } x \in P$$

$$\mu(x,y) = -\sum_{x \leq z < y} \mu(x,z) \text{ for all } x < y \in P$$

Finally we have the following proposition (Proposition 3.7.1 in [**Sta97**]):

PROPOSITION 1.26. Möbius inversion formula: *Let $(P, \leq)$ be a locally finite poset such that for all $x \in P$ the set $\{y \in P;\ y \leq x\}$ is finite and let $g, f : P \to \mathbb{C}$. Then:*

$$g(x) = \sum_{y \leq x} f(y) \text{ for all } x \in P$$

if and only if:

$$f(x) = \sum_{y \leq x} g(y)\mu(y,x) \text{ for all } x \in P$$

1.5.2. Stirling's approximation. We will also often apply Stirling's approximation. We shall not only need the approximation itself but we need to know the error in the approximation as well. To this end we have the following theorem by Robbins:

THEOREM 1.27. [**Rob55**] *Let $n \in \mathbb{N}$ and $n \neq 0$. Let $\lambda_n \in \mathbb{R}$ such that:*

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\lambda_n}$$

then:

$$\frac{1}{12n+1} \leq \lambda_n \leq \frac{1}{12n}$$

1.5.3. Partitions. In order to apply the Diaconis-Shahshahani upper bound later on, we will need to study the representation theory of the symmetric and alternating group. Much of this will rely on partitions of natural numbers:

DEFINITION 1.31. Let $N \in \mathbb{N}$. A *partition* of N is a sequence $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{N}^k$ such that:

$$0 < \lambda_k \leq \lambda_{k-1} \leq \ldots \leq \lambda_1 \text{ and } \sum_{i=1}^{k} \lambda_i = N$$

If λ is a partition of N, we will write $\lambda \models N$. The numbers $\lambda_1, \ldots, \lambda_k$ are called the *parts* of the partition.

We will need an upper bound on the number of partitions of a number $N \in \mathbb{N}$. To us it will only matter that this upper bound is subexponential in N. For reference, we include the following theorem (that can be found as Theorem 14.5 in [**Apo76**]):

THEOREM 1.28. *Let $p(N)$ be the number of partitions of the number $N \in \mathbb{N}$. Then:*

$$p(N) < \exp\left(\pi\sqrt{\frac{2N}{3}}\right)$$

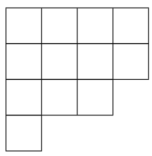
1.6. Group characters

In this final section of the preliminaries we will gather some facts on the group representations of S_N and A_N. For a comprehensive treatment of the representation theory of the symmetric and alternating group we refer the reader to [**dBR61**] and [**JK81**].

1.6.1. The symmetric group. As for any finite group, the irreducible representations of S_N are in bijection with the conjugacy classes of S_N. The nice feature of S_N is that there is a natural bijection between these two sets.

We recall that the conjugacy classes of S_N are labeled by partitions $\lambda \models N$. We will denote the corresponding conjugacy class by $K(\lambda)$.

A partition λ can be represented by what is called a *Young diagram.* If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ then the corresponding Young diagram is formed by k left aligned rows of boxes where row i has length λ_i. For example, if $\lambda = (4, 4, 3, 1)$ then the corresponding Young diagram is:



When we fill the boxes of such a diagram with the numbers $1, 2, \ldots, N$, it is called a *Young tableau.* In general we will not make a distinction between a partition, its corresponding Young diagram or a Young tableau corresponding to that. A permutation acts on a Young tableau by permuting the numbers in its boxes. So for example, we have:

$$(1\ 4\ 2) \cdot \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = \begin{array}{|c|c|}\hline 4 & 1 \\ \hline 3 & 5 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Note that a permutation does not change the shape of the tableau.

One obtains a representation for every $\lambda \models N$ by taking the Young tableaux of shape λ as a basis for a $\mathbb{C}$-vector space and extending the action linearly. In general these representations are not irreducible, but there is a procedure to obtain exactly one distinct irreducible representation as a subrepresentation of each of them. We will denote the corresponding vector space V^λ, the representation $\rho^\lambda : \mathrm{S}_N \to \mathrm{GL}(V^\lambda)$, its character $\chi^\lambda : \mathrm{S}_N \to \mathbb{C}$ and its dimension $f^\lambda = \dim\left(V^\lambda\right)$. By construction these representations form the complete set of irreducible representations of S_N.

In order to apply the Diaconis-Shahshahani upper bound lemma we need to obtain bounds on the dimension and the characters of all the irreductible representations. For the dimension we need the notion of the *hook length* $h(b)$ of a box $b \in \lambda$. This is simply 1 plus the number of boxes to the right of b plus the number of boxes below b. As an example the tableau below is filled with the hook lengths of the corresponding boxes:

$$\begin{array}{|c|c|c|c|}\hline 6 & 4 & 2 & 1 \\ \hline 3 & 1 & \multicolumn{2}{c}{} \\ \cline{1-2} 1 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}$$

We have the following classical theorem for the numbers f^λ (see for instance Equation 2.37 on page 44 of [**dBR61**]):

THEOREM 1.29. (Hook length formula) *Let $\lambda \models N$. Then:*

$$f^\lambda = \frac{N!}{\prod\limits_{b \in \lambda} h(b)}$$

Next, we need to gather some facts about the characters χ^λ. The first one is the Murnaghan-Nakayama rule (Lemma 4.15 and Equation 4.21 on pages 77 and 78 of [**dBR61**]):

THEOREM 1.30. (Murnaghan-Nakayama rule) *Let $g \in \mathrm{S}_N$ be such that:*

$$g = hc$$

where $h \in \mathrm{S}_{N-m}$ and c an m-cycle disjoint from h. Then:

$$\chi^\lambda(g) = \sum_\mu (-1)^{r(\lambda,\mu)} \chi^\mu(h)$$

where the sum above runs over all tableaux μ that can be obtained from λ by removing a continuous region on the boundary of λ consisting of m boxes (called a skew m hook *or a* rim hook*). And $r(\lambda, \mu)$ is the number of rows in the skew m hook that needs to be removed from λ to obtain μ minus one.*

As an example, the starred boxes below form a skew 3 hook in a tableau for S_7:

$$\begin{array}{|c|c|c|}\hline & & * \\ \hline & * & * \\ \hline & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

in this case we have $r(\lambda, \mu) = 1$.

Also note that it follows from Theorem 1.30 that if we cannot remove a skew m hook from λ (i.e. there is no tableau μ that can be obtained by removing such a skew hook) and $g \in \mathrm{S}_N$ contains an m cycle then:

$$\chi^\lambda(g) = 0$$

From the two theorems above, one can derive that (Theorem 4.56 of [**dBR61**]):

THEOREM 1.31. *If $a \in \mathrm{S}_M$ and b is a product of k cycles, each of length m, that leaves $\{1, \ldots, a\}$ fixed pointwise. And λ a partition of $N = M + km$ out of which exactly k skew m hooks are removable then:*

$$\chi^\lambda(ab) = \sigma f_m^\lambda \chi^{\tilde{\lambda}}(a)$$

where $\tilde{\lambda}$ is what is left over of λ after the removal of k skew m hooks and is independent of the order of removal. Furthermore:

$$\sigma = (-1)^{\sum_{i=1}^{k} r(\mu_{i-1}, \mu_i)}$$

where $\mu_0 = \lambda$, $\mu_k = \tilde{\lambda}$ and μ_i is any tableau that is obtainable from μ_{i-1} by the removal of a skew m hook for $i = 1, \ldots, k$. Finally f_m^λ is the number of ways to consecutively remove k skew m hooks from λ.

We will be interested in the case where there might be more skew m hooks removable from a tableau λ then there are m cycles in the element $g \in \mathrm{S}_N$ (note again that if there are fewer skew m hooks removable from λ than m cycles in g then $\chi^\lambda(g) = 0$). We have the following lemma:

LEMMA 1.32. *Let $g \in \mathrm{S}_N$ contain k cycles of length m. Then:*

$$\left|\chi^\lambda(g)\right| \leq \max\left\{\left|\chi^\mu(a)\right|;\ a \in \mathrm{S}_{N-km},\ \mu \textit{ a partition of } N - km\right\} f_m^\lambda$$

PROOF. We write $g = ab$ where b a product of k m cylces and a contains no such cycle. Then we have:

$$\chi^\lambda(g) = \sum_\mu \sigma_\mu \chi^\mu(a)$$

where the sum is over diagrams μ that can be obtained from λ by removing k skew m hooks and σ_μ is the power of -1 that comes out of Theorem 1.30. This means that:

$$\left|\chi^\lambda(g)\right| \leq \max\left\{\left|\chi^\mu(a)\right|;\ a \in \mathrm{S}_{N-km},\ \mu \text{ a partition of } N - km\right\} f_{k,m}^\lambda$$

where $f_{k,m}^\lambda$ is the number of ways to remove k skew m hooks from λ. We have:

$$f_{k,m}^\lambda \leq f_m^\lambda$$

and hence:

$$\left|\chi^\lambda(g)\right| \leq \max\left\{\left|\chi^\mu(a)\right|;\ a \in \mathrm{S}_{N-km},\ \mu \text{ a partition of } N - km\right\} f_m^\lambda$$

$\square$

We also need to define a family of maps

$$\mathbb{Y} \to \mathbb{Y}^r$$

where $\mathbb{Y}$ is the set of Young tableaux and $r > 0$ is an integer. Before we can define this map we need the notion of a (r, s)-node:

DEFINITION 1.32. Let λ be a partition of N and $(i,j) \in \lambda$ such that $r|h(i,j)$. If:

$$\lambda_i - i \equiv s \mod r$$

then we call (i,j) an (r,s) node. The set of (r,s) nodes in λ will be denoted λ_r^s

Note that it is not immediately clear that λ_r^s is a Young tableau, it might not be right aligned. However, we claim that this defines a map $\mathbb{Y} \to \mathbb{Y}^r$:

$$\lambda \mapsto \left(\lambda_r^0, \lambda_r^1, \ldots, \lambda_r^{r-1}\right)$$

In fact, we have the following theorem (Theorem 4.46 in [**Rob55**])

THEOREM 1.33. *Let λ be a partition of N and $r > 0$ an integer. The set of nodes in $(i,j) \in \lambda$ with $r|h(i,j)$ can be divided into disjoint sets whose (r,s) nodes constitute right aligned Young tableaux:*

$$\lambda_r^s$$

for $s = 0, \ldots, r-1$. Furthermore, if $(i,j) \in \lambda$ is an (r,s) node then its hook length with respect to λ_r^s is given by:

$$h^{(s)}(i,j) = \frac{1}{r} h(i,j)$$

The map above is sometimes called the *star construction.*

1.6.1.1. *Removing skew hooks.* The last fact about the characters of S_N we need is the following theorem by Fomin and Lulov.

THEOREM 1.34. [**FL95**] *Let λ be a partition of N. Then: Let $\lambda \models N = M + km$ such that exactly k skew m hooks can be removed from λ. Then:*

$$f_m^\lambda \leq \frac{k!\, m^k}{(N!)^{1/m}} \left(f^\lambda\right)^{1/m}$$

In fact, Fomin and Lulov state the theorem only in the case $M = 0$. Hence, for completeness we include a proof, which is their proof verbatim, but starting from a slightly more general set up. We will prove the theorem in small steps. We start with the generalisation of Corollary 2.2 of [**FL95**]:

PROPOSITION 1.35. *Let λ be a partition of $N = M + km$ such that exactly k skew m hooks can be removed from λ. Then:*

$$f_m^\lambda = \frac{k!}{\prod\limits_{\substack{b \in \lambda \\ m|h(b)}} \frac{h(b)}{m}}$$

Proof. We combine Theorems 1.29, 1.31 and 1.33 to obtain:

$$\begin{aligned} f_m^\lambda &= \binom{k}{k_0, k_1, \dots k_{m-1}} f^{\lambda_m^0} f^{\lambda_m^1} \cdots f^{\lambda_m^{m-1}} \\ &= \binom{k}{k_0, k_1, \dots k_{m-1}} \frac{k_0!}{\prod\limits_{b\in\lambda_m^0} h^{(0)}(b)} \frac{k_1!}{\prod\limits_{b\in\lambda_m^1} h^{(1)}(b)} \cdots \frac{k_{m-1}!}{\prod\limits_{b\in\lambda_m^{m-1}} h^{(m-1)}(b)} \\ &= k! \frac{1}{\prod\limits_{b\in\lambda_m^0} \frac{h(b)}{m}} \frac{1}{\prod\limits_{b\in\lambda_m^1} \frac{h(b)}{m}} \cdots \frac{1}{\prod\limits_{b\in\lambda_m^{m-1}} \frac{h(b)}{m}} \end{aligned}$$

The boxes in the product above are exactly those boxes whose hook length in λ is divisible by m. □

Now we need to investigate the product $\prod\limits_{\substack{b\in\lambda \\ m|h(b)}} \frac{h(b)}{m}$. To this end we define a partial order on λ. We set:

$$(k,l) \le (i,j) \Leftrightarrow k \ge i \text{ and } l \ge j$$

So, in words: a box b_1 is 'bigger' than a box b_2 if one can get from b_2 to b_1 by moving to the left and/or upwards.

We define the function $p_\lambda : \lambda \to \mathbb{Z}$ by:

$$p_\lambda(b) = \begin{cases} -m+1 & \text{if } m|h(b) \\ 1 & \text{otherwise} \end{cases}$$

We have the following theorem (Theorem 2.7.40 from [**JK81**]):

Theorem 1.36. *The number of skew m hooks that can be removed from a tableau λ is equal to the number of hook lengths of λ divisible by m.*

This implies the following:

Lemma 1.37. *Let $\lambda \models N$. For all $b \in \lambda$ we have:*

$$\sum_{v\le b} p_\lambda(v) \ge 0$$

Proof. First of all note that for every $b \in \lambda$ the set $\lambda_b \{v \in \lambda;\ v \le b\}$ forms a Young tableau. Write $n = |\lambda_b|$ Now suppose the number of hook lengths of λ_b divisible by m is k. By the theorem above this means that we can remove k skew m hooks from λ_b and hence that $k \le n/m$. So:

$$\begin{aligned} \sum_{v\le b} p_\lambda(v) &= n - k - (m-1)k \\ &\ge n - \frac{n}{m} - m\frac{n}{m} + \frac{n}{m} \\ &= 0 \end{aligned}$$

□

Let us also compute the Möbius function of the partial order we have created. It is not difficult to see that:

$$\mu(b,v)=\begin{cases}1 & \text{if } b=v \text{ or } (b,v)=((i,j),(i-1,j-1))\\ -1 & \text{if } (b,v)=((i,j),(i-1,j)) \text{ or } (b,v)=((i,j),(i,j-1))\\ 0 & \text{otherwise}\end{cases}$$

Now we can state the theorem we want to prove:

THEOREM 1.38. *Let λ be a partition of N. Then:*

$$\prod_{b\in\lambda} h(b)^{p_\lambda(b)}\leq 1$$

PROOF. By Proposition 1.26 we have:

$$\prod_{b\in\lambda} h(b)^{p_\lambda(b)}=\prod_{b\in\lambda} h(b)^{\sum\limits_{v\leq b}\mu(v,b)\sum\limits_{w\leq v}p_\lambda(w)}=\prod_{v\in\lambda}\left(\prod_{v\leq b\in\lambda} h(b)^{\mu(v,b)}\right)^{\sum\limits_{w\leq v}p_\lambda(w)}$$

Because $\sum\limits_{w\leq v} p_\lambda(w)\geq 0$ for every $v\in\lambda$, it suffices to show that for for any $v\neq(1,1)$:

$$\prod_{v\leq b\in\lambda} h(b)^{\mu(v,b)}\leq 1$$

If we work out the product above then generically we have:

$$\prod_{v\leq b\in\lambda} h(b)^{\mu(v,b)}=\frac{h(v)h(v_{tl})}{h(v_t)h(v_l)}$$

where v_t is the box above v, v_l is the box to the left of v and v_{tl} us the box to the top of v_l. It might be that v is already entirely at the top or entirely on the left of λ but in both these cases it is clear that the expression above is less than 1. Excluding these possibilities, we have:

$$h(v_{tl})+h(v)=h(v_l)+h(v_t)$$

and:

$$h(v_l)>h(v),\ h(v_t)>h(v)$$

Hence:

$$\begin{aligned}\frac{h(v)h(v_{tl})}{h(v_t)h(v_l)}&=\frac{h(v)(h(v_t)+h(v_l)-h(v))}{h(v_t)h(v_l)}\\&=\frac{h(v)}{h(v_l)}+\frac{h(v)}{h(v_t)}\left(1-\frac{h(v)}{h(v_l)}\right)\\&\leq 1\end{aligned}$$

□

As a consequence of this we get Theorem 1.34, which we repeat for the reader's convenience:

THEOREM 1.34. [**FL95**] *Let λ be a partition of N. Then: Let λ be a partition of $N = M + km$ such that exactly k skew m hooks can be removed from λ. Then:*

$$f_m^\lambda \leq \frac{k!\, m^k}{(N!)^{1/m}} \left(f^\lambda\right)^{1/m}$$

PROOF. We have:

$$\begin{aligned} f_m^\lambda &= \frac{k!}{\prod\limits_{\substack{b\in\lambda \\ m|h(b)}} \frac{h(b)}{m}} \\ &= k!\, m^k \left(\prod_{b\in\lambda} h(b)^{p_\lambda(b)}\right)^{1/m} \left(\frac{1}{\prod\limits_{b\in\lambda} h(b)}\right)^{1/m} \\ &\leq k!\, m^k \left(\frac{1}{\prod\limits_{b\in\lambda} h(b)}\right)^{1/m} \\ &= \frac{k!\, m^k}{(N!)^{1/m}} \left(f^\lambda\right)^{1/m} \end{aligned}$$

$\square$

1.6.1.2. *Sums of inverse dimensions.* We shall also need the following proposition (which appears as Theorem 1.1 in [**LS04**] and Proposition 4.2 in [**Gam06**]):

PROPOSITION 1.39. [**LS04**][**Gam06**] *For any $t > 0$ and $m \in \mathbb{N}$ we have:*

$$\sum_{\substack{\lambda\models N \\ \lambda\neq(N),(1,1,\ldots,1) \\ \lambda_1,\lambda_1'\leq N-m}} \left(f^\lambda\right)^{-t} = \mathcal{O}\left(N^{-mt}\right)$$

as $N \to \infty$.

The notation '$f(N) = \mathcal{O}(g(N))$ as $N \to \infty$' for real-valued functions $f, g : \mathbb{N} \to \mathbb{R}$ means that:

$$\limsup_{N\to\infty} \frac{f(N)}{g(N)} < \infty$$

1.6.1.3. *A character table.* In one of our proofs we shall use some specific character values and dimensions of S_N representations. We have tabulated those we need below. This table can be found as part of Table 1 in [**Gam06**]. The two rightmost columns are meant to indicate the abolute value of the corresponding characters whenever $N/2$ and $N/3$ are integers.

λ	f^λ	$\left\|\chi^\lambda\left(\left(K\left(2^{N/2}\right)\right)\right\|\right.$	$\left\|\chi^\lambda\left(\left(K\left(2^{N/3}\right)\right)\right\|\right.$
$(N-1,1)$	$N-1$	1	1
$(N-2,2)$	$\frac{N(N-3)}{2}$	$\frac{N}{2}$	0
$(N-2,1,1)$	$\frac{N(N-2)}{2}$	$\frac{N}{2}+1$	1
$(N-3,2,1)$	$\frac{N(N-2)(N-4)}{3}$	0	$\frac{N}{3}+1$
$(N-3,1,1,1)$	$\frac{N(N-2)(N-3)}{3}$	$\frac{N}{2}+1$	$\frac{N}{3}-1$
$(N-3,3)$	$\frac{N(N-1)(N-5)}{6}$	$\frac{N}{2}+2$	$\frac{N}{3}+1$

TABLE 1. Dimensions and characters of S_N representations.

We furthermore note that if a partition λ' is obtained by reflecting the partition λ in its main diagonal, then it follows from the Hook length formula (Theorem 1.29) that:

$$f^{\lambda'} = f^\lambda$$

λ' will be called the *associated partition* of λ. If $\lambda' = \lambda$ then we will call λ *self-associated.*

1.6.2. The alternating group. Finally, we briefly consider the representations of the alternating group. We start with the conjugacy classes of A_N (Lemma 1.2.10 of [**JK81**]):

LEMMA 1.40. *Let λ be a partition of N. Then:*

- *if λ contains an odd number of even parts then:*

$$K(\lambda) \cap \mathrm{A}_N = \emptyset$$

- *if the parts of $\lambda = (\lambda_1, \dots, \lambda_k)$ are all pairwise different and odd then $K(\lambda) \cap \mathrm{A}_N$ splits into two conjugacy classes $K(\lambda)^+$ and $K(\lambda)^-$ of equal size. By convention we take:*

$$(1 \dots \lambda_1)(\lambda_1 + 1 \dots \lambda_1 + \lambda_2) \cdots (N - \lambda_k + 1 \dots N) \in K(\lambda)^+$$

- *otherwise $K(\lambda) \cap \mathrm{A}_N$ is a conjugacy class of A_N.*

If a vector space V is a representation of S_N then the restriction of V to A_N will be denoted $V \downarrow_{\mathrm{A}_N}$. We have the following theorem (Theorem 2.5.7 of [**JK81**]):

THEOREM 1.41. *Suppose λ is a partition of N then:*

- *If $\lambda \neq \lambda'$ then $V^\lambda \downarrow_{\mathrm{A}_N} = V^{\lambda'} \downarrow_{\mathrm{A}_N}$ is an irreducible representation of A_N.*
- *If $\lambda = \lambda'$ then $V^\lambda \downarrow_{\mathrm{A}_N} = V^{\lambda'} \downarrow_{\mathrm{A}_N}$ splits into two inequivalent irreducible representations V^λ_+ and V^λ_- of equal dimension.*

It follows from counting conjugacy classes that the theorem above gives us a complete list of irreducible representations.

Now we want to express the characters associated to these representations in terms of the characters of S_N. It follows immediately from the theorem above that the characters for non self-associated partitions are identical. To avoid confusion we will however denote the A_N-character corresponding to λ by ζ^λ. For self-associated partitions we have the following theorem (Theorem 2.5.13 of [**JK81**]):

THEOREM 1.42. *If λ is a partition of N such that $\lambda = \lambda'$ and the main diagonal of λ has length k. We write:*

$$H^+(\lambda) = K(h(1,1),\dots,h(k,k))^+ \text{ and } H^-(\lambda) = K(h(1,1),\dots,h(k,k))^-$$

Then for $\pi \in \mathrm{A}_N$ the A_N-characters corresponding to λ are given by:

$$\zeta^\lambda_\pm(\pi) = \begin{cases} \frac{1}{2}\left((-1)^{(N-k)/2} \pm \sqrt{(-1)^{(N-k)/2}\prod_{i=1}^{k} h(i,i)}\right) & \text{if } \pi \in H^+(\lambda) \\ \frac{1}{2}\left((-1)^{(N-k)/2} \mp \sqrt{(-1)^{(N-k)/2}\prod_{i=1}^{k} h(i,i)}\right) & \text{if } \pi \in H^-(\lambda) \\ \frac{1}{2}\chi^\lambda(\pi) & \text{otherwise} \end{cases}$$

where $\chi^\lambda(\pi)$ is the character of π as an element of S_N.

Finally, we have the following lemma about the values of the S_N-characters of self-associated partitions (Lemma 2.5.12 of [**JK81**]):

LEMMA 1.43. *If λ is a partition of N such that $\lambda = \lambda'$ and the main diagonal of λ has length k. Then:*

$$\chi^\lambda\left(K(h(1,1),\dots,h(k,k))\right) = (-1)^{(N-k)/2}$$

CHAPTER 2

Random surfaces

After all these preliminaries we can finally get to the main subject of this text: random surfaces. In this chapter we will first introduce the model we will study. Then we will discuss connections to random graphs and random permutations. After this, we explain how to read off the topological and geometric properties of the surface from the data given by the model and how to restrict to sets of surfaces containing specific subsurfaces. After that, we discuss the results on random surfaces that are already known and we finish this chapter with a short section on alternative models for random surfaces.

2.1. The model

A random surface will be a random gluing of $2N$ triangles (where $N \in \mathbb{N}$) along their sides. In order to make this idea rigorous, we need to find a way to encode the gluing of the triangles. In fact, when we know which side of which triangle is glued to which other side of which other triangle and how each triangle is oriented on the resulting surface then this uniquely defines the gluing. This means that we can define the following probability space:

Definition 2.1. Let $N \in \mathbb{N}$. We define the *probability space of random surfaces built out of* $2N$ *triangles* to be the probability space Ω_N with probability measure $\mathbb{P}_N$, where:

$$\Omega_N = \{\text{Partitions of } \{1, 2, \ldots, 6N\} \text{ into pairs}\}$$

and:

$$\mathbb{P}_N [A] = \frac{|A|}{|\Omega_N|} \text{ for all } A \subset \Omega_N$$

We have:

$$|\Omega_N| = (6N-1)(6N-3)\cdots 3\cdot 1 = \frac{(6N)!}{2^{3N}(3N)!}$$

The surface corresponding to $\omega \in \Omega_N$ is obtained as follows: we label the triangles $1, 2, \ldots 2N$ and the sides $1, 2, \ldots, 6N$ in such a way that the sides 1,2 and 3 correspond to triangle 1, sides 4, 5 and 6 to triangle 2, and so forth. Furthermore, the cyclic order in these labelings defines an orientation on the triangle. Topologically there is a unique way to glue the triangles along their sides as prescribed by ω such that the resulting surface is oriented with orientation corresponding to the orientation on the triangles. This will be the surface $S(\omega)$.

We note that the order in which we pick the pairs of the partition has no influence on the resulting surface. This means that from the point of surfaces, we get an equivalent model for random surfaces by considering the set:

$$\Omega_N^o = \{\text{Ordered partitions of } \{1, 2, \ldots, 6N\} \text{ into pairs}\}$$

With a probability measure $\mathbb{P}_N : 2^{\Omega_N^o} \to [0, 1]$ given by:

$$\mathbb{P}_N [A] = \frac{|A|}{|\Omega_N^o|} \text{ for all } A \subset \Omega_N^o$$

Sometimes it will turn out to make computations easier to use Ω_N^o instead of Ω_N. We note that Ω_N^o contains exactly $(3N)!$ copies of every element in Ω_N. So we obtain:

$$|\Omega_N^o| = \frac{(6N)!}{2^{3N}}$$

2.2. Ribbon graphs

Random surfaces can also be described using cubic ribbon graphs: cubic (also trivalent or 3-regular) graphs equipped with an orientation. Here an orientation is a cyclic order at every vertex of the edges emanating from this vertex.

We will depict the orientation at a vertex with an arrow as in the figure below:

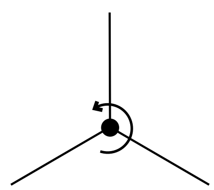


FIGURE 2.1. A vertex of a cubic ribbon graph.

This orientation gives a notion of turning left (opposite to the direction of the orientation) or right (following the direction of the orientation) when traversing a vertex on the graph.

The number of vertices of a cubic ribbon graph $\Gamma = (V, E)$, or in fact of any cubic graph, must be even. This follows from the fact that for cubic graphs we have: $3|V| = 2|E|$. Hence we can assume that our graph has $2N$ vertices for some $N \in \mathbb{N}$.

Cubic ribbon graphs on $2N$ vertices can also be described by partitions of the set $\{1, 2, \ldots, 6N\}$. We label the vertices with the numbers $1, 2, \ldots 2N$ and the half-edges emanating from the vertices $1, 2, \ldots, 6N$, in such a way that the cyclic order of the half-edges at the vertex i corresponds to $(3i-2, 3i-1, 3i)$ for all $1 \leq i \leq 2N$, as in Figure 2.2 below:

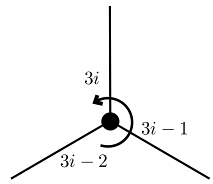


FIGURE 2.2. Vertex i of a labelled cubic ribbon graph.

A partition $\omega \in \Omega_N$ then corresponds to a cubic ribbon graph $\Gamma(\omega)$ by connecting every pair of half-edges that forms a pair in the partition. For a general $\omega \in \Omega_N$ the graph $\Gamma(\omega)$ will be a *multigraph*: it might contain loops or double edges. For most of this text we will not make a distinction between graphs and multigraphs. When we do need to make a distiction, like in Chapter 7, we will speak of *simple* graphs and multigraphs.

It is not difficult to see that for every cubic ribbon graph Γ on $2N$ vertices there exists an $\omega \in \Omega_N$ such that $\Gamma = \Gamma(\omega)$. In fact, because of the labelling, we even obtain many isomorphic copies of every graph. Furhtermore, we note that also the properties of the graph do not depend on the ordering of the pairs. So events depending only on graph theoretic properties, just like those depending only on the properties of the resulting surface, have the same probability in Ω_N^o and Ω_N.

The cubic ribbon graph corresponding to a partition is dual to the triangulation of the surface corresponding to this same partition. That is to say, we can obtain the graph by adding a vertex to every triangle of the triangulation and then adding an edge between two vertices if the corresponding triangles share a side. If we label the graph in the same way as the triangles on the surface, we obtain the cubic ribbon graph corresponding to the partition defining the surface.

The construction above describes an embedding of the random cubic ribbon graph into the random surface corresponding to the same partition. Figure 2.3 shows what the graph looks like on the surface:

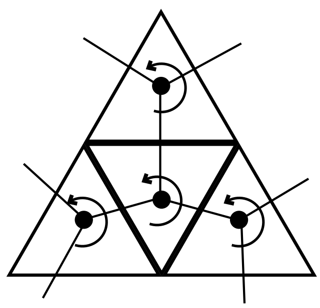


FIGURE 2.3. A part of a triangulation and its dual graph.

We will often think of the graph $\Gamma(\omega)$ a embedded in $S(\omega)$ without mentioning it.

2.3. Permutations

Random surfaces can also be described by random elements of symmetric groups. This is done by associating a permutation $\sigma \in \mathrm{S}_{6N}$ to the vertices of the corresponding random graph and a permutation $\tau \in \mathrm{S}_{6N}$ to the edges.

σ labels the left hand turns at every vertex. So if a vertex has half-edges i_1, i_2 and i_3 emanating from it and the left hand turns at this vertex are of the form (i_1, i_2), (i_2, i_3) and (i_3, i_1) then we add the cycle $(i_1\ i_2\ i_3)$ to σ, as in Figure 2.4 below:

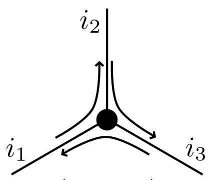


FIGURE 2.4. A 3-cycle corresponding to a vertex. The arrows indicate the left hand turns.

So σ is a product of $2N$ disjoint 3-cycles.

τ records which half-edge is glued to which other half-edge in the graph. If half-edge i_1 is glued to half-edge i_2 in the graph then we add a cycle $(i_1\ i_2)$ to τ as in Figure 2.5 below:

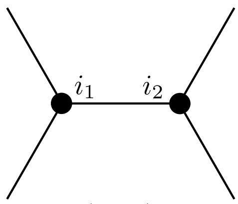


FIGURE 2.5. A 2-cycle corresponding to an edge.

So τ is a product of $3N$ disjoint 2-cycles.

Recall that for $\lambda \models 6N$, the corresponding conjugacy class in S_{6N} is denoted $K(\lambda)$. So we have:

$$\sigma \in K\left(3^{2N}\right),\ \tau \in K\left(2^{3N}\right)$$

Where $1^{i_1}2^{i_2}\ldots(6N)^{i_{6N}}$ denotes the partition of $6N$ with i_1 parts equal to 1, i_2 parts equal to 2, and so forth.

This means that we can identify the set of random surfaces with $K\left(3^{2N}\right) \times K\left(2^{3N}\right)$. Using the counting measure, we can turn this set into a probability space again.

Note that this set is a lot larger than Ω_N, we get many copies of every element of Ω_N, corresponding to different choices of σ. For example, in Ω_N half-edges 1, 2 and 3 always emanate from the same vertex on the corresponding graph. In $K\left(3^{2N}\right) \times K\left(2^{3N}\right)$ there could be a vertex whose half-edges are labeled 1, 2 and 7. We could of course choose to fix $\sigma = (1\ 2\ 3) \ldots (6N-2\ 6N-1\ 6N)$ so that we get Ω_N back.

However, these extra choices again do not influence the topology or geometry (if we relabel τ along with σ), so from the point of view of random surfaces, the two probability measures are the same. Sometimes it is convenient to also randomly pick σ, so we will not always fix it and work with the probability measure on $K\left(3^{2N}\right) \times K\left(2^{3N}\right)$. Because from the point of view of random surfaces it is equivalent, we will denote this measure by $\mathbb{P}_N$ as well. Finally, given a pair (σ, τ), we will sometimes denote the corresponding surface by $S(\sigma, \tau)$ and the corresponding graph by $\Gamma(\sigma, \tau)$.

2.4. The topology of random surfaces

In this section we describe how the topology of a random surface can be read off of the graph and permutation corresponding to the surface. If we suppose that the surface is connected then this comes down to determining the Euler characteristic. It will turn out that the probability that a random surface is connected tends to 1 when $N \to \infty$ (see Theorem 2.8 below), so this is not a big restriction.

We start with the graph. So given the graph we want to recover the numbers of triangles, sides and corners in the triangulation. The number of triangles is the number of vertices of the graph, which is given and equal to $2N$. Likewise, the number of sides can easily be recovered, this is equal to the number of edges and from the earlier mentioned fact that $3\,|V| = 2\,|E|$ we get that this is equal to $3N$. The difficult thing to recover is the number of corners. Around a corner the surface looks like the figure below:

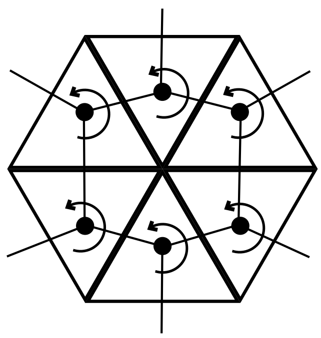


FIGURE 2.6. A part of a triangulation around a corner.

The sides of all the triangles around the corner have to be ordered consistently, because these orders have to correspond to the orientation on the surface. This means that if we walk along the cycle[1] on the graph around this corner on the surface, we turn in the same direction at every vertex of the graph with respect to the orientation on the graph. Whether this direction is a constant 'left' or 'right' depends on the direction in which we traverse the cycle. So we can conclude that corners correspond to left hand turn cycles on the graph. Thus, if we denote this number by $LHT(\omega)$ then we get that the genus of the surface is given by:

$$g(\omega) = 1 + \frac{N}{2} - \frac{LHT(\omega)}{2}$$

In the description using permutations $LHT(\omega)$ is equal to the number of cycles in $\sigma\tau$ for any choice of $(\sigma, \tau) \in K\left(3^{2N}\right) \times K\left(2^{3N}\right)$ corresponding to ω. This is because the permutation $\sigma\tau$ describes what happens to a given half-edge after consecutively traversing one edge and then taking a left hand turn.

We will sometimes restrict to certain sets of surfaces, which we will select based upon their genus. Sometimes we will want these sets to be non-negligible in an appropriate sense. This is what the following definition is for:

DEFINITION 2.2. A sequence of subsets $D_N \subset \mathbb{N}$ for $N \in \mathbb{N}$ will be called *non-negligible with respect to the genus* if:

$$\liminf_{N\to\infty} \mathbb{P}_N \left[g \in D_N\right] > 0$$

Note that formally there is a problem with this definition: the genus is only defined when the surface is connected. However, because asymptotically the set of disconnected surfaces form a probability 0 set, this is not an issue.

Finally we gather some topological facts about closed curves on surfaces. The first fact is that curves on the surface are homotopic to curves on the graph, where with curves on the graph we mean curves on the graph embbeded into the surface in the way described above. This homotopy can be realized as follows: we divide the curve in pieces, such that every piece corresponds to the curve entering and leaving one specific triangle exactly once. We then homotope these entry and exit points to the midpoints of the corresponding sides of the triangle (where the edge of the graph cuts the side). After that we homotope the curve onto the two half-edges connecting the sides to the vertex corresponding to the triangle in every piece of the curve. Figure 2.7 shows an example:

[1] In this text we use the convention that a cycle on a graph is any closed walk. If we want to consider a closed walk that visits every vertex and edge at most once we will speak of a circuit.

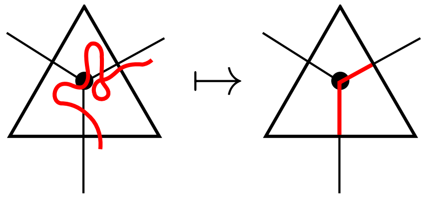


FIGURE 2.7. Homotopy.

This means that we can use results on curves on random graphs for the study of curves on random surfaces. This will be very helpful in the study of the length spectrum of a random surface.

There is one problem: when we homotope a curve on the surface to a curve on the graph, it does not necessarily maintain its properties. For instance, a simple curve on the surface does not always homotope to a simple curve on the graph. However, the following proposition tells us that the homotopic image of a non null homotopic curve does always contain a circuit.

PROPOSITION 2.1. *Let γ be a non null homotopic curve on the surface corresponding to a partition $\omega \in \Omega_N$ and γ' a homotopic image of γ on $\Gamma(\omega)$ then γ' contains a non null homotopic circuit.*

PROOF. We argue by contradiction. Suppose that γ' contains no homotopocially non-trivial circuits. That means that we can contract all circuits and γ' is homotopic to a subtree of $\Gamma(\omega)$. Trees are homotopically trivial, which concludes the proof. □

2.5. The geometry of random surfaces

In order to turn our random surfaces into geometric objects, we need to define metrics on them. In this section we will describe two ways of doing this.

2.5.1. Ideal triangulations. The first way of putting a metric on the surface we consider uses ideal hyperbolic triangles, as in [**BM04**].

Let $\mathbb{H}^2 = \{z \in \mathbb{C};\ \mathrm{Im}(z) > 0\}$ be the upper half plane model of the hyperbolic plane. We will use $2N$ isometric copies of the triangle $T \subset \mathbb{H}^2$, given by the vertices 0,1 and ∞, shown in the picture below:

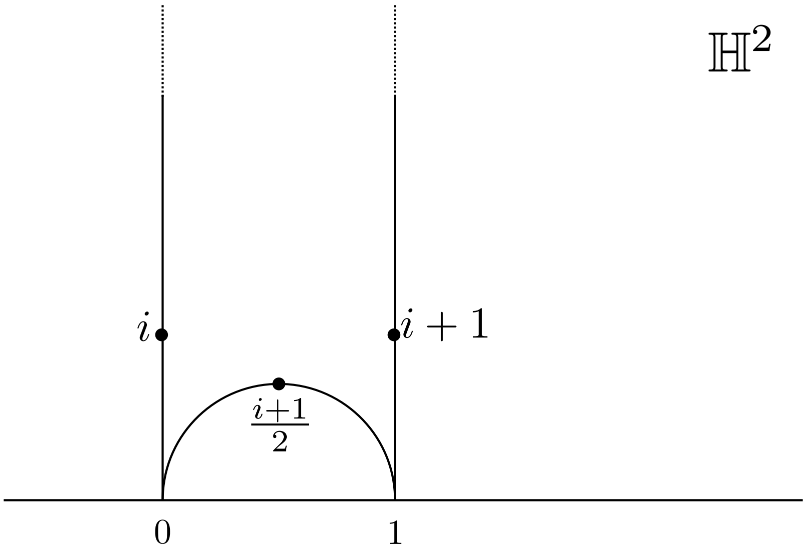


FIGURE 2.8. The triangle T.

There are many isometries between two sides of a pair of ideal triangles. So, to properly define a triangulation gluing we need one extra parameter per pair of sides in the gluing called the *shear* of the gluing. This parameter measures the signed distance between the midpoints of the two sides. Here the midpoint of a side of a triangle is determined by where the orthogonal from the corner opposite this side hits the side (these are the points $i+1$ and $\frac{i+1}{2}$ in the figure above). Figure 2.9 below illustrates a gluing of two ideal hyperbolic triangles with shear in the Poincaré disk model:

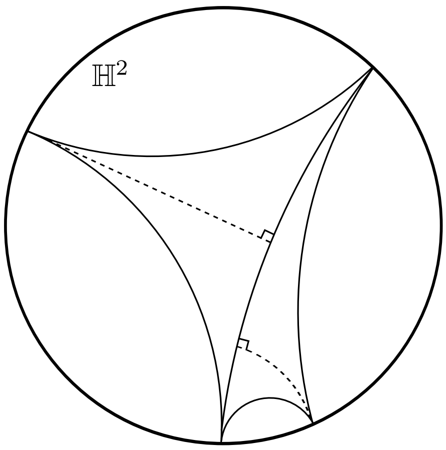


FIGURE 2.9. Shear along a common side of two triangles in the Poincaré disk model of the hyperbolic plane.

The sign of the shear can be defined using the orientation on the surface. In this text all gluings will have shear coordinate 0 at every pair of sides. This means that the triangles will always be glued such that the orthogonals from the two corners opposite a side meet each other, or equivalently midpoints will be glued to midpoints.

If $\omega \in \Omega_N$, the surface obtained by gluing together isometric copies of T will be denoted $S_O(\omega)$. Note that on this surface the corners of the triangles turn into punctures. Furthermore, because T has area π we immediately obtain that for any $\omega \in \Omega_N$ we have:

$$\text{area}\,(S_O(\omega)) = 2N\pi$$

To understand the geometry of curves on random surfaces we will need the following two 2×2 matrices:

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

The set of all words in L and R will be denoted $\{L, R\}^*$. Elements in this set will sometimes be interpreted as matrices and sometimes as strings in two letters. It will be clear from the context which of the two is the case. We need to define and equivalence relation on this set:

DEFINITION 2.3. Two words $w \in \{L, R\}^*$ and $w' \in \{L, R\}^*$ will be called *equivalent* if one of following two conditions holds:
- w' is a cyclic permutation of w
- w' is a cyclic permutation of w^*, where w^* is the word obtained by reading w backwards and replacing every L with an R and vice versa.

If $w \in \{L, R\}^*$, we will use $[w]$ to denote the set of words equivalent to w.

The reason this has anything to do with the geometry of curves on random surfaces is the following. Given an essential closed curve γ on such a surface, it follows from Theorem 1.9 in Chapter 1 that it is homotopic to a unique closed geodesic $\tilde{\gamma}$. If we trace this geodesic and record whether it turns left or right at every triangle it passes (which is well defined by the orientation on the surface) this gives a word $w_{\tilde{\gamma}} \in \{L, R\}^*$. Note however that this word is only defined up to the equivalence defined above.

It follows from Proposition 1.7 that the length of $\tilde{\gamma}$ is given by:

$$\ell(\tilde{\gamma}) = 2\cosh^{-1}\left(\frac{\text{tr}\,(w_{\tilde{\gamma}})}{2}\right)$$

This implies that the number of curves on the punctured surface of a fixed length is given by the number of appearances of all the possible words in L and R with the corresponding trace. This leads us to the following definition:

DEFINITION 2.4. Let $N \in \mathbb{N}$ and $w \in \{L, R\}^*$. Define $Z_{N,[w]} : \Omega_N \to \mathbb{N}$ by:

$$Z_{N,[w]}(\omega) = |\{\gamma;\ \gamma \text{ a circuit on } \Gamma(\omega),\, \gamma \text{ carries } w\}|$$

So if we understand the probability distribution of the random variables $Z_{N,[w]}$ for all $w \in \{L,R\}^*$ we understand the probability distribution of the length spectrum of the punctured random surfaces.

In general, closed geodesics on $S_O(\omega)$ correspond to cycles on $\Gamma(\omega)$. The systole of $S_O(\omega)$ however will always be a circuit. This follows from the following proposition:

PROPOSITION 2.2. *Let $\omega \in \Omega_N$. Suppose γ is a non null homotopic curve on $S_O(\omega)$ and γ' the curve on $\Gamma(\omega)$ corresponding to the geodesic representative of γ. Then γ' contains a homotopically non-trivial circuit γ'' with $\ell(\gamma'') \leq \ell(\gamma') \leq \ell(\gamma)$.*

PROOF. Proposition 2.1 tells us that γ' contains a homotopically non-trivial circuit γ''. We will prove that $\ell_{\mathbb{H}}(\gamma'') \leq \ell_{\mathbb{H}}(\gamma')$.

Because $\gamma'' \subset \gamma'$, the word in L and R on γ' can be obtained by inserting letters into the word on γ''. So, suppose the word in L and R on γ'' is $w = w_1 w_2 \cdots w_k$ and the word on γ' is $v = v_1 \cdots v_l$. Then we have $k \leq l$ and there are $1 \leq i_1 < i_2 < \ldots < i_k \leq l$ such that $w_j = v_{i_j}$ for all $1 \leq j \leq k$.

To prove that $\ell_{\mathbb{H}}(\gamma'') \leq \ell_{\mathbb{H}}(\gamma')$ we will prove that $\mathrm{tr}\,(w) \leq \mathrm{tr}\,(v)$ and to prove this we will prove that if we add letters to a word, the trace of the corresponding matrix increases (the rest will then follow by induction on the number of letters in the word).

So, let $w = w_1 w_2 \cdots w_n$ with $w_i \in \{L,R\}$ for $1 \leq i \leq n$ and $w' = w_1 \cdots w_i x w_{i+1} \cdots w_n$ with $x \in \{L,R\}$ and $1 \leq i \leq n$. We have:

$$\begin{aligned} \mathrm{tr}\,(w) &= \mathrm{tr}\,(w_1 \cdots w_i w_{i+1} \cdots w_n) \\ &= \mathrm{tr}\,(w_{i+1} \cdots w_n w_1 \cdots w_i) \end{aligned}$$

Likewise, we have:

$$\mathrm{tr}\,(w') = \mathrm{tr}\,(w_{i+1} \cdots w_n w_1 \cdots w_i x)$$

Write:

$$w_{i+1} \cdots w_n w_1 \cdots w_i = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Because $w_i \in \{L,R\}$ for $1 \leq i \leq n$ we have $a_{11}, a_{12}, a_{21}, a_{22} \geq 0$. So:

$$\mathrm{tr}\,(w) = a_{11} + a_{22}$$

and:

$$\begin{aligned} \mathrm{tr}\,(w') &= \begin{cases} a_{11} + a_{22} + a_{21} & \text{if } x = L \\ a_{11} + a_{22} + a_{12} & \text{if } x = R \end{cases} \\ &\geq \mathrm{tr}\,(w) \end{aligned}$$

□

2.5.2. Compactifcation. Using the cusped surfaces we obtain, we can also create compact hyperbolic surfaces. This goes through a conformal compactification that essentially consists of adding points in the cusps. That is, if $\omega \in \Omega_N$ then there is a unique closed Riemann surface $S_C(\omega)$ with a set of points $\{p_1, \ldots p_n\} \subset S_C(\omega)$ such that:

$$S_O(\omega) \simeq S_C(\omega) \backslash \{p_1, \ldots p_n\}$$

conformally. It follows from the Uniformization Theorem (Theorem 1.5) that when $g(S_O(\omega)) \geq 2$ we can find a unique complete hyperbolic structure that is conformally equivalent to this given conformal structure. A particularly nice feature of the set of surfaces we obtain like this is the following theorem by Belyĭ [**Bel80**]:

THEOREM 2.3. [**Bel80**] *For all $g \in \mathbb{N}$ with $g \geq 2$ the set:*

$$\left\{ S_C(\omega);\ \omega \in \bigcup_{N=1}^{\infty} \Omega_N \right\} \bigcap \mathcal{M}\left(\Sigma_g\right)$$

is a dense set in $\mathcal{M}\left(\Sigma_g\right)$.

In fact, the theorem as we state it here is a weaker version of Belyĭ's original theorem, which is also not stated in terms of random surfaces.

The problem is that the hyperbolic geometry of $S_C(\omega)$ is difficult to deduce from the combinatorial data (as opposed to the geometry of $S_O(\omega)$). However, a theorem from Brooks [**Bro04**] tells us that the geometries of $S_C(\omega)$ and $S_O(\omega)$ are close if the cusps of $S_C(\omega)$ are 'large enough'. This idea is formalised by the following definition:

DEFINITION 2.5. Let $\omega \in \Omega_N$, $n \in \mathbb{N}$ such that $S_O(\omega)$ has cusps $\{C_i\}_{i=1}^n$ and $L \in (0, \infty)$. Then $S_O(\omega)$ is said to have *cusp length* $\geq L$ if there exists a set of horocycles $\{h_i\}_{i=1}^n \subset S_O(\omega)$ such that:

- h_i is a horocycle around C_i that does not self-intersect for $1 \leq i \leq n$.
- $\ell_{\mathbb{H}}(h_i) \geq L$ for $1 \leq i \leq n$.
- $h_i \cap h_j = \emptyset$ for $1 \leq i \neq j \leq n$.

We also need some notation for disks. If $\omega \in \Omega_N$, $r \in (0, \infty)$ and $p \in S_C(\omega)$ then $B_r(p) \subset S_C(\omega)$ denotes the hyperbolic disk of radius r around p. If C is one of the cusps of $S_O(\omega)$ then $B_r(C) \subset S_O(\omega)$ denotes the neighborhood of C bounded by the horocycle of length r around C.

Now we can state the comparison theorem:

THEOREM 2.4. [**Bro04**] *For every $\varepsilon > 0$ there exist $L \in (0,\infty)$ and $r \in (0,\infty)$ such that: If $\omega \in \Omega_N$ such that:*

- *$S_C(\omega)$ carries a hyperbolic metric $ds^2_{S_C(\omega)}$*
- *$S_O(\omega)$ carries a hyperbolic metric $ds^2_{S_O(\omega)}$*
- *$S_O(\omega)$ has cusps $\{C_i\}_{i=1}^n$ and cusp length $\geq L$.*

Then: outside $\bigcup_{i=1}^n B_L(C_i)$ and $\bigcup_{i=1}^n B_r(p_i)$ we have:

$$\frac{1}{1+\varepsilon} ds^2_{S_O(\omega)} \leq ds^2_{S_C(\omega)} \leq (1+\varepsilon) ds^2_{S_O(\omega)}$$

This theorem implies the following lemma by Brooks:

LEMMA 2.5. [**Bro04**] *For $L \in (0,\infty)$ sufficiently large there is a constant $\delta(L)$ with the following property: Let $\omega \in \Omega_N$ such that $S_O(\omega)$ has cusp length $\geq L$. Then for every geodesic γ in $S_C(\omega)$ there is a geodesic γ' in $S_O(\omega)$ such that the image of γ' is homotopic to γ, and:*

$$\ell(\gamma) \leq \ell(\gamma') \leq (1+\delta(L))\ell(\gamma)$$

Furthermore, $\delta(L) \to 0$ as $L \to \infty$.

2.5.3. Riemannian metrics. The second type of metrics we will study is actually a collection of metrics. The idea is just to assume that we are given a fixed triangle with a metric on it. We will however make some assumptions on this metric. Because we need to apply Gromov's systolic inequality for surfaces at some point, we need the metric on the surface to be Riemannian up to a finite set of points, which means that we need to make some symmetry and smoothness assumptions. One of these models is the model using equilateral Euclidean triangles that was studied in [**GPY11**]. The goal of this section is to explain the type of metrics we are talking about.

Since we will be gluing triangles, we can define all our metrics on the standard 2-simplex given by:

$$\Delta = \left\{ t_1e_1 + t_2e_2 + t_3e_3;\ (t_1,t_2,t_3) \in [0,1]^3,\ t_1+t_2+t_3 = 0 \right\}$$

where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$. Note that this description of the triangle also gives us a natural midpoint of the triangle and natural midpoints of the sides. So, given a random surface made of these triangles, we get a natural embedding of the corresponding cubic ribbon graph.

We will assume that we have a metric $d : \Delta \times \Delta \to [0,\infty)$ that comes from a Riemannian metric g, that we will describe by four smooth functions on $g_{ij} : \Delta \to \mathbb{R}$, $i,j = 1,2$.

We will also assume some symmetry. Basically, we want that permuting the corners of the triangle is an isometry of the sides and that all the derivatives of the metric at the boundary of Δ in directions normal to the boundary vanish. Recall that the symmetric group on k letters is denoted by S_k. We have the following definition:

Definition 2.6. (Δ, g) will be called *symmetric in the sides* if:

$$g_{ij}(te_i + (1-t)e_j) = g_{ij}(te_{\sigma(i)} + (1-t)e_{\sigma(j)})$$

for all $\sigma \in \mathrm{S}_3$ (the symmetric group of order 6), $t \in [0,1]$ and $i, j \in \{1,2\}$ and:

$$\frac{\partial^k}{\partial n^k}\Big|_x g_{ij} = 0$$

for all $k \geq 1$, $i, j \in \{1,2\}$, $x \in \partial\Delta$ and n normal to $\partial\Delta$ at x.

These two symmetries are necessary to turn the 'obvious' gluing maps into isometries. It is obvious that the sides have to be isometric. The fact that we want the metric to be symmetric under reflection in the midpoint of a side comes from the fact that when we glue two triangles along a side, we might glue them together with an opposite orientation on the two sides. The condition on the derivatives guarantees that the metric is not only continuous but also smooth. Finally, note that these conditions do not imply that g has a central symmetry on Δ as a whole (as opposed to the ideal hyperbolic triangles). An example of a metric that satisfies the conditions above is the metric of an equilateral Eucliedean triangle with side lengths 1.

For estimates on the systole later on we need to define a rough minimal and maximal ratio between the length of a curve on the surface and the number of edges of its representant on the graph. To this end we have the following definition:

Definition 2.7. Let $d : \Delta \times \Delta \to [0, \infty)$ be a metric. We define:

$$m_1(d) = \min\left\{ d(s,s');\ \begin{array}{c} s, s' \text{ opposite sides of a gluing of two copies of} \\ (\Delta, d) \text{ along one side} \end{array} \right\}$$

and:

$$m_2(d) = \max\left\{ d\left(\frac{e_i+e_j}{2}, \frac{e_k+e_l}{2}\right);\ i,j,k,l \in \{1,2,3\},\ i \neq j,\ k \neq l \right\}$$

A simple Euclidean geometric argument shows that in the case of a Euclidean triangle we have $m_1(d) = 1$ and $m_2(d) = \frac{1}{2}$.

We also note that in the Riemannian setting the systole of a random surface does not necessarily correspond to a circuit on the graph. We will describe an example of this. Let us consider the graph (with some loose half-edges attached) in the figure below:

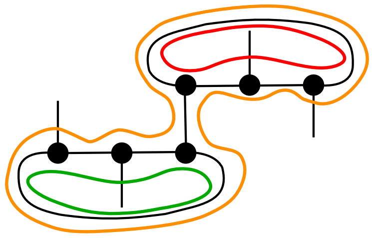


FIGURE 2.10. A graph with three cycles on it.

The surface (with boundary corresponding to the the loose half-edges) coming from to this graph (taking the orientation from the orientation of the plane) is formed by two cylinders that share half of one of each of their boundary components, as in Figure 2.11 below:

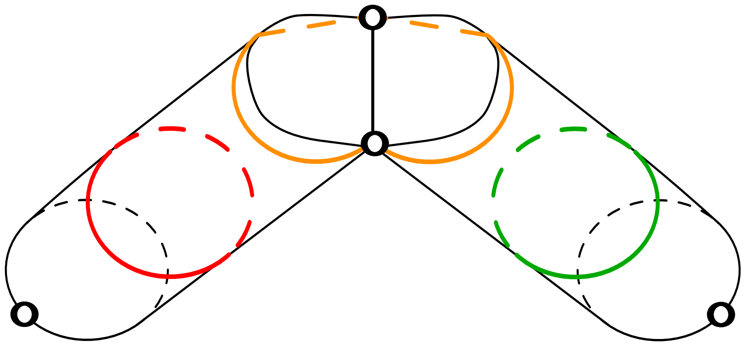


FIGURE 2.11. The surface corresponding to the graph from Figure 2.10.

We will triangulate this surface in such a way that the orange cycle is shorter than the two (red and green) circuits.

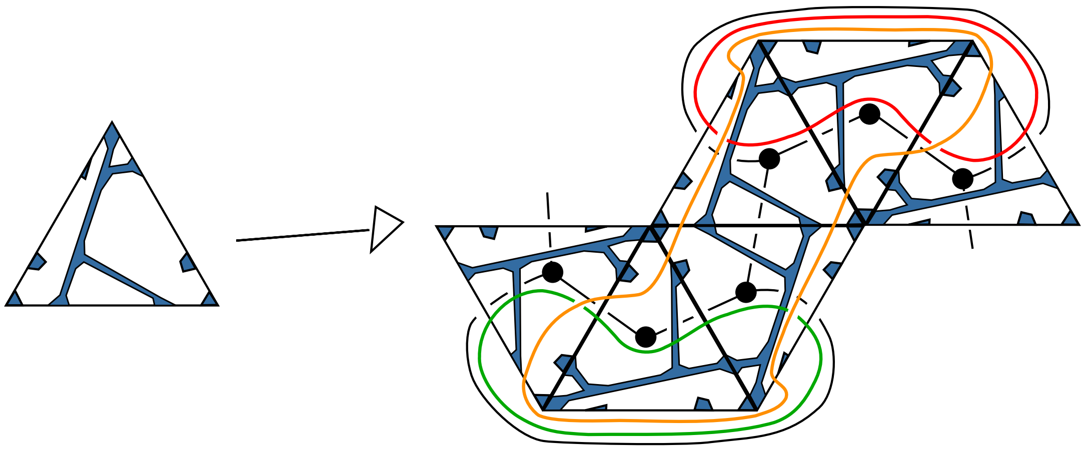


FIGURE 2.12. Triangulating the graph from Figure 2.10.

The Riemannian metric we choose on the triangle is such that the white regions on the triangle are 'cheap' and the dark blue regions 'expensive'. This can be achieved by choosing a Euclidean metric on the whole triangle and multiplying it with a large factor in the blue parts (and then smoothing it). If this factor is large enough then the orange curve, which does not cross any of the blue parts, is shorter than the red curve and the green curve that both do cross the blue parts. Furthermore, the red and green curve cannot be homotoped to curves that do not cross the blue parts. Hence, on this surface with boundary, the shortest closed curve is not homotopic to a circuit. An example of a closed surface that has this property can be obtained by gluing two copies (circled) of the surface above along their boundary as follows:

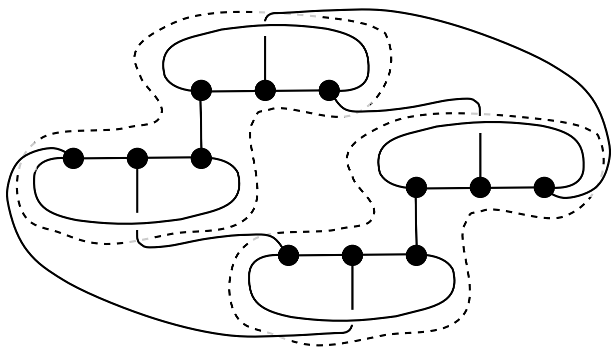


FIGURE 2.13. Two copies of the graph from Figure 2.10 glued together.

This graph represents a surface of genus 3. All the other circuit intersect the blue parts essentially, which means that the orange curve on the subsurface is still shorter than any circuit, even though it is not a circuit itself.

Another feature that this example illustrates is that in the Riemannian setting the central symmetry of the triangle is lost. That means that not only the cyclic orientation at the vertices of the corresponding graph matters, but actually which half-edge gets identified with which half-edge. Luckily, our probability space already encodes this information, so another way of looking at this is that there is less redundancy in the probability space in the Riemannian setting.

2.6. Restricting to surfaces containing a specific subsurface

In order to compute conditional probabilities later on, we will develop a procedure for restricting to random surfaces containing a fixed labelled subsurface. In what follows we will explain how this procedure works.

A 'fixed labelled subsurface' is described by a labelled oriented graph H, which consists of the following data:

- An even number of distinct labels in $\{1,\ldots,6N\}$
- The *edges* of H: a partition of these labels into pairs. Every pair is called an edge.
- The *vertices* of H: a partition of these labels such that each part of this partition is a set of at most 3 elements. Every part is called a vertex.
- The *orientation* of H:
  - A cyclic order of the labelled half-edges at every vertex of degree 3 in H.
  - The direction of the turn (L or R) from one label to the other at every vertex of degree 2.

In short this data describes the edges of H, which of these edges share a vertex and how they are oriented at this vertex.

In terms of permutations, the condition that $H \subset \Gamma(\sigma,\tau)$ means that:

- The edges in H describe a set of pairs that need to appear as 2-cycles in τ
- The vertices describe labels that need to appear as (parts of) 3 cycles in σ.
- The orientation describes in which cyclic order these labels should appear in their 3-cycles in σ.

We treat a simple example. Suppose that we want to restrict to surfaces that contain the oriented labelled subgraph H depicted in Figure 2.14 below, where the orientation on the graph is induced by the orientation of the plane:

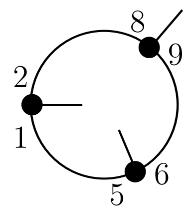


FIGURE 2.14. An example of an oriented labelled subgraph H.

This means that in this case we want to restrict to pairs of permutations (σ,τ) such that:

- σ contains three cycles of the form $(1\ 2\ \star)$, $(5\ \star\ 6)$ and $(8\ \star\ 9)$
- τ contains the cycles $(1\ 5)$, $(2\ 8)$ and $(6\ 9)$.

In general, given an oriented labelled graph H in which every vertex has degree at most 3, we want to be able to compute probabilities of the form:

$$\mathbb{P}_N\left[A\right| H \subset \Gamma]$$

For some $A \subset \Omega_N$.

We are particularly interested in the distribution of the number of left hand turn cycles under the condition that a graph contains a fixed labelled subgraph. In this particular case there is a way to modify the underlying probability space so that we can actually compute such probabilities. In order for this to work, it is however necessary that H contains no left hand turn cycles of itself. So we will assume this from here on.

2.6.1. Modifying the probability space. Because the procedure of modifying the probability space involves a lot of notation, it is good to keep an example in mind. So, let us consider the graph in the figure below (in which we haven't drawn the labels in H itself, the orientation comes from the plane again):

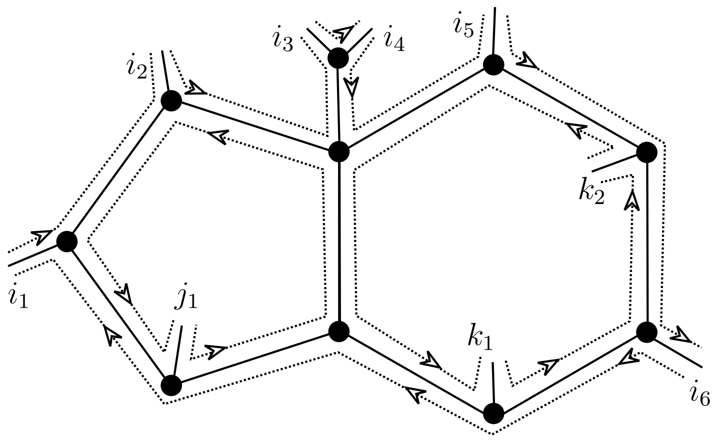


FIGURE 2.15. A subgraph H and its 'remaining' half-edges.

In this figure H should be imagined to be embedded in a larger graph $\Gamma(\sigma, \tau)$. What we have labelled in Figure 2.15 are the half-edges emanating from H. These are the half-edges that connect H to the rest of the graph $\Gamma(\sigma, \tau)$. These labelled half-edges at the vertices of H are *not* part of the subgraph H. So in particular these labels depend on the choice of σ. The orientation of these edges with respect to the other edges in H is however contained in the data that describes H.

The dotted curves indicate what we will call the *left hand turn segments* that go through H and the arrows on these segments indicate the direction in which they are left hand turn segments (as opposed to right hand turn segments). These left hand turn segments will be crucial in the construction afterwards. We make the following two observations about them:

1. All the left hand turn segments can be obtained by a walk of the following form:
   - Travel into H over one of the emanating half-edges and turn left.
   - Traverse edges and make left hand turns until another half-edge emanating from H is reached.

   The trajectory of this walk is a left hand turn segment.
2. The left hand turn segments fall into cycles. In Figure 2.15 these are $(i_1\ i_2\ i_3\ i_4\ i_5\ i_6)$, (j_1) and $(k_1\ k_2)$.

Suppose the number of vertices of H is k. The probability space we want to understand is the space of all possible (labelled) gluings of the half-edges emanating from H and $2N-k$ tripods. As a set this is:

$$\left\{(\sigma,\tau)\in K\left(3^{2N}\right)\times K\left(2^{3N}\right);\ H\subset\Gamma(\sigma,\tau)\right\}$$

We will now explain a procedure of creating a different probability space in which the number of left hand turn cycles has the same distribution as in the space we want to understand. This procedure relies on the following observation:

OBSERVATION. We want to count the number of left hand turn paths in any graph glued out of H and some fixed number of other labelled tripods. To be able to do this, all we need to know is how the left hand turn segments in H are glued to the left hand turns of these tripods. So, if all we are interested in is the number of left hand turn cycles, we can forget about the internal structure of H. Note that it is crucial here that H contains no left hand turn cycles of its own.

There is also a more topological way to see this. Consider the figure below:

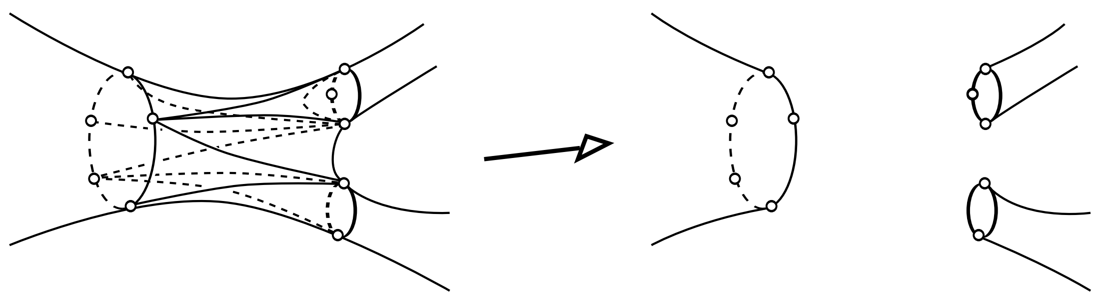


FIGURE 2.16. Replacing a subsurface.

What the figure shows is that if we cut out a trianglulated subsurface and fill the holes that this leaves by polygons we obtain a new surface 'triangulated' by triangles and these polygons. The number of vertices in the triangulation of this new surface the same number of vertices as the triangulated surface we started with. This relies on the subsurface having no interior vertices, or equivalently the dual graph to the triangulation in this subsurface having no left hand turn paths. So, if we want to count the number of vertices of the original surface, we can just as well count the number of vertices on the modified surface (regardless of the fact that the topology of the modified surface is different). This means that we need to count the number of cycles formed by the left hand turns in the inserted polygons and those in the remaining triangles. The left hand turns in these polygons correspond one to one to the left hand turn segments in the dual graph to the triangulation.

We will formalize this in Lemma 2.6 below.

2.6.2. Formal description. We will describe the probability space of random surfaces containing H by pairs of permutations $(\tilde{\sigma}, \tilde{\tau})$ where:

- $\tilde{\sigma}$ records the configuration of the left hand turn segments in H and the left hand turns in the remaining tripods.
- $\tilde{\tau}$ records how H and these remaining tripods are glued together.

Some information about the pair (σ, τ) will be lost in this description. We will however still be able to recover the number of left hand turn cycles in $\Gamma(\sigma, \tau)$ (or equivalently the number of cycles in $\sigma\tau$) from $(\tilde{\sigma}, \tilde{\tau})$.

Concretely we define a map:

$$F_{H,N} : \left\{(\sigma, \tau) \in K\left(3^{2N}\right) \times K\left(2^{3N}\right);\ H \subset \Gamma(\sigma, \tau)\right\} \to \mathrm{S}_{6N-2k} \times \mathrm{S}_{6N-2k}$$

by:

$$(\sigma, \tau) \mapsto (\tilde{\sigma}, \tilde{\tau})$$

where $\tilde{\tau}$ is obtained by simply forgetting all the pairs of half-edges in H and $\tilde{\sigma}$ is obtained by the following procedure:

1. Repeat the following two steps until all half-edges outside H have been visited:
   a. Pick a labelled half-edge l_0 outside of H and we turn left.
      - If the resulting label is not in H, we record it.
      - If this label does appear in H then we traverse an edge and turn left, which we keep repeating until we reach a label outside H, which we then record.

      We repeat this until we reach l_0 again.

   b. We turn the labels we have recorded into a cycle (in the order in which we have recorded them) and record this cycle.
2. The product of the cycles we have recorded is $\tilde{\sigma}$.

Let us apply this procedure to our example. So suppose we are given a pair $(\sigma, \tau) \in K\left(3^{2N}\right) \times K\left(2^{3N}\right)$ such that $H \subset \Gamma(\sigma, \tau)$. We will focus on $\tilde{\sigma}$. First of all note that every vertex outside H just contributes the standard 3-cycle associated to that vertex to $\tilde{\sigma}$. From the half-edges emanating from H we obtain the cycles:

$$(i_1\ i_2\ i_3\ i_4\ i_5\ i_6)(j_1)(k_1\ k_2)$$

$\tilde{\sigma}$ will be the concatenation of the cycles above and the aforementioned 3-cycles.

Note that technically the image of $F_{H,N}$ lies in $\mathrm{S}_{\{1,\ldots,6N\}\setminus\{\text{the labels in } H\}}$. However, because this group is isomorphic to S_{6N-2k} and we are not interested in the labelling in the end, we will continue to write S_{6N-2k}.

We now have the following lemma:

Lemma 2.6. *Let $N \in \mathbb{N}$ and H an oriented labelled graph of k vertices, that all have degree at most* 3*, such that H contains no left hand turn cycles. Then:*

(a) There is a conjugacy class $K(H,N) \subset \mathrm{S}_{6N-2k}$ such that:

$$F_{H,N}\left(\left\{(\sigma,\tau) \in K\left(3^{2N}\right) \times K\left(2^{3N}\right);\ H \subset \Gamma(\sigma,\tau)\right\}\right) \subset K(H,N) \times K(2^{3N-k})$$

(b) The map

$$F_{H,N} : \left\{(\sigma,\tau) \in K\left(3^{2N}\right) \times K\left(2^{3N}\right);\ H \subset \Gamma(\sigma,\tau)\right\} \to K(H,N) \times K(2^{3N-k})$$

is surjective and $\left|F_{H,N}^{-1}(\tilde{\sigma},\tilde{\tau})\right|$ depends only on H and N (and not on $(\tilde{\sigma},\tilde{\tau}) \in K(H,N) \times K(2^{3N-k})$).

(c) The number of cycles in $\tilde{\sigma}\tilde{\tau}$ is equal to the number of cycles in $\sigma\tau$.

Proof. For (a) it is clear that the conjugacy class of $\tilde{\tau}$ is constant for a fixed H. For the conjugacy class of $\tilde{\sigma}$ we have already noted that every vertex outside of H contributes a 3-cycle to $\tilde{\sigma}$, this is not particular to the example. Furthermore, the other cycles in $\tilde{\sigma}$ come from the internal structure of H (they are cycles of left hand turn segments), which is fixed.

For (b) surjectivity will follow from 'reconstructing' (σ,τ) from its image $(\tilde{\sigma},\tilde{\tau})$: given $(\tilde{\sigma},\tilde{\tau}) \in K(H,N) \times K\left(2^{3N-k}\right)$ we will construct a pair $(\sigma,\tau) \in K\left(3^{2N}\right) \times K\left(2^{3N}\right)$ such that $F_{H,N}(\sigma,\tau) = (\tilde{\sigma},\tilde{\tau})$.

Reconstructing τ is easy, if we want that $F_{H,N}(\sigma,\tau) = (\tilde{\sigma},\tilde{\tau})$ then we must concatenate $\tilde{\tau}$ with the pairs of half-edges in H. Because this is the only possibility, the map $F_{H,N}$ is in fact injective on the τ-coordinate.

For the reconstruction of σ we do have some choices to make. We can reconstruct σ as follows:

1.(a) All of the vertices of H already have some (1, 2 or 3) labels assigned to them, coming from the data of H. We start the construction of σ by labelling the remaining half-edges emanating from H. This goes as follows. We have already seen that these emanating half-edges are grouped in cycles of left hand turn segments. If we want to find a σ such that $F_{H,N}(\sigma,\tau) = (\tilde{\sigma},\tilde{\tau})$ then each of these cycles of left hand turn segments needs to contribute one cycle to $\tilde{\sigma}$. So, to construct σ, we assign a distinct cycle c from $\tilde{\sigma}$ (of the right length) to each of these cycles of left hand turn segments s in H. Because of the orientation of H, the cyclic order of the left hand turn segments in s needs to correspond to the cyclic order of the labels in c (again, if we want that $F_{H,N}(\sigma,\tau) = (\tilde{\sigma},\tilde{\tau})$). We choose any assignment of labels to emanating half-edges that satisfies this requirement.

1.(b) H and its emanating half-edges are now entirely labelled. In particular around each vertex of H we now have 3 labels. Because H is oriented, these labels have a natural cyclic order. This is the 3-cycle we record.

2. After step 1 the cycles in $\tilde{\sigma}$ that have not yet been used are exactly $2N-k$ cycles of length 3. The concatenation of these and the 3-cycles from step 1 gives us a σ such that $F_{H,N}(\sigma,\tau)=(\tilde{\sigma},\tilde{\tau})$.

All the possible choices in steps 1.(a) and 1.(b) above give us distinct pairs (σ,τ). Furthermore, we see every pre image of $(\tilde{\sigma},\tilde{\tau})$. This means that the number of elements in $F_{H,N}^{-1}(\tilde{\sigma},\tilde{\tau})$ is equal to the number of choices in step 1. This depends only on H and N, because what we have to choose is:

- Which cycle of $\tilde{\sigma}$ is assigned to which cycle of left hand turn segments in H. If H contains multiple cycles of left hand turn segments of the same length then this generates multiple possibilities and if H contains 3-cycles of left hand turn segments this generates multiple possibilities as well (because in $\tilde{\sigma}$ we cannot see which of the 3-cycles come from cycles of left hand turn segments in H and which come from tripods).
- Once we have chosen which cycle of $\tilde{\sigma}$ is associated with which cycle of left hand turn segments in H, the orientation of H determines the cyclic order in which these cycles need to be assigned. It does however not dictate more than that, so all cyclic permutations of these assignments are allowed.

After these choices have been made, the pre image of $(\tilde{\sigma},\tilde{\tau})$ is determined.

Finally, (c) essentially follows from our obeservation above. $\tilde{\sigma}\tilde{\tau}$ describes what happens to a label after consecutively traversing of an edge and then either turning left or traversing a left hand turn segment in H. A sequence of such moves closes up if and only if a full left hand turn cycle on $\Gamma(\sigma,\tau)$ is completed. This means that the number of cycles in $\tilde{\sigma}\tilde{\tau}$ is equal to the number of left hand turn cycles in $\Gamma(\sigma,\tau)$ and hence equal to the number of cycles in $\sigma\tau$. □

Note that this lemma says that some information might be lost when we apply $F_{H,N}$, it is only surjective, not injective. For instance, the cycles in $\tilde{\sigma}\tilde{\tau}$ that contain left hand turn segments in H are shorter than the corresponding cycles in $\sigma\tau$. However, from point (c) we get that the topology of the original surface can still be recovered.

2.6.3. The consequence of Lemma 2.6. The main consequence of Lemma 2.6 is the following:

Proposition 2.7. *Let H be an oriented labelled graph of k vertices, that have degree at most 3, that contains no left hand turn cycles. Then for any $m\in\mathbb{N}$:*

$$\mathbb{P}_N\left[\mathrm{LHT}=m\,|\,H\subset\Gamma\right]=\frac{\left|\left\{(\tilde{\sigma},\tilde{\tau})\in K(H,N)\times K\left(2^{3N-k}\right);\ \tilde{\sigma}\tilde{\tau}\ \textit{has } m \textit{ cycles}\right\}\right|}{\left|K(H,N)\times K\left(2^{3N-k}\right)\right|}$$

Proof. To lighten the notation we will write:

$$X_N=K\left(3^{2N}\right)\times K\left(2^{3N}\right)\ \text{ and }\ Y_N=K(H,N)\times K\left(2^{3N-k}\right)$$

Furthermore, if $y = (\tilde{\sigma}, \tilde{\tau}) \in Y_N$ then we will write $\mathrm{LHT}(y)$ for the number of cycles in $\tilde{\sigma}\tilde{\tau}$.

The proof of the proposition will consist of applying the properties of $F_{H,N} : X_N \to Y_N$. We will write $C = \left|F_{H,N}^{-1}(y)\right|$ for any (and by Lemma 2.6 (b) all) $y \in Y_N$. We have:

$$\mathbb{P}_N\left[\mathrm{LHT} = m \middle| H \subset \Gamma\right] = \frac{\left|\{x \in X_N;\ \mathrm{LHT}(x) = m,\ H \subset \Gamma(x)\}\right|}{\left|\{x \in X_N;\ H \subset \Gamma(x)\}\right|}$$

Using Lemma 2.6 (b) we get that:

$$\left|\{x \in X_N;\ H \subset \Gamma(x)\}\right| = C\left|Y_N\right|$$

Furthermore, using Lemma 2.6 (c) and then (b) we obtain:

$$\begin{aligned}\left|\{x \in X_N;\ \mathrm{LHT}(x) = m,\ H \subset \Gamma(x)\}\right| &= \left|\{x \in X_N;\ \mathrm{LHT}(F_{H,N}(x)) = m,\ H \subset \Gamma(x)\}\right| \\ &= C\left|\{y \in Y_N;\ \mathrm{LHT}(y) = m\}\right|\end{aligned}$$

Filling these in we obtain the proposition □

2.6.4. Restricting to graphs carrying given words in $\{L, R\}$. Proposition 2.7 can be used to restrict to graphs carrying a given set of equivalence classes of words in $\{L, R\}$ (unequal to $[L^k]$ for any k) as circuits. We recall that a circuit to us is a cycle that goes through each of its vertices and edges once. We shall now apply the machinery from above to this situation.

So, in this situation graph H will be a disconnected union of labelled oriented circuits. The orientation of these circuits is such that one circuit corresponds to one equivalence class of words in L and R. Let us introduce some notation: W will denote the set of equivalence classes of words represented by H. And for each $w \in W$ we will denote the number of circuits in H corresponding to w by m_w. This means that the number of vertices of H is equal to:

$$\sum_{w \in W} m_w\left|w\right|$$

where $|w|$ denotes the number of letters in the word w.

Every emanating half-edge contributes one left hand turn segment to the total number of such segments in H. Because all the vertices in a circuit have degree 2, each circuit in H has as many emanating half-edges as vertices. The number of vertices is in turn equal to the number of letters of the corresponding word. So, a circuit corresponding to a word $w \in W$ contributes $|w|$ left hand turn segments.

These left hand turn segments form two cycles at every circuit. This can be seen as follows: draw a circuit in the plane such that all emanating half-edges that lie to the right of the circuit point inwards and all the emanating half-edges that lie to the left of the circuit point outwards. If one now draws the left hand turn segments in this circuit, one observes that the segments inside the circuit form a cycle and those outside the circuit form a cycle. Also note that it follows from this reasoning that the length of the inside cycle of left hand turn segments is equal to the number of L's in w and the length of the outside cycle is equal to the number of R's in w. We will denote these numbers by l_w and r_w respectively. We have:

$$l_w + r_w = |w|$$

We now apply $F_{H,N}$ to any (σ, τ) containing H (and hence carrying these words in a fixed labelled way). Because each circuit (or class of words w) contributes two cycles of lengths l_w and r_w respectively to $\tilde{\sigma}$, we obtain that:

$$K(H,N) = K\left(3^{2N-\sum\limits_{w\in W} m_w|w|} \cdot \prod_{w\in W} l_w^{m_w} \cdot \prod_{w\in W} r_w^{m_w}\right)$$

In other words, in this case the image of $F_{H,N}$ is:

$$K\left(3^{2N-\sum\limits_{w\in W} m_w|w|} \cdot \prod_{w\in W} l_w^{m_w} \cdot \prod_{w\in W} r_w^{m_w}\right) \times K\left(2^{3N-\sum\limits_{w\in W} m_w|w|}\right)$$

Because both conjugacy classes above are determined by the pair (W, m) we shall sometimes denote them by $K_3(W, m)$ and $K_2(W, m)$ respectively. To further shorten notation, we will write:

$$M = M(W, m) = \sum_{w\in W} m_w\,|w|$$

So, both $\tilde{\sigma}$ and $\tilde{\tau}$ are elements in S_{6N-2M}.

2.7. Results

In this section we will summarize the results that were already known about random surfaces. Most of them we will not prove in this text.

2.7.1. Topology. The first question that needs to be answered is what random surfaces look like topologically. The following theorem was orignially stated as a theorem about random regular graphs. It was proved independently by Wormald and Bollobás and settles the problem of connectivity.

THEOREM 2.8. [**Wor81.1**] [**Bol85**] *We have:*

$$\mathbb{P}_N\left[S \textit{ is connected }\right] \to 1 \textit{ as } N \to \infty$$

The theorem as we stated is actually a consequence of the original theorem of Wormald and Bollobás. Since their results, the theorem has also been proved in more general settings, see [**Wor99**] for more information.

One of the consequences of the theorem above is that understanding the distribution of the genus of a random surface comes down to understanding the distribution of the number of left hand turn cycles in the dual graph. Many authors have worked on the study of this distribution. Estimates for the asymptotics of the expected value of the genus were obtained by Gamburd and Makover [**GM02**], Brooks and Makover [**BM04**], Pippenger and Schleich [**PS06**] and Dunfield and Thurston [**DT06**]. By now the full asymptotic distribution of the number of left hand turn cycles is known by a result of Gamburd:

THEOREM 2.9. [**Gam06**] *Let $\lambda_N(\sigma,\tau) = (\lambda_1,\ldots,\lambda_k)$ denote the partition describing the cycle type of $\sigma\tau$ for $(\sigma,\tau) \in K\left(3^{2N}\right) \times K\left(2^{3N}\right)$. Define $\tilde{\lambda}_N(\sigma,\tau) = (\tilde{\lambda}_1,\ldots,\tilde{\lambda}_k)$ by:*

$$\tilde{\lambda}_i = \lambda_i/N$$

Then $\tilde{\lambda}_N$ converges in distribution to a Poisson-Dirichlet distribution as $N \to \infty$.

The proof of this Theorem relies on proving that asymptotically the element $\sigma\tau$ behaves like a uniformly chosen element in the alternating group. This is done using the Diaconis-Shahshahani upper bound lemma (Lemma 1.25). We will use this method ourselves later on in Chapter 4. As such, our Theorem 4.6 can be used to derive the theorem above as well. Gamburd derives the following corollary from this:

COROLLARY 2.10. [**Gam06**] *We have:*

$$\mathbb{E}_N\left[g\right] \sim \frac{N}{2} \text{ and } g \overset{d}{\sim} 1 + \frac{N}{2} + \mathcal{N}\left(\log(2N), \sqrt{\log(2N)}\right)$$

as $N \to \infty$. Furthermore, let $\mathcal{L} : \Omega_N \to \mathbb{N}$ denote the length of the largest left hand turn cycle. We have:

$$\lim_{N\to\infty} \mathbb{E}_N\left[\frac{\mathcal{L}}{2N}\right] = \int_0^\infty \exp\left(-x - \int_x^\infty \frac{\exp(y)}{y} dy\right) dx \approx 0.6243$$

Because $g(\omega) \leq \frac{N+1}{2}$ for any $\omega \in \Omega_N$, this essentially tells us that on average random surfaces are surfaces of genus close to the maximal possible genus.

2.7.2. Geometry. About the geometry of random surfaces also many results are known. The first one, which is actually a list of results, is the following by Brooks and Makover about the hyperbolic setting:

THEOREM 2.11. [**BM04**] *In the hyperbolic setting we have:*

(a) For every $L > 0$:

$$\mathbb{P}_N\left[S_O \text{ has cusp length } \geq L\right] \to 1$$

as $N \to \infty$.

(b) There exists a constant C_1 such that the first non-trivial eigenvalue of the Laplacian λ_1 satisfies:

$$\mathbb{P}_N\left[\lambda_1(S_C) > C_1\right] \to 1$$

as $N \to \infty$.

(c) There exists a constant C_2 such that the Cheeger constant h satisfies:

$$\mathbb{P}_N\left[h(S_C) > C_2\right] \to 1$$

as $N \to \infty$.

(d) There exists a constant C_3 such that the systole satisfies:

$$\mathbb{P}_N\left[\mathrm{sys}(S_C) > C_3\right] \to 1$$

as $N \to \infty$.

(e) There exists a constant C_4 such that the diameter satisfies:

$$\mathbb{P}_N\left[\mathrm{diam}(S_C) < C_4 \log\left(g(S_C)\right)\right] \to 1$$

as $N \to \infty$.

Part (a) follows from analyzing the distribution of left hand turn cycles on random graphs. This, together with Theorem 2.4 and Lemma 2.5 implies that the geometry of the punctured and compactified surfaces are 'close' with asymptotic probability 1. This means that one can transfer results about the geometry of the punctured surfaces to the closed surfaces. Hence, parts (b)-(e) are proved by analyzing the geometry of the punctured surfaces, which in turn comes down to studying the combinatorics and geometry of the dual graph. In this text we will see sharper versions of parts (a) and (d).

Using an argument from [**BM04**], one can also conclude the following from Corollary 2.10:

COROLLARY 2.12. [**Gam06**] *Let $\mathcal{E} : \Omega \to \mathbb{R}$ denote area of the largest embedded ball in S_C. We have:*

$$\mathbb{E}_N\left[\frac{\mathcal{E}}{\mathrm{area}(S_C)}\right] \geq \frac{1}{2\pi}\int_0^\infty \exp\left(-x - \int_x^\infty \frac{\exp(y)}{y}dy\right)dx - \varepsilon(N) \approx 0.0994 - \varepsilon(N)$$

where $\varepsilon(N) \to 0$ as $N \to \infty$.

Finally, Guth, Parlier and Young obtained the following result about pants decompositions of random surfaces built out of equilateral Euclidean triangles:

Theorem 2.13. [**GPY11**] *For random surfaces built out of equilateral Euclidean triangles we have:*

$$\mathbb{P}_N\left[S \text{ has a pants decomposition of total length } \leq N^{7/6-\varepsilon}\right] \to 0 \text{ as } N \to \infty$$

2.8. Other types of random surfaces

The combinatorial random surfaces from the previous sections form the main subject of study in this text. There are however also many interesting results on other types of random surfaces. For some context, we will gather a (very small) selection of these results in this section. We will discuss random surfaces coming from the Weil-Petersson metric on moduli space and random surfaces coming from random mapping classes.

2.8.1. Weil-Petersson random surfaces.

Because moduli space has finite volume in the Weil-Petersson metric, we can define a probability measure on it by setting:

$$\mathbb{P}_{WP}[A] = \frac{\mathrm{vol}_{WP}(A)}{\mathrm{vol}_{WP}(\mathcal{M}_g)} \text{ for all measurable } A \subset \mathcal{M}_g$$

where $\mathrm{vol}_{WP}(A)$ denotes the Weil-Petersson volume of $A \subset \mathcal{M}_g$.

Using her work on Weil-Petersson volumes of moduli spaces, Mirzakhani proved the following:

Theorem 2.14. [**Mir13**] *There exists a constant $C \in (0,\infty)$ such that for all $\varepsilon > 0$ small enough and all g:*

$$\frac{\varepsilon^2}{C} \leq \mathbb{P}_{WP}[S \in \mathcal{M}_g \text{ has systole} < \varepsilon] \leq C\varepsilon^2$$

Furthermore, it turns out that in this setting short curves tend not to be separating:

Theorem 2.15. [**Mir13**] *There exists constant $D \in (0,\infty)$ such that:*

$$\frac{\log(g)}{D} \leq \mathbb{E}_{WP}[\text{The shortest separating curve on } S \in \mathcal{M}_g] \leq D\log(g)$$

for all g large enough.

We shall see an analogue of this theorem for graphs in the next chapter.

Furthermore, Guth, Parlier and Young also proved the Weil-Petersson analogue of their theorem:

Theorem 2.16. [**GPY11**] *For every $\varepsilon > 0$:*

$$\mathbb{P}_{WP}\left[S \in \mathcal{M}_g \text{ has a pants decomposition of total length } \leq g^{7/6-\varepsilon}\right] \to 0 \text{ as } g \to \infty$$

2.8.2. Random mapping classes. Another option is to consider random deformations of a surface. This can be done by choosing a random element in the corresponding mapping class group. The mapping class group $\mathrm{Mod}(\Sigma_g)$ is a finitely generated group (it can for example be generated by a finite number of Dehn twists). This means that we can define a random walk on it by starting at the identity element and then composing with a uniformly randomly chosen generator for every step. This procedure gives rise to a stochastic process $\{\varphi_L\}_{L=1}^{\infty}$ with values in $\mathrm{Mod}(\Sigma_g)$.

Before we look at what happens when we let φ_L act on Teichmüller space, we state an example of a result on the element $\varphi_L \in \mathrm{Mod}(\Sigma_g)$ itself. Namely, the following theorem by Maher which says that generic mapping classes are what is called *pseudo-Anosov*: they and their powers do not fix any finite set of curves on Σ_g:

THEOREM 2.17. [**Mah11**] *We have:*

$$\mathbb{P}_N\left[\varphi_L \in \mathrm{Mod}(\Sigma_g) \text{ is pseudo-Anosov}\right] \to 1 \text{ as } L \to \infty$$

The result of Maher in fact holds for more general random walks on $\mathrm{Mod}(\Sigma_g)$.

Because the mapping class group acts on Teichmüller space, we obtain a notion of a random surface from these random walks. The first question to ask is: how are these random surfaces distributed in Teichmüller space? Of course this depends on the choice of a metric on Teichmüller space. If we for instance choose the Weil-Petersson metric, then it follows from a more general result by Karlsson and Margulis [**KM99**] that random walks approximate geodesics (see also [**Tio14**]). That is, for almost every sample path there exists a Weil-Petersson geodesic such that the sample path tracks that geodesic sublinearly.

The analogous result in the case of the Teichmüller metric $d_T : \mathcal{T}_g \times \mathcal{T}_g \to [0,\infty)$, which is a metric that measures quasi-conformal distortion between surfaces, is due to Duchin in the thick part of Teichmüller space and generalized to the entire Teichmüller space by Tiozzo:

THEOREM 2.18. [**Duc05**] [**Tio14**] *Let $X \in \mathcal{T}_g$. There exists a constant $A \in (0,\infty)$ such that for almost every sample path $\{\varphi_L X\}_{L=1}^{\infty}$ there exists a Teichmüller geodesic ray*
$\gamma : [0,\infty) \to \mathcal{T}_g$ with $\gamma(0) = X$ such that:

$$\lim_{L\to\infty} \frac{d_T(\phi_L X, \gamma(AL))}{L} = 0$$

Besides random surfaces, these random mapping classes also give rise to a notion of random 3-manifolds. Given $\varphi_L \in \mathrm{Mod}(\Sigma_g)$, the associated 3-manifold is given by identifying the boundaries of two genus g handlebodies using φ_L. For more information on these manifolds we refer the reader to [**DT06**].

CHAPTER 3

Random graphs

The model for random cubic graphs coming from random surfaces actually coincides with a well studied model for random cubic graphs. In this chapter we will summarize some known results and also prove some new results on random cubic graphs. There are more general versions of these results for k-regular graphs, but because we only need cubic graphs here, we shall restrict to the cubic case. The main result we prove in this chapter is Theorem 3.18 about short separating circuits.

3.1. Counting cubic graphs

The first result we will need does not directly sound like a result on random graphs, but it follows from counting the number of labeled cubic graphs, which was done by Bender and Canfield in [**BC78**] and then considering the probability that such a graph carries a non-trivial automorphism, which was done independently by Bollobás in [**Bol82**] and McKay and Wormald in [**MW82**]. A *simple* graph is a graph without loops and without multiple edges.

THEOREM 3.1. [**BC78**], [**Bol82**], [**MW82**] *Let I_N^* denote the number of isomorphism classes of simple cubic graphs on $2N$ vertices. Then:*

$$I_N^* \sim \frac{1}{e^2\sqrt{2\pi N}} \left(\frac{3N}{2e}\right)^N$$

for $N \to \infty$.

We will also use the result on automorphisms, so we state it separately. $\mathrm{Aut}(\Gamma)$ will denote the automorphism group of the graph Γ.

THEOREM 3.2. [**Bol82**], [**MW82**] *We have:*

$$\lim_{N\to\infty} \mathbb{P}_N\left[\mathrm{Aut} \neq \{e\}\right] \ = \ 0$$

The next classical result we will need describes the distribution of the number of circuits (recall that by a circuit we mean a cycle that traverses every edge at most once) of k edges (or k-circuits). Note that a 1-circuit is a loop and a 2-circuit is a multiple edge. We begin by defining the corresponding set of random variables. We let:

$$X_{N,k} : \Omega_N \to \mathbb{N}$$

denote the random variable that counts the number of k-circuits for all $k \in \mathbb{N}$.

We will use the following theorem by Bollobás:

THEOREM 3.3. [**Bol80**] *Let $m \in \mathbb{N}$. Then:*

$$X_{N,i} \to X_i \text{ in distribution for } N \to \infty \text{ for all } i = 1, \ldots, m$$

where:

- *X_i is a Poisson distributed random variable with mean $\lambda_i = 2^i/2i$.*
- *The random variables $X_1, \ldots, X_m$ are mutually independent.*

In the proof of the theorem above one uses the fact that fixed graphs that are not circuits appear with probability tending to 0. We will need this fact and state it as a theorem:

THEOREM 3.4. [**Bol80**] *Let H be a graph that is not a circuit or a tree. Then:*

$$\mathbb{E}_N \left[\text{number of copies of } H \text{ in } \Gamma\right] = \mathcal{O}\left(N^{-1}\right)$$

for $N \to \infty$.

Finally, most graphs we will consider will actually be multigraphs. That means that we need to know the cardinality of the set of multigraphs with a fixed number of vertices as well. We have the following theorem by Wormald, of which we provide a new proof:

THEOREM 3.5. [**Wor81.2**] *Let I_N denote the number of isomorphism classes of cubic multigraphs on $2N$ vertices. Then:*

$$I_N \sim \frac{e^2}{\sqrt{2\pi N}} \left(\frac{3N}{2e}\right)^N$$

for $N \to \infty$.

PROOF. To prove this, we define the following two sets:

$$\mathcal{G}_N = \{\text{Cubic multigraphs with vertex set } \{1, \ldots, 2N\}\}$$

$$\mathcal{U}_N = \{\text{Isomorphism classes of cubic multigraphs on } 2N \text{ vertices}\}$$

We have $I_N = |\mathcal{U}_N|$. We will use the two natural forgetful maps:

$$\Omega_N \xrightarrow{\pi_1} \mathcal{G}_N \xrightarrow{\pi_2} \mathcal{U}_N$$

If $\Gamma \in \mathcal{U}_N$ and $G \in \mathcal{G}_N$ have k 1-circuits and l 2-circuits, then we have:

$$\left|\pi_1^{-1}(G)\right| = \frac{6^{2N}}{2^k 2^l} \text{ and } \left|\pi_2^{-1}(\Gamma)\right| = \frac{(2N)!}{|\mathrm{Aut}(\Gamma)|}$$

We know $|\Omega_N|$. We will first count $|\mathcal{G}_N|$ using π_1 and after that we will use π_2 to count I_N.

Because $X_1(\omega) \leq 2N$ and $X_2(\omega) \leq 2N$ for all $\omega \in \Omega_N$ we have:

$$\begin{aligned} |\mathcal{G}_N| &= \frac{|\Omega_N|}{6^{2N}} \sum_{k,l=0}^{2N} 2^k 2^l \mathbb{P}_N \left[X_1 = k,\ X_2 = l \right] \\ &= \frac{|\Omega_N|}{6^{2N}} \mathbb{E}_N \left[2^{X_1+X_2} \right] \\ &= \frac{|\Omega_N|}{6^{2N}} \mathbb{E}_N \left[\sum_{k=0}^{X_1+X_2} \binom{X_1+X_2}{k} \right] \end{aligned}$$

Because $X_1 + X_2 \leq 4N$ we have that:

$$\binom{X_1+X_2}{k} = 0 \text{ for all } k > 4N$$

So we obtain:

$$\begin{aligned} |\mathcal{G}_N| &= \frac{|\Omega_N|}{6^{2N}} \mathbb{E}_N \left[\sum_{k=0}^{4N} \binom{X_1+X_2}{k} \right] \\ &= \frac{|\Omega_N|}{6^{2N}} \sum_{k=0}^{4N} \frac{1}{k!} \mathbb{E}_N \left[(X_1+X_2)_k \right] \end{aligned}$$

The number $(X_1 + X_2)_k$ is equal to the number of ordered k-tuples of 1- and 2-circuits. We define the set:

$\mathcal{C}_k = \{k\text{-tuples of 1- and 2-circuits, with half-edges labeled with labels in } \{1, \ldots, 6N\}\}$

and:

$$\mathcal{C}_{k,i} = \{c \in \mathcal{C}_k;\ c \text{ contains } i \text{ 1-circuits}\}$$

So we get:

$$|\mathcal{G}_N| = \frac{|\Omega_N|}{6^{2N}} \sum_{k=0}^{4N} \frac{1}{k!} \cdot \frac{1}{|\Omega_N|} \sum_{\omega \in \Omega_N} \sum_{i=0}^{k} \sum_{c \in \mathcal{C}_{k,i}} \chi_c(\omega)$$

where:

$$\chi_c(\omega) = \begin{cases} 1 & \text{if } c \subset \omega \text{ as partitions} \\ 0 & \text{otherwise} \end{cases}$$

So:

$$|\mathcal{G}_N| = \frac{|\Omega_N|}{6^{2N}} \sum_{k=0}^{4N} \frac{1}{k!} \sum_{i=0}^{k} \sum_{c \in \mathcal{C}_{k,i}} \mathbb{P}_N \left[c \subset \omega \right]$$

Because $\mathbb{P}_N \left[c \subset \omega \right]$ depends only on the number of edges in c, it is constant on $\mathcal{C}_{k,i}$ (see also Lemma 4.1 below). So we fix $c_{k,i} \in \mathcal{C}_{k,i}$ and write:

$$|\mathcal{G}_N| = \frac{|\Omega_N|}{6^{2N}} \sum_{k=0}^{2N} \frac{1}{k!} \sum_{i=0}^{k} |\mathcal{C}_{k,i}| \cdot \mathbb{P}_N \left[c_{k,i} \subset \omega \right]$$

We have:

$$|\mathcal{C}_{k,i}| = \frac{3^{2(k-i)}3^i 2^{k-i} k!}{2^{k-i} i!(k-i)!} 2N(2N-1)\cdots(2N-(i+2(k-i))+1)$$
$$= \binom{k}{i} \cdot 3^{2k-i} \cdot 2N(2N-1)\cdots(2N-(2k-i)+1)$$

and:

$$\mathbb{P}_N\left[c_{k,i} \subset \omega\right] = \frac{1}{(6N-1)(6N-3)\cdots(6N-2\cdot(2k-i)+1)}$$

So:

$$|\mathcal{G}_N| = \frac{|\Omega_N|}{6^{2N}} \sum_{k=0}^{4N} \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} \cdot 3^{2k-i} \cdot \frac{2N(2N-1)\cdots(2N-(2k-i)+1)}{(6N-1)(6N-3)\cdots(6N-2\cdot(2k-i)+1)}$$

We have:

$$\frac{2N-m}{6N-2m-1} \leq \frac{2N}{6N-1} \text{ for all } m = 0, \ldots 2k-i$$

So, using dominated convergence, we obtain:

$$|\mathcal{G}_N| \sim \frac{|\Omega_N|}{6^{2N}} \sum_{k=0}^{4N} \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} \cdot 3^{2k-i} \cdot \frac{1}{3^{2k-i}}$$
$$= \frac{|\Omega_N|}{6^{2N}} \sum_{k=0}^{4N} \frac{1}{k!} 2^k$$
$$\sim e^2 \frac{|\Omega_N|}{6^{2N}}$$

as $N \to \infty$.

We will now use this, combined with π_2 to compute I_N. In particular, we will show that we can ignore graphs with automorphisms. We have:

$$\frac{|\{G \in \mathcal{G}_N;\ \mathrm{Aut}(G) \neq \{e\}\}|}{|\mathcal{G}_N|} = \frac{\sum\limits_{k,l} 2^{k+l}\mathbb{P}_N\left[X_1 = k, X_2 = l, \mathrm{Aut} \neq \{e\}\right]}{6^{2N}} \frac{|\Omega_N|}{|\mathcal{G}_{\mathcal{N}}|}$$

Theorem 3.3 tells us that there exists a $C > 0$ such that for all $k, l \in \mathbb{N}$:

$$2^{k+l}\mathbb{P}_N\left[X_1 = k, X_2 = l, \mathrm{Aut} \neq \{e\}\right] \leq\ 2^{k+l}\mathbb{P}_N\left[X_1 = k, X_2 = l\right]$$
$$\leq\ C \cdot 2^{k+l} \frac{\lambda_1^k e^{-\lambda_1} \lambda_2^l e^{-\lambda_2}}{k!\ l!}$$

which is summable and does not depend on N. Hence by the dominated convergence theorem we can take the limits of the terms to compute the limit of the sum. We have:

$$\lim_{N\to\infty} \mathbb{P}_N\left[X_1 = k, X_2 = l, \mathrm{Aut} \neq \{e\}\right] \ \leq\ \lim_{N\to\infty} \mathbb{P}_N\left[\mathrm{Aut} \neq \{e\}\right] \ =\ 0$$

by Theorem 3.2. So we obtain that:

$$\lim_{N\to\infty} \frac{|\{G \in \mathcal{G}_N;\ \mathrm{Aut}(G) \neq \{e\}\}|}{|\mathcal{G}_N|} = 0$$

and:

$$\frac{|\mathcal{G}_N|}{|\mathcal{U}_N|} \sim (2N)!$$

for $N \to \infty$. Combining the above and applying Stirling's approximation (Theorem 1.27 in Chapter 1) now gives the theorem. $\square$

3.2. Maps with a small defect

We will also need to control the number of graphs that carry a map which distorts the adjacency structure at at most a fixed number of edges. This is a slight generalization of an automorphism of a graph. In Theorem 3.2 we have already seen that the probability that a random cubic graph carries a non-trivial automorphism tends to 0 for $N \to \infty$ (cf. also [**KSV02**], [**Wor86**]). In [**KSV02**], Kim, Sudakov and Vu also consider maps of small distortion, but for regular graphs with growing vertex degrees.

We note that the set of bijections of the vertex set V of a graph can be identified with the symmetric group S_V. The distortion we were speaking about is the edge defect defined below:

Definition 3.1. Let $\Gamma = (V, E)$ be a graph and $\pi \in \mathrm{S}_V$. The *edge defect* of π on Γ is the number:

$$\mathrm{ED}_\pi(\Gamma) = |\{e \in E;\ \pi(e) \notin E\}|$$

This definition is similar to, but different from, the definition of the defect of a permutation by Kim, Sudakov and Vu in [**KSV02**].

We will also need to consider the action of an element $\pi \in \mathrm{S}_{2N}$ on the edges K_{2N}, the complete graph on $2N$ vertices. From hereon an *edge orbit* of an element $\pi \in \mathrm{S}_{2N}$ will mean the orbit of an edge in K_{2N} under π.

We want to bound the probability that a cubic graph carries a non-trivial map with edge defect $\leq k$ for a fixed $k \in \mathbb{N}$. To do this, we will adapt the proof of Wormald in [**Wor86**] of the fact that a random regular graph asymptotically almost surely carries no automorphisms to our situation. The key ingredient is the following lemma (Equations 2.3 and 2.7 in [**Wor86**]):

LEMMA 3.6. [**Wor86**] *There exists a constant $C > 0$ such that: given $N \in \mathbb{N}$, $a \in \mathbb{N}$, $s_i \in \mathbb{N}$ for $i = 2, \ldots, 6$ such that $2s_2 + 3s_3 + \ldots 6s_6 \leq 2N$ and $e_1 \in \mathbb{N}$ such that $e_1 \leq s_2$ and $f \in \mathbb{N}$. Let $\mathcal{H}_N(a, s_1, \ldots, s_6, e_1, f, k)$ be the set of pairs (π, H) such that:*
- *$\pi \in \mathrm{S}_{2N}$, H a graph on $\{1, \ldots, 2N\}$.*
- *$\pi \neq \mathrm{id}$ and π has s_i i-cycles for $i = 2, \ldots 6$.*
- *The support of π is A, $|A| = a$ and π fixes H as a graph.*
- *$\deg_H(x) = 3$ for all $x \in A$.*
- *At least one end of every edge in H is moved by π.*
- *The subgraph of H induced by A can be written as the union of f edge-orbits of π and f is minimal in this respect.*
- *e_1 edges of H are fixed edge-wise by π.*
- *The subgraph of H induced by A has k edges.*

Then:

$$\sum_{(\pi,H)\in\mathcal{H}_N(a,s_1,\ldots,s_6,e_1,f,k)} \mathbb{P}_N\left[H \subset \Gamma\right] \leq C^a N^{-a/2} a^{3a/8}$$

Using this, we can prove the following, where we write $a(\pi)$ for the number of elements in the support $A(\pi)$ of $\pi \in \mathrm{S}_{2N}$:

PROPOSITION 3.7. *Let $n, k \in \mathbb{N}$. There exists a $C > 0$ such that:*

$$\mathbb{P}_N\left[\begin{array}{c}\exists\ \mathrm{id} \neq \pi_1, \ldots, \pi_n \in \mathrm{S}_{2N} \text{ such that } \mathrm{ED}_{\pi_i}(\Gamma) \leq k \text{ and } a(\pi_i) \geq k-1 \\ \forall i = 1, \ldots, n \text{ and } \Gamma \text{ has } < Cn \text{ circuits of length } \leq k\end{array}\right] \to 0$$

for $N \to \infty$.

PROOF. The idea of the proof is as follows: we consider a set of graphs $\mathcal{H}'_N$ such that any graph with no circuits of less than k edges that carries a map $\pi \in \mathrm{S}_{2N}$ with edge defect $\leq k$ contains at least one graph in the set $\mathcal{H}'_N$ as a subgraph. If we can then prove that:

$$\sum_{H'\in\mathcal{H}'_N} \mathbb{P}_N\left[H' \subset \Gamma\right] \to 0$$

for $N \to \infty$, then we have proved the proposition, because every circuit adds at most a finite number of maps with a bounded edge defect, which is where the constant C comes from.

We will construct a part of $\mathcal{H}'_N$ out of the set $\mathcal{H}_N$, which is the set of subgraphs out of which a graph with a non-trivial automorphism must contain at least one. This last set is given by:

$$\mathcal{H}_N = \bigcup_{\substack{\pi\in\mathrm{S}_{2N} \\ \pi\neq\mathrm{id},\ a(\pi)\geq k-1}} \mathcal{H}_N(\pi)$$

where:

$$\mathcal{H}_N(\pi) = \left\{ H \text{ graph on } \{1, \ldots, 2N\};\ \begin{array}{c}\text{every edge in } H \text{ has at least on end in } A \\ \deg_H(x) = 3\ \forall x \in A \\ \pi H = H\end{array}\right\}$$

where A is the support of π in $\{1,\ldots,2N\}$. The condition $\pi H = H$ is equivalent to the fact that H can be written as a union of edge-orbits of π.

We are interested in graphs that carry a non-trivial 'almost-automorphism'. Suppose we have a graph Γ for which the map $\pi \in \mathrm{S}_{2N}$ with support $A \subset \{1,\ldots,2N\}$ is such an almost-automorphism. We consider the graph $H \subset \Gamma$ that consists of all edges that have at least one end in A. All but k images of the edges of H (seen as a subgraph of K_N) under π should be edges in Γ again. This means that H can be written as a union of edge orbits of π out of which k edges have been removed and replaced by different edges. We are not interested in these replacement edges and consider the graph H' that consists of H minus these edges. We define:

$$\mathcal{H}'_N(\pi) = \left\{ H' \text{ graph on } \{1,\ldots,2N\};\ \begin{array}{c} \exists H \in \mathcal{H}_N(\pi) \text{ such that } H\backslash H' \text{ has } k \text{ edges} \\ H' \text{ contains no circuits of length } \leq k \end{array} \right\}$$

What we have argued is that for $\pi \in \mathrm{S}_{2N}$:

$$\Gamma \text{ has no circuits of length } \leq k \text{ and } \mathrm{ED}_\pi(\Gamma) \leq k \ \Rightarrow\ \exists H' \in \mathcal{H}'_N(\pi) \text{ with } H' \subset \Gamma$$

We now want to apply Lemma 3.6. This means that we need relate the cardinalities of $\mathcal{H}_N(\pi)$ and $\mathcal{H}'_N(\pi)$ and the probabilities $\mathbb{P}_N[H \subset \Gamma]$ and $\mathbb{P}_N[H' \subset \Gamma]$, where H' is the graph obtained from $H \in \mathcal{H}_N(\pi)$ by removing k edges.

Because a graph in $\mathcal{H}_N(\pi)$ has at most $3a/2$ edges we get:

$$|\mathcal{H}'_N(\pi)| \leq \frac{1}{2}\left(\frac{3a}{2}\right)^k \cdot |\mathcal{H}_N(\pi)|$$

and because H' contains k edges fewer than H, we have:

$$\mathbb{P}_N[H' \subset \Gamma] \leq C' N^k \cdot \mathbb{P}_N[H \subset \Gamma]$$

where $C' > 0$ is independent of H,H' and N.

It turns out that the bounds above in combination with Lemma 3.6 are only small enough when the support of π is large enough, i.e. when it contains at least $2k+1$ elements. This means that we need to cut the sum over subgraphs into two pieces. Recall that $a(\pi)$ denotes the number of elements in the support $A(\pi)$ of $\pi \in \mathrm{S}_{2N}$. Define:

$$T_1(N) = \sum_{\substack{\pi \in \mathrm{S}_N \\ k-1 \leq a(\pi) \leq 2k, \pi \neq \mathrm{id}}} \sum_{H' \in \mathcal{H}'_N(\pi)} \mathbb{P}_N[H' \subset \Gamma]$$

$$T_2(N) = \sum_{\substack{\pi \in \mathrm{S}_n \\ a(\pi) > 2k}} \sum_{H' \in \mathcal{H}'_N(\pi)} \mathbb{P}_N[H' \subset \Gamma]$$

So now we need to prove that both $T_1(N)$ and $T_2(N)$ tend to 0 for $N \to \infty$.

We start with $T_2(N)$. We have:

$$\begin{aligned}
T_2(N) &\leq \sum_{\substack{\pi\in\mathrm{S}_N \\ a(\pi)>2k}} C'' \cdot a(\pi)^2 N^k \sum_{H\in\mathcal{H}_N(\pi)} \mathbb{P}_N\left[H\subset\Gamma\right] \\
&\leq \sum_{\substack{a,s_1,\ldots,s_6 \\ e_1,f}} C''\cdot a^2 N^k \sum_{(\pi,H)\in\mathcal{H}_N(a,s_1,\ldots,s_6,e_1,f)} \mathbb{P}_N\left[H\subset\Gamma\right] \\
&\leq \sum_{a=2k+1}^{N} (C''')^a N^{-a/2+k}
\end{aligned}$$

for some constants $C'', C''' > 0$ independent of a and N, where we have used Lemma 3.6 and the fact that the number of choices for the variables other than a is polynomial in a in the last step. The final expression tends to 0 for $N\to\infty$.

To prove that $T_1(N)$ also tends to 0 we will need to use the assumptions on short circuits and the support of the supposed almost-automorphisms. Recall that we are summing over all permutations $\pi\in\mathrm{S}_N$ with $k-1\leq a(\pi)\leq 2k$. We first note that the set of isomorphism classes of graphs we are summing over is finite (they are graphs of bounded degree on at most $2k$ vertices). Because none of these are circuits by assumption, Theorem 3.4 tells us that if they are not subtrees, they have asymptotic probability 0 of appearing in the graph.

So, the only subgraphs we need to worry about are small subtrees. Because we assume that the support of π contains at least $k-1$ vertices, this means that we need to consider subtrees of at least $k-1$ vertices. A counting argument shows that such a tree needs to be connected to the rest of the graph by at least $k+1$ edges. This leaves two options. Either π could have edge defect $k+1$, in which case we are done, or at least two of these edges connect to the same vertex, in which case our graph needs to contain a subgraph that is either a short circuit or a more complicated graph and then we can apply the same reasoning as above.

This means that also:

$$T_1(N)\to 0$$

as $N\to\infty$. □

3.3. Circuits

Our treatment of the short curves in both models for random surfaces will rely heavily on the distribution of the number of circuits of a fixed length in a random graph. We have already seen the asymptotic probability distribution of this number. We will also need some bounds on this distribution for dominated convergence arguments later on. These we shall prove in this section

We will prove two upper bounds on the probability distribution of $X_{N,k}$. The first one will be an upper bound on $\mathbb{P}_N\left[X_{N,k}=0\right]$ for all $k\in\mathbb{N}$ and uniform in N, which will be given by Proposition 3.11. The second one will be an upper bound on $\mathbb{P}[X_{N,k} = i]$ for all $i \in \mathbb{N}$ for fixed k and uniform in N and will be given by Lemma 3.12.

The first of these two requires the most effort and will need some preparation. We start with the following lemma:

LEMMA 3.8. *Let e be an edge in a cubic graph Γ. e is part of at most $2^{\lfloor \frac{k}{2} \rfloor}$ circuits of length k.*

PROOF. First we assume that k is even. Suppose that $e = \{v_1, v_2\}$. We look at all the vertices at distance $\frac{k}{2} - 1$ from v_1 and v_2 respectively, like in Figure 3.1 below:

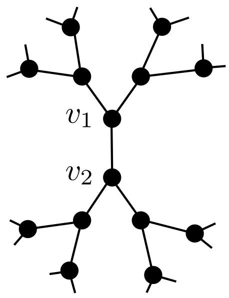


FIGURE 3.1. The vertices at distance ≤ 2 from v_1 and v_2.

A circuit of length k containing e corresponds to an edge between a vertex at distance $\frac{k}{2} - 1$ from v_1 and a vertex at distance $\frac{k}{2} - 1$ from v_2. There are $2^{\frac{k}{2}-1}$ vertices at distance $\frac{k}{2} - 1$ from v_1 and 2 half-edges emanating from each of them. This means that in a given graph there can be at most $2 \cdot 2^{\frac{k}{2}-1} = 2^{\frac{k}{2}}$ edges between these two sets of vertices and hence at most $2^{\frac{k}{2}} = 2^{\lfloor \frac{k}{2} \rfloor}$ length k circuits passing through e.

If k is odd then a length k circuit corresponds to an edge between a vertex at distance $\frac{k}{2} - \frac{1}{2}$ from v_1 and a vertex at distance $\frac{k}{2} - \frac{3}{2}$ from v_2. There can be at most $2^{\frac{k}{2}-\frac{1}{2}} = 2^{\lfloor \frac{k}{2} \rfloor}$ such edges in a given graph. $\square$

We will also need the following definition and theorem from [**Wor99**]:

DEFINITION 3.2. Let $\omega, \omega' \in \Omega_N$. We say ω and ω' *differ by a simple switching* when ω' can be obtained by taking two pairs $\{p_1, p_2\}, \{p_3, p_4\} \in \omega$ and replacing them by $\{p_1, p_3\}$ and $\{p_2, p_4\}$.

So on the level of graphs a simple switching looks like the picture below:

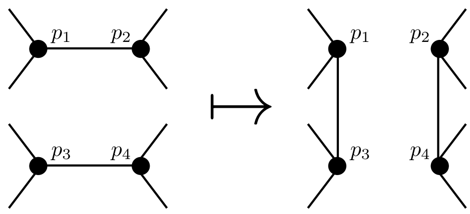


FIGURE 3.2. A simple switching.

The reason we are interested in simple switchings is the following theorem about the behavior of random variables that do not change too much after a simple switching:

THEOREM 3.9. [**Wor99**] *Let $c \in (0, \infty)$. If $Z_N : \Omega_N \to \mathbb{R}$ is a random variable such that $|Z_N(\omega) - Z_N(\omega')| < c$ whenever ω and ω' differ by a simple switching then:*

$$\mathbb{P}_N\left[|Z_N - \mathbb{E}_N\left[Z_N\right]| \geq t\right] \leq 2\exp\left(-\frac{t^2}{6Nc^2}\right)$$

In the proof of the upper bound we need to control the number of possible ways two circuits of fixed length can interesect. We will be interested in pairs of k-circuits that intersect in i vertices such that the intersection has j connected components. To avoid having to keep repeating this phrase, we have the following definition:

DEFINITION 3.3. An (i, j, k) *double circuit* is a graph consisting of two k-circuits that intersect in i vertices such that the intersection has j connected components and such that every vertex of the graph has degree at most 3.

Figure 3.3 gives an example:

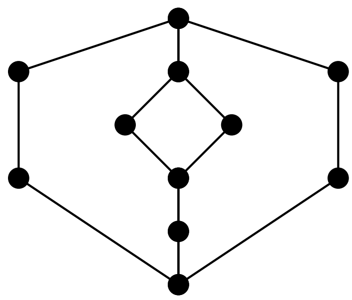


FIGURE 3.3. A $(5, 2, 8)$ double circuit.

Note that because the degree of the graph is not allowed to exceed 3, a connected component of the intersection of an (i, j, k) double circuit contains at least 2 vertices and 1 edge. This means that we can assume that $j \leq \left\lfloor \frac{i}{2} \right\rfloor$.

We have the following lemma about this type of graphs:

LEMMA 3.10. *Let $i, j, k \in \mathbb{N}$ such that $1 \leq i \leq k$ and $j \leq \left\lfloor \frac{i}{2} \right\rfloor$ then there are at most*

$$2^{j-1}(j-1)!\binom{i-j-1}{j-1}\binom{k-i+j-1}{j-1}^2$$

isomorphism classes of (i, j, k) double circuits.

PROOF. To prove this we will deconstruct the graphs we are considering into building blocks. We will count how many possible building blocks there are and in how many ways we can put these blocks together.

Every (i, j, k) double circuit can be constructed from the following building blocks:

- j connected components of the intersection of lengths $l_1, l_2, \ldots, l_j$. Where the n^{th} connected component consists of a line of l_n vertices with two edges emanating from each end of the line.
- j segments of the first circuit of lengths $l_{j+1}, l_{j+2}, \ldots, l_{2j}$.
- j segments of the second circuit of lengths $l_{2j+1}, l_{2j+2}, \ldots, l_{3j}$.

such that:

$$\sum_{n=1}^{j} l_n = i \tag{1}$$

$$\sum_{n=j+1}^{2j} l_n = k - i \tag{2}$$

$$\sum_{n=2j+1}^{3j} l_n = k - i \tag{3}$$

and:

$$l_i \geq \begin{cases} 2 & \text{if } 1 \leq i \leq j \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Figure 3.4 depicts these building blocks:

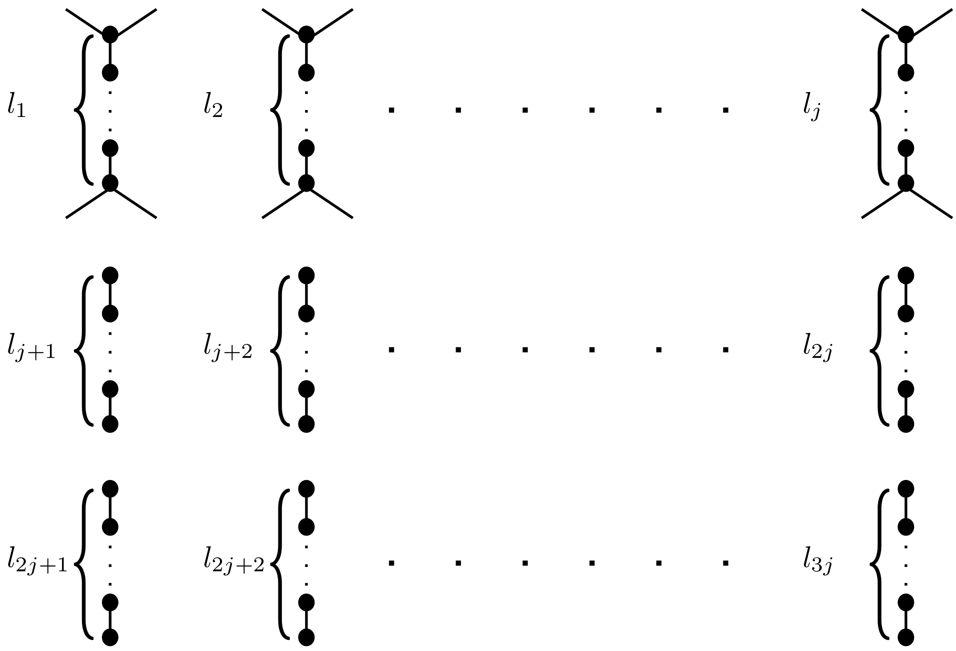


FIGURE 3.4. Building blocks for an (i, j, k) double circuit.

We can construct an (i, j, k) double circuit out of these blocks in the following way:

- we start by connecting the j 'intersection' blocks by joining them with the first j 'line blocks' (we use one loose edge on either side of each intersection block). So, between every pair of intersection blocks there needs to be a line block. From this construction we obtain one circuit with loose edges. The conditions on the lengths l_i above guarantee that we get a k-circuit.
- after this we make the second circuit with the remaining line blocks and 'open edges' of the intersection blocks, again such that the blocks alternate.

Every (i, j, k) double circuit can be constructed like this. This means that the number of ways this construction can be carried out times the number of sequences $(l_1, l_2, \ldots, l_{3j})$ satisfying conditions (1), (2), (3) and (4) gives an upper bound for the number of isomorphism classes of (i, j, k) double circuits.

We start with the factor accounting for the number of sequences $(l_1, l_2, \ldots, l_{3j})$. This factor is:

$$\binom{i-j-1}{j-1}\binom{k-i+j-1}{j-1}^2 \tag{5}$$

To prove this, we use the fact that the number of positive integer solutions to the equation:

$$a_1 + a_2 + \ldots + a_m = n$$

is equal to:

$$\binom{n-1}{m-1}$$

(see for instance [**Sta97**]). To get the first binomial coefficient we note that

$$a_1 + a_2 + \ldots + a_m = n - m$$

if and only if:

$$(a_1 + 1) + (a_2 + 1) + \ldots + (a_m + 1) = n$$

So the number of integer solutions to:

$$b_1 + b_2 + \ldots + b_m = n$$

with $b_i \geq 2$ for $1 \leq i \leq m$ is equal to the number of positive integer solutions to:

$$a_1 + a_2 + \ldots + a_m = n - m$$

which is:

$$\binom{n-m-1}{m-1}.$$

This gives us the first binomial coefficient in (5). The same trick works for the second binomial coefficient: because:

$$a_1 + a_2 + \ldots + a_m = n + m$$

if and only if:

$$(a_1 - 1) + (a_2 - 1) + \ldots + (a_m - 1) = n$$

the number of integer solutions to :

$$c_1 + c_2 + \ldots + c_m = n$$

with $c_i \geq 0$ for $1 \leq i \leq m$ is:

$$\binom{n+m-1}{m-1},$$

which explains the quadratic part in (5).

The factor accounting for the number of gluings is equal to:

$$2^{j-1}(j-1)!$$

We get to this number by using the fact the gluing is determined by two things: we can choose the order of the blocks in each circuit (as long as the alternation of intersections and lines is maintained) and we can choose the orientation of the intersection blocks in each circuit (i.e. to which of the two ends of the block we glue the line).

We start with first circuit that we glue. When we reorder the blocks in this circuit, the only thing we do is interchange the lengths $l_1, l_2, \ldots l_{2n}$. This has already been accounted for in the factor for the lengths. Also the choice in orientation of the intersection blocks does not matter for the isomorphism class of the circuit with loose edges we obtain at the end. So for the number of constructions of the first circuit is 1.

However, for the second circuit the order and orientation of the intersection blocks do matter for the isomorphism class of the (i, j, k) double circuit we obtain (one could say that only the relative order and orientation of the intersections matter). Figure 3.5 below illustrates how the change of orientation can affect the isomorphism class:

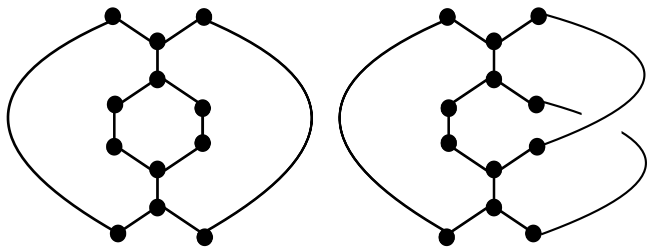


FIGURE 3.5. Two non-isomorphic $(4, 2, 8)$ double circuits.

The order of the line blocks in the second circuit again corresponds to a changing of lengths that has already been accounted for.

We count the orders as follows: first of all, we note that because the order of the blocks in the first circuit does not matter we can choose it. We choose an order that cyclically corresponds to $l_1, l_{j+1}, l_2, l_{j+2}, \ldots, l_j, l_{2j}$. To construct the second circuit we start with one of the loose edges of the intersection block corresponding to l_1 (it does not matter which edge we choose). We glue the first line block to it and for the other end of the line block we have a choice of $j-1$ intersection blocks and 2 loose edges per intersection block to glue it to. After we have chosen the intersection block and the edge we glue it to, we glue the second line block to the other loose edge of this intersection block and repeat the process. Like this we pick up a factor of:

$$2^{j-1}(j-1)!$$

Combining this with the formula for the number of lengths we see that the number of (i, j, k) double circuits is at most:

$$2^{j-1}(j-1)!\binom{i-j-1}{j-1}\binom{k-i+j-1}{j-1}^2$$

□

Now we are ready to prove the following upper bound:

PROPOSITION 3.11. *There exists a* $D \in (0, \infty)$ *such that:*

$$\mathbb{P}\left[X_{N,k} = 0\right] \leq Dk^8 \left(\frac{3}{8}\right)^k$$

for all $N \in \mathbb{N}$ *with* $2N \geq k$.

PROOF. We will consider two cases for the upper bound on $\mathbb{P}\left[X_{N,k} = 0\right]$, namely $N \leq \frac{1}{12k^2}\left(\frac{8}{3}\right)^k$ and $N > \frac{1}{12k^2}\left(\frac{8}{3}\right)^k$.

First suppose that $N \leq \frac{1}{12k^2}\left(\frac{8}{3}\right)^k$. In this case we will use Theorem 3.9. We first want to compute the expected value of $X_{N,k}$. To do this, we recapitulate part of the proof of Theorem 3.3. We have:

$$\mathbb{E}_N\left[X_{N,k}\right] = a_{N,k} p_{N,k}$$

where $a_{N,k}$ counts the number of possible distinct labelings a k-circuit as a set of k pairs of half-edges can have and $p_{N,k}$ is the probability that an element of Ω_N contains a given set of k pairs of half-edges. $p_{N,k}$ is given by (we will actually give a short proof of this in Chapter 4):

$$p_{N,k} = \frac{1}{(6N-1)(6N-3)\cdots(6N-2k+1)}$$

To count $a_{N,k}$ we reason as follows: we have $6^k(2N-1)(2N-2)\cdots(2N-k+1)$ ways to consistently assign half-edges to a k-circuit. However the dihedral group $\mathcal{D}_k$ of order $2k$ acts on the labelings, so we get:

$$a_{N,k} = \frac{6^k}{2k}(2N-1)(2N-2)\cdots(2N-k+1)$$

So, we obtain:

$$\mathbb{E}_N\left[X_{N,k}\right] = \frac{6^k}{2k}\frac{2N(2N-1)\cdots(2N-k+1)}{(6N-1)(6N-3)\cdots(6N-2k+1)}$$

If $X_{N,k} = 0$ then $\left|X_{N,k} - \mathbb{E}_N\left[X_{N,k}\right]\right| = \mathbb{E}_N\left[X_{N,k}\right]$. Hence:

$$\mathbb{P}\left[X_{N,k} = 0\right] \leq \mathbb{P}\left[\left|X_{N,k} - \mathbb{E}_N\left[X_{N,k}\right]\right| \geq \mathbb{E}_N\left[X_{N,k}\right]\right]$$

By Lemma 3.8 there are at most $2^{\lfloor \frac{k}{2} \rfloor}$ k-circuits going through an edge. This means that a simple switching can change the number of k-circuits by at most $2 \cdot 2^{\lfloor \frac{k}{2} \rfloor} \leq 2^{\frac{k}{2}+1}$. So, using Theorem 3.9 we get:

$$\mathbb{P}\left[X_{N,k} = 0\right] \leq 2\exp\left(-\frac{\mathbb{E}_N\left[X_{N,k}\right]^2}{24N \cdot 2^k}\right)$$

Because we are interested in an upper bound for this expression, we need to find a lower bound for $\mathbb{E}_N\left[X_{N,k}\right]$. We claim that $\mathbb{E}_N\left[X_{N,k}\right]$ is increasing in N, which means that we can get a lower bound by looking at $\mathbb{E}_N\left[X_{\lceil \frac{k}{2} \rceil, k}\right]$.

The fact that $\mathbb{E}_N[X_{N,k}]$ is increasing in N follows from differentiating it with respect to N:

$$\begin{aligned}\frac{\partial}{\partial N}\left(\mathbb{E}_N[X_{N,k}]\right) &= \mathbb{E}_N[X_{N,k}]\left(\sum_{i=0}^{k-1}\frac{2}{2N-i}-\frac{6}{6N-2i-1}\right)\\ &= \mathbb{E}_N[X_{N,k}]\left(\frac{-2}{2N(6N-1)}+\frac{2}{(2N-2)(6N-5)}\right.\\ &\qquad\left.+\sum_{i=3}^{k-1}\frac{2i-2}{(2N-i)(6N-2i-1)}\right)\end{aligned}$$

We have $2N(6N-1)\geq(2N-2)(6N-5)$, so $\frac{-2}{2N(6N-1)}\geq\frac{-2}{(2N-2)(6N-5)}$. Hence:

$$\frac{\partial}{\partial N}\left(\mathbb{E}_N[X_{N,k}]\right)\geq\mathbb{E}_N[X_{N,k}]\left(\sum_{i=3}^{k-1}\frac{2i-2}{(2N-i)(6N-2i-1)}\right)$$

We have $\mathbb{E}_N[X_{N,k}]\geq 0$ and every term in the sum above is also non negative. So:

$$\frac{\partial}{\partial N}\left(\mathbb{E}_N[X_{N,k}]\right)\geq 0$$

So, indeed:

$$\begin{aligned}\mathbb{E}_N[X_{N,k}] &\geq \mathbb{E}_N\left[X_{\lceil\frac{k}{2}\rceil,k}\right]\\ &= \frac{6^k}{2k}\frac{2\lceil\frac{k}{2}\rceil(2\lceil\frac{k}{2}\rceil-1)\cdots(2\lceil\frac{k}{2}\rceil-k+1)}{(6\lceil\frac{k}{2}\rceil-1)(6\lceil\frac{k}{2}\rceil-3)\cdots(6\lceil\frac{k}{2}\rceil-2k+1)}\end{aligned}$$

We will now look at the asymptotic behavior of this expression for $k\to\infty$. We get:

$$\frac{6^k}{2k}\frac{2\lceil\frac{k}{2}\rceil(2\lceil\frac{k}{2}\rceil-1)\cdots(2\lceil\frac{k}{2}\rceil-k+1)}{(6\lceil\frac{k}{2}\rceil-1)(6\lceil\frac{k}{2}\rceil-3)\cdots(6\lceil\frac{k}{2}\rceil-2k+1)}\sim\sqrt{\frac{2\pi}{k}}\left(\frac{4}{\sqrt{3}}\right)^k$$

where we have used Stirling's approximation. So for $k\to\infty$ we get:

$$\begin{aligned}\mathbb{P}[X_{N,k}=0] &\leq 2\exp\left(-\frac{\mathbb{E}_N[X_{N,k}]^2}{24N\cdot 2^k}\right)\\ &\leq 2\exp\left(-\frac{12k^2\mathbb{E}_N\left[X_{\lceil\frac{k}{2}\rceil,k}\right]^2}{24\cdot\left(\frac{8}{3}\right)^k 2^k}\right)\\ &\sim 2\exp(-\pi k)\end{aligned}$$

So for $N\leq\frac{1}{12k^2}\left(\frac{8}{3}\right)^k$ there exists a $C\in(0,\infty)$ such that:

$$\mathbb{P}[X_{N,k}=0]\leq C\exp(-\pi k)$$

Because $\exp(-\pi)<\frac{3}{8}$ this implies that there exists a $D\in(0,\infty)$ such that:

$$\mathbb{P}[X_{N,k}=0]\leq Dk^8\left(\frac{3}{8}\right)^k \tag{6}$$

for all $\lceil\frac{k}{2}\rceil\leq N\leq\frac{1}{12k^2}\left(\frac{8}{3}\right)^k$.

Now suppose that $N > \frac{1}{12k^2}\left(\frac{8}{3}\right)^k$. The goal will be to use Lemma 1.20, which tells us that:

$$\mathbb{P}\left[X_{N,k}=0\right] \leq 1-\frac{\mathbb{E}_N\left[X_{N,k}\right]^2}{\mathbb{E}_N\left[X_{N,k}^2\right]}$$

Also, we will again use some of the ideas from the proof of Theorem 3.3. We will need an upper bound on $\mathbb{E}_N\left[X_{N,k}\right]$, and because N is larger, we can also get a better lower bound on $\mathbb{E}_N\left[X_{N,k}\right]$. Furthermore we need to get an upper bound on $\mathbb{E}_N\left[X_{N,k}^2\right]$.

We start with the bounds on $\mathbb{E}_N\left[X_{N,k}\right]$. Because $\mathbb{E}_N\left[X_{N,k}\right]$ has non-negative derivative with respect to N, we get:

$$\mathbb{E}_N\left[X_{N,k}\right] \leq \frac{2^k}{2k}$$

For the lower bound on $\mathbb{E}_N\left[X_{N,k}\right]$ we have:

$$\begin{aligned}\mathbb{E}_N\left[X_{N,k}\right] &= \frac{2^k}{2k}\prod_{i=0}^{k-1}\left(1-\frac{i-1}{6N-2i-1}\right)\\ &\geq \frac{2^k}{2k}\left(1-\frac{k-2}{\max\{\frac{1}{2k^2}\left(\frac{8}{3}\right)^k,3k\}-2k+1}\right)^k\end{aligned}$$

Because $k \geq 2$ we have $\max\{\frac{1}{2k^2}\left(\frac{8}{3}\right)^k,3k\}-2k+1 \geq \frac{1}{4k^2}\left(\frac{8}{3}\right)^k$, so we get:

$$\mathbb{E}_N\left[X_{N,k}\right] \geq \frac{2^k}{2k}\left(1-\left(4k^3-4k^2\right)\left(\frac{3}{8}\right)^k\right)^k$$

The last and longest step is the upper bound on $\mathbb{E}_N\left[X_{N,k}^2\right]$. We will compute $\mathbb{E}_N\left[X_{N,k}^2\right]$ using the expected value of $(X_{N,k})_2 = X_{N,k}(X_{N,k}-1)$. Note that $(X_{N,k})_2$ counts the number of ordered pairs of distinct k-circuits. We write:

$$(X_{N,k})_2 = Y'_{N,k}+Y''_{N,k}$$

where $Y'_{N,k}$ counts the number of ordered pairs of vertex disjoint k-circuits and $Y''_{N,k}$ counts the number of ordered pairs of vertex non-disjoint k-circuits.

To compute $\mathbb{E}_N\left[Y'_{N,k}\right]$ we use a similar argument as for $\mathbb{E}_N\left[X_{N,k}\right]$. Now we have $2k$ vertices and $2k$ pairs of half-edges to label and $\mathcal{D}_k\times\mathcal{D}_k$ acts on the labelings (note that we cannot exchange the two circuits, because we are dealing with ordered pairs of circuits). So we get:

$$\mathbb{E}_N\left[Y'_{N,k}\right] = \frac{6^{2k}}{(2k)^2}\frac{2N(2N-1)\cdots(2N-2k+1)}{(6N-1)(6N-3)\cdots(6N-4k+1)}$$

Because the number of factors in the numerator in this expression is equal to the number of factors in its denominator, a similar argument as before shows that this expression again has non negative derivative with respect to N. So:

$$\mathbb{E}_N\left[Y'_{N,k}\right] \leq \frac{2^{2k}}{(2k)^2}$$

To compute $\mathbb{E}_N\left[Y''_{N,k}\right]$ we will use Lemma 3.10. So we split up $Y''_{N,k}$ and write:

$$Y''_{N,k} = \sum_{i=2}^{k}\sum_{j=1}^{\lfloor\frac{i}{2}\rfloor} Y''_{N,i,j,k}$$

where $Y''_{N,i,j,k}$ counts the number of ordered (i,j,k) double circuits. We will start by giving an upper bound for $\mathbb{E}_N\left[Y''_{N,i,j,k}\right]$. Let $\mathcal{I}(i,j,k)$ be the set of isomorphism classes of (i,j,k) double circuits. Furthermore, given $c \in \mathcal{I}(i,j,k)$ let a_c be the number of non-isomorphic labelings of c and let p_c be the probability of finding a fixed labeled (i,j,k) double circuit isomorphic to c in a random cubic graph. Then we have:

$$\begin{aligned}\mathbb{E}_N\left[Y''_{N,i,j,k}\right] &= \sum_{c\in\mathcal{I}(i,j,k)} 2a_cp_c \\ &\leq 2\left|\mathcal{I}(i,j,k)\right| \max\left\{a_cp_c;\ c\in\mathcal{I}(i,j,k)\right\} \\ &\leq 2\cdot 2^{j-1}(j-1)!\binom{i-j-1}{j-1}\binom{k-i+j-1}{j-1}^2 \max\left\{a_cp_c;\ c\in\mathcal{I}(i,j,k)\right\}\end{aligned}$$

Note the extra factor 2 coming from the fact that we are counting oriented double circuits. Also observe that we have used Lemma 3.10 in the last step. So now we need to find an upper bound on $\max\left\{a_cp_c;\ c\in\mathcal{I}(i,j,k)\right\}$. Note that an (i,j,k) double circuit consits of $2k-i$ vertices and $2k-i+j$ edges (because in each connected component of the intersection the number of pairs of vertices that are identified is one more than the number of pairs of edges that are identified). There might also be some symmetry in the double circuit, but because we are looking for an upper bound we disregard this. So if $c \in \mathcal{I}(i,j,k)$ we get:

$$a_c \leq 6^{2k-i}2N(2N-1)\cdots(2N-2k+i+1)$$

and:

$$p_c = \frac{1}{(6N-1)(6N-3)\cdots(6N-4k+2i-2j-1)}$$

So we get:

$$\begin{aligned}\mathbb{E}_N\left[Y''_{N,k}\right] &\leq \sum_{i=2}^{k}\sum_{j=1}^{\lfloor\frac{i}{2}\rfloor}\Bigg(2\cdot 2^{j-1}(j-1)!\binom{i-j-1}{j-1}\binom{k-i+j-1}{j-1}^2 \\ &\quad\cdot\frac{6^{2k-i}2N(2N-1)\cdots(2N-2k+i+1)}{(6N-1)(6N-3)\cdots(6N-4k+2i-2j-1)}\Bigg) \\ &= \sum_{i=2}^{k}\sum_{j=1}^{\lfloor\frac{i}{2}\rfloor}\frac{2^{2k-i+j}(j-1)!\binom{i-j-1}{j-1}\binom{k-i+j-1}{j-1}^2\prod_{m=0}^{2k-i}\frac{6N-3m}{6M-2m-1}}{(6N-4k+2i-1)(6N-4k+2i-3)\cdots(6N-4k+2i-2j-1)}\end{aligned}$$

Now we use the fact that:

$$\prod_{m=0}^{2k-i} \frac{6N-3m}{6M-2m-1} \leq 1$$

which can be proved with the same derivative argument that we used for $\mathbb{E}_N\left[X_{N,k}\right]$. So we get:

$$\begin{aligned}
\mathbb{E}_N\left[Y''_{N,k}\right] &\leq \sum_{i=2}^{k} \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor} \frac{2^{2k-i+j}(j-1)!\binom{i-j-1}{j-1}\binom{k-i+j-1}{j-1}^2}{(6N-4k-1)^j} \\
&= 2^{2k} \sum_{i=2}^{k} \frac{1}{2^i} \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor} \frac{2^j (j-1)!(i-j-1)!((k-i+j-1)!)^2}{(j-1)!(i-2j)!((j-1)!)^2((k-i)!)^2(6N-4k-1)^j}
\end{aligned}$$

We have $\frac{(j-1)!(i-j-1)!}{(j-1)!(i-2j)!} \leq (i-j-1)^{j-1} \leq i^{j-1}$ and similarly $\frac{(k-i-1)!}{(k-i-j)!} \leq k^{j-1}$, so:

$$\begin{aligned}
\mathbb{E}_N\left[Y''_{N,k}\right] &\leq 2^{2k} \sum_{i=2}^{k} \frac{1}{2^i} \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor} \frac{2^j i^{j-1}(k-i+j-1)^{2j-2}}{((j-1)!)^2(6N-4k-1)^j} \\
&\leq 2^{2k} \sum_{i=2}^{k} \frac{1}{2^i} \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor} \frac{2^j k^j k^{2j}}{((j-1)!)^2(6N-4k-1)^j}
\end{aligned}$$

Now using the fact that $N \geq \frac{1}{12k^2}\left(\frac{8}{3}\right)^k$ we get:

$$\mathbb{E}_N\left[Y''_{N,k}\right] \leq 2^{2k} \sum_{i=2}^{k} \frac{1}{2^i} \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor} \frac{1}{((j-1)!)^2} \left(\frac{2k^3}{\frac{1}{2k^2}\left(\frac{8}{3}\right)^k - 4k - 1} \right)^j$$

The $j = 1$ term in the sum over j is the largest, thus:

$$\mathbb{E}_N\left[Y''_{N,k}\right] \leq 2^{2k} \sum_{i=2}^{k} \frac{\lfloor \frac{i}{2} \rfloor}{2^i} \frac{2k^3}{\frac{1}{2k^2}\left(\frac{8}{3}\right)^k - 4k - 1}$$

We apply the same trick again to obtain:

$$\mathbb{E}_N\left[Y''_{N,k}\right] \leq \frac{k^4 2^{2k}}{\frac{1}{2k^2}\left(\frac{8}{3}\right)^k - 4k - 1}$$

As such, there must be a $C' \in (0, \infty)$ satisfying:

$$\begin{aligned}
\mathbb{E}_N\left[Y''_{N,k}\right] &\leq C' \frac{k^6 2^{2k}}{\left(\frac{8}{3}\right)^k} \\
&= C' k^6 \left(\frac{3}{2}\right)^k
\end{aligned}$$

Now we have all the bounds we need and we can put everything together. We have:

$$\begin{aligned}
\mathbb{P}_N\left[X_{N,k}=0\right] &\leq 1-\frac{\mathbb{E}_N\left[X_{N,k}\right]^2}{\mathbb{E}_N\left[X_{N,k}^2\right]} \\
&= 1-\frac{\mathbb{E}_N\left[X_{N,k}\right]^2}{\mathbb{E}_N\left[(Y'_{N,k})\right]+\mathbb{E}_N\left[(Y''_{N,k})\right]+\mathbb{E}_N\left[(X_{N,k})\right]} \\
&\leq 1-\frac{\frac{2^{2k}}{(2k)^2}\left(1-\frac{k-1}{6\cdot 2^{k/2}-2k+1}\right)^{2k}}{\frac{2^{2k}}{(2k)^2}+C'k^6\left(\frac{3}{2}\right)^k+\frac{2^k}{2k}} \\
&\sim 4C'k^8\left(\frac{3}{8}\right)^k
\end{aligned}$$

for $k \to \infty$. This implies that there exists a $D \in (0,\infty)$ such that:

$$\mathbb{P}\left[X_{N,k}=0\right] \leq Dk^8\left(\frac{3}{8}\right)^k$$

for all $N > \frac{1}{12k^2}\left(\frac{8}{3}\right)^k$. Putting this together with (6) proves the proposition. □

We also need a bound on the probability that $X_{N,k}$ grows very large for fixed k. This is given by the following lemma:

Lemma 3.12. *Let $k \geq 2$. Then there exists a $C_k \in (0,\infty)$ such that:*

$$\mathbb{P}\left[X_{N,k}=i\right] \leq \frac{C_k}{i^2}$$

for all $N \in \mathbb{N}$ and all $i \in \mathbb{N}$.

Proof. Chebyshev's inequality (Corollary 1.19) tells us that:

$$\mathbb{P}_N\left[\left|X_{N,k}-\mathbb{E}_N\left[X_{N,k}\right]\right| \geq i\right] \leq \frac{\mathbb{V}\mathrm{ar}\left[X_{N,k}\right]}{i^2}$$

where:

$$\mathbb{V}\mathrm{ar}\left[X_{N,k}\right] = \mathbb{E}_N\left[X_{N,k}^2\right]-\mathbb{E}_N\left[X_{N,k}\right]^2$$

In the proof of Proposition 3.11 we have seen that $\mathbb{E}_N\left[X_{N,k}\right] \leq \frac{2^k}{2k}$ and $\mathbb{E}_N\left[X_{N,k}^2\right] \leq D2^{2k}$ for all N and some $D \in (0,\infty)$, which proves the lemma. □

3.4. Short separating circuits

In this section we will study separating circuits on graphs, our motivation of course being separating curves on surfaces. Note however that we have not proved that a circuit that is separating on a graph is separating on the corresponding surface as well. In fact, this is not the case and Figure 3.6 gives a counter example:

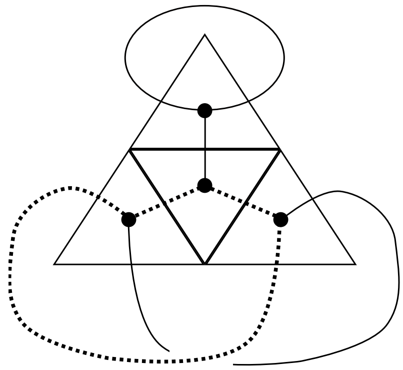


FIGURE 3.6. A circuit (dotted) that is separating on the graph but not on the corresponding surface.

What we do need is that a circuit that is separating as a closed curve on the surface is also separating on the graph. Stated as such this is not true, it is however not far from the truth. In fact we have the following situation: suppose γ is such a separating circuit, then:

- either γ is homotopic to a corner shared by some number of triangles and corresponds to a left hand turn circuit on the corresponding graph
- or γ is separating on the graph as well.

Because left hand turn circuits are always homotopically trivial on the surface corresponding to our given random graph, we will be able to exclude these via other methods. In other words: the probability that a random graph has short separating circuits is an upper bound on the probability that a random graph has short circuits that are separating on the corresponding surface.

Now we forget about surfaces again and return to random graphs.

It turns out that for the computations in this section it is easier to work with ordered partitions. This means that as a probability space we will use Ω_N^o.

So in this section we are going to study the following set:

$$G_{N,k}^o = \{\omega \in \Omega_N^o;\ \Gamma(\omega) \text{ has a separating circuit of length } k\}$$

where the length of a circuit in this case means the number of edges in that circuit.

What we want is to count the number of elements in $G_{N,k}^o$ and study the asymptotics of this number. However, we will not count the cardinality of $G_{N,k}^o$ directly. This turns out to be too difficult. Instead we will count the number of graphs we obtain by cutting open the graphs in $G_{N,k}^o$ along a separating circuit of length k. This will give us an upper bound for the number of elements in this set.

The plan of the rest of this section is as follows: in Section 3.4.1 we explain how this cutting along a separating circuit of length k works and how this gives an upper bound for $\left|G^o_{N,k}\right|$. After that we count the cardinality of the set of graphs that are cut open along a separating circuit of length k in Section 3.4.2. Because the expression we obtain is rather difficult to handle, we compute an upper bound for it in Section 3.4.3 and in Section 3.4.4 we use that to prove that the probability of $G^o_{N,k}$ tends to 0 as $N \to \infty$ for k up to $C\log_2(N)$ for any $C \in (0,1)$.

3.4.1. Cutting along a separating circuit. If we cut a graph along a separating circuit of length k (as shown in Figure 3.7 below), we obtain a graph with k degree 1 vertices and, because the circuit along which we cut is separating, more than one connected component. The degree 1 vertices will also be spread over different connected components.

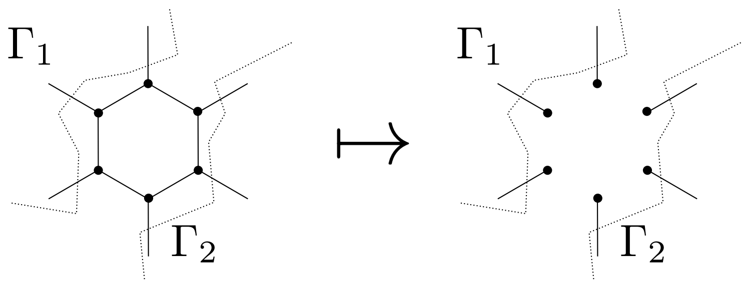


FIGURE 3.7. Cutting a graph along a separating circuit such that it splits into two connected components Γ_1 and Γ_2 both of which contain degree 1 vertices.

We need some notation for the set of graphs, or rather to say ordered partitions, we obtain by this procedure. First of all we need to look at more general sets of disjoint pairs out of $\{1,\ldots,6N\}$ than just partitions. We define:

$$\Theta_N = \left\{\omega \subset 2^{\{1,\ldots,6N\}};\ |p| = 2\ \forall p \in \omega,\ p_1 \cap p_2 = \emptyset\ \forall p_1 \neq p_2 \in \omega\right\}$$

where, as before, $2^{\{1,\ldots,6N\}}$ denotes the power set of $\{1,\ldots,6N\}$. We will denote the corresponding set of ordered sequences of pairs with the same properties by Θ^o_N. Furthermore, recall that for a partition $\omega \in \Omega^o_N$ the corresponding graph is denoted by $\Gamma(\omega)$. An element $\omega \in \Theta^o_N$ also naturally defines a graph which we will denote by $\Gamma(\omega)$ as well. Finally, we will denote the vertex set and edge set of a graph Γ by $V(\Gamma)$ and $E(\Gamma)$ repsectively.

First we define the set of partitions that represent graphs with $2N$ vertices out of which $2N-k$ have degree 3 and k have degree 1:

$$\Omega^o_{N,k} = \left\{\omega \in \Theta^o_N;\ \begin{array}{c} |\omega| = 3N-k,\ |V(\Gamma(\omega))| = 2N, \\ |\{v \in V(\Gamma(\omega));\ \deg(v) = 3\}| = 2N-k, \\ |\{v \in V(\Gamma(\omega));\ \deg(v) = 1\}| = k \end{array}\right\}$$

The reason we define this set only for an even number of vertices is that the set of partitions satisfying the properties above but corresponding to an odd number of vertices is empty. This follows from what is sometimes called the handshaking lemma: suppose Γ is a graph with vertex set $V(\Gamma)$ consisting vertices of degree 1 and 3 only and edge set $E(\Gamma)$. Then the handshaking lemma tells us that:

$$\begin{aligned} 2 \cdot |E(\Gamma)| &= |\{v \in V(\Gamma);\ \deg(v) = 1\}| + 3 \cdot |\{v \in V(\Gamma);\ \deg(v) = 3\}| \\ &= |V(\Gamma)| + 2 \cdot |\{v \in V(\Gamma);\ \deg(v) = 3\}| \end{aligned}$$

which means that $|V(\Gamma)|$ must be even.

The graph we obtain by forgetting the pairs that formed the separating circuit is an element the set:

$$D^o_{N,k} = \left\{ \omega \in \Omega^o_{N,k};\ \begin{array}{l} \Gamma(\omega) \text{ not connected, } \Gamma(\omega) \text{ has more than one} \\ \text{component with degree 1 vertices} \end{array} \right\}$$

Now we want an upper bound for the cardinality of $G^o_{N,k}$ in terms of that of $D^o_{N,k}$. This is given by the following lemma:

LEMMA 3.13.

$$\left|G^o_{N,k}\right| \leq 2^k (k-1)! \frac{(3N)!}{(3N-k)!} \left|D^o_{N,k}\right|$$

PROOF. To get to an upper bound of this form we need a map $f : G^o_{N,k} \to D^o_{N,k}$. Because then:

$$\left|G^o_{N,k}\right| \leq \max\left\{\left|f^{-1}(\omega)\right|;\ \omega \in D^o_{N,k}\right\} \left|D^o_{N,k}\right|$$

'Cutting along a separating circuit of length k' is a good candidate for such a map. However, this is not a well-defined map: if a graph in $G^o_{N,k}$ has more than one such circuit we have to choose which one to cut. So we fix such a choice and call it f.

Now we need to know how many graphs in $G^o_{N,k}$ can land on the same graph in $D^o_{N,k}$ under f.

First of all, given $\omega \in G^o_{N,k}$ note that the degree 1 vertices of $f(\omega)$ must correspond to the separating circuit that was cut open in ω by f, also if the original has another separating circuit of length k. This means that we can give an upper bound for $\max\left\{|f^{-1}(\omega)|;\ \omega \in D^o_{N,k}\right\}$ by looking at how many circuits we can construct from the degree 1 vertices in $D^o_{N,k}$ and then multiplying by a factor that accounts for the order of picking the pairs in the partition.

Given a graph in $D^o_{N,k}$ we can make $(k-1)!$ different circuits out of the k vertices with degree 1 with 2^k different orientations on these vertices. Furthermore, an element in $D^o_{N,k}$ consists of only $3N-k$ pairs of half-edges, while an element of $G^o_{N,k}$ consists of $3N$ pairs of half-edges. This means that, because we are making a distinction between the different orders of picking the pairs of half-edges, we get a factor of $\frac{(3N)!}{(3N-k)!}$. □

3.4.2. The cardinality of $D^o_{N,k}$. What remains is to compute the cardinality of $D^o_{N,k}$. Let us first compute the number of graphs with $2N$ vertices, where $2N-k$ have degree 3 and k degree 1 (so we drop the assumption of the graph having more than one connected component). This means that we want to know the cardinality of $\Omega^o_{N,k}$. We have the following lemma:

LEMMA 3.14.

$$\left|\Omega^o_{N,k}\right| = 6^k \binom{2N}{k} \frac{(6N-2k)!}{2^{3N}}$$

PROOF. Basically, the formula consists of two factors. The first one counts the number of ordered partitions into pairs of a set of $6N-2k$ labeled half-edges. The second factor comes from the fact that we have to choose which vertices will be degree 1 and which half-edge of these vertices we include, so in principle this factor comes from the labeling of the half-edges.

The first factor is $\frac{(6N-2k)!}{2^{3N-k}}$, there are k vertices with degree 1, so in a graph there are $3N-k$ pairs of formed out of $6N-2k$ half-edges. There are $\frac{(6N-2k)!}{2^{3N-k}}$ ordered ways to choose these pairs.

Now we have to count the number of labeled sets of half-edges we can choose. First of all, we have to choose k degree 1 vertices, which gives a factor of $\binom{2N}{k}$. After that, we have to include 1 out of 3 half-edges per degree 1 vertex, this gives a factor of 3^k. So we get:

$$\begin{aligned}\left|\Omega^o_{N,k}\right| &= 3^k \binom{2N}{k} \frac{(6N-2k)!}{2^{3N-k}} \\ &= 6^k \binom{2N}{k} \frac{(6N-2k)!}{2^{3N}}\end{aligned}$$

□

With this number, we can compute the cardinality of $D_{N,k}$.

LEMMA 3.15.

$$\left|D^o_{N,k}\right| = \frac{1}{2}\sum_{L=1}^{N-1}\sum_{l=1}^{k-1} \binom{2N}{k}\binom{k}{l}\binom{2N-k}{2L-l}\binom{3N-k}{3L-l}\frac{6^k}{2^{3N}}(6L-2l)!(6(N-L)-2(k-l))!$$

PROOF. The idea behind the proof is that if $\omega \in D^o_{N,k}$, then we can write:

$$\Gamma(\omega) = \Gamma(\omega_1) \sqcup \Gamma(\omega_2)$$

with $\omega_1 \in \Omega^o_{L,l}$ and $\omega_2 \in \Omega^o_{N-L,k-l}$ for some appropriate $L \in \{1, \ldots, N-1\}$ and $l \in \{1, \ldots, k\}$, as depicted in Figure 3.7. So we can count the cardinality of $D^o_{N,k}$ by counting all the possible combinations of smaller components.

If the first component has $2L$ vertices out of which l have degree 1 and the second component has $2(N-L)$ vertices out of which $k-l$ have degree 1, we have

$$\left|\Omega^o_{L,l}\right| \cdot \left|\Omega^o_{N-L,k-l}\right|$$

possibilities for the graph. There are $\binom{2N}{2L}$ ways of choosing $2L$ out of $2N$ triangles and $\binom{3N-k}{3L-l}$ ways to re-order the picking of the pairs. Also, we are counting everything double: we are making an artificial distinction between the 'first' and 'second' component. So we get:

$$\begin{aligned}\left|D^o_{N,k}\right| &= \frac{1}{2}\sum_{L=1}^{N-1}\sum_{l=1}^{k-1}\binom{2N}{2L}\binom{3N-k}{3L-l}\left|\Omega^o_{L,l}\right|\left|\Omega^o_{N-L,k-l}\right| \\ &= \frac{6^k \sum_{L=1}^{N-1}\sum_{l=1}^{k-1}\binom{2N}{k}\binom{k}{l}\binom{2N-k}{2L-l}\binom{3N-k}{3L-l}(6L-2l)!(6(N-L)-2(k-l))!}{2^{3N+1}}\end{aligned}$$

□

Note that there is some redundancy in the expression for $\left|D^o_{N,k}\right|$, that is to say, for certain combinations of L and l we have that $\binom{2N-k}{2L-l}=0$. There are two cases where this happens. The first one is when $2L-l<0$, so when $L<\left\lceil\frac{l}{2}\right\rceil$. The second one is when $2N-k<2L-l$ so when $L>N-\left\lceil\frac{1}{2}(k-l)\right\rceil$. So we can also write:

$$\left|D^o_{N,k}\right| = \frac{6^k \sum_{l=1}^{k-1}\sum_{L=\left\lceil\frac{l}{2}\right\rceil}^{N-\left\lceil\frac{1}{2}(k-l)\right\rceil}\binom{2N}{k}\binom{k}{l}\binom{2N-k}{2L-l}\binom{3N-k}{3L-l}(6L-2l)!(6(N-L)-2(k-l))!}{2^{3N+1}}$$

Sometimes we will write $\mathbb{P}\left[D^o_{N,k}\right] = \frac{\left|D^o_{N,k}\right|}{\left|\Omega^o_N\right|}$, which is a slight abuse of notation, because technically it is not defined ($D^o_{N,k}$ is not a subset of Ω^o_N). So, we get:

$$\mathbb{P}\left[D^o_{N,k}\right] = \frac{6^k}{2}\frac{\sum_{l=1}^{k-1}\sum_{L=\left\lceil\frac{l}{2}\right\rceil}^{N-\left\lceil\frac{1}{2}(k-l)\right\rceil}\binom{2N}{k}\binom{k}{l}\binom{2N-k}{2L-l}\binom{3N-k}{3L-l}\binom{6N-2k}{6L-2l}^{-1}}{6N(6N-1)\cdots(6N-2k+1)} \tag{7}$$

Note that $\left\lceil\frac{l}{2}\right\rceil \leq L \leq N-\left\lceil\frac{1}{2}(k-l)\right\rceil$ implies that $\binom{6N-2k}{6L-2l}\neq 0$, so the right hand side of the equation above is well defined.

3.4.3. An upper bound for $\mathbb{P}\left[D^o_{N,k}\right]$. Even though we have a closed expression for $\mathbb{P}\left[D^o_{N,k}\right]$, it does not immediately give much insight into how 'large' the probability is when we let N grow. The goal of this section is to obtain an upper bound for this probability, using the given expression.

Because of the binomial coefficients in the expression we will work with Stirling's approximation. Using Theorem 1.27 we can write:

$$\binom{n}{k} = \sqrt{\frac{n}{2\pi(n-k)k}} \frac{n^n}{(n-k)^{n-k}k^k} e^{\lambda_n - \lambda_{n-k} - \lambda_k}$$

with the bounds on λ_n, λ_{n-k} and λ_k from the theorem above for $n, k \in \mathbb{N}$ such that $0 < k < n$.

The 'difficult part' of the expression for $\mathbb{P}\left[D^o_{N,k}\right]$ is the double sum over the product of binomial coefficients, so we will focus on finding an upper bound for that. For $2L - l > 0$ and $2L - l < 2N - k$ we will write:

$$\frac{\binom{2N-k}{2L-l}\binom{3N-k}{3L-l}}{\binom{6N-2k}{6L-2l}} = F_1(N,k,L,l)F_2(N,k,L,l)F_3(N,k,L,l)$$

where:

$$F_1(N,k,L,l) = \sqrt{\frac{2N-k}{\pi(2(N-L)-(k-l))(2L-l)}}$$

$$F_2(N,k,L,l) = \frac{(2N-k)^{2N-k}(6(N-L)-2(k-l))^{3(N-L)-(k-l)}(6L-2l)^{3L-l}}{(2(N-L)-(k-l))^{2(N-L)-(k-l)}(2L-l)^{2L-l}(6N-2k)^{3N-k}}$$

and:

$$F_3(N,k,L,l) = \frac{\exp(\lambda_{2N-k} + \lambda_{2N-k} + \lambda_{6(N-L)-2(k-l)} + \lambda_{6L-2l})}{\exp(\lambda_{2(N-L)-(k-l)} + \lambda_{2L-l} + \lambda_{3(N-L)-(k-l)} + \lambda_{3L-l} + \lambda_{6N-2k})}$$

Note that we have to treat the cases where $2L-l = 0$ and $2L-l = 2N-k$ seperately, because there this way of writing the binomial coefficients does not work.

We want upper bounds for F_1, F_2 and F_3 for N, k, L, l in the appropriate ranges. We will start with F_1 and F_3 because those are the two easiest expressions.

Because we're assuming that $2L - l > 0$ and $2(N-L) - (k-l) > 0$ and at least one of these two expressions must be greater than or equal to $\frac{1}{2}(2N-k)$ we get:

$$F_1(N,k,L,l) \leq \frac{1}{\sqrt{2\pi}}$$

From the fact that $\frac{1}{12n+1} \leq \lambda_n \leq \frac{1}{12n}$ and the assumption that $2L - l > 0$ and $2(N-L) - (k-l) > 0$ and as a consequence also:

$$3L - l > 0,\ 3(N-L) - (k-l) > 0,\ 6L - 2l > 0 \text{ and } 6(N-L) - 2(k-l) > 0$$

we get that:

$$F_3(N,k,L,l) \leq e^{-\frac{2}{39}}$$

For both of these bounds we are not really interested in the exact constants, only the fact that such constants exist is important.

For F_2 we have the following lemma:

LEMMA 3.16. *Let $N > 0$, $k > 0$ and $1 \leq l \leq k-1$. Then the function*

$$\Phi_{N,k,l} : \left[\left\lceil \frac{l}{2} \right\rceil, N - \left\lceil \frac{1}{2}(k-l) \right\rceil\right] \to \mathbb{R}$$

defined by:

$$\Phi_{N,k,l}(x) = F_2(N,k,x,l)$$

for all $x \in \left[\left\lceil \frac{l}{2} \right\rceil, N - \left\lceil \frac{1}{2}(k-l) \right\rceil\right]$ is a convex function.

PROOF. To prove this we will look at the derivative of $F_2(N,k,x,l)$ with respect to x and show that it is monotonely increasing. To shorten the expressions a bit, we will only look at the factor in $F_2(N,k,x,l)$ that actually depends on x. So we set:

$$G_2(N,k,x,l) = \frac{(6(N-x)-2(k-l))^{3(N-x)-(k-l)}(6x-2l)^{3x-l}}{(2(N-x)-(k-l))^{2(N-x)-(k-l)}(2x-l)^{2x-l}}$$

We can write:

$$G_2(N,k,x,l) = \frac{e^{(3(N-x)-(k-l))\log(6(N-x)-2(k-l))+(3x-l)\log(6x-2l)}}{e^{(2(N-x)-(k-l))\log(2(N-x)-(k-l))+(2x-l)\log(2x-l)}}$$

So we get:

$$\frac{\frac{\partial}{\partial x}G_2(N,k,x,l)}{G_2(N,k,x,l)} = \left(3\log\left(\frac{6x-2l}{6(N-x)-2(k-l)}\right) + 2\log\left(\frac{2(N-x)-(k-l)}{2x-l}\right)\right)$$

Because $G_2(N,k,x,l) > 0$ we see that the derivative turns from negative (when x is small compared to N) to positive. This means that $G_2(N,k,x,l)$ turns from monotonely decreasing to monotonely increasing. The second factor in the derivative is monotonely increasing and the point where it turns from negative to positive is exactly the point where $G_2(N,k,x,l)$ turns from decreasing to increasing. This means that the product of the two (the derivative) is still monotonely increasing, which concludes the proof. ☐

The reason that this is interesting is that this implies that the maxima of $F_2(N,k,\cdot,l)$ on
$\left[\left\lceil \frac{l}{2} \right\rceil, N - \left\lceil \frac{1}{2}(k-l) \right\rceil\right]$ lie on the edges of this interval. The same is true if we look for the maxima of $F_2(N,k,\cdot,l)$ on the subinterval

$$\left[\left\lceil \frac{l}{2} \right\rceil + 1, N - \left\lceil \frac{1}{2}(k-l) \right\rceil - 1\right] \subset \left[\left\lceil \frac{l}{2} \right\rceil, N - \left\lceil \frac{1}{2}(k-l) \right\rceil\right]$$

The lemma allows us to prove the following:

PROPOSITION 3.17. *There exists a $R \in (0,\infty)$ such that for all $N, k \in \mathbb{N}$ with $k \geq 2$ and $N \geq k^2$ we have:*

$$\mathbb{P}\left[D^o_{N,k}\right] \leq \frac{1}{N}\frac{Rk^3}{k!}\frac{(3N-k)!}{(3N)!}$$

PROOF. The proof consists of applying Lemma 3.16 to the sum of binomial coefficients that appears in the expression we have for $\mathbb{P}\left[D^o_{N,k}\right]$. We have:

$$\begin{aligned}\sum_{L=\lceil\frac{l}{2}\rceil}^{N-\lceil\frac{1}{2}(k-l)\rceil} \frac{\binom{2N-k}{2L-l}\binom{3N-k}{3L-l}}{\binom{6N-2k}{6L-2l}} \leq{}& \frac{\binom{2N-k}{2\lceil\frac{l}{2}\rceil-l}\binom{3N-k}{3\lceil\frac{l}{2}\rceil-l}}{\binom{6N-2k}{6\lceil\frac{l}{2}\rceil-2l}} + \frac{\binom{2N-k}{2(N-\lceil\frac{1}{2}(k-l)\rceil)-l}\binom{3N-k}{3(N-\lceil\frac{1}{2}(k-l)\rceil)-l}}{\binom{6N-2k}{6(N-\lceil\frac{1}{2}(k-l)\rceil)-2l}}\\
&+ N\frac{e^{-\frac{2}{39}}}{\sqrt{2\pi}}\max\left\{F_2(N,k,\lceil\tfrac{l}{2}\rceil+1,l), F_2(N,k,N-\lceil\tfrac{1}{2}(k-l)\rceil-1,l)\right\} \qquad (8)\end{aligned}$$

So we need to study the four terms that appear on the right hand side above. We will only treat the two terms that correspond to $L = \lceil\frac{l}{2}\rceil$ and $L = \lceil\frac{l}{2}\rceil + 1$. The analysis for the other two terms is analogous, the only difference is that $k-l$ takes over the role of l.

If l is even we have:

$$\begin{aligned}\frac{\binom{2N-k}{2\lceil\frac{l}{2}\rceil-l}\binom{3N-k}{3\lceil\frac{l}{2}\rceil-l}}{\binom{6N-2k}{6\lceil\frac{l}{2}\rceil-2l}} &= \frac{\binom{3N-k}{\frac{l}{2}}}{\binom{6N-2k}{l}}\\
&= \frac{l!}{(\frac{l}{2})!(6N-2k-1)(6N-2k-3)\cdots(6N-2k-l+1)}\\
&\leq \frac{l(l-1)\cdots(\frac{l}{2}+1)}{(6N-3k)^{\frac{l}{2}}}\\
&\leq \frac{l(l-1)\cdots(\frac{l}{2}+1)}{(6N-3k)k^{l-1}}\\
&\leq \frac{k\cdot l!}{(6N-3k)k(k-1)\cdots(k-l+1)}\\
&= \frac{k}{(6N-3k)\binom{k}{l}}\end{aligned}$$

where we have used the assumption that $N \geq k^2$ and hence $6N-3k \geq k^2$.

Similarly, for odd l we have:

$$\frac{\binom{2N-k}{2\lceil\frac{l}{2}\rceil-l}\binom{3N-k}{3\lceil\frac{l}{2}\rceil-l}}{\binom{6N-2k}{6\lceil\frac{l}{2}\rceil-2l}} \leq \frac{(k+3)(k+2)}{(6N-3k)\binom{k}{l}}$$

For the second term we have:

$$\begin{aligned}F_2(N,k,\left\lceil\frac{l}{2}\right\rceil+1,l) &= \frac{(2N-k)^{2N-k}6(N-\lceil\frac{l}{2}\rceil)-2(k-l)-6)^{3(N-\lceil\frac{l}{2}\rceil)-(k-l)-3}}{(2(N-\lceil\frac{l}{2}\rceil)-(k-l)-2)^{2(N-\lceil\frac{l}{2}\rceil)-(k-l)-2}}\\
&\quad\cdot\frac{((6\lceil\frac{l}{2}\rceil-2l+6)^{3\lceil\frac{l}{2}\rceil-l+3}}{(6N-2k)^{3N-k}(2\lceil\frac{l}{2}\rceil-l+2)^{2\lceil\frac{l}{2}\rceil-l+2}}\end{aligned}$$

So if l is even we get:

$$F_2(N,k,\left\lceil\frac{l}{2}\right\rceil+1,l)=\frac{(2N-k)^{2N-k}(6N-2k-l-6)^{3N-k-\frac{1}{2}l-3}(l+6)^{\frac{l}{2}+3}}{(2N-k-2)^{2N-k-2}2^2(6N-2k)^{3N-k}}$$

For all $x\in\mathbb{R}$ we have $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x$. This means that for all $x\in\mathbb{R}$ there exists a constant $M_x\in(0,\infty)$ such that $\left(1+\frac{x}{n}\right)^n\leq M_x\,e^x$ for all $n\in\mathbb{N}$. So:

$$\begin{aligned}F_2(N,k,\left\lceil\frac{l}{2}\right\rceil+1,l)&\leq\frac{M_2\,e^2}{4}\frac{(2N-k)^2(6N-2k-l-6)^{3N-k-\frac{1}{2}l-3}(l+6)^{\frac{l}{2}+3}}{(6N-2k)^{3N-k}}\\&\leq\frac{M_2\,e^2}{4}\frac{(l+6)^{\frac{l}{2}+3}}{(6N-2k)^2k^{l-2}}\end{aligned}$$

Using Theorem 1.27 we see that there exists a constant $K\in(0,\infty)$ such that $(l+6)^{\frac{l}{2}+3}\leq K\cdot l!$. So there exists a constant $K'\in(0,\infty)$ such that for even l:

$$F_2(N,k,\left\lceil\frac{l}{2}\right\rceil+1,l)\leq\frac{K'}{(6N-2k)^2\binom{k}{l}}$$

The proof for odd l is analogous and as we have already mentioned, so are the corresponding proofs for $L=N-\left\lceil\frac{1}{2}(k-l)\right\rceil$. Hence there exists a constant $K''\in(0,\infty)$ such that:

$$\max\left\{F_2(N,k,\left\lceil\frac{l}{2}\right\rceil+1,l),F_2(N,k,N-\left\lceil\frac{1}{2}(k-l)\right\rceil-1,l)\right\}\leq\frac{K''}{(6N-2k)^2\binom{k}{l}}$$

So if we fill in equation 8 we get:

$$\sum_{L=\left\lceil\frac{l}{2}\right\rceil}^{N-\left\lceil\frac{1}{2}(k-l)\right\rceil}\frac{\binom{2N-k}{2L-l}\binom{3N-k}{3L-l}}{\binom{6N-2k}{6L-2l}}\leq\frac{k}{(6N-3k)\binom{k}{l}}+\frac{(k+3)(k+2)}{(6N-3k)\binom{k}{l}}+N\frac{e^{-\frac{2}{39}}}{\sqrt{2\pi}}\frac{K''}{(6N-2k)^2\binom{k}{l}}$$

So there exists a constant $B\in(0,\infty)$ such that:

$$\sum_{l=1}^{k-1}\binom{k}{l}\sum_{L=\left\lceil\frac{l}{2}\right\rceil}^{N-\left\lceil\frac{1}{2}(k-l)\right\rceil}\frac{\binom{2N-k}{2L-l}\binom{3N-k}{3L-l}}{\binom{6N-2k}{6L-2l}}\leq\frac{Bk^3}{N}$$

Now we can fill in this bound in the expression we have for $\mathbb{P}\left[D^o_{N,k}\right]$ in equation 7 and we get:

$$\begin{aligned}\mathbb{P}\left[D^o_{N,k}\right]&\leq\frac{6^k}{2}\frac{\binom{2N}{k}Bk^3}{N\cdot 6N(6N-1)\cdots(6N-2k+1)}\\&=\frac{3^k}{2\cdot k!}\frac{2N}{6N-2k+1}\left(\prod_{i=1}^{k}\frac{2N-i}{6N-2i+1}\right)\frac{Bk^3}{N\cdot 3N(3N-1)\cdots(3N-k+1)}\end{aligned}$$

We have $3^{k-1}\prod_{i=1}^{k}\frac{2N-i}{6N-2i+1}=\prod_{i=1}^{k}\frac{6N-3i}{6N-2i+1}\leq 1$, because every factor in the product is at most equal to 1. Thus:

$$\mathbb{P}\left[D^o_{N,k}\right]\leq\frac{3}{2\cdot k!}\frac{2N}{6N-2k+1}\frac{Bk^3}{N\cdot 3N(3N-1)\cdots(3N-k+1)}$$

So there exists an $R\in(0,\infty)$ such that:

$$\mathbb{P}\left[D^o_{N,k}\right]\leq\frac{1}{N}\frac{Rk^3}{k!3N(3N-1)\cdots(3N-k+1)}$$

which is what we wanted to prove. □

3.4.4. The limit. Recall that $G^o_{N,k}\subset\Omega^o_{N,k}$ denotes the set of cubic graphs on $2N$ vertices that contain a separating circuit of k edges. Using Lemma 3.13 and Proposition 3.17 we are now able to prove the following:

THEOREM 3.18. *Let $C\in(0,1)$. For every $\varepsilon>0$:*

$$\mathbb{P}_N\left[\bigcup_{2\leq k\leq C\log_2(N)}G^o_{N,k}\right]=\mathcal{O}\left(N^{1-C-\varepsilon}\right)\ \text{as } N\to\infty$$

PROOF. We have:

$$\begin{aligned}\mathbb{P}_N\left[\bigcup_{2\leq k\leq C\log_2(N)}G^o_{N,k}\right]&\leq\sum_{2\leq k\leq C\log_2(N)}\mathbb{P}_N\left[G^o_{N,k}\right]\\&\leq\sum_{2\leq k\leq C\log_2(N)}2^k(k-1)!\frac{(3N)!}{(3N-k)!}\mathbb{P}_N\left[D^o_{N,k}\right]\end{aligned}$$

where we have used Lemma 3.13 in the last step. Proposition 3.17 now tells us that there exists an $R\in(0,\infty)$ such that:

$$\mathbb{P}_N\left[\bigcup_{2\leq k\leq C\log_2(N)}G^o_{N,k}\right]\leq\sum_{2\leq k\leq C\log_2(N)}2^k\frac{Rk^2}{N}$$

The last term (corresponding to $k=\lfloor C\log_2(N)\rfloor$) in the sum above is the biggest term, so we get:

$$\begin{aligned}\mathbb{P}_N\left[\bigcup_{2\leq k\leq C\log_2(N)}G^o_{N,k}\right]&\leq\lfloor C\log_2(N)\rfloor\,2^{\lfloor C\log_2(N)\rfloor}\frac{R(\lfloor C\log_2(N)\rfloor)^2}{N}\\&\leq\frac{R(C\log_2(N))^3}{N^{1-C}}\end{aligned}$$

which proves the statement. □

CHAPTER 4

Subsurfaces

In this chapter we will compute probabilities of the form:

$$\mathbb{P}_N\left[X \subset S\right] \text{ and } \mathbb{P}_N\left[X \subset S \middle|\, g \in D_N\right]$$

where $D_N \subset \mathbb{N}$ and X is some fixed labelled triangulated surface (with boundary). Often X will be a triangulated annulus with a circuit as a dual graph. The main result of this chapter will be that under suitable conditions on the genus, the probability on the right hand side will asymptotically be the same as the corresponding unconditional probability (Theorems 4.7 and 4.11). The examples we will see are conditions on the genus so that the resulting set is non-negligible (Theorem 4.7) and maximal genus (Theorem 4.11).

4.1. The unconditional case

We start with computing the probability that a random surface contains a fixed labelled subsurface, without any restrictions on the surface itself. Such a subsurface is completely determined by side pairings. So what we are looking for is the probability that a partition $\omega \in \Omega_N$ contains a fixed set of pairs (or equivalently that a random cubic graphs contains a fixed set of labelled edges). We have the following lemma, which is standard in the theory of random graphs but which we will prove for completeness.

LEMMA 4.1. *Let ω' be a partition of $2k$ elements of $\{1,\ldots,6N\}$ into pairs, then:*

$$\mathbb{P}_N\left[\omega' \subset \omega \in \Omega_N\right] = \frac{1}{(6N-1)(6N-3)\cdots(6N-2k+1)}$$

PROOF. We have:

$$\begin{aligned}\mathbb{P}_N\left[\omega' \subset \omega \in \Omega_N\right] &= \frac{\left|\{\omega \in \Omega_N;\ \omega' \subset \omega\}\right|}{|\Omega_N|} \\ &= \frac{(6N-2k-1)(6N-2k-3)\cdots 1}{(6N-1)(6N-3)\cdots 1} \\ &= \frac{1}{(6N-1)(6N-3)\cdots(6N-2k+1)}\end{aligned}$$

□

Like we said above, the goal of the next two sections is to show that under two types of suitable conditions Lemma 4.1 asymptotically still holds.

4.2. Non-negligible restrictions on the genus

The first such condition is a condition on the genus such that the resulting set of surfaces is non-negligible in the sense of Definition 2.2. To prove the analogue of Lemma 4.1 in this setting we will reason in a somewhat reverted order. That is, we will prove that the distribution of the genus conditioned on our random surface containing a fixed labelled subsurface is the same as the unconditional distribution. After this we will 'invert' the distribution to get the result we want. This inversion is the reason we need to assume the non-negligibility, without this assumption the proof would not work.

In order to prove that the genus distribution does not change we invoke the Diaconis-Shahshahani upper bound lemma (Lemma 1.25) in a way similar to Gamburd's proof of Theorem 2.9 in [**Gam06**]. Gamburd applies this lemma to the distribution of σ and τ as elements of the alternating group. It turns out that this switch to the alternating group is essential, as one has to avoid the sign representation of the symmetric group.

In our case σ and τ will be elements of conjugacy classes depending on the underlying subsurface. Unfortunately, as such they do not generally lie in the alternating group. This problem will be solved by the following two lemmas (the notation of which can be found in Section 2.6). The first of the two relates the size of the boundary of and oriented graph and the genus of this graph as a surface to the number of edges in the given graph:

LEMMA 4.2. *Let N be even and let H be a connected labelled oriented graph in which every vertex has degree at most 3. Furthermore suppose that H contains M edges and no left hand turn cycles. Finally, let g be the genus of the triangulated surface with boundary associated to H, b its number of boundary components and ℓ_i the number of sides of triangles in its i^{th} boundary component for $i = 1, \ldots, b$. Then:*

$$M = 3(b - 2 + 2g) + \sum_{i=1}^{b} \ell_i$$

PROOF. We have:

$$2M = n_1 + 2n_2 + 3n_3$$

where n_i is the number of vertices of degree i in H for $i = 1, 2, 3$. Furthermore:

$$\sum_{i=1}^{b} \ell_i = 2n_1 + n_2$$

because every vertex of degree i in H contributes $3 - i$ sides to the boundary of the associated surface. If we fill the boundary of the surface associated to H with polygons, we get a closed surface of genus g. From the Euler characteristic we then get:

$$V - E + F = 2 - 2g$$

We have:

$$V = \sum_{i=1}^{b} \ell_i, \quad E = M + \sum_{i=1}^{b} \ell_i \text{ and } F = n_1 + n_2 + n_3 + b = \frac{2M + \sum_{i=1}^{b} \ell_i}{3} + b$$

So we get:

$$-\frac{1}{3}M + \frac{1}{3}\sum_{i=1}^{b} \ell_i + b = 2 - 2g$$

which implies the lemma. $\square$

We can now prove that the product $\sigma\tau$ always lies in the alternating group:

LEMMA 4.3. *Let N be even and let H be a labelled oriented graph in which every vertex has degree at most 3. Furthermore suppose that H contains M edges and no left hand turn cycles. If $\sigma \in K_3(H,N)$ and $\tau \in K_2(H,N)$ then $\sigma\tau \in \mathrm{A}_{6N-2M}$.*

PROOF. Suppose the triangulated surface corresponding to H consists of L triangles and has b boundary components of ℓ_i sides for $i = 1, \ldots, b$ respectively. If we suppose that H is connected, then as a triangulated surface H contains

$$3(b - 2 + 2g) + \sum_{i=1}^{b} \ell_i$$

inner diagonals. That is, as a graph it has $3(b-2+2g) + \sum_{i=1}^{b} \ell_i$ edges. This means that:

$$\sigma \in K\left(\left(\prod_{i=1}^{b} \ell_i\right) \cdot 3^{2N-L}\right) \text{ and } \tau \in K\left(2^{3N-3(b-2+2g)-\sum_{i=1}^{b} \ell_i}\right)$$

So $\sigma \in \mathrm{A}_{6N-2M}$ if and only if

$$|\{i;\ \ell_i \text{ is even}\}| \in 2\mathbb{N}$$

Furthermore, $\tau \in \mathrm{A}_{6N-2M}$ if and only if

$$3(b - 2 + 2g) + \sum_{i=1}^{b} \ell_i \in 2\mathbb{N}$$

These two conditions are equivalent. This means that either both σ and τ lie in A_{6N-2M} in which case their product does as well or they both lie in $\mathrm{S}_{6N-2M} \backslash \mathrm{A}_{6N-2M}$ in which case their product also lies in A_{6N-2k}. If H is disconnected we see from the proof above that each connected component of H adds either an even number of even cycles to both σ and τ or an odd number of even cycles to both σ and τ. $\square$

This lemma implies that the probability measure of the product $\sigma\tau$ can be seen as a probability measure on A_{6N-2M} when N is even. We will denote the probability measure by $\mathbb{P}_{3\star 2,H,N}$. This is also a probability measure on S_{6N-2M}, and as such we have:

$$\mathbb{P}_{3\star 2,H,N} = \mathbb{P}_{3,H,N} \star \mathbb{P}_{2,H,N}$$

where $\mathbb{P}_{3,H,N}$ and $\mathbb{P}_{2,H,N}$ are the uniform probability measures on $K_3(H,N)$ and $K_2(H,N)$ repsectively and $\star$ denotes the convolution product. Recall that the randomly chosen element in $K_3(H,N)$ can be interpreted as describing the orientation at the triangles and polygons determined by H and that the element in $K_2(H,N)$ describes which side is glued to which other side.

Because of the lemma above we will assume that N is even for the remainder of this section.

4.2.1. Some inequalities for self associated tableaux. In our proof, which relies on the Diaconis-Shahshahani upper bound lemma and hence the character theory of A_N, we will be relating the characters of the alternating group to those of the symmetric group. Theorem 1.42 shows us that the characters corresponding to self associated partitions might cause a problem. To solve this, we have the following upper bounds, for which we recall that the numbers $h(i,j)$ denote hook lengths and the numbers f^λ denote the dimensions of irreducible S_N-representations associated to partitions $\lambda \models N$:

Proposition 4.4. *There exists a constant $A > 0$ independent of N such that for any partition λ of N with $\lambda' = \lambda$ we have:*

$$\frac{\prod_{i=1}^{d} h(i,i)}{f^\lambda} \leq A^N N^{\sqrt{N}-N/2}$$

where d is the number of boxes in the main diagonal of λ.

Proof. The hook length formula (Theorem 1.29) gives us:

$$\frac{\prod_{i=1}^{d} h(i,i)}{f^\lambda} = \frac{\prod_{i=1}^{d} h(i,i) \prod_{i,j=1}^{r} h(i,j)}{N!}$$

where r is the number of rows in λ. Hence, by the arithmetic-geometric mean inequality we get:

$$\frac{\prod_{i=1}^{d} h(i,i)}{f^\lambda} \leq \frac{\left(\frac{1}{N+d}\left(\sum_{i=1}^{d} h(i,i) + \sum_{i,j=1}^{r} h(i,j)\right)\right)^{N+d}}{N!} = \frac{\left(\frac{2}{N+d}\left(\sum_{i\leq j=1}^{r} h(i,j)\right)\right)^{N+d}}{N!}$$

where the last step follows from the fact that $\lambda = \lambda'$ and hence that $h(i,j) = h(j,i)$ for all $i, j = 1, \ldots, r$. From the fact that $\lambda' = \lambda$ we also get that:

$$h(i,j) \leq \frac{1}{2}\left(h(i,i) + h(j,j)\right)$$

for $i, j = 1, \ldots, r$, where we set $h(i,i) = 0$ if $(i,i) \notin \lambda$. Note that we have equality if and only if both $(i,i) \in \lambda$ and $(j,j) \in \lambda$. So:

$$\begin{aligned}
\frac{\prod_{i=1}^{d} h(i,i)}{f^\lambda} &\leq \frac{\left(\frac{1}{N+d}\left(\sum_{i\leq j=1}^{r} h(i,i) + h(j,j)\right)\right)^{N+d}}{N!} \\
&= \frac{\left(\frac{1}{N+d}\left(\sum_{i=1}^{d}(d-i+1)h(i,i) + \sum_{i=1}^{d} ih(j,j)\right)\right)^{N+d}}{N!} \\
&= \frac{\left(\frac{(d+1)N}{N+d}\right)^{N+d}}{N!} \\
&\leq \frac{(d+1)^{N+d}}{N!}
\end{aligned}$$

We have $d \leq \sqrt{N}$, because a tableau with a main diagonal of d boxes must contain a $d \times d$ square. Hence:

$$\begin{aligned}
\frac{\prod_{i=1}^{d} h(i,i)}{f^\lambda} &\leq \frac{(\sqrt{N}+1)^{N+\sqrt{N}}}{N!} \\
&\leq A^N \frac{\sqrt{N}^{N+\sqrt{N}}}{N^N} \\
&\leq A^N \frac{1}{N^{N/2-\sqrt{N}}}
\end{aligned}$$

for some constant $A > 0$ independent of N, which comes out of Stirling's approximation. Note that we could get explicit constants A and B, but since they won't be needed and will only complicate the formulas, we choose not to compute them. □

Furthermore, we will need the following:

Proposition 4.5. *For any partition λ of N with $\lambda' = \lambda$ we have:*

$$\frac{1}{f^\lambda} \leq \frac{\left(\sqrt{N}+1\right)^N}{N!}$$

PROOF. From the arithmetic-geometric mean inequality we obtain:

$$\begin{aligned}
\frac{1}{f^\lambda} &= \frac{\prod\limits_{(i,j)\in\lambda} h(i,j)}{N!} \\
&\leq \frac{\left(\frac{1}{N}\sum\limits_{(i,j)\in\lambda} h(i,j)\right)^N}{N!} \\
&= \frac{\left(\frac{1}{N}\left(2\sum\limits_{i<j,(i,j)\in\lambda} h(i,j)+\sum\limits_{i=1}^{d} h(i,i)\right)\right)^N}{N!}
\end{aligned}$$

Reasoning in a similar way to the previous proof we get:

$$\begin{aligned}
\frac{1}{f^\lambda} &\leq \frac{\left(\frac{1}{N}\left(dN+N\right)\right)^N}{N!} \\
&= \frac{(d+1)^N}{N!} \\
&\leq \frac{\left(\sqrt{N}+1\right)^N}{N!}
\end{aligned}$$

□

4.2.2. Proof of the main theorem. Now we can prove the following theorem, which is the main theorem of this section and will imply the analogue of Lemma 4.1. Recall that the notation '$\overset{d}{\sim}$' means that the total variational distance of the given two sequences of random variables tends to zero.

THEOREM 4.6. *Let N be even and let H be a labelled oriented graph in which every vertex has degree at most 3. Furthermore suppose that H contains M edges and no left hand turn cycles. Then:*

$$\mathbb{P}_{3\star 2,H,N} \overset{d}{\sim} \mathbb{U}_{H,N} \text{ as } N\to\infty$$

where $\mathbb{U}_{H,N}$ denotes the uniform probability measure on A_{6N-2M}.

PROOF. To lighten notation we are going to drop the subscripts in $\mathbb{P}_{3\star 2,H,N}$ and $\mathbb{U}_{H,N}$ and we will write $r = 6N-2M$. Furthermore, characters denoted with a ζ will always be A_r-characters and characters denoted with a χ will always be S_r-characters.

The Diaconis-Shahshahani upper bound lemma (Lemma 1.25) in combination with Lemma 4.3, which tells us that when $\sigma \in K_3(H,N)$ and $\tau \in K_2(H,N)$ then their product $\sigma\tau$ lies in A_r, gives us:

$$\begin{aligned} d(\mathbb{P},\mathbb{U})^2 &\leq \frac{1}{4}\sum_{\substack{\rho\in\widehat{\mathrm{A}_r}\\ \rho\neq\mathrm{id}}} \dim(\rho)\mathrm{tr}\left(\hat{\mathbb{P}}(\rho)\overline{\hat{\mathbb{P}}(\rho)}\right) \\ &= \frac{1}{4}\sum_{\substack{\rho\in\widehat{\mathrm{A}_r}\\ \rho\neq\mathrm{id}}} \frac{1}{\dim(\rho)} \sum_{\substack{K,L \text{ conjugacy}\\ \text{classes of } \mathrm{A}_r}} \mathbb{P}[K]\,\mathbb{P}[L]\,|K|\,|L|\,\zeta^\rho(K)\zeta^\rho(L) \end{aligned}$$

where we have used the fact that:

$$\hat{\mathbb{P}}(\rho) = \frac{1}{\dim(\rho)} \sum_{K \text{ conjugacy class of } \mathrm{A}_r} \mathbb{P}[K]\,|K|\,\zeta^\rho(K) I_{\dim(\rho)}$$

where $\mathbb{P}[K] = \mathbb{P}[\pi]$ for any $\pi \in K$ (and is not to be confused with the probability of obtaining an element in K, which is equal to $\mathbb{P}[K]\,|K|$). This follows from the fact that $\mathbb{P}$ is constant on conjugacy classes and Lemma 1.24.

If K is a conjugacy class of S_r such that $K \cap \mathrm{A}_r = \emptyset$ then it follows from Lemma 4.3 that $\mathbb{P}[K] = 0$. This means that we can add all these conjugacy classes to the sum above. Furthermore, Theorem 1.42 tells us how to relate A_r-characters to S_r-characters, so we get:

$$\begin{aligned} d(\mathbb{P},\mathbb{U})^2 \leq\ & \frac{1}{2}\sum_{\substack{\lambda\vDash r,\lambda\neq\lambda',\\ \lambda\neq(r),(1,1,\ldots,1)}} \sum_{\substack{K,L \text{ conjugacy}\\ \text{classes of } \mathrm{S}_r}} \frac{\mathbb{P}[K]\,\mathbb{P}[L]\,|K|\,|L|\,\chi^\lambda(K)\chi^\lambda(L)}{f^\lambda} \\ & + \sum_{\substack{\lambda\vDash r\\ \lambda=\lambda'}} \sum_{\substack{L \text{ conjugacy}\\ \text{class of } \mathrm{S}_r\\ L\neq K(\lambda)}} \frac{\mathbb{P}[H^+(\lambda)]\,\mathbb{P}[L]\,|H^+(\lambda)|\,|L|\,\zeta^\lambda(H^+(\lambda))\chi^\lambda(L)}{f^\lambda} \\ & + \sum_{\substack{\lambda\vDash r\\ \lambda=\lambda'}} \sum_{\substack{L \text{ conjugacy}\\ \text{class of } \mathrm{S}_r\\ L\neq K(\lambda)}} \frac{\mathbb{P}[H^-(\lambda)]\,\mathbb{P}[L]\,|H^-(\lambda)|\,|L|\,\zeta^\lambda(H^-(\lambda))\chi^\lambda(L)}{f^\lambda} \\ & + \sum_{\substack{\lambda\vDash r\\ \lambda=\lambda'}} \sum_{i,j=\pm} \frac{\mathbb{P}[H^i(\lambda)]\,|H^i(\lambda)|\,\mathbb{P}[H^j(\lambda)]\,|H^j(\lambda)|\,\zeta^\lambda(H^i(\lambda))\zeta^\lambda(H^j(\lambda))}{f^\lambda} \end{aligned}$$

We now use the fact that value of a S_r character of a self associated partition λ on $H^{\pm}(\lambda)$ is a power of -1 (Lemma 1.43) to obtain:

$$
\begin{aligned}
d(\mathbb{P},\mathbb{U})^2 \leq & \frac{1}{2} \sum_{\substack{\lambda\models r,\lambda\neq\lambda' \\ \lambda\neq(r),(1,1,\ldots,1)}} f^{\lambda}\mathrm{tr}\left(\widehat{\mathbb{P}(\lambda)}\overline{\widehat{\mathbb{P}(\lambda)}}\right) \\
& + \sum_{\substack{\lambda\models r \\ \lambda=\lambda'}} \mathbb{P}\left[H^{+}(\lambda)\right]\left|H^{+}(\lambda)\right|\zeta^{\lambda}(H^{+}(\lambda))\left(\mathrm{tr}\left(\widehat{\mathbb{P}(\lambda)}\right)+\frac{2}{f^{\lambda}}\right) \\
& + \sum_{\substack{\lambda\models r \\ \lambda=\lambda'}} \mathbb{P}\left[H^{-}(\lambda)\right]\left|H^{-}(\lambda)\right|\zeta^{\lambda}(H^{-}(\lambda))\left(\mathrm{tr}\left(\widehat{\mathbb{P}(\lambda)}\right)+\frac{2}{f^{\lambda}}\right) \\
& + \sum_{\substack{\lambda\models r \\ \lambda=\lambda'}} \sum_{i,j=\pm} \frac{\mathbb{P}\left[H^{i}(\lambda)\right]\left|H^{i}(\lambda)\right|\mathbb{P}\left[H^{j}(\lambda)\right]\left|H^{j}(\lambda)\right|\zeta^{\lambda}(H^{i}(\lambda))\zeta^{\lambda}(H^{j}(\lambda))}{f^{\lambda}}
\end{aligned}
$$

We first want to get rid of the last three sums, because the first sum is the analogue of one that appears in the proof of Theorem 2.9 by Gamburd in [**Gam06**]. For this we are going to use Theorem 1.42, Propositions 4.4 and 4.5 and estimates similar to the ones in the proofs of these propositions. For a self associated partition λ with d blocks on its main diagonal we have:

$$
\begin{aligned}
\left|\mathbb{P}\left[H^{\pm}(\lambda)\right]\left|H^{\pm}(\lambda)\right|\zeta^{\lambda}(H^{\pm}(\lambda))\right| &\leq \left|\zeta^{\lambda}(H^{+}(\lambda))\right| \\
&\leq 1+\prod_{i=1}^{d} h(i,i)
\end{aligned}
$$

Using the arithmetic-geometric mean inequality we get:

$$
\begin{aligned}
\left|\mathbb{P}\left[H^{\pm}(\lambda)\right]\left|H^{\pm}(\lambda)\right|\zeta^{\lambda}(H^{\pm}(\lambda))\right| &\leq 1+\left(\frac{1}{d}\sum_{i=1}^{d} h(i,i)\right)^{d} \\
&= 1+\left(\frac{N}{d}\right)^{d}
\end{aligned}
$$

Furthermore, because now we are working in the symmetric group and the Fourier transform turns convolution into ordinary multiplication (see for instance Lemma 1 of [**DS81**]), we have:

$$
\begin{aligned}
\mathrm{tr}\left(\widehat{\mathbb{P}(\lambda)}\right) &= \mathrm{tr}\left(\widehat{\mathbb{P}_3(\lambda)\mathbb{P}_2(\lambda)}\right) \\
&= \frac{\chi^{\lambda}(K_3)\chi^{\lambda}(K_2)}{f^{\lambda}}
\end{aligned}
$$

In Lemma 1.32 we have already seen that if an element $g\in\mathrm{S}_N$ contains k cycles of length m then:

$$
\left|\chi^{\lambda}(g)\right| \leq \max\left\{\left|\chi^{\mu}(a)\right|;\ a\in\mathrm{S}_{N-km},\ \mu\models N-km\right\} f^{\lambda}_{m}
$$

This means that:

$$\left|\chi^\lambda(K_3)\right| \leq \max\left\{\left|\chi^\mu(a)\right|;\ a \in \mathrm{S}_M,\ \mu \models M\right\} f_3^\lambda$$

and:

$$\left|\chi^\lambda(K_2)\right| = f_2^\lambda$$

because τ contains only 2-cycles. Note that the first factor in the upper bound for σ does not depend on N but only on the finite set of words W we fix. We will write:

$$\left|\chi^\lambda(K_3)\right| \leq C f_3^\lambda$$

So we obtain:

$$\begin{aligned}\sum_{\substack{\lambda\models r\\ \lambda=\lambda'}} & \mathbb{P}\left[H^+(\lambda)\right]\left|H^+(\lambda)\right|\zeta^\lambda(H^+(\lambda))\left(\mathrm{tr}\left(\widehat{\mathbb{P}(\lambda)}\right)+\frac{2}{f^\lambda}\right)\\ &\leq \sum_{\substack{\lambda\models r\\ \lambda=\lambda'}} \mathbb{P}\left[H^+(\lambda)\right]\left|H^+(\lambda)\right|\zeta^\lambda(H^+(\lambda))\frac{Cf_3^\lambda f_2^\lambda+2}{f^\lambda}\end{aligned}$$

Now we are going to use the upper bound on f_2^λ and f_3^λ of Theorem 1.34, which gives us:

$$\frac{Cf_3^\lambda f_2^\lambda+2}{f^\lambda} \leq \frac{C\frac{k_3!\,3^{k_3}}{(r!)^{1/3}}\left(f^\lambda\right)^{1/3}\frac{k_2!\,2^{k_2}}{(r!)^{1/2}}\left(f^\lambda\right)^{1/2}+2}{f^\lambda}$$

Where k_2 and k_3 are the numbers of skew 2 and 3 hooks that can be removed from λ. We have: $k_2 \leq r/2$ and $k_3 \leq r/3$. Hence:

$$\begin{aligned}\frac{Cf_3^\lambda f_2^\lambda+2}{f^\lambda} &\leq \frac{C(r/3)!\;3^{r/3}\;(r/2)!\;2^{r/2}}{(r!)^{5/6}\left(f^\lambda\right)^{1/6}}+\frac{2}{f^\lambda}\\ &\leq C'\frac{\sqrt{r}\sqrt{r}2^{r/2}3^{r/3}\left(\frac{r}{2e}\right)^{r/2}\left(\frac{r}{3e}\right)^{r/3}}{r^{5/12}\left(\frac{r}{e}\right)^{5r/6}}\frac{1}{\left(f^\lambda\right)^{1/6}}+\frac{2}{f^\lambda}\\ &= C'r^{7/12}\frac{1}{\left(f^\lambda\right)^{1/6}}+\frac{2}{f^\lambda}\end{aligned}$$

where the second inequality comes from Stirling's approximation. We will now use the upper bound for $\frac{1}{f^\lambda}$ for λ self associated from Proposition 4.5. Combining this with all the above, we get:

$$\begin{aligned}
&\sum_{\substack{\lambda\vDash r\\ \lambda=\lambda'}} \mathbb{P}\left[H^{\pm}(\lambda)\right]\left|H^{\pm}(\lambda)\right|\zeta^{\lambda}(H^{\pm}(\lambda))\left(\mathrm{tr}\left(\widehat{\mathbb{P}(\lambda)}\right)+\frac{2}{f^{\lambda}}\right)\\
&\leq \sum_{\substack{\lambda\vDash r\\ \lambda=\lambda'}}\left(1+\left(\frac{r}{d_\lambda}\right)^{d_\lambda}\right)\left(C' r^{7/2}\left(\frac{\left(\sqrt{r}+1\right)^r}{r!}\right)^{1/6}+2\frac{\left(\sqrt{r}+1\right)^r}{r!}\right)\\
&\leq p(r)\left(1+(r)^{\sqrt{r}}\right)\left(C' r^{7/2}\left(\frac{\left(\sqrt{r}+1\right)^r}{r!}\right)^{1/6}+2\frac{\left(\sqrt{r}+1\right)^r}{r!}\right)\\
&\leq C'' A^r r^{\sqrt{r}}\left(\frac{1}{r^{r/12}}+2\frac{1}{r^{r/2}}\right)\\
&\leq C''' A^r r^{\sqrt{r}-r/12}
\end{aligned}$$

for constants $A, C'', C''' > 0$ independent of r. For $r\to\infty$ this tends to 0 . For the final term of the sum above we need Proposition 4.4 and Theorem 1.42. We have:

$$\begin{aligned}
\frac{\left|\mathbb{P}\left[H^i(\lambda)\right]\left|H^i(\lambda)\right|\mathbb{P}\left[H^j(\lambda)\right]\left|H^j(\lambda)\right|\zeta^{\lambda}(H^i(\lambda))\zeta^{\lambda}(H^j(\lambda))\right|}{f^{\lambda}} &\leq \frac{\left|\zeta^{\lambda}(H^i(\lambda))\zeta^{\lambda}(H^j(\lambda))\right|}{f^{\lambda}}\\
&\leq C'\frac{\prod_{i=1}^{d} h(i,i)}{f^{\lambda}}
\end{aligned}$$

for some $C' > 0$ independent of r, where we have used Theorem 1.42 for the final step. Now we apply Proposition 4.4 and we get:

$$\frac{\left|\mathbb{P}\left[H^i(\lambda)\right]\left|H^i(\lambda)\right|\mathbb{P}\left[H^j(\lambda)\right]\left|H^j(\lambda)\right|\zeta^{\lambda}(H^i(\lambda))\zeta^{\lambda}(H^j(\lambda))\right|}{f^{\lambda}} \leq C' A^r r^{\sqrt{r}-r/2}$$

for some $A > 0$ independent of r. So the only term in the Diaconis-Shahshahani upper bound we are concerned with now is:

$$\frac{1}{2}\sum_{\substack{\lambda\vDash r,\lambda\neq\lambda'\\ \lambda\neq(r),(1,1,\ldots,1)}} f^{\lambda}\mathrm{tr}\left(\widehat{\mathbb{P}(\lambda)}\overline{\widehat{\mathbb{P}(\lambda)}}\right)=\frac{1}{2}\sum_{\substack{\lambda\vDash r,\lambda\neq\lambda'\\ \lambda\neq(r),(1,1,\ldots,1)}}\left(\frac{\chi^{\lambda}(K_3)\chi^{\lambda}(K_2)}{f^{\lambda}}\right)^2$$

We have:

$$\left(\frac{\chi^{\lambda}(K_3)\chi^{\lambda}(K_2)}{f^{\lambda}}\right)^2 \leq C^2\left(\frac{f_3^{\lambda}f_2^{\lambda}}{f^{\lambda}}\right)^2$$

Now we use Theorem 1.34 again in combination with the fact that at most r/m skew m hooks can be removed from a tableau of r boxes to obtain:

$$\begin{aligned}\left(\frac{\chi^\lambda(K_3)\chi^\lambda(K_2)}{f^\lambda}\right)^2 &\leq C^2\left(\frac{\frac{(r/3)!\,3^{r/3}}{(r!)^{1/3}}\,\frac{(r/2)!\,2^{r/2}}{(r!)^{1/2}}}{\left(f^\lambda\right)^{1/6}}\right)^2 \\ &\leq \frac{B\cdot C^2 2^{1/12}}{3^{1/2}}(\pi r)^{7/12}\frac{1}{\left(f^\lambda\right)^{1/3}}\end{aligned}$$

for some $B \in (0,\infty)$ coming from Stirling's approximation. Finally we apply Proposition 1.39 which tells us that:

$$\sum_{\substack{\lambda\models r\\ \lambda\neq(r),(1,1,\ldots,1)\\ \lambda_1,\lambda_1'\leq r-4}} \frac{1}{\left(f^\lambda\right)^{1/3}} = \mathcal{O}\left(r^{-\frac{4}{3}}\right)$$

To estimate the remaining terms we need to make use Table 1 from Chapter 1. It turns out that the only partitions that give us a problem are the partitions $(r-1,1)$ and $(2,1,\ldots,1)$. All the other partitions in Table 1 have dimensions quadratic in r and hence add a term $r^{-2/3}$ in total.

We have:

$$f^{(r-1,1)} = f^{(2,1,\ldots,1)} = r-1$$

and a straight forward application of the Murnaghan-Nakayama rule (Theorem 1.30) gives us:

$$\left|\chi^{(r-1,1)}(K_3)\right| \leq n_1+1 \text{ and } \left|\chi^{(2,1,\ldots,1)}(K_3)\right| \leq n_1+1$$

where n_1 is the number of singleton cycles in σ, which is a constant in our considerations (because it only depends on W and m). And:

$$\left|\chi^{(r-1,1)}(K_2)\right| = \left|\chi^{(2,1,\ldots,1)}(K_2)\right| = 1$$

Hence these partitions add a term $\frac{n_1+1}{r-1}$. Note that all the terms we found limit to 0 as $r\to\infty$.

Summing all the estimates above concludes the proof. $\square$

4.2.3. The analogue of Lemma 4.1. The analogue of Lemma 4.1 is the following:

THEOREM 4.7. *Let H be a labelled oriented graph in which every vertex has degree at most 3. Furthermore suppose that H contains M edges and no left hand turn cycles. Finally, let the sequence of subsets $D_N \subset \mathbb{N}$ be non-negligible with respect to the genus. Then:*

$$\mathbb{P}_N\left[H \subset \Gamma \middle| g \in D_N\right] \sim \frac{1}{(6N-1)(6N-3)\cdots(6N-2M+1)} \quad \text{as } N \to \infty$$

where the limit has to be taken over even N.

PROOF. We have:

$$\begin{aligned}\mathbb{P}_N\left[H \subset \Gamma \middle| g \in D_N\right] &= \frac{\mathbb{P}_N\left[H \subset \Gamma \text{ and } g \in D_N\right]}{\mathbb{P}_N\left[g \in D_N\right]} \\ &= \frac{\mathbb{P}_N\left[g \in D_N \middle| H \subset \Gamma\right]}{\mathbb{P}_N\left[g \in D_N\right]}\mathbb{P}_N\left[H \subset \Gamma\right]\end{aligned}$$

By Proposition 2.7 we have:

$$\mathbb{P}_N\left[g \in D_N \middle| H \subset \Gamma\right] = \mathbb{P}_{3\star 2,H,N}\left[g \in D_N\right]$$

From Theorem 4.6 in combination Theorem 2.9 we know that:

$$\left|\mathbb{P}_N\left[g \in D_N \middle| H \subset \Gamma\right] - \mathbb{P}_N\left[g \in D_N\right]\right| \to 0$$

for $N \to \infty$. Furthermore, because we have assumed that the sequence $D_N \subset \mathbb{N}$ is non-negligble, we have:

$$\liminf_{N\to\infty} \mathbb{P}_N\left[g \in D_N\right] > 0$$

So:

$$\frac{\mathbb{P}_N\left[H \subset \Gamma \middle| g \in D_N\right]}{\mathbb{P}_N\left[H \subset \Gamma\right]} \to 1$$

as $N \to \infty$. Filling in Lemma 4.1 now gives the desired result. □

4.3. Maximal genus

We would of course like to have an analogue of Lemma 4.1 for any restriction on the topology whitout having to assume the non-negligibility. It is clear however that it is not possible to adapt the method of proof of Theorem 4.7. to this level of generality. This can for instance be seen from the fact that we could restrict to random surfaces that are disjoint unions of N once punctured tori. It is clear that in this case the probabilities of containing specific subsurfaces change.

In this section we will focus on the case of maximal genus. We will consider the case of odd N, which means that the maximal genus we can attain is:

$$g = \frac{N+1}{2}$$

In terms of the symmetric group description this means that $\sigma\tau$ contains a single cycle of full length.

We want to understand the probability that a random surface contains a fixed subsurface, under the condition that the genus of the surface is maximal. So, given an oriented labelled graph H containing no left hand turn cycles, we want to count the number of elements in the set:

$$\left\{\omega \in \Omega_N;\ H \subset \Gamma(\omega) \text{ and } g(\omega) = \frac{N+1}{2}\right\}$$

Using Proposition 2.7, this is equivalent to counting the number:

$$|\{(\sigma,\tau) \in K_3(H,N) \times K_2(H,N);\ \sigma\tau \text{ has 1 cycle}\}|$$

We have also already seen that in this number we count the same random surface many times, corresponding to the relabeling of vertices, or equivalently the choice of σ. This means that we can also fix a $\sigma \in K_3(H,N)$ and count the number:

$$n(H,N) = |\{\tau \in K_2(H,N);\ \sigma\tau \text{ has 1 cycle}\}|$$

which is what we will do. We will use methods similar to those of Appendix 6 in [**BIZ80**], where a similar number for gluings of quadrilaterals is counted (see also the appendix of [**Pen92**]).

For an element $\pi \in \mathrm{S}_N$ we denote the conjugacy class of π by $K(\pi)$ and for two conjugacy classes $K, K' \subset \mathrm{S}_N$ we write:

$$\delta_{K,K'} = \begin{cases} 1 & \text{if } K = K' \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, we set $K_3 := K_3(H,N)$ and $K_2 := K_2(H,N)$. Now:

$$n(H,N) = \sum_{\tau \in \mathrm{S}_{6N-2M}} \delta_{K(\tau),K_2}\delta_{K(\sigma\tau),K(6N-2M)}$$

where M is again the number of edges in H.

We start with the following lemma:

Lemma 4.8. *Let W be a finite set of words in L and R and $m \in \mathbb{N}^W$. Then:*

$$n(H,N) = \frac{|K_2| \cdot |K(6N-2M)|}{(6N-2M)!} \sum_{p=0}^{6N-2M-1} \frac{(-1)^p}{f^p} \chi^p(K_2)\chi^p(K_3)$$

Proof. For any two elements $\alpha, \beta \in \mathrm{S}_{6N-2M}$ we have:

$$\sum_{\lambda \models 6N-2M} \chi^\lambda(\alpha)\chi^\lambda(\beta) = \frac{(6N-2M)!}{|K(\alpha)|}\delta_{K(\alpha),K(\beta)}$$

So:

$$n(H,N) = \frac{|K_2| \cdot |K(6N-2M)|}{((6N-2M)!)^2} \sum_{\substack{\tau \in \mathrm{S}_{6N-2M}, \\ \lambda,\mu \models 6N-2M}} \chi^\lambda(\tau)\chi^\lambda(K_2)\chi^\mu(\sigma\tau)\chi^\mu(K(6N-2M))$$

We have:

$$\sum_{\tau \in \mathrm{S}_{6N-2M}} \chi^\lambda(\tau)\chi^\mu(\sigma\tau) = \delta_{\lambda,\mu}\frac{(6N-2M)!}{f^\lambda}\chi^\lambda(\sigma)$$

This means that:

$$n(H,N) = \frac{|K_2| \cdot |K(6N-2M)|}{(6N-2M)!} \sum_{\lambda \models 6N-2M} \frac{1}{f^\lambda}\chi^\lambda(K_2)\chi^\lambda(K(6N-2M))\chi^\lambda(\sigma)$$

The characters $\chi^\lambda(K(6N-2M))$ can be computed using Theorem 1.30. We have:

$$\chi^\lambda(K(6N-2M)) = \begin{cases} (-1)^p & \text{if } \lambda = (6N-2M-p, 1^p) \\ 0 & \text{otherwise} \end{cases}$$

Furthermore:

$$\chi^\lambda(\sigma) = \chi^\lambda(K_3)$$

Because the sum above is now over all $\lambda \models 6N-2M$ of the form $(6N-2M-p, 1^p)$, we will replace all indices λ by indices p. So we get:

$$n(H,N) = \frac{|K_2| \cdot |K(6N-2M)|}{(6N-2M)!} \sum_{p=0}^{6N-2M-1} \frac{(-1)^p}{f^p}\chi^p(K_2)\chi^p(K_3)$$

□

We have:

$$f^p = \binom{6N-2M-1}{p}$$

and:

$$\chi^p(K_2) = (-1)^{\lceil \frac{p}{2} \rceil}\binom{3N-M-1}{\lfloor \frac{p}{2} \rfloor}$$

So far we have adapted the computation of [**BIZ80**] to the trivalent case. For the next part of the computation we will need to use different methods.

To compute the characters $\chi^p(K_3)$ we will use the Murnaghan-Nakayama rule (Theorem 1.30). First we need some notation. Suppose that as a surface H consists of L triangles and has b boundary components of ℓ_i sides for $i = 1, \ldots, b$ respectively. We write $\Lambda = \sum_{i=1}^{b} \ell_i$ and:

$$K_\Lambda = K\left(\prod_{i=1}^{b} \ell_i\right) \subset \mathrm{S}_\Lambda$$

We have the following lemma:

Lemma 4.9. *Let $0 \leq p \leq 6N-2M$. Then:*

$$\chi^p(K_3) = \sum_{\substack{0 \leq r \leq \min\{\Lambda-1,p\} \\ 3 | p-r}} \chi^r(K_\Lambda)\binom{2N-L}{\frac{p-r}{3}}$$

where χ^r is the character of S_M corresponding to the partition $(M-r, 1^r)$.

Proof. The idea is to remove skew 3 hooks from every $\lambda_p = (6N-2M-p, 1^p)$ until we arrive at a Young tableau for S_Λ. This will allow us to express $\chi^p(K_3)$ in terms of the characters of S_Λ. We have:

$$\chi^{(3)}(K(3)) = \chi^{(1,1,1)}(K(3)) = 1$$

and these are the only skew 3 hooks we can remove from λ_p. There are $2N-M$ skew 3 hooks to remove, so repeated application of the Murnaghan-Nakayama rule will yield a sum over all possible sequences of removing copies of the two skew 3 hooks of length $2N-M$. For such a sequence $s \in \left\{ \ydiagram{1,1,1}, \ydiagram{3} \right\}^{2N-M}$ we define the partition $\mu_p^s \models M$ to be the partition coming from λ_p by consecutive removal of skew 3 hooks as dictated by s. So we get:

$$\chi^p(K_3) = \sum_s \chi^{\mu_p^s}(K_M)$$

where some care has to be taken: μ_p^s does not make sense for every sequence s, the numbers of copies of $\ydiagram{1,1,1}$ and $\ydiagram{3}$ that can be removed respectively are limited by functions of p.

Next we need to know how often we obtain the same partition of Λ in the sum above. First of all note that we only obtain tableaux of the form $(\Lambda - r, 1^r)$ for some $r \geq 0$. Furthermore to obtain $(\Lambda - r, 1^r)$ from λ_p we certainly need that $p - r$ is positive and divisible by 3. It is not difficult to see that we obtain $\binom{2N-L}{\frac{p-r}{3}}$ copies of each tableau that satisfies the conditions above. So we get:

$$\chi^p(K_3) = \sum_{\substack{0 \leq r \leq \min\{\Lambda-1, p\} \\ 3 | p-r}} \chi^r(K_\Lambda) \binom{2N-L}{\frac{p-r}{3}}$$

which is the desired result. $\square$

We write:

$$s(H,N) = \sum_{p=0}^{6N-2M-1} \frac{(-1)^p}{f^p} \chi^p(K_2)\chi^p(K_3)$$

So we have:

$$s(H,N) = \sum_{p=0}^{6N-2M-1} (-1)^{\lfloor \frac{p}{2} \rfloor} \sum_{\substack{0 \leq r \leq \min\{\Lambda-1, p\} \\ 3 | p-r}} \chi^r(K_\Lambda) \frac{\binom{2N-L}{\frac{p-r}{3}} \binom{3N-M-1}{\lfloor \frac{p}{2} \rfloor}}{\binom{6N-2M-1}{p}}$$

Lemma 4.10. *Let H be an oriented graph in which every vertex has degree at most three. Futhermore suppose that H does not contain any left hand turn cycles. Then:*

$$\lim_{N \to \infty} s(H,N) = 2$$

PROOF. First we look at the terms in the sum corresponding to $p = 0$ and $p = 6N - 2M - 1$. The sum of these two terms is equal to:

$$\chi^0(K_\Lambda) + (-1)^{3N-M-1}\chi^{\Lambda-1}(K_\Lambda)$$

Recall from Lemma 4.2 that $M = 3(b - 2 + 2g) + \Lambda$, where g is the genus of the surface associated to H. Furthermore note that $\chi^{\Lambda-1}$ corresponds to the sign representation of S_Λ and hence has values in $\{-1, 1\}$. If we now go through the four possible combinations of the value of $\chi^{\Lambda-1}(K_\Lambda)$ and the parity of Λ, then we see that M is even if and only if $\chi^{\Lambda-1}(K_\Lambda) = 1$. Because N is odd, this means that:

$$\chi^0(K_\Lambda) + (-1)^{3N-M-1}\chi^{\Lambda-1}(K_\Lambda) = 2$$

So what we need to prove is that the limit of the remaining terms is 0. This follows easily from the fact that the p^{th} term in the sum is of the order $\mathcal{O}(N^{-1})$ for $p = 1$ and $p = 6N - 2M - 2$, $\mathcal{O}(N^{-2})$ for $p = 2$ and $p = 6N - 2M - 3$ and smaller than the $p = 2$ term for all $3 \leq p \leq 6N - 2M - 4$ $\square$

From this we obtain the following:

THEOREM 4.11. *Let H be a labelled oriented graph in which every vertex has degree at most 3. Furthermore suppose that H contains M edges and no left hand turn cycles. Then:*

$$\mathbb{P}_N\left[H \subset \Gamma |\, g = \frac{N+1}{2}\right] \sim \frac{1}{(6N-1)(6N-3)\cdots(6N-2M+1)} \quad \textit{as } N \to \infty$$

where the limit has to be taken over odd N.

PROOF. Filling in Lemma 4.10 in the expression for $n(H, N)$ gives us that:

$$n(H, N) \sim 2\frac{|K_2| \cdot |K(6N - 2M)|}{(6N - 2M)!}$$

as $N \to \infty$. General formulas for the cardinalities of conjugacy classes are known. These give:

$$|K_2| = \frac{(6N - 2M)!}{2^{3N-M}(3N - M)!}, \; |K(6N - 2M)| = (6N - 2M - 1)!$$

So we get:

$$n(H, N) \sim \frac{(6N - 2M)!!}{3N - M}$$

as $N \to \infty$, where for $t \in 2\mathbb{N}$ the number $t!!$ is given by $(t-1)(t-3)\cdots 1$. We have:

$$\mathbb{P}_N\left[H \subset \Gamma |\, g = \frac{N+1}{2}\right] = \frac{n(H, N)}{n(\emptyset, N)}$$

Filling this in gives the theorem. $\square$

CHAPTER 5

Lengths of curves on hyperbolic random surfaces

In this chapter we investigate the length spectra of hyperbolic random surfaces. In Chapter 2 we have explained that this comes down to understanding the probability distributions of the random variables $Z_{N,[w]} : \Omega_N \to \mathbb{N}$ which count the number of appearances of a given class words $[w]$ as circuits in the corresponding graph.. We will determine the asymptotic probability distributions of these random variables, with and without conditions on the genus of the underlying surface (Theorem 5.1). After this, we will use these to compute the asymptotic probability distribution of the systole in the hyperbolic setting (Corollary 5.2). This we will then use to compute the limit of the expected value of the systole in the unconditional case (Theorems 5.5 and 5.7). Along the way we will also see that the asymptotic genus distribution is independent of the sytole (Corollary 5.3).

5.1. Finite length spectra

We have the following theorem which in the unrestricted case is very similar to Theorem 3.3, both in statement and in proof.

For the restricted case, the notations $Z_{N,[w]}\big|_{g\in D_N}$ and $Z_{N,[w]}\big|_{g=\frac{N+1}{2}}$ mean the random variables $Z_{N,[w]}$ restricted to the sets of surfaces satisfying the respective conditions on the genus.

Furthermore, recall that d denotes the total variational distance between two random variables (Definition 1.23). In the case of $\mathbb{N}$-valued random variables X and Y this is given by:

$$d(X,Y) = \sup\left\{\left|\mathbb{P}\left[X\in A\right] - \mathbb{P}\left[Y \in A\right]\right| ;\ A \subset \mathbb{N}\right\}$$

Finally, for $[w] \in \{L,R\}^*/\sim$:

- the number $|[w]|$ denotes the number of elements in $[w]$ as a subset of $\{L,R\}^*$
- and $|w|$ denotes the number of letters in w.

THEOREM 5.1. *Let $W \subset \{L, R\}^*/\sim$ be a finite set of equivalence classes of words. Then we have:*

$$\lim_{N\to\infty} d(Z_{[w]}, Z_{N,[w]}) = 0$$

for all $[w] \in W$, where:

- *$Z_{[w]} : \mathbb{N} \to \mathbb{N}$ is a Poisson distributed random variable with mean $\lambda_{[w]} = \frac{|[w]|}{2|w|}$ for all $w \in W$.*
- *The random variables $Z_{[w]}$ and $Z_{[w']}$ are independent for all $[w], [w'] \in W$ with $[w] \neq [w']$.*

The same holds for the random variables:

$$Z_{N,[w]}\Big|_{g\in D_N} \text{ and } Z_{N,[w]}\Big|_{g=\frac{N+1}{2}}$$

where the sequence $\{D_N \subset \mathbb{N}\}$ is non-negligible with respect to the genus and the limit has to be taken over even N in the first case and odd N in the second.

PROOF. To prove this we will adapt the proof of Theorem 3.3 as it can be found in [**Bol82**] (Theorem II.4 16). What we will do here is explain the ideas of this proof and how we will change them to obtain the result we need.

We start with the unrestricted case. The basic tool in the proof of Theorem 3.3 is the method of moments (Theorem 1.21). More specifically: one looks at the limits of all the joint factorial moments of the variables $\left\{Z_{N,[w]}\right\}_{w\in W}$. That is, we consider:

$$(Z_{N,[w]})_m = Z_{N,[w]}(Z_{N,[w]} - 1)\cdots(Z_{N,[w]} - m + 1)$$

and prove that:

$$\mathbb{E}_N\left[\prod_{[w]\in W}(Z_{N,[w]})_{m_{[w]}}\right] \to \prod_{[w]\in W}\lambda_{[w]}^{m_{[w]}}$$

for all $(m_{[w]})_{[w]\in W} \in \mathbb{N}^W$, where $\lambda_{[w]} = \frac{|[w]|}{2|w|}$ for all $w \in W$.

First we will look at $\mathbb{E}_N\left[Z_{N,[w]}\right]$ we will write:

$$\mathbb{E}_N\left[Z_{N,[w]}\right] = a_{N,[w]}p_{N,[w]}$$

where $a_{N,[w]}$ counts the number of possible distinct labelings a $[w]$-circuit as a set of $|w|$ pairs of half-edges can have and $p_{N,[w]}$ is the probability that an element of Ω_N contains a given set of $|w|$ pairs of half-edges.

To count $a_{N,[w]}$ we will count the number of possible distinct labelings of a directed $[w]$-circuit with a start vertex. Because we are fixing a start vertex and direction what we are actually counting is $2\,|w|\,a_{N,[w]}$. If we write:

$$w = w_1w_2\cdots w_{|w|}$$

where $w_i \in \{L, R\}$ for $i = 1, 2, \ldots |w|$ then a directed w-circuit with a start vertex corresponds to a list:

$$((x_1, w_1 x_1), (x_2, w_2 x_2), \ldots (x_{|w|}, w_{|w|} x_{|w|}))$$

where x_i is a half-edge and $w_i x_i$ is the half-edge left from x_i at the same vertex if $w_i = L$ and right from x_i otherwise, for all $1 \leq i \leq |w|$. Because $x_1, x_2, \ldots, x_{|w|}$ must all be half-edges from different vertices the number of such lists for the word w is:

$$3^{|w|} 2N(2N-1)\ldots(2N-|w|+1)$$

and because we get these lists for all the representatives of $[w]$ we have:

$$a_{N,[w]} = \frac{|[w]|}{2\,|w|} 3^{|w|} 2N(2N-1)\ldots(2N-|w|+1)$$

Like in the case of the k-circuits the probability $p_{N,[w]}$ depends only on the number of pairs of half-edges, so:

$$p_{N,[w]} = \frac{1}{(6N-1)(6N-3)\cdots(6N-2\,|w|+1)}$$

This means that:

$$\mathbb{E}_N\left[Z_{N,[w]}\right] = \frac{|[w]|}{2\,|w|} 3^{|w|} \frac{2N(2N-1)\ldots(2N-|w|+1)}{(6N-1)(6N-3)\cdots(6N-2\,|w|+1)}$$

so:

$$\lim_{N\to\infty} \mathbb{E}_N\left[Z_{N,[w]}\right] = \frac{|[w]|}{2\,|w|}$$

The next moment to consider is $(Z_{N,[w]})_2$, which counts the number of ordered pairs of $[w]$-circuits. Analogously to the proof of Theorem 3.3 we will write:

$$(Z_{N,[w]})_2 = Y'_{N,[w]} + Y''_{N,[w]}$$

where $Y'_{N,[w]}$ counts the number of ordered pairs of non-intersecting $[w]$-circuits and $Y''_{N,[w]}$ counts the number of ordered pairs of intersecting $[w]$-circuits. A similar argument as before tells us that:

$$\lim_{N\to\infty} \mathbb{E}_N\left[Y'_{N,[w]}\right] = \left(\frac{|[w]|}{2\,|w|}\right)^2$$

Furthermore we have $Y''_{N,[w]} \leq Y''_{N,|w|}$, where the latter counts the number of ordered pairs of intersecting $|w|$-circuits. We already know from Theorem 3.4 that $\mathbb{E}_N\left[Y''_{N,|w|}\right] = \mathcal{O}(N^{-1})$ for $N \to \infty$, which implies that the same is true for $\mathbb{E}_N\left[Y''_{N,[w]}\right]$. So we get that:

$$\lim_{N\to\infty} \mathbb{E}_N\left[(Z_{N,[w]})_2\right] = \left(\frac{|[w]|}{2\,|w|}\right)^2$$

As in the proof of Theorem 3.3 a similar argument works for the higher and combined moments.

Now we consider the case where we restrict the random variables to surfaces of genus $g \in D_N$ for a sequence $\{D_N \subset \mathbb{N}\}_N$ that is non-negligible with respect to the genus. We have:

$$\mathbb{E}_N\left[\prod_{w\in W}\left(Z_{[w]}\right)_{m_w} \mid g \in D_N\right] = a_{N,(W,m)} \cdot \mathbb{P}_N\left[\Gamma_{(W,m)} \subset \Gamma \mid g \in D_N\right]$$

where $a_{N,(W,m)}$ counts the number of ways of realizing (W,m) as a graph $\Gamma_{(W,m)}$, which is a disjoint union of oriented circuits, each representing a word in W. We note that the number $a_{N,(W,m)}$ is independent of the restrictions on the genus of a random surface. So when we apply Theorem 4.7 we get:

$$\begin{aligned}\mathbb{E}_N\left[\prod_{w\in W}\left(Z_{[w]}\right)_{m_w} \mid g \in D_N\right] &= a_{N,(W,m)} \cdot \mathbb{P}_N\left[\Gamma_{(W,m)} \subset \Gamma \mid g \in D_N\right] \\ &= \mathbb{E}_N\left[\prod_{w\in W}\left(Z_{[w]}\right)_{m_w}\right] \cdot \frac{\mathbb{P}_N\left[\Gamma_{(W,m)} \subset \Gamma \mid g \in D_N\right]}{\mathbb{P}_N\left[\Gamma_{(W,m)} \subset \Gamma\right]} \\ &\sim \mathbb{E}_N\left[\prod_{w\in W}\left(Z_{[w]}\right)_{m_w}\right]\end{aligned}$$

as $N \to \infty$. In the above we have cheated slightly, we have skipped over realizations of (W,m) as intersecting circuits, but asymptotically these do not contribute anything. In fact, because our condition on the genus is non-negligible, Theorem 3.4 gives us an $\mathcal{O}\left(N^{-1}\right)$ upper bound for the contribution of intersecting circuits in the restricted case as well.

Finally, for the case of maximal genus we can again ignore representations of (W,m) by intersecting circuits (one can show that these contribute a term of the order $\mathcal{O}(N^{-1})$). So, applying Theorem 4.11, we obtain:

$$\begin{aligned}\mathbb{E}_N\left[\prod_{w\in W}\left(Z_{[w]}\right)_{m_w} \mid g = \frac{N+1}{2}\right] &= a_{N,(W,m)} \cdot \mathbb{P}_N\left[\Gamma_{(W,m)} \subset \Gamma \mid g = \frac{N+1}{2}\right] \\ &\sim \mathbb{E}_N\left[\prod_{w\in W}\left(Z_{[w]}\right)_{m_w}\right]\end{aligned}$$

as $N \to \infty$, which finishes the proof. $\square$

5.2. The systole

In this section we investigate the systole of random surfaces. Recall from Definition 1.7 that the systole of a surface is the length of a shortest homotopically non-trivial and non-peripheral curve of this surface. Its length defines a function:

$$\mathrm{sys} : \Omega_N \to \mathbb{R}_+$$

We will use this notation both for the systole in the hyperbolic case (both for the punctured and the compactified surfaces) and the Riemannian case, it will be clear from the context which case we are speaking about. Furthermore note that in order to define the systole, the surface needs to contain non-trivial non-peripheral curves, which is not always the case. In the cases where the underlying surface does not contain any such curve we set the systole to 0. We will now start with the asymptotic probability distribution of the systole in the hyperbolic case.

Before we can state the results in the hyperbolic setting we need to fix some notation. We need to order equivalence classes of words in L and R by their traces. To this end we define the sets:

$$A_k = \{[w] \in \{L,R\}^* / \sim;\ \mathrm{tr}\,(w) = k\}$$

for all $k \in \mathbb{N}$. Note that because the trace of a product of matrices is invariant under cyclic permutation and transposes, these sets are well defined.

5.2.1. The probability distribution.

The probability distribution of the systole is now given by the following corollary to Theorem 5.1:

Corollary 5.2. *Let $\varepsilon > 0$ small enough and $k \in \mathbb{N}$ with $k \geq 3$ then in for both the punctured and compactified hyperbolic surfaces we have:*

$$\lim_{N\to\infty} \mathbb{P}_N\left[\left|\mathrm{sys}(S) - 2\cosh^{-1}\left(\frac{k}{2}\right)\right| \leq \varepsilon\right] = \left(\prod_{[w]\in \bigcup_{i=3}^{k-1} A_i} e^{-\frac{|[w]|}{2|w|}}\right)\left(1 - \prod_{[w]\in A_k} e^{-\frac{|[w]|}{2|w|}}\right)$$

The same holds for the conditional probabilities:

$$\mathbb{P}_N\left[\left|\mathrm{sys}(S) - 2\cosh^{-1}\left(\frac{k}{2}\right)\right| \leq \varepsilon \,\middle|\, g \in D_N\right]$$

where the sequence $\{D_N \subset \mathbb{N}\}$ is non-negligible with respect to the genus and the limit has to be taken over even N, and:

$$\mathbb{P}_N\left[\left|\mathrm{sys}(S) - 2\cosh^{-1}\left(\frac{k}{2}\right)\right| \leq \varepsilon \,\middle|\, g = \frac{N+1}{2}\right]$$

where the limit has to be taken over odd N.

Note that in the punctured case we need not add the ε in the corollary, in this case the systole takes values in $\left\{2\cosh^{-1}\left(\frac{k}{2}\right)\right\}_{k=3}^{\infty}$. Furthermore, using essentially the same proof as the one below, we can also compute the corresponding probability distribution for the i^{th} shortest curve on the surface for any finite i. The resulting formula does however become very long as i grows larger.

Proof. We will consider the random variables $Z^{\circ}_{N,[w]} : \Omega_N \to \mathbb{N}$ that count the number of appearances of $[w]$ as a circuit such that the corresponding curve on the compactified surface S_C is homotopically non-trivial. The sets we will study are:

$$C^{\circ}_{N,k} = \left\{ \omega \in \Omega_N;\ Z^{\circ}_{N,[w]}(\omega) = 0 \ \ \forall [w] \in \bigcup_{i=3}^{k-1} A_i \ \text{ and } \ \exists [w] \in A_k \text{ s.t. } Z^{\circ}_{N,[w]}(\omega) > 0 \right\}$$

So we want to compute the probability:

$$\lim_{N\to\infty} \mathbb{P}_N \left[C^{\circ}_{N,k} \right]$$

Before we can use Theorem 5.1 we have to show that the probability that a word is carried by a contractible circuit tends to 0, because the probability above depends on $Z^{\circ}_{N,[w]}$ rather than $Z_{N,[w]}$. This follows from the fact that a circuit corresponding to w can only be contractible if it is separating (because $[w] \neq [L^k]$ for all $k \in \mathbb{N}$). However, a circuit carrying $[w]$ is of a fixed finite number of edges (i.e. $|w|$). By Theorem 3.18 the probability that such circuits are separating tends to 0. So we get:

$$\lim_{N\to\infty} \mathbb{P}_N \left[C^{\circ}_{N,k} \right] = \lim_{N\to\infty} \mathbb{P}_N \left[C_{N,k} \right]$$

where the sets $C_{N,k}$ have the same definition as $C^{\circ}_{N,k}$ but with the random variables $Z^{\circ}_{N,[w]}$ replaced by $Z_{N,[w]}$.

Seeing how the statement in the definition of $C_{N,k}$ is a statement about a finite number of equivalence classes of words, we can apply Theorem 5.1. So, using both the formula for the Poisson distribution and the independence we get:

$$\begin{aligned}
\mathbb{P}_N \left[C_{N,k} \right] \to & \left(\prod_{[w] \in \bigcup\limits_{i=3}^{k-1} A_i} \mathbb{P}_N \left[Z_{[w]} = 0 \right] \right) \left(1 - \prod_{[w] \in A_k} \mathbb{P}_N \left[Z_{[w]} = 0 \right] \right) \\
= & \left(\prod_{[w] \in \bigcup\limits_{i=3}^{k-1} A_i} \exp\left(-\frac{|[w]|}{2\,|w|} \right) \right) \left(1 - \prod_{[w] \in A_k} \exp\left(-\frac{|[w]|}{2\,|w|} \right) \right)
\end{aligned}$$

as $N \to \infty$. The probability on the right hand side above is the probability we are after in the punctured setting (in fact, in the punctured setting we could have just considered the variables $Z_{N,[w]}$). To see that the same holds in the compactified setting we need to invoke Lemma 2.5 in combination with Theorem 2.11 (a). ☐

5.2.2. The genus distribution. We also obtain an independence statement in the opposite direction to Corollary 5.2. That is, if we consider only surfaces that satisfy certain conditions on the systole then the limits of the probabilities that these surface have a given genus do not change.

COROLLARY 5.3. *Let $D_N \subset \mathbb{N}$ for all $N \in \mathbb{N}$ be a sequence of subsets such that the probability $\mathbb{P}_N[g \in D_N]$ converges for $N \to \infty$ and let $x \in (2\log((3+\sqrt{5})/2), \infty)$. Then both in the punctured and compactified hyperbolic setting we have:*

$$\lim_{N\to\infty} \mathbb{P}_N[g \in D_N|\, \mathrm{sys} \leq x] = \lim_{N\to\infty} \mathbb{P}_N[g \in D_N]$$

and:

$$\lim_{N\to\infty} \mathbb{P}_N[g \in D_N|\, \mathrm{sys} \geq x] = \lim_{N\to\infty} \mathbb{P}_N[g \in D_N]$$

PROOF. We first note that the conditions 'sys $\leq x$' and 'sys $\geq x$' can be expressed in terms of a finite number of $Z_{[w]}$-variables. We prove the corollary for sys $\leq x$.

We first assume that $\lim_{N\to\infty} \mathbb{P}_N[g \in D_N] > 0$. This means that:

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{P}_N[g \in D_N|\, \mathrm{sys} \leq x] &= \lim_{N\to\infty} \frac{\mathbb{P}_N[g \in D_N \text{ and } \mathrm{sys} \leq x]}{\mathbb{P}_N[\mathrm{sys} \leq x]} \\
&= \lim_{N\to\infty} \frac{\mathbb{P}_N[g \in D_N \text{ and } \mathrm{sys} \leq x]}{\mathbb{P}_N[g \in D_N]\,\mathbb{P}_N[\mathrm{sys} \leq x]} \mathbb{P}_N[g \in D_N] \\
&= \lim_{N\to\infty} \frac{\mathbb{P}_N[\mathrm{sys} \leq x|\, g \in D_N]}{\mathbb{P}_N[\mathrm{sys} \leq x]} \mathbb{P}_N[g \in D_N] \\
&= \lim_{N\to\infty} \mathbb{P}_N[g \in D_N]
\end{aligned}$$

where the last step follows from the fact that the sequence D_N is non-negligble with respect to the genus and Corollary 5.2.

If $\lim_{N\to\infty} \mathbb{P}_N[g \in D_N] = 0$ then we have:

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{P}_N[g \in D_N|\, \mathrm{sys} \leq x] &\leq \lim_{N\to\infty} \frac{\mathbb{P}_N[g \in D_N]}{\mathbb{P}_N[\mathrm{sys} \leq x]} \\
&= 0
\end{aligned}$$

where we have used that $\lim_{N\to\infty} \mathbb{P}_N[\mathrm{sys} \leq x] > 0$, which follows from the fact that $x \geq 2\log((3+\sqrt{5})/2)$. $\square$

Note that in the proof of Corollary 5.3 we have only used the fact that our condition can be expressed in a finite number of $Z_{[w]}$-variables. This means that the corollary holds for all such conditions.

5.2.3. The expected value. The next thing we want to do is to use the probabilities in Corollary 5.2 to compute the limit of the expected value of the systole of a hyperbolic random surface. The reason we cannot immediately use the limiting probabilities above is that the expected value of the systole will be a sum over all the possible values of the systole. So to show that this sum converges to the sum of the limits of its terms we need a dominated convergence argument.

This will be a three step process. First we look at the non-compact case and we prove that we can (in the appropriate sense) ignore random surfaces with a certain set of properties. After that we use the dominated convergence theorem for the sum that remains in the non-compact case. Finally we prove that the probability that a random surface has small cusps decreases fast enough (we need to sharpen Theorem 2.11 (a)), which will imply that the expression in the compact case is the same as in the non-compact case.

We start with describing the set of random surfaces that we want to exclude. Let $N \in \mathbb{N}$, recall that the genus of $S_O(\omega)$ for $\omega \in \Omega_N$ is denoted by $g(\omega)$. We define the following random variable:

Definition 5.1. Let $N \in \mathbb{N}$. Define $m_\ell : \Omega_N \to \mathbb{N}$ by:

$$m_\ell(\omega) = \begin{cases} \min \{|\gamma|\,;\ \gamma \text{ a circuit on } \Gamma(\omega), \text{ non-contractible on } S_C(\omega)\} & \text{if } g(\omega) > 0 \\ 0 & \text{otherwise} \end{cases}$$

The set of surfaces we want to ignore is the following set:

$$B_N = \left\{ \omega \in \Omega_N;\ g(\omega) \leq \frac{N}{3} \text{ or } m_\ell(\omega) > C\log_2(N) \text{ or } \omega \in \bigcup_{2 \leq k \leq C\log_2(N)} G_{N,k} \right\}$$

where we have chosen some constant $C \in (0, 1)$ that we will keep fixed until the end of this section.

Like we said, we want to exclude the surfaces in B_N. We want to do this in the following way: we have:

$$\mathbb{E}_N[\text{sys}] = \frac{1}{|\Omega_N|} \sum_{\omega \in \Omega_N} \text{sys}(\omega)$$

and in the sum above we want to forget about the $\omega \in B_N$. We can prove the following:

Proposition 5.4. *In the hyperbolic model we have:*

$$\lim_{N \to \infty} \frac{1}{|\Omega_N|} \sum_{\omega \in B_N} \text{sys}(\omega) = 0$$

Proof. Basically B_N consists of 3 subsets (with some overlap): surfaces of small genus, surfaces with a short separating curve and surfaces with large m_ℓ. We will prove the seemingly stronger result that the restrictions of the sum to each of these subsets tend to 0.

We start with surfaces with small genus. For these we will use Gromov's systolic inequality (Theorem 1.10), Markov's inequality (Theorem 1.18) and Corollary 2.10. We have $g(\omega) \leq \frac{N+1}{2}$ for all $\omega \in \Omega_N$. This means that $\frac{N+1}{2} - g$ is a non-negative random variable and we can apply Markov's inequality to it. We have $g(\omega) \leq \frac{N}{3}$ if and only if $\frac{N+1}{2} - g(\omega) \geq \frac{N}{6} + 1$. So we get:

$$\begin{aligned}\mathbb{P}_N\left[g \leq \frac{N}{3}\right] &= \mathbb{P}_N\left[\frac{N+1}{2} - g \geq \frac{N}{6} + 1\right] \\ &= \frac{\frac{N+1}{2} - \mathbb{E}_N\left[g\right]}{\frac{N}{6} + 1}\end{aligned}$$

Now we apply Corollary 2.10, which tells us that there exists a constant $C \in (0, \infty)$ such that:

$$\begin{aligned}\mathbb{P}_N\left[g \leq \frac{N}{3}\right] &\leq \frac{\frac{N+1}{2} - 1 - \frac{N}{2} + C\log(N)}{\frac{N}{6} + 1} \\ &= \frac{K\log(N)}{N}\end{aligned}$$

for some $K \in (0, \infty)$. We want to apply Gromov's systolic inequality now. The problem is that we need a closed surface to apply this and our surface has cusps. However, we can do the following: at each cusp we cut off a horocycle neighborhood and replace it with a Euclidean hemisphere with an equator of the same length as the horocycle, as in Figure 5.1 below.

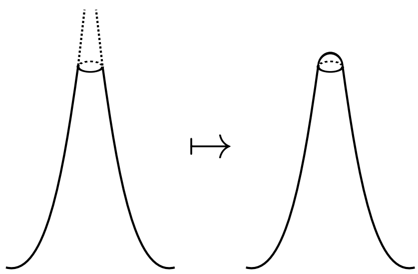


FIGURE 5.1. Cutting off the cusps.

When we do this with all the cusps we get a compact surface with a Riemannian metric on it. If we shorten the length of the horocycle, the area of the neighborhood we cut off gets smaller and so does the area of the hemispheres we glue. Recall that the area of a random surface of $2N$ ideal hyperbolic triangles is $2\pi N$. So, given $\varepsilon > 0$, we can choose the horocycles such that the area of the resulting surface is at most $2\pi N + \varepsilon$. Furthermore, we want that the systole on the resulting surface is at least as long as the systole on the surface with cusps, so we need to be sure that the systole on the resulting surface does not pass through any of the added hemispheres. This again comes down to choosing the horocycles small enough. So when we apply Theorem 1.10 we get:

$$\begin{aligned}\frac{1}{|\Omega_N|}\sum_{\omega\in\Omega_N,\, g(\omega)\leq\frac{N}{3}}\mathrm{sys}(\omega) &\leq \frac{A}{|\Omega_N|}\sum_{\omega\in\Omega_N,\, g(\omega)\leq\frac{N}{3}}\sqrt{2\pi N+\varepsilon}\\ &= \mathbb{P}_N\left[g\leq\frac{N}{3}\right]A\sqrt{2\pi N+\varepsilon}\\ &\leq AK\frac{\sqrt{2\pi N+\varepsilon}\log(N)}{N}\end{aligned}$$

for some $A\in(0,\infty)$. Note that we have only used the fact that the ratio $\frac{\mathrm{sys}(\omega)^2}{2\pi N+\varepsilon}$ is bounded and not how this bound behaves with respect to the genus of ω. So we get:

$$\lim_{N\to\infty}\frac{1}{|\Omega_N|}\sum_{\omega\in\Omega_N,\, g(\omega)\leq\frac{N}{3}}\mathrm{sys}(\omega)=0 \tag{9}$$

The next part we treat is the set of random surfaces with short separating curves. To keep the notation simple we will write: $G_N=\bigcup_{2\leq k\leq C\log_2(N)}G_{N,k}$. So G_N is the set of random surfaces that contain a separating circuit with fewer than $C\log_2(N)$ edges. We have:

$$\begin{aligned}\frac{1}{|\Omega_N|}\sum_{\omega\in G_N}\mathrm{sys}(\omega) &= \frac{1}{|\Omega_N|}\sum_{\omega\in G_N,\, g(\omega)\leq\frac{N}{3}}\mathrm{sys}(\omega)+\frac{1}{|\Omega_N|}\sum_{\omega\in G_N,\, g(\omega)>\frac{N}{3}}\mathrm{sys}(\omega)\\ &\leq \frac{1}{|\Omega_N|}\sum_{\omega\in\Omega_N,\, g(\omega)\leq\frac{N}{3}}\mathrm{sys}(\omega)+\frac{1}{|\Omega_N|}\sum_{\omega\in G_N,\, g(\omega)>\frac{N}{3}}\mathrm{sys}(\omega)\end{aligned}$$

We already know that the first of these two terms tends to 0. For the second term we will use Gromov's systolic inequality again, so we will again apply the trick of replacing the cusps with small hemispheres. This means that there exists a $K'\in(0,\infty)$ such that:

$$\begin{aligned}\sum_{\omega\in G_N,\, g(\omega)>\frac{N}{3}}\mathrm{sys}(\omega) &\leq \sum_{\omega\in G_N,\, g(\omega)>\frac{N}{3}}\sup\left\{\frac{\mathrm{sys}(S_{g(\omega)},ds^2)}{\sqrt{\mathrm{area}(S_{g(\omega)},ds^2)}};\ ds^2\right\}\sqrt{2\pi N+\epsilon}\\ &\leq \sum_{\omega\in G_N,\, g(\omega)>\frac{N}{3}}\frac{K'\log(\frac{N}{3})}{\sqrt{\pi\frac{N}{3}}}\sqrt{2\pi N+\epsilon}\\ &= |\Omega_N|\cdot\mathbb{P}_N\left[\omega\in G_N\right]\frac{K'\log(\frac{N}{3})}{\sqrt{\pi\frac{N}{3}}}\sqrt{2\pi N+\epsilon}\end{aligned}$$

From Theorem 3.18 we know that for all $\varepsilon>0$ there exists an $R\in(0,\infty)$ such that $\mathbb{P}_N\left[\omega\in G_N\right]\leq\frac{R}{N^{1-C-\varepsilon}}$ for all $N\in\mathbb{N}$. So we get:

$$\frac{1}{|\Omega_N|}\sum_{\omega\in G_N,\, g(\omega)>\frac{N}{3}}\mathrm{sys}(\omega)\leq\frac{R}{N^{1-C-\varepsilon}}\frac{K'\log(\frac{N}{3})}{\sqrt{\pi\frac{N}{3}}}\sqrt{2\pi N+\epsilon}$$

so:

$$\lim_{N\to\infty}\frac{1}{|\Omega_N|}\sum_{\omega\in G_N}\mathrm{sys}(\omega)=0 \tag{10}$$

The last part is surfaces with large m_ℓ. For these surfaces we can restrict to the surfaces with large genus and without short separating circuits. That is:

$$\begin{aligned}\frac{1}{|\Omega_N|}\sum_{\omega\in\Omega_N,\ m_\ell(\omega)>C\log_2(N)}\mathrm{sys}(\omega) &= \frac{1}{|\Omega_N|}\sum_{\substack{\omega\in\Omega_N-G_N,\\ m_\ell(\omega)>C\log_2(N),\ g(\omega>\frac{N}{3}}}\mathrm{sys}(\omega)\\ &\leq \mathbb{P}_N\left[{}_{m_\ell(\omega)>C\log_2(N),\ g(\omega)>\frac{N}{3}}^{\omega\in\Omega_N-G_N,}\right]\frac{K'\log(\frac{N}{3})}{\sqrt{\pi\frac{N}{3}}}\sqrt{2\pi N+\epsilon}\end{aligned}$$

The reason we want to restrict to $\Omega_N - G_N$ is that it makes null homotopic circuits easier: if a circuit is null homotopic it is either separating or it cuts off a cusp, in which case it carries a word of type L^k for some k. If we restrict to $\Omega_M - G_N$, the first case does not appear. This means that if the shortest non null homotopic circuit on a random surface has k edges (i.e. $m_\ell = k$) then there are either no $k-1$ circuits or there are $k-1$ circuits that carry a word of the type L^{k-1}. So if we set:

$$J_N=\left\{\omega\in\Omega_N-G_N;\ \begin{array}{c} g(\omega)>\dfrac{N}{3}, X_{N,\lfloor C\log_2(N)\rfloor}(\omega)=0 \text{ or } X_{N,\lfloor C\log_2(N)\rfloor}(\omega)>0 \text{ and}\\ \text{all } \lfloor C\log_2(N)\rfloor\text{-circuits carry words of the type } L^{\lfloor C\log_2(N)\rfloor}\end{array}\right\}$$

then we obtain:

$$\begin{aligned}\mathbb{P}_N\left[{}_{m_\ell(\omega)>C\log_2(N),\ g(\omega)>\frac{N}{3}}^{\omega\in\Omega_N-G_N,}\right] &= \mathbb{P}_N\left[J_N\right]\\ &\leq \mathbb{P}_N\left[X_{N,\lfloor C\log_2(N)\rfloor}=0\right]+\mathbb{P}_N\left[J'_N\right]\end{aligned}$$

where:

$$J'_N=\left\{\omega\in\Omega_N;\ \begin{array}{c} X_{N,\lfloor C\log_2(N)\rfloor}(\omega)>0 \text{ and all } \lfloor C\log_2(N)\rfloor\text{-circuits}\\ \text{carry words of the type } L^{\lfloor C\log_2(N)\rfloor}\end{array}\right\}$$

For the first of the two we will use Proposition 3.11, which says that there exists a constant $D\in(0,\infty)$ such that $\mathbb{P}_N\left[X_{N,k}=0\right]\leq Dk^8\left(\frac{3}{8}\right)^k$. This means that:

$$\begin{aligned}\mathbb{P}_N\left[X_{N,\lfloor C\log_2(N)\rfloor}=0\right] &\leq D(\lfloor C\log_2(N)\rfloor)^8\left(\frac{3}{8}\right)^{\lfloor C\log_2(N)\rfloor}\\ &\leq K''\log(N)^8N^{C\log_2(\frac{3}{8})}\end{aligned}$$

Just to give an idea: $\log_2(\frac{3}{8})\approx -1.4$. For the second probability we set:

$$J'_{N,i}=\left\{\omega\in J_N;\ X_{N,\lfloor C\log_2(N)\rfloor}(\omega)=i\right\}$$

Hence we obtain:

$$\mathbb{P}_N\left[J'_N\right]=\sum_{i=1}^{\infty}\mathbb{P}_N\left[J'_{N,i}\right]$$

The value of $X_{N,\lfloor C\log_2(N)\rfloor}$ only depends on the graph and not on the orientation on the graph. Because every orientation has equal probability, we can work with the ratio of orientations on a $\lfloor C\log_2(N)\rfloor$-circuit that correspond to a $L^{\lfloor C\log_2(N)\rfloor}$ type word. If a graph has one $\lfloor C\log_2(N)\rfloor$-circuit then this ratio is $\frac{2}{2^{\lfloor C\log_2(N)\rfloor}}$ (there are 2 words of type $L^{\lfloor C\log_2(N)\rfloor}$: the word itself and $R^{\lfloor C\log_2(N)\rfloor}$ and there are $2^{\lfloor C\log_2(N)\rfloor}$ possible orientations on a $\lfloor C\log_2(N)\rfloor$-circuit). If a graph has more than one $\lfloor C\log_2(N)\rfloor$-circuit, the ratio is at most $\frac{2}{2^{\lfloor C\log_2(N)\rfloor}}$, which we get from considering the ratio on just one circuit. So we get:

$$\begin{aligned}\mathbb{P}_N\left[J_N'\right] &\leq \sum_{i=1}^{\infty}\frac{2}{2^{\lfloor C\log_2(N)\rfloor}}\mathbb{P}_N\left[X_{N,\lfloor C\log_2(N)\rfloor}=i\right]\\ &\leq \frac{4}{N^C}\end{aligned}$$

So we obtain:

$$\frac{1}{|\Omega_N|}\sum_{\substack{\omega\in\Omega_N,\\ m_\ell(\omega)>C\log_2(N)}}\mathrm{sys}(\omega)\leq\frac{\left(K''\log(N)^8N^{C\log_2(\frac{3}{8})}+\frac{4}{N^C}\right)K'\log(\frac{N}{3})\sqrt{2\pi N+\epsilon}}{\sqrt{\pi\frac{N}{3}}}$$

Hence:

$$\lim_{N\to\infty}\frac{1}{|\Omega_N|}\sum_{\omega\in\Omega_N,\, m_\ell(\omega)>C\log_2(N)}\mathrm{sys}(\omega)=0 \tag{11}$$

When we put (9), (10) and (11) together we get the desired result. □

We can now compute the limit of the expected value in the non-compact case:

THEOREM 5.5. *In the non-compact hyperbolic setting we have:*

$$\lim_{N\to\infty}\mathbb{E}_N\left[\mathrm{sys}\right]=\sum_{k=3}^{\infty}2\left(\prod_{[w]\in\bigcup_{i=3}^{k-1}A_i}e^{-\frac{|[w]|}{2|w|}}\right)\left(1-\prod_{[w]\in A_k}e^{-\frac{|[w]|}{2|w|}}\right)\cosh^{-1}\left(\frac{k}{2}\right)$$

PROOF. Of course we will use Proposition 5.4. This means that we can write:

$$\lim_{N\to\infty}\mathbb{E}_N\left[\mathrm{sys}\right]=\lim_{N\to\infty}\sum_{k=3}^{\infty}\mathbb{P}_N\left[S_{N,k}\right]\cdot 2\cosh^{-1}\left(\frac{k}{2}\right)$$

where

$$S_{N,k}=\left\{\omega\in\Omega_N-B_N;\ \begin{array}{c}Z^\circ_{N,[w]}(\omega)=0\ \forall[w]\in\bigcup_{i=3}^{k-1}A_i\text{ and}\\ \exists[w]\in A_k\text{ such that }Z^\circ_{N,[w]}(\omega)>0\end{array}\right\}$$

supposing the left hand side above exists. We know the pointwise limits of the probabilities on the right hand side, these are the same as the ones for $\omega\in\Omega_N$, because the probability that $\omega\in B_N$ tends to 0. We will apply the dominated convergence theorem to prove the fact that we can use those pointwise limits. This means that we need a uniform upper bound on the probabilities in the sum on the right hand side of the equation.

The point is that given the trace of the word in L and R corresponding to the systole, we get a lower bound on the number of edges on the circuit corresponding to the systole. Because if we want the trace of a word in L and R to increase we need to increase the number of letters in this word. More concretely, the distance between the midpoints on the triangle T is $\log\left(\frac{3+\sqrt{5}}{2}\right)$, this means that $k\log\left(\frac{3+\sqrt{5}}{2}\right)$ is an upper bound for the hyperbolic length of a circuit with k edges[1].

This means that if the systole is $2\cosh^{-1}\left(\frac{k}{2}\right)$ then $m_\ell \geq \frac{2\cosh^{-1}\left(\frac{k}{2}\right)}{\log\left(\frac{3+\sqrt{5}}{2}\right)}$. So we get:

$$\mathbb{P}_N\left[S_{N,k}\right] \leq \mathbb{P}_N\left[\omega \in \Omega_N - B_N \text{ s.t. } m_\ell(\omega) \geq \frac{2\cosh^{-1}\left(\frac{k}{2}\right)}{\log\left(\frac{3+\sqrt{5}}{2}\right)}\right]$$

Now we use the fact that we can ignore surfaces with short separating curves and big m_ℓ. That is, if $\omega \notin B_N$ and $m_\ell(\omega) \geq \frac{2\cosh^{-1}\left(\frac{k}{2}\right)}{\log\left(\frac{3+\sqrt{5}}{2}\right)}$ then either $\Gamma(\omega)$ has no circuits of

$$r(k) := \left\lfloor \frac{2\cosh^{-1}\left(\frac{k}{2}\right)}{\log\left(\frac{3+\sqrt{5}}{2}\right)} - 1 \right\rfloor$$

edges, or circuits of this length all carry a word consisting of only L's (or only R's, depending on the direction in which we read the word). So we get:

$$\begin{aligned}\mathbb{P}_N\left[S_{N,k}\right] &\leq \mathbb{P}_N\left[X_{N,r(k)} = 0\right] + 2\left(\frac{1}{2}\right)^{r(k)} \\ &\leq D\left(r(k)\right)^8\left(\frac{3}{8}\right)^{r(k)} \\ &\quad + 2\left(\frac{1}{2}\right)^{r(k)}\end{aligned}$$

for some $D \in (0,\infty)$ independent of N and k. We have:

$$\cosh^{-1}\left(\frac{k}{2}\right) = \log\left(\frac{k}{2} + \sqrt{\frac{k^2}{4} - 1}\right) \geq \log\left(\frac{k}{2}\right)$$

So:

$$r(k) \geq \frac{2\log\left(\frac{k}{2}\right)}{\log\left(\frac{3+\sqrt{5}}{2}\right)} - 2 = \frac{2\log\left(\frac{3}{8}\right)}{\log\left(\frac{3+\sqrt{5}}{2}\right)}\log_{\frac{3}{8}}\left(\frac{k}{2}\right) - 2 =: c_1\log_{\frac{3}{8}}\left(\frac{k}{2}\right) - 2$$

[1]It is noteworthy that the length on the surface of a curve that corresponds to the word $(LR)^k$ is also $\log\left(\frac{3+\sqrt{5}}{2}\right)2k$. This implies that, among all the words of $2k$ letters, $(LR)^k$ is 'the word of greatest geodesic length'.

and likewise:

$$r(k) \geq \frac{2\log\left(\frac{1}{2}\right)}{\log\left(\frac{3+\sqrt{5}}{2}\right)} \log_{\frac{1}{2}}\left(\frac{k}{2}\right) - 2 =: c_2 \log_{\frac{1}{2}}\left(\frac{k}{2}\right) - 2$$

Hence:

$$\mathbb{P}_N\left[S_{N,k}\right] \leq D \cdot r(k)^8 \left(\frac{8}{3}\right)^2 k^{c_1} + 8k^{c_2}$$

So there exist constants D' and D'' (independent of N and k) such that:

$$\mathbb{P}_N\left[S_{N,k}\right] \cdot 2\cosh^{-1}\left(\frac{k}{2}\right) \leq D' \cdot \log(k)^9 \cdot k^{c_1} + D'' \cdot \log(k) \cdot k^{c_2}$$

We have $c_1 \approx -2.0$ and $c_2 \approx -1.4$. So the right hand side above is a summable function. This means that we can apply the dominated convergence theorem. In combination with Corollary 5.2 this gives the desired result. □

It will turn out that the limit of the expected value of the systole in the compact case is the same. This will follow from the fact that asymptotically the non-compact surfaces have large cusps with high probability, which implies that the metrics on the compact and non-compact surfaces are comparable. Given $L \in (0,\infty)$, we write:

$$E_{L,N} = \{\omega \in \Omega_N;\ S_O(\omega) \text{ has cusp length } < L\}$$

In Theorem 2.11(a), Brooks and Makover proved that random surfaces have large cusps with probability tending to 1 for $N \to \infty$. However, we also need to know how fast this probability tends to 1, so we sharpen their result as follows:

Proposition 5.6. *Let $L \in (0,\infty)$. We have:*

$$\mathbb{P}_N\left[E_{L,N}\right] = \mathcal{O}(N^{-1})$$

for $N \to \infty$.

Proof. The idea of the proof is similar to that of Theorem 2.11(a): if $S_O(\omega)$ has cusp length $< L$, that means that we cannot find non-intersecting horocycles of length L around its cusps. So, there must be two circuits in $\Gamma(\omega)$ that are close. So, we must have subgraph of the form shown in Figure 5.2 below: two circuits of lengths $0 < \ell_1 < L$ and $0 < \ell_2 < L$ joined by a path of $0 \leq d \leq d_{\max}(L)$ edges. Where $d_{\max}(L)$ is determined by the fact that if the distance d between the two circuits becomes too big, it will be possible to choose horocycles of length L around the corresponding cusps. Furthermore, the case $d = 0$ will be interpreted as two intersecting circuits.

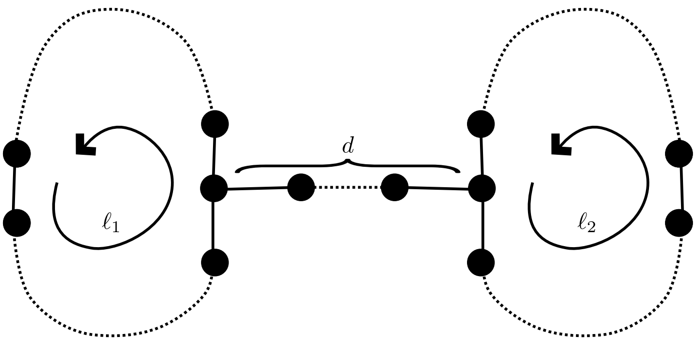


FIGURE 5.2. Two circuits joined by a path.

The set M_L of such graphs is finite. Given $\Gamma \in M_L$, let $X_{\Gamma,N} : \Omega_N \to \mathbb{N}$ denote the random variable that counts the number of appearances of Γ in $\Gamma(\omega)$. By Markov's inequality we have:

$$\begin{aligned}\mathbb{P}_N\left[X_{\Gamma,N} > 0\right] &= \mathbb{P}_N\left[X_{\Gamma,N} \geq \frac{1}{2}\right] \\ &\leq 2\mathbb{E}_N\left[X_{\Gamma,N}\right]\end{aligned}$$

So:

$$\begin{aligned}\mathbb{P}_N\left[E_{L,N}\right] &\leq \sum_{\Gamma\in M_L} \mathbb{P}_N\left[X_{\Gamma,N} \geq 0\right] \\ &\leq 2\sum_{\Gamma\in M(L)} \mathbb{E}_N\left[X_{\Gamma,N}\right]\end{aligned}$$

Every graph $\Gamma \in M_L$ has at least one more edge than it has vertices. This implies that $\mathbb{E}_N\left[X_{\Gamma,N}\right] = \mathcal{O}(N^{-1})$ (Theorem 3.4) for all $\Gamma \in M_L$. Because M_L is finite and does not depend on N we get that:

$$\mathbb{P}_N\left[E_{L,N}\right] = \mathcal{O}(N^{-1})$$

□

With this and Lemma 2.5 we can prove:

THEOREM 5.7. *In the compact hyperbolic setting we have:*

$$\lim_{N\to\infty} \mathbb{E}_N\left[\mathrm{sys}\right] = \sum_{k=3}^{\infty} 2\left(\prod_{[w]\in\bigcup_{i=3}^{k-1} A_i} e^{-\frac{|[w]|}{2|w|}}\right)\left(1 - \prod_{[w]\in A_k} e^{-\frac{|[w]|}{2|w|}}\right)\cosh^{-1}\left(\frac{k}{2}\right)$$

PROOF. We want to compare the compact with the non-compact setting. To make a distinction between the two, we will write sys_C for the systole in the compact setting and sys_O for the systole in the non-compact setting. Given $L \in (0,\infty)$, we will split off the set of surfaces in Ω_N that have cusp length $< L$ in the non-compact setting.

We have:

$$\mathbb{E}_N\left[\mathrm{sys}_C\right] = \frac{1}{|\Omega_N|} \sum_{\omega\in\Omega_N\setminus E_{L,N}} \mathrm{sys}_C(\omega) + \frac{1}{|\Omega_N|} \sum_{\omega\in E_{L,N}} \mathrm{sys}_C(\omega)$$

Using Theorem 1.10 and Proposition 5.6 we obtain:

$$\begin{aligned}\frac{1}{|\Omega_N|} \sum_{\omega\in E_{L,N}} \mathrm{sys}_{N,C}(\omega) &\leq \mathbb{P}_N\left[E_{L,N}\right]\sqrt{2\pi N} \\ &= \mathcal{O}(N^{-\frac{1}{2}})\end{aligned}$$

So:

$$\lim_{N\to\infty} \mathbb{E}\left[\mathrm{sys}_C\right] = \lim_{N\to\infty} \frac{1}{|\Omega_N|} \sum_{\omega\in\Omega_N\setminus E_{L,N}} \mathrm{sys}_C(\omega)$$

Using Lemma 2.5, we get:

$$\lim_{N\to\infty} \mathbb{E}_N\left[\mathrm{sys}_O\right] \leq \lim_{N\to\infty} \mathbb{E}_N\left[\mathrm{sys}_{N,C}\right] \leq (1+\delta(L)) \lim_{N\to\infty} \mathbb{E}_N\left[\mathrm{sys}_O\right]$$

Because $\delta(L) \to 0$ for $L \to \infty$ and we can choose L as big as we like, the two limits are actually equal. $\square$

5.2.3.1. *A numerical value.* Theorems 5.5 and 5.7 from the previous section give us a formula for the limit of the expected value of the systole in the hyperbolic model. The problem is that the formula is rather abstract and it is not clear how to determine the sets A_k for all $k = 3, 4, \ldots$. To get to a numerical value for the limit, we can however compute the first couple of terms (because it is not difficult to determine the sets A_k for k up to any finite value) and then give an upper bound for the remainder of the sum. To simplify notation we write:

$$p_k = \left(\prod_{[w]\in \bigcup_{i=3}^{k-1} A_i} \exp\left(-\frac{|[w]|}{2\,|w|}\right)\right)\left(1 - \prod_{[w]\in A_k} \exp\left(-\frac{|[w]|}{2\,|w|}\right)\right)$$

for $k = 3, 4, \ldots$. So:

$$\lim_{N\to\infty} \mathbb{E}_N\left[\mathrm{sys}\right] = \sum_{k=3}^{\infty} p_k 2\cosh^{-1}\left(\frac{k}{2}\right)$$

We have the following lemma:

LEMMA 5.8. *Let $k, n \in \mathbb{N}$ such that $4 \leq n \leq k$ then:*

$$p_k \leq \frac{p_n}{e^{k-n}\left(1 - \prod_{[w]\in A_n} \exp\left(-\frac{|[w]|}{2|w|}\right)\right)}$$

PROOF. We have:

$$\begin{aligned}
p_k &\leq \left(\prod_{[w]\in \bigcup_{i=3}^{k-1} A_i} \exp\left(-\frac{|[w]|}{2\,|w|}\right) \right) \\
&= \left(\prod_{[w]\in \bigcup_{i=3}^{n-1} A_i} \exp\left(-\frac{|[w]|}{2\,|w|}\right) \right) \left(\prod_{[w]\in \bigcup_{i=n}^{k-1} A_i} \exp\left(-\frac{|[w]|}{2\,|w|}\right) \right) \\
&= \frac{p_n}{1-\prod_{[w]\in A_n} \exp\left(-\frac{|[w]|}{2|w|}\right)} \left(\prod_{[w]\in \bigcup_{i=n}^{k-1} A_i} \exp\left(-\frac{|[w]|}{2\,|w|}\right) \right)
\end{aligned}$$

We know that $[L^{i-2}R] \in A_i$. It is not difficult to see that $|[L^{i-2}R]| = 2(i-1)$ when $i > 3$. So:

$$\prod_{[w]\in A_i} \exp\left(-\frac{|[w]|}{2\,|w|}\right) \leq \exp(-1)$$

Because $\prod_{[w]\in \bigcup_{i=n}^{k-1} A_i} \exp\left(-\frac{|[w]|}{2|w|}\right) = \prod_{i=n}^{k-1} \prod_{[w]\in A_i} \exp\left(-\frac{|[w]|}{2|w|}\right)$ we get:

$$p_k \leq \frac{p_n}{e^{k-n}\left(1-\prod_{[w]\in A_n} \exp\left(-\frac{|[w]|}{2|w|}\right)\right)}$$

which is what we wanted to prove. $\square$

We will write:

$$S_n = \sum_{k=3}^{n} p_k 2\cosh^{-1}\left(\frac{k}{2}\right)$$

for $n = 3, 4, \ldots$. So S_n is the approximation of the limit of the expected value of the systole by the fist $n-2$ terms of the sum.

We have the following proposition:

PROPOSITION 5.9. *For all $n = 3, 4, \ldots$ we have:*

$$S_n \leq \lim_{N\to\infty} \mathbb{E}_N[\mathrm{sys}] \leq S_n + 2\frac{p_n}{1-\prod_{[w]\in A_n} \exp\left(-\frac{|[w]|}{2|w|}\right)} \frac{\left(\frac{\log(n+1)^{\frac{1}{n+1}}}{e}\right)^{n+1}}{1-\frac{\log(n+1)^{\frac{1}{n+1}}}{e}}$$

PROOF. The inequality on the left hand side is trivial, so we will forcus on the inequality on the right hand side. We have:

$$\begin{aligned}\lim_{N\to\infty} \mathbb{E}_N\left[\text{sys}\right] - S_n &= \sum_{k=n+1}^{\infty} 2p_k \cosh^{-1}\left(\frac{k}{2}\right) \\ &\leq \sum_{k=n+1}^{\infty} 2p_k \log(k)\end{aligned}$$

Now we use Lemma 5.8 and we get:

$$\begin{aligned}\lim_{N\to\infty} \mathbb{E}_N\left[\text{sys}\right] - S_n &\leq \frac{2e^n p_n}{1 - \prod_{[w]\in A_n} \exp\left(-\frac{|[w]|}{2|w|}\right)} \sum_{k=n+1}^{\infty} \frac{\log(k)}{e^k} \\ &\leq \frac{2e^n p_n}{1 - \prod_{[w]\in A_n} \exp\left(-\frac{|[w]|}{2|w|}\right)} \sum_{k=n+1}^{\infty} \left(\frac{\log(n+1)^{\frac{1}{n+1}}}{e}\right)^k \\ &= \frac{2e^n p_n}{1 - \prod_{[w]\in A_n} \exp\left(-\frac{|[w]|}{2|w|}\right)} \frac{\left(\frac{\log(n+1)^{\frac{1}{n+1}}}{e}\right)^{n+1}}{1 - \frac{\log(n+1)^{\frac{1}{n+1}}}{e}}\end{aligned}$$

which proves the proposition. $\square$

So now approximating $\lim_{N\to\infty} \mathbb{E}_N\left[\text{sys}\right]$ is just a matter of filling in the proposition above. For instance with $n = 7$ we obtain:

$$2.48432 \leq \lim_{N\to\infty} \mathbb{E}_N\left[\text{sys}\right] \leq 2.48434$$

CHAPTER 6

The systole of a Riemannian random surface

In the Riemannian case we lose the nice combinatorial description of lengths of curves. However, using circuits in the dual graph to the triangulation we can still get bounds on the distribution of the number of curves of given lengths (Corollary 6.3). Using these we can also derive bounds on the limit infimum and limit supremum of the expected value of the systole in this setting (Theorem 6.5).

6.1. The shortest non-trivial curve on the graph

Recall that m_ℓ measures the number of edges in the shortest homotopically non-trivial circuit in the dual graph. The goal of this section is to compute the following limiting probability:

$$\lim_{N\to\infty} \mathbb{P}_N \left[m_\ell = k \middle| \, g \in D_N \right]$$

where D_N is either non-negligible with respect to the genus or $\left\{\frac{N+1}{2}\right\}$. Note that the first of these includes the case of no restrictions on the genus. For sequences D_N that form an actual restriction, we will sometimes speak of *proper restrictions*.

The idea behind the computation is again to split the probability space Ω_N up into two subsets. In this case means we will split off the surfaces with short non-trivial curves that are separating and surfaces with pairs of intersecting short non-trivial curves. So we define the following set:

$$H_{N,k} = \{\omega \in \Omega_N;\ \omega \text{ contains two intersecting circuits both with } \leq k \text{ edges}\}$$

We have the following lemma about this set:

Lemma 6.1.

$$\lim_{N\to\infty} \mathbb{P}_N \left[H_{N,k} \right] = 0$$

Proof. This is a direct application of Theorem 3.4.

Let $Y_{N,k} : \Omega_N \to \mathbb{N}$ be the random variable that counts the number of distinct pairs of intersecting circuits of length at most k. So:

$$H_{N,k} = \{\omega \in \Omega_N;\ Y_{N,k}(\omega) \geq 1\}$$

So Markov's inequality implies that:

$$\mathbb{P}_N \left[H_{N,k} \right] \leq \mathbb{E}_N \left[Y_{N,k} \right]$$

Theorem 3.4 tells us that $\mathbb{E}_N \left[Y_{N,k} \right]$ is $\mathcal{O}(N^{-1})$ for $N \to \infty$. ∎

We now have the following proposition:

Proposition 6.2. *Let $D_N \subset \mathbb{N}$ for all $N \in \mathbb{N}$ be a sequence of subsets such that one of the following holds:*

1. *The sequence is non-negligible with respect to the genus.*
2. *$D_N = \left\{\frac{N+1}{2}\right\}$ for all odd N*

Then for all $k \in \mathbb{N}$ we have:

$$\lim_{N\to\infty} \mathbb{P}_N \left[m_\ell = k \middle| \, g \in D_N \right] = e^{-\sum\limits_{j=1}^{k-1} \frac{2^{j-1}-1}{j}} - e^{-\sum\limits_{j=1}^{k} \frac{2^{j-1}-1}{j}}$$

If $D_N \subset \mathbb{N}$ is a proper restriction and non-negligible with respect to the genus then the limit has to be taken over even N. If $D_N = \left\{\frac{N+1}{2}\right\}$ then the limit has to be taken over odd N.

Proof. First of all we have:

$$\mathbb{P}_N \left[m_\ell = k\right] = \mathbb{P}_N \left[\omega \in \Omega_N - H_{N,k} \text{ and } m_\ell(\omega) = k\right] + \mathbb{P}_N \left[\omega \in H_{N,k} \text{ and } m_\ell(\omega) = k\right]$$

Because $\mathbb{P}_N \left[m_\ell = k, \ \omega \in H_{N,k}\right] \leq \mathbb{P}_N \left[H_{N,k}\right]$, Lemma 6.1 tells us that:

$$\lim_{N\to\infty} \mathbb{P}_N \left[m_\ell = k\right] = \lim_{N\to\infty} \mathbb{P}_N \left[\omega \in \Omega_N - H_{N,k} \text{ and } m_\ell(\omega) = k\right]$$

With a similar argument and Theorem 3.18 we can also exclude separating circuits. Recall that $G_{N,i}$ denotes the set of partitions $\omega \in \Omega_N$ such that $\Gamma(\omega)$ has a separating circuit of i edges. We have:

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{P}_N \left[m_\ell = k\right] &= \lim_{N\to\infty} \left(\mathbb{P}_N \left[\omega \in \Omega_N - H_{N,k} - \bigcup_{i=2}^{k} G_{N,i} \text{ and } m_\ell(\omega) = k \right] \right. \\
&\qquad \left. + \, \mathbb{P}_N \left[\omega \in \bigcup_{i=2}^{k} G_{N,i} - H_{N,k} \text{ and } m_\ell(\omega) = k \right] \right) \\
&= \lim_{N\to\infty} \mathbb{P}_N \left[\omega \in \Omega_N - H_{N,k} - \bigcup_{i=2}^{k} G_{N,i} \text{ and } m_\ell(\omega) = k \right]
\end{aligned}$$

So, in our computation we only need to consider non-separating curves that do not intersect each other. Recall that the random variable $X_{N,j}$ counts the number of circuits of length j on elements of Ω_N. Given $i_1, \ldots, i_k \in \mathbb{N}$, we write:

$$J_{N,i_1,\ldots,i_k} = \left\{ \omega \in \Omega_N - H_{N,k} - \bigcup_{i=2}^{k} G_{N,i}; \ \begin{array}{c} m_\ell(\omega) = k \text{ and} \\ X_{N,j}(\omega) = i_j \text{ for } j = 1, \ldots k \end{array} \right\}$$

in order to split the probability above up into a sum over the possible values of $X_{N,j}$:

$$\lim_{N\to\infty} \mathbb{P}_N \left[m_\ell = k\right] = \lim_{N\to\infty} \sum_{i_1,\ldots,i_{k-1}=0}^{\infty} \sum_{i_k=1}^{\infty} \mathbb{P}_N \left[J_{N,i_1,\ldots,i_k}\right]$$

Because a surface in the Riemannian setting still induces an orientation on the corresponding graph, we can still assign words in L and R to circuits on the graph. These words no longer have a geometric meaning, but they still tell us whether or not a circuit turns around a corner on the surface. In fact, a non-separating curve is (non-)trivial if and only if the word on the corresponding curve on the graph is (un)equal to L^j or R^j, where j is the length of this curve. So if $\omega \in \Omega_N - H_{N,k} - \bigcup_{i=2}^{k} G_{N,i}$ then the condition that $m_\ell(\omega) = k$ is equivalent to: all circuits γ on $\Gamma(\omega)$ of less than k edges carry a word equivalent to L^j where j is the number of edges of γ and there is at least one circuit of k edges on $\Gamma(\omega)$ that carries a word that is inequivalent to L^k.

Furthermore, we observe that if a graph has no intersecting curves of length less than k, the words of these curves are independent: any combination of words on the curves is possible and equally probable. This means that we can just count the fraction of surfaces with the 'right words' on short curves. If $X_{N,j} = i_j$ for $j = 1, \ldots k$, this fraction is:

$$\left(1 - \frac{2^{i_k}}{2^{i_k k}}\right)\left(\prod_{j=1}^{k-1} \frac{2^{i_j}}{2^{i_j j}}\right)$$

So:

$$\lim_{N\to\infty} \mathbb{P}_N\left[m_\ell = k\right] = \lim_{N\to\infty} \sum_{\substack{i_1,\ldots,\\ i_{k-1}=0,\\ i_k=1}}^{\infty} \mathbb{P}_N\left[\substack{\omega\in\Omega_N - H_N^k - \bigcup_{i=2}^{k} G_N^i \\ X_N^j(\omega)=i_j}\right]\left(1 - \frac{2^{i_k}}{2^{i_k k}}\right)\left(\prod_{j=1}^{k-1} \frac{2^{i_j}}{2^{i_j j}}\right)$$

We will now use the fact (i.e. Theorem 3.3) that the random variables $X_{N,j}$ converge to Poisson distributions in the sup-norm on $\mathbb{N}$. Even though this theorem is about the convergence on the entire probability space Ω_N, the fact that:

$$\mathbb{P}_N\left[H_N^k \cup \bigcup_{i=2}^{k} G_N^i\right] \to 0$$

as $N \to \infty$ tells us that the limit of $\mathbb{P}_N\left[\substack{\omega\in\Omega_N - H_N^k - \bigcup_{i=2}^{k} G_N^i \\ X_N^j(\omega)=i_j}\right]$ is the same as that of $\mathbb{P}_N\left[\omega\in\Omega_N, X_N^j(\omega)=i_j\right]$ for $N \to \infty$.

We also need to prove that we can actually use these limits. For this we have Lemma 3.12 in combination with the dominated convergence theorem. Given $i_1 \in \mathbb{N}$ we have:

$$\sum_{\substack{i_2,\ldots,\\ i_{k-1}=0,\\ i_k=1}}^{\infty} \mathbb{P}_N\left[\substack{\omega\in\Omega_N - H_N^k - \bigcup_{i=2}^{k} G_N^i \\ X_N^j(\omega)=i_j}\right]\left(1 - \frac{2^{i_k}}{2^{i_k k}}\right)\left(\prod_{j=1}^{k-1} \frac{2^{i_j}}{2^{i_j j}}\right) \leq \mathbb{P}_N\left[X_N, 1 = i_1\right] \leq \frac{C_1}{i_1^2}$$

for some $C_1 \in (0, \infty)$ independent of i_1. This is a summable function, so we have:

$$\lim_{N\to\infty} \mathbb{P}_N\left[m_\ell = k\right] = \sum_{i_1=0}^{\infty} \lim_{N\to\infty} \sum_{\substack{i_2,\ldots,\\ i_{k-1}=0,\\ i_k=1}}^{\infty} \mathbb{P}_N\left[\substack{\omega\in\Omega_N - H_N^k - \bigcup_{i=2}^{k} G_N^i \\ X_N^j(\omega)=i_j}\right]\left(1-\frac{2^{i_k}}{2^{i_k}k}\right)\left(\prod_{j=1}^{k-1}\frac{2^{i_j}}{2^{i_j}j}\right)$$

We can apply this trick k times and we get:

$$\lim_{N\to\infty} \mathbb{P}_N\left[m_\ell = k\right] = \sum_{\substack{i_1,\ldots,\\ i_{k-1}=0,\\ i_k=1}}^{\infty} \lim_{N\to\infty} \mathbb{P}_N\left[\substack{\omega\in\Omega_N - H_N^k - \bigcup_{i=2}^{k} G_N^i \\ X_N^j(\omega)=i_j}\right]\left(1-\frac{2^{i_k}}{2^{i_k}k}\right)\left(\prod_{j=1}^{k-1}\frac{2^{i_j}}{2^{i_j}j}\right)$$

This means that:

$$\lim_{N\to\infty} \mathbb{P}_N\left[m_\ell = k\right] = \sum_{i_1,\ldots,i_{k-1}=0}^{\infty}\sum_{i_k=1}^{\infty} \frac{\left(\frac{2^j}{2j}\right)^{i_j} e^{-\frac{2^j}{2j}}}{i_j!}\left(1-\frac{2^{i_k}}{2^{i_k}k}\right)\left(\prod_{j=1}^{k-1}\frac{2^{i_j}}{2^{i_j}j}\right)$$

For $1 \leq j < k$ we compute:

$$\begin{aligned}\sum_{i_j=0}^{\infty} \frac{\left(\frac{2^j}{2j}\right)^{i_j} e^{-\frac{2^j}{2j}}}{i_j!}\frac{2^{i_j}}{2^{i_j}j} &= e^{-\frac{2^j}{2j}}\sum_{i_j=0}^{\infty}\frac{1}{i_j!}\frac{1}{j^{i_j}} \\ &= e^{-\frac{2^{j-1}-1}{j}}\end{aligned}$$

and for $j = k$ we have:

$$\begin{aligned}\sum_{i_k=1}^{\infty} \frac{\left(\frac{2^k}{2k}\right)^{i_k} e^{-\frac{2^k}{2k}}}{i_k!}\left(1-\frac{2^{i_k}}{2^{i_k}k}\right) &= e^{-\frac{2^k}{2k}}\sum_{i_k=1}^{\infty}\frac{1}{i_k!}\left(\left(\frac{2^k}{2k}\right)^{i_k}-\frac{1}{k^{i_k}}\right) \\ &= e^{-\frac{2^k}{2k}}\left(e^{\frac{2^k}{2k}}-1-e^{\frac{1}{k}}+1\right) \\ &= 1-e^{-\frac{2^{k-1}-1}{k}}\end{aligned}$$

So we get:

$$\begin{aligned}\lim_{N\to\infty} \mathbb{P}_N\left[m_\ell = k\right] &= \left(1-e^{-\frac{2^{k-1}-1}{k}}\right)\prod_{j=1}^{k-1} e^{-\frac{2^{j-1}-1}{j}} \\ &= e^{-\sum_{j=1}^{k-1}\frac{2^{j-1}-1}{j}} - e^{-\sum_{j=1}^{k}\frac{2^{j-1}-1}{j}}\end{aligned}$$

The reason that all this also works in the restricted case is that the arguing relies on subsurfaces (or, equivalently, subgraphs), which by Theorems 4.7 and 4.11 behave the same way under the given restrictions. □

6.2. The probability distribution of the systole

Using Proposition 6.2 above, we can get the bounds on the asymptotic probability distribution of the systole in the Riemannian case. Recall that $m_1(d)$ is the minimal distance between two opposite sides of a square glued out of two triangles with the metric d and $m_2(d)$ the maximal distance between the midpoints of the sides one such triangle.

COROLLARY 6.3. *Let $D_N \subset \mathbb{N}$ for all $N \in \mathbb{N}$ be a sequence of subsets such that one of the following holds:*

1. *The sequence is non-negligible with respect to the genus.*
2. *$D_N = \left\{\frac{N+1}{2}\right\}$ for all odd N*

Then:

$$\lim_{N\to\infty} \mathbb{P}_N\left[\text{sys} < m_1(d) \middle|\, g \in D_N\right] = 0$$

and for all $x \in [0,\infty)$:

$$\limsup_{N\to\infty} \mathbb{P}_N\left[\text{sys} \geq x \middle|\, g \in D_N\right] \leq 1 - \sum_{k=2}^{\lfloor x/m_2(d) \rfloor} \left(e^{-\sum\limits_{j=1}^{k-1} \frac{2^{j-1}-1}{j}} - e^{-\sum\limits_{j=1}^{k} \frac{2^{j-1}-1}{j}} \right)$$

If the sequence D_N is a proper restriction then both limits have to be taken over even N in the first case and over odd N in the second case

PROOF. The first inequality follows from the fact that if the surface actually contains homotopically essential curves then the systole needs to cross at least two triangles and cannot turn around a vertex, as in Figure 6.1 below:

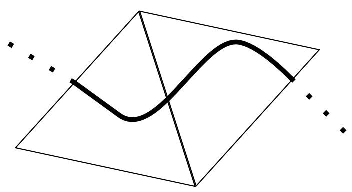


FIGURE 6.1. A segment that crosses two triangles through the opposite sides.

This implies that for any $\omega \in \Omega_N$ with $g(\omega) > 0$ we have:

$$\text{sys}(\omega) \geq m_1(d)$$

Hence:

$$\mathbb{P}_N\left[\text{sys} < m_1(d) \middle|\, g \in D_N\right] = \mathbb{P}_N\left[g = 0 \middle|\, g \in D_N\right]$$

which tends to 0 as $N \to \infty$ in all cases (unrestricted and restricted).

To prove the second inequality we note that if the graph dual to the triangulation of a random surface contains a circuit of k edges that cannot be contracted to a point on the surface we have:

$$\mathrm{sys}(\omega) \leq m_2(d) \cdot k$$

So we get:

$$\mathbb{P}_N \left[\mathrm{sys} \geq x \,|\, g \in D_N \right] \leq \mathbb{P}_N \left[\begin{array}{c} \Gamma \text{ contains no essential} \\ \text{circuit of } \leq \left\lfloor \dfrac{x}{m_2(d)} \right\rfloor \text{ edges} \end{array} \middle| \, g \in D_N \right]$$

This is a finite condition in the sense that m_ℓ needs to be larger than some finite $k \in \mathbb{N}$. So we can apply Proposition 6.2, which gives us the formula we want. □

6.3. The expected value

To prove an upper bound on the limit of the expected value of the systole in the Riemannian model we proceed in the same way as in the hyperbolic model: we start by showing that we can ignore a certain set of surfaces in our computation and after that we will use dominated convergence to prove a formula for what remains.

Proposition 6.4. *In the Riemannian model we have:*

$$\lim_{N \to \infty} \frac{1}{|\Omega_N|} \sum_{\omega \in B_N} \mathrm{sys}(\omega) = 0$$

Proof. The proof of this proposition is identical to that of Proposition 5.4, except that we use different constants here: the area of a random surface with $2N$ triangles is $2N \cdot \mathrm{area}(\Delta, d)$ instead of $2\pi N$. □

Using this proposition we can prove the following theorem:

Theorem 6.5. *In the Riemannian we have:*

$$m_1(d) \leq \liminf_{N \to \infty} \mathbb{E}_N \left[\mathrm{sys} \right]$$

and

$$\limsup_{N \to \infty} \mathbb{E}_N \left[\mathrm{sys} \right] \leq m_2(d) \sum_{k=2}^{\infty} k \left(e^{-\sum\limits_{j=1}^{k-1} \frac{2^{j-1}-1}{j}} - e^{-\sum\limits_{j=1}^{k} \frac{2^{j-1}-1}{j}} \right)$$

Proof. The lower bound follows immediately from Corollary 6.3. For the upper bound Proposition 6.4 tells us that:

$$\limsup_{N \to \infty} \mathbb{E}_N \left[\mathrm{sys} \right] = \limsup_{N \to \infty} \frac{1}{|\Omega_N|} \sum_{\omega \in \Omega_N - B_N} \mathrm{sys}(\omega)$$

We want to use our results on m_ℓ, so we split the sum on the right hand side up over m_ℓ and we get:

$$\limsup_{N\to\infty} \mathbb{E}_N[\text{sys}] = \limsup_{N\to\infty} \frac{1}{|\Omega_N|} \sum_{k=2}^{\infty} \sum_{\substack{\omega\in\Omega_N-B_N,\\ m_\ell(\omega)=k}} \text{sys}(\omega)$$

Given $\omega \in \Omega_N$ with $m_\ell(\omega) = k$ we know that $\text{sys}(\omega) \leq m_2(d)k$, so:

$$\begin{aligned}\limsup_{N\to\infty} \mathbb{E}_N[\text{sys}] &\leq \limsup_{N\to\infty} \frac{1}{|\Omega_N|} \sum_{k=2}^{\infty} \sum_{\substack{\omega\in\Omega_N-B_N,\\ m_\ell(\omega)=k}} m_2(d)k \\ &= \limsup_{N\to\infty} m_2(d) \sum_{k=2}^{\infty} \mathbb{P}_N\left[\substack{\omega\in\Omega_N-B_N,\\ m_\ell(\omega)=k}\right] k\end{aligned}$$

The limit on the right hand side we can compute. We will show that $\mathbb{P}_N\left[\substack{\omega\in\Omega_N-B_N,\\ m_\ell(\omega)=k}\right] k$ is universally bounded by a summable function of k. This implies we can apply the dominated convergence theorem in combination with the pointwise limits that we already know from Proposition 6.2.

To get an upper bound on $\mathbb{P}_N\left[\substack{\omega\in\Omega_N-B_N,\\ m_\ell(\omega)=k}\right] k$ we reason as follows: if $\omega \notin B_N$ and $m_\ell(\omega) = k$ then there are either no circuits of $k-1$ edges on ω or there are some of these circuits of $k-1$ that all carry a word of the type L^{k-1} (again the third option would be that there is some circuit of $k-1$ edges that cuts off a disk and hence is separating, but because $\omega \notin B_N$ this is impossible). So we get:

$$\begin{aligned}\mathbb{P}_N\left[\substack{\omega\in\Omega_N-B_N,\\ m_\ell(\omega)=k}\right] k &\leq \mathbb{P}_N\left[X_{N,k-1}=0\right] k + \mathbb{P}_N\left[\substack{X_{N,k-1}>0 \text{ and all } k-1\text{-circuits}\\ \text{carry a word of the type } L^{k-1}}\right] k \\ &\leq D(k-1)^8\left(\frac{3}{8}\right)^{k-1} k + \frac{2}{2^{k-1}}k\end{aligned}$$

for some $D \in (0,\infty)$, where we have used Proposition 3.11 for the first term. This is a summable function that is independent of N. So the dominated convergence theorem implies that we can fill in the pointwise limits of the terms, which completes the proof. □

The expression on the right hand side of Theorem 6.5 is something we can compute up to a finite number of digits. We have:

$$\limsup_{N\to\infty} \mathbb{E}_N[\text{sys}] \leq 2.87038 \cdot m_2(d)$$

We already noted that in the equilateral Euclidean case we have $m_1(d) = 1$ and $m_2(d) = \frac{1}{2}$, so we get:

$$1 \leq \liminf_{N\to\infty} \mathbb{E}_N[\text{sys}]$$

and:

$$\limsup_{N\to\infty} \mathbb{E}_N[\text{sys}] \leq 1.43519$$

It is not difficult to see that this last inequality is not optimal. We can however construct a sequence of metrics that does come close to the upper bound in Theorem 6.5, which we will do in the next section.

6.4. Sharpness of the upper bound

The goal of this section is to show that the upper bound in Corollary 5.3 and Theorem 6.5 is sharp. We have the following proposition:

PROPOSITION 6.6. *For every $\varepsilon > 0$ and every $M \in (0,\infty)$ there exists a Riemannian metric $d : \Delta \times \Delta \to [0,\infty)$ such that:*

$$m_2(d) = M$$

and:

$$\liminf_{N\to\infty} \mathbb{P}_N \left[\mathrm{sys} \geq x \middle| \, g \in D_N\right] \geq 1 - \sum_{k=2}^{\lfloor x/m_2(d)\rfloor} \left(e^{-\sum\limits_{j=1}^{k-1} \frac{2^{j-1}-1}{j}} - e^{-\sum\limits_{j=1}^{k} \frac{2^{j-1}-1}{j}} \right) - \varepsilon$$

where the sequence D_N is either non-negligible with respect to the genus or $D_N = \left\{\frac{N+1}{2}\right\}$ and if $\{D_N\}_N$ forms a proper restriction the limits have to be taken over even N and odd N respectively. Furthermore:

$$m_2(d) \sum_{k=2}^{\infty} k \left(e^{-\sum\limits_{j=1}^{k-1} \frac{2^{j-1}-1}{j}} - e^{-\sum\limits_{j=1}^{k} \frac{2^{j-1}-1}{j}} \right) - \varepsilon \leq \liminf_{N\to\infty} \mathbb{E}_N \left[\mathrm{sys}\right]$$

PROOF. The idea of the proof is simply to construct the metric d. Recall that an element $x \in \Delta$ can be expressed as $x = t_1 e_1 + t_2 e_2 + t_3 e_3$ with $t_1, t_2, t_3 \in [0,1]$ and $t_1 + t_2 + t_3 = 1$. We can write $t_i = \langle x, e_i\rangle$ for $i = 1,2,3$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^3$.

We define the following two subsets of Δ:

$$\mathcal{P}_1 = \left\{ x \in \Delta;\ \exists i \in \{1,2,3\} \text{ such that } \langle x, e_i\rangle = \frac{1}{2} \right\}$$

and for a given $\delta > 0$:

$$\mathcal{P}_2 = \left\{ x \in \Delta;\ \exists i \in \{1,2,3\} \text{ such that } \langle x, e_i\rangle = \frac{1 \pm \delta}{2} \right\}$$

as depicted in Figure 6.2 below:

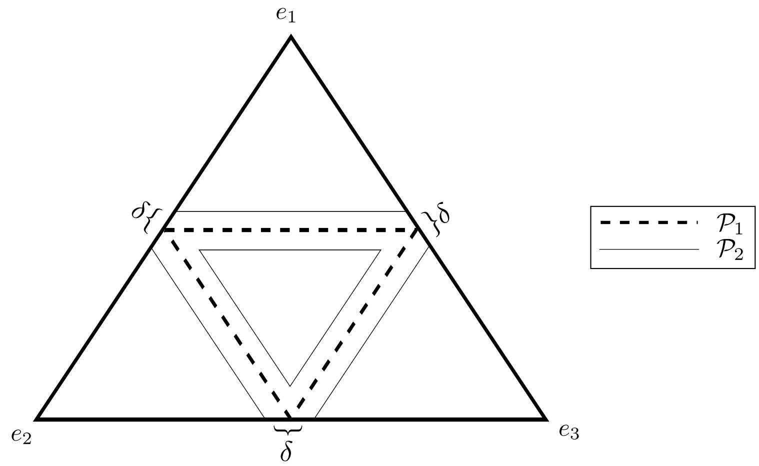


FIGURE 6.2. Δ, $\mathcal{P}_1$ and $\mathcal{P}_2$.

Furthermore, given $D \in (1, \infty)$, we construct a function $\rho_{D,\delta} : \Delta \to [1, D]$ that satisfies the following properties:

I. $(\rho_{D,\delta})\big|_{\mathcal{P}_1} = 1$.
II-a. $\rho_{D,\delta}(x) = 1$ when $\langle x, e_i \rangle \leq \frac{1}{2} - \delta$ for all $i \in \{1, 2, 3\}$.
II-b. $\rho_{D,\delta}(x) = 1$ when there exists an $i \in \{1, 2, 3\}$ such that $\langle x, e_i \rangle \geq \frac{1}{2} + \delta$.
III-a. $\int_\gamma \rho_{D,\delta} \geq D$ when $\gamma : [0, 1] \to \Delta$ is a curve such that there exists an $i \in \{1, 2, 3\}$ and $s_1, s_2 \in [0, 1]$ such that $\langle \gamma(s_1), e_i \rangle \leq \frac{1}{2} - \delta$ and $\langle \gamma(s_2), e_i \rangle \geq \frac{1}{2}$.
III-b. $\int_\gamma \rho_{D,\delta} \geq D$ when $\gamma : [0, 1] \to \Delta$ is a curve such that there exists an $i \in \{1, 2, 3\}$ and $s_1, s_2 \in [0, 1]$ such that $\langle \gamma(s_1), e_i \rangle \leq \frac{1}{2}$ and $\langle \gamma(s_2), e_i \rangle \geq \frac{1}{2} + \delta$.
IV. $\rho_{D,\delta} \in C^\infty(\Delta)$ and $\frac{\partial^k}{\partial n^k}\big|_x \rho_{D,\delta} = 0$ for all $k \geq 1$, all $x \in \partial\Delta$ and all n normal to $\partial\Delta$ at x.

Property II-a says that $\rho_{D,\delta}$ has to be equal to 1 far enough from $\mathcal{P}_2$ inside the middle triangle in Δ and property II-b says the same about the triangles in the corners. Property III-a means that when a curve goes between the middle triangle and $\mathcal{P}_1$ then the area lying under the function $\rho_{D,\delta}$ on this curve is at least D. Property III-b states the equivalent for the triangles in the corner.

A candidate for such a function is the function $\tilde{\rho}_{D,\delta} : \Delta \to [1, \infty)$ given by:

$$\tilde{\rho}_{D,\delta}(x) = \min\left\{ D\left(1 - \frac{4}{\delta} d_{\text{Eucl}}\left(x, \mathcal{P}_2\right)\right), 1 \right\}$$

where d_{Eucl} denotes the standard Euclidean metric on Δ. It is easy to see that $\tilde{\rho}_{D,\delta}$ satisfies all the properties except property IV. So if we let $\rho_{D,\delta}$ be a smoothing of $\tilde{\rho}_{D,\delta}$ we get the type of function we are looking for. Figure 6.3 below shows a cross section of $\rho_{D,\delta}$ around $(t_1, t_2, t_3) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$:

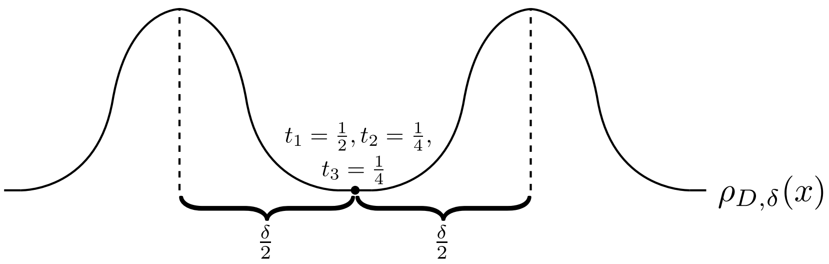


FIGURE 6.3. A cross section of $\rho_{D,\delta}$.

$\rho_{D,\delta}$ gives us a Riemannian metric $g_{D,\delta} : T\Delta \times T\Delta \to \mathbb{R}$ by:

$$g_{D,\delta} = \rho_{D,\delta} g_{\text{Eucl}}$$

Where g_{Eucl} denotes the standard Riemannian metric on Δ that induces d_{Eucl}. $g_{D,\delta}$ induces a metric $d_{D,\delta} : \Delta \times \Delta \to (0, \infty)$, given by:

$$d_{D,\delta}(x, y) = \inf \left\{ \int_\gamma \rho_{D,\delta};\ \gamma : [0, 1] \to \Delta \text{ continuous},\ \gamma(0) = x,\ \gamma(1) = y \right\}$$

Note that:

$$m_2(d_{D,\delta}) = \frac{1}{\sqrt{2}}$$

The idea behind the metric $d_{D,\delta}$ is that when D grows it is very 'expensive' to go from one of the smaller triangles to another. So the shortest paths between the different sides of the triangle avoid the region around $\mathcal{P}_2$.

Now suppose that γ is a closed curve on a random surface, not homotopic to a corner. Then somewhere γ must cross a quadrilatereal (i.e. two triangles glued together) through both the opposite sides (the curve cannot turn in the same direction on every triangle, because then it would be homotopic to a corner) as in Figure 6.4 below:

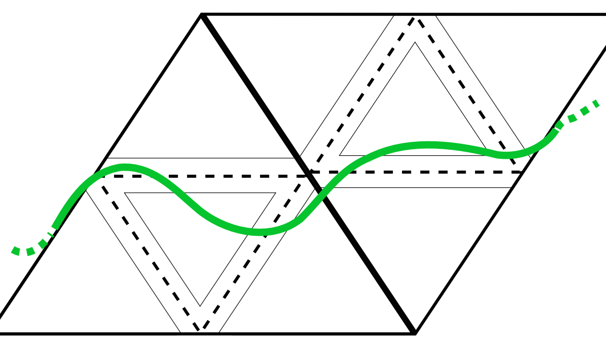


FIGURE 6.4. Two consecutive triangles that are crossed by a path.

It is clear that if D is chosen large enough then the 'cheapest' way to do this is without crossing the thin lines in the picture and staying on the dotted line.

It is not difficult to see that the systole of a random surface equipped with the metric coming from $d_{D,\delta}$ has to be homotopic to a circuit. So if $\omega \in \Omega_N$ and $m_\ell(\omega) \leq D/m_2(d_{D,\delta})$ then the systole stays between the thin lines (so at Euclidean distance at most $\frac{\delta}{2}$ from the dotted lines).

Because $\rho_{D,\delta}(x) \geq 1$ for all $x \in \Delta$ we have:

$$\ell_{\text{Eucl}}(\gamma) \leq \ell_{D,\delta}(\gamma)$$

for every curve γ in every random surface, where $\ell_{D,\delta}$ denotes the length with respect to the metric $d_{D,\delta}$ and ℓ_{Eucl} denotes the length with respect to the Euclidean metric. Also note that for a curve travelling over the dotted lines we have equality.

Suppose we are given $\omega \in \Omega_N$ with $m_\ell(\omega) \leq D/m_2(d_{D,\delta})$ and we look at the systole. On every triangle the systole passes, it must travel at least the (Euclidean) length of the middle subtriangle between the dotted lines (before and after this middle subtriangle there is some possibility to 'cut the corners'). This length is $\frac{1}{\sqrt{2}} - \sqrt{2}\delta = m_2(d_{D,\delta}) - \sqrt{2}\delta$. So for the probability we get that for all $x \leq D$:

$$\mathbb{P}_N\left[\text{sys} \geq x \middle| g \in D_N\right] \geq \mathbb{P}_N\left[m_\ell \geq \left\lfloor \frac{x}{m_2(d_{D,\delta}) - \sqrt{2}\delta}\right\rfloor \middle| g \in D_N\right]$$

When we fill in Proposition 6.2 for the right hand side we get the expression we want up to an error depending on δ. When we choose $\delta = \delta(\varepsilon)$ small enough this error is smaller than ε. Furthermore, if we choose $D = D(\varepsilon)$ large enough then the probability for $> D$ is smaller than ε. This gives us the first part of the proposition for $m_2(d) = \frac{1}{\sqrt{2}}$.

For the expected value we obtain the following:

$$\mathbb{E}_N\left[\text{sys}\right] \geq \sum_{2 \leq k \leq D/m_2(d_{D,\delta})} \mathbb{P}_N\left[m_\ell = k\right](m_2(d_{D,\delta}) - \sqrt{2}\delta)k$$

Because the expression on the right is a finite sum we get:

$$\begin{aligned}\liminf_{N\to\infty} \mathbb{E}_N\left[\text{sys}\right] &\geq \liminf_{N\to\infty} \sum_{2 \leq k \leq D/m_2(d_{D,\delta})} \mathbb{P}_N\left[m_\ell = k\right](m_2(d_{D,\delta}) - \sqrt{2}\delta)k \\ &\geq (m_2(d_{D,\delta}) - \sqrt{2}\delta) \sum_{2 \leq k \leq D/m_2(d_{D,\delta})} k\left(e^{-\sum\limits_{j=1}^{k-1} \frac{2^{j-1}-1}{j}} - e^{-\sum\limits_{j=1}^{k} \frac{2^{j-1}-1}{j}}\right)\end{aligned}$$

We know that the sum on the right converges for $D \to \infty$, so by increasing D it gets arbitrarily close to its limit. Furthermore we have chosen δ and D such that they do not depend on each other, which means that we can make δ arbitrarily small. So this proves the proposition for $m_2(d) = \frac{1}{\sqrt{2}}$.

To get the result for any other prescribed m_2 we can put a constant factor in front of $\rho_{D,\delta}$. $\square$

CHAPTER 7

Curve, pants and flip graphs

In this final chapter, which contains results obtained from joint work with Hugo Parlier, we will make a slight change of subject. We will study the topology of curve, pants and flip graphs. Some of the proofs will however use results from the previous chapters. Thus, instead of an object of study in itself, random surfaces will be a tool in this chapter. It is also important to note that in this chapter we will need to distinguish between multigraphs and simple graphs.

Concretely, the goal of this chapter is to determine the asymptotic behavior of the genus of the curve, pants and flip graph (Theorems 7.1, 7.3 and 7.9) and their quotients by the mapping class group (Theorems 7.4 and 7.10).

7.1. The genus of the (modular) curve graph

For the curve graphs we will not need the random graph approach. Instead we rely on the two classical theorems by Ringel and Ringel-Youngs we have seen in Section 1.3.

We start with the genus of the full curve graph.

THEOREM 7.1. *Let $g \geq 2$. We have:*

$$\gamma(\mathcal{C}(\Sigma_g)) = \infty$$

PROOF. We need to prove that there exists no finite genus surface Σ_g. We will do this by embedding $K_{m,m}$ into $\mathcal{C}(\Sigma_g)$ for arbitrarily large m. We consider curves α_1, β_1, α_2 and β_2 on $\Sigma_{g,n}$ such that:

$$i(\alpha_i, \beta_j) = \delta_{ij},\ i(\alpha_1, \alpha_2) = 0 \text{ and } i(\beta_1, \beta_2) = 0$$

Curves with the properties of α_1, β_1, α_2 and β_2 exist because of the assumption on the genus. Now let D_{β_i} denote the Dehn twist around β_i for $i = 1, 2$. Then the vertices:

$$\left\{\alpha_1, D_{\beta_1}\alpha_1, \ldots, D_{\beta_1}^{m-1}\alpha_1, \alpha_2, D_{\beta_2}\alpha_2, \ldots D_{\beta_2}^{m-1}\alpha_2\right\}$$

induce a $K_{m,m}$. So Theorem 1.17 implies the statement. ☐

In Chapter 1 we have already noted that the modular curve graph of a closed surface is a complete graph with loops added to all or all but one of the vertices. These loops do not change the genus of a graph, so we can ignore them.

In Section 1.3 we have seen that the genus of a complete graph is known. As such, in the case of the modular curve graph all we need to do is to compute its number of vertices. This is given by the number of homeomorphism types of curves on Σ_g which is equal to

$$\left\lfloor \frac{g}{2} \right\rfloor + 1$$

Thus as a corollary to the Ringel-Youngs theorem (Theorem 1.16), we obtain a closed formula for the genus of the modular curve graph.

Corollary 7.2. *Let $g \in \mathbb{N}$. Then:*

$$\gamma(\mathcal{MC}(\Sigma_g)) = \begin{cases} \left\lceil \frac{(\lfloor \frac{g}{2} \rfloor - 2)(\lfloor \frac{g}{2} \rfloor - 3)}{12} \right\rceil & \text{if } g \geq 10 \\ 0 & \text{otherwise} \end{cases}$$

7.2. The genus of the (modular) pants graph

Just like the curve graph, the pants graph has infinite genus in general:

Theorem 7.3. *Let $g \geq 2$, then:*

$$\gamma(\mathcal{P}(\Sigma_g)) = \infty$$

Proof. We will prove this by constructing a subgaph $Z_m \subset \mathcal{P}(\Sigma_g)$ with $\gamma(Z_m) \gtrsim m^2/6$ for arbitrarily large m. We consider a pants decomposition P of Σ_g that contains two curves that cut off distinct one holed tori. So P has two curves, which we denote α_1 and α_2, that are found inside these one holed tori (the curves look like those in Figure 1.4). We choose two curves β_1 and β_2 that also lie entirely in the one holed tori and that satisfy

$$i(\beta_i, \alpha_j) = \delta_{ij}$$

Let $m \in \mathbb{N}$. We look at the set of pairs of curves:

$$\{\alpha_1, \beta_1, D_{\alpha_1}\beta_1, \ldots, D_{\alpha_1}^m \beta_1\} \times \{\alpha_2, \beta_2, D_{\alpha_2}\beta_2, \ldots, D_{\alpha_2}^m \beta_2\}$$

Each pair (a_1, a_2) in this set corresponds to a pants decomposition $P(a_1, a_2)$ by replacing α_1 in P with a_1 and α_2 in P with a_2. We will identify this set of pairs of curves with the corresponding set of pants decompositions and consider the subgraph $Z_m \subset \mathcal{P}(\Sigma_g)$ it induces. We have:

$$p(Z_m) = (m+2)^2$$

Furthermore:

$$\deg\left(\alpha_1, D_{\alpha_2}^k \beta_2\right) = \deg\left(D_{\alpha_1}^k \beta_1, \alpha_2\right) = m + 4$$

and

$$\deg\left(D_{\alpha_1}^k \beta_1, D_{\alpha_2}^l \beta_2\right) = 6 \text{ for } k, l = 1, \ldots, m-1$$

This means that:

$$q(Z_m) \geq \frac{1}{2}\left(2(m-1)(m+4)+6(m-1)^2\right) \geq 4m^2-1$$

Finally, we have $h(Z_m)=3$. Proposition 1.15 now gives the desired result. $\square$

To compute the genus of the modular pants graph, we will use the random graph tools from Chapter 3. The reason we can use these results is that pants decompositions of Σ_g are in one to one correspondence with cubic graphs on $2g-2$ vertices. The correspondence is given by the dual graph to a pants decomposition. This is the graph with the pairs of pants as vertices and an edge between every pair of pairs of pants per curve they share. Note that this dual graph can in fact be a multigraph; it can contain loops corresponding to pairs of pants like the one in Figure 1.4 and it can contain a 2-cycle when two distinct pairs of pants share two curves.

So, the vertices of $\mathcal{MP}(\Sigma_g)$ can be seen as homeomorphism types of cubic multigraphs. When counting the edges of $\mathcal{MP}(\Sigma_g)$, we only need to consider elementary moves that make a difference on the graph level. On the graph level the elementary moves in Figures 1.2, 1.3 and 1.4 look as follows:

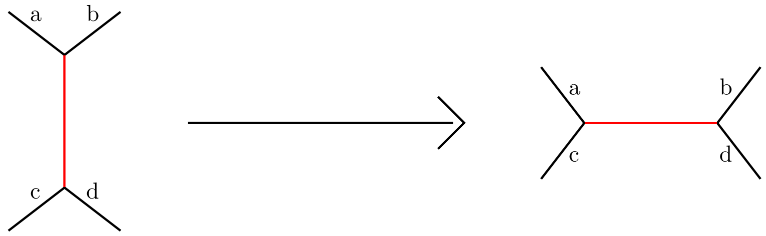


FIGURE 7.1. Elementary move 1. Half-edges a,b,c and d represent pants curves.

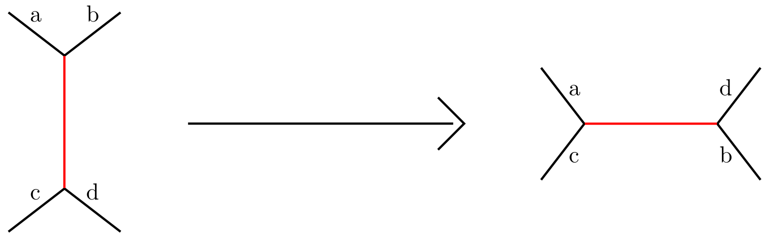


FIGURE 7.2. Elementary move 2. Half-edges a,b,c and d represent pants curves.

FIGURE 7.3. Elementary move 3. Half-edge a represents a pants curve.

The labels in Figures 7.1 and 7.2 are meant to indicate the difference between the two corresponding moves. Also note that move 3 does not do anything to the homeomorphism type of the pants decomposition. This means that the corresponding edge in $\mathcal{P}(\Sigma_g)$ projects to a loop in $\mathcal{MP}(\Sigma_g)$.

Before we state out theorem, we define the following notation.

DEFINITION 7.1. Let $f, g : \mathbb{N} \to \mathbb{N}$ and $c_1, c_2 > 0$. If

$$\liminf_{n\to\infty} \frac{f(n)}{g(n)} \geq c_1 \text{ and } \limsup_{n\to\infty} \frac{f(n)}{g(n)} \leq c_2$$

we write

$$f(n) \sim_{c_1,c_2} g(n) \text{ for } n \to \infty$$

We can now state our theorem.

THEOREM 7.4. *We have:*

$$\gamma(\mathcal{MP}(\Sigma_g)) \sim_{c_1,c_2} \frac{1}{\sqrt{2g-2}} \cdot \left(\frac{6g-6}{4e}\right)^g$$

for $g \to \infty$, where $c_1 = \frac{1}{3e\sqrt{\pi}}$ and $c_2 = \frac{e^3}{\sqrt{\pi}}$.

PROOF. We start with the upper bound. In every pants decomposition P there are $3g-3$ curves on which we can at most perform 2 elementary moves that make a difference and are distinct on the graph level (those of Figures 7.1 and 7.2). This means that in $\mathcal{MP}(\Sigma_g)$:

$$\deg(P) \leq 6g - 6$$

The upper bound now follows directly by applying Theorem 3.5 and Proposition 1.15.

The lower bound is more difficult to obtain. What we need is a lower bound on the number of edges emanating from a 'generic' pants decomposition in $\mathcal{MP}(\Sigma_g)$. More precicely, given $m \in \mathbb{N}$ we want to understand the ratio:

$$\frac{|\{P \in \mathcal{MP}(\Sigma_g)\,;\ \deg(P) \geq 6g-6-m\}|}{|\mathcal{MP}(\Sigma_g)|}$$

To do this we will use the results from random cubic graphs from Chapter 3. We define the following two sets:

$$\mathcal{G}_{2g-2} = \{\text{Cubic graphs with vertex set } \{1,\ldots,2g-2\}\}$$

$$\mathcal{U}_{2g-2} = \{\text{Isomorphism classes of cubic graphs on } 2g-2 \text{ vertices}\}$$

$\mathcal{U}_{2g-2}$ can be identified with the vertex set of $\mathcal{MP}(\Sigma_g)$ and we have two forgetful maps:

$$\Omega_{2g-2} \xrightarrow{\pi_1} \mathcal{G}_{2g-2} \xrightarrow{\pi_2} \mathcal{U}_{2g-2}$$

Furthermore, if $\Gamma \in \mathcal{U}_{2g-2}$ and $G \in \mathcal{G}_{2g-2}$ have k 1-circuits and l 2-circuits, then we have:

$$\left|\pi_1^{-1}(G)\right| = \frac{6^{2g-2}}{2^k 2^l} \text{ and } \left|\pi_2^{-1}(\Gamma)\right| = \frac{(2g-2)!}{|\mathrm{Aut}(\Gamma)|}$$

We want to relate probabilities in Ω_{2g-2} to probabilities in $\mathcal{U}_{2g-2}$. From the two equations above we see that 1-circuits, 2-circuits and automorphisms cause trouble when relating these probabilities. To avoid this, we will consider only graphs in $\mathcal{U}_{2g-2}$ that don't have any such circuits. We define:

$$\mathcal{U}^*_{2g-2} := \left\{\Gamma \in \mathcal{U}_{2g-2};\ \begin{array}{c}\Gamma \text{ carries no non-trivial automorphisms} \\ \text{and has no 1- and 2-circuits}\end{array}\right\}$$

If $A \subset \mathcal{U}^*_{2g-2}$ then:

$$\begin{aligned}\frac{|A|}{\left|\mathcal{U}^*_{2g-2}\right|} &= \frac{\left|\pi_1^{-1} \circ \pi_2^{-1}(A)\right|}{\left|\pi_1^{-1} \circ \pi_2^{-1}(\mathcal{U}^*_{2g-2})\right|} \\ &\geq \frac{\left|\pi_1^{-1} \circ \pi_2^{-1}(A)\right|}{|\Omega_{2g-2}|} \\ &= \mathbb{P}_{2g-2}\left[\pi_1^{-1} \circ \pi_2^{-1}(A)\right]\end{aligned}$$

So what remains is to translate the statement 'deg$(P) \geq 6g-6-m$' into the language of graphs. More precisely, we need to find a necessary condition for a vertex in $\mathcal{MP}(\Sigma_g)$ to have less than $6g-6-m$ edges and this condition needs to be phrased in terms of the graph dual to the pants decomposition of that vertex. There are essentially two ways in which a pants decomposition P can 'lose' an outgoing edge:

(i) P shares an edge with a pants decomposition P' that is isomorphic to P.
(ii) P shares edges with two pants decompositions P_1 and P_2 that are isomorphic to each other.

We start with situation (i). The first possibility in this case is that P contains a curve on which no elementary move makes a difference in the local adjacency structure. This means that it comes from a loop in the dual graph, which we will exclude by considering $\mathcal{U}^*_{2g-2}$. If this is not the case, we proceed as follows. We label the pairs of pants in P and P' with the labels 1,2,...,$2g-2$ consistently (i.e. the pairs of pants that are 'the same' in P and P' get the same label) and consider the map

$$F : P \to P'$$

that sends vertex i to vertex i for all $i = 1, ..., 2g-2$. Because P and P' are related by an elementary move, F preserves the adjacency structure everywhere except around the curve on which the move was performed. Here F changes 2 edges. By assumption we have an isomorphism $\varphi : P' \to P$. Because the adjacency structure around the curve on which the move is performed is different in P and P', the map

φ acts non-trivially on the labels. This means that the map:

$$\varphi \circ F : P \to P$$

is a non-trivial element in S_{2g-2} with:

$$\mathrm{ED}_{\varphi \circ F}(P) \leq 2$$

This is the graph theoretic description we need.

In situation (ii) we exclude loops again and we consider the maps $F_1 : P \to P_1$ and $F_2 : P \to P_2$ and the isomorphism $\varphi : P_1 \to P_2$. The map:

$$F_2^{-1} \circ \varphi \circ F_1 : P \to P$$

is now a non-trivial map with edge defect:

$$\mathrm{ED}_{F_2^{-1} \circ \varphi \circ F_1}(P) \leq 4$$

The map $F_2^{-1} \circ \varphi \circ F_1$ can also not be a two cycle, because it has to move the two vertices around the edge on which the move was performed and cannot interchange them. So in particular, its support contains at least 3 vertices.

Every such map collapses at most 1 of the $6g-6$ edges. So if we define:

$$J_{2g-2,m} = (\pi_2 \circ \pi_1)^{-1} \left(\left\{ \Gamma \in \mathcal{U}^*_{2g-2};\ \begin{array}{c} \exists \sigma_1, \ldots, \sigma_m \in \mathrm{S}_{2g-2} \backslash \{\mathrm{id}\} \text{ such that} \\ \mathrm{ED}_{\sigma_i} \leq 4 \text{ and } a(\sigma_i) \geq 3\ \forall i \text{ and} \\ X_{2g-2,1} = X_{2g-2,1} = 0 \text{ and } \mathrm{Aut}(\Gamma) = \{e\} \end{array} \right\} \right)$$

where $a(\sigma_i)$ denotes the number of elements in the suport of σ_i,then we get that:

$$\mathbb{P}_{2g-2}\left[\deg(P) \geq 6g-6-m\right] \geq 1 - \mathbb{P}_{2g-2}\left[J_{2g-2,m}\right]$$

We now use Proposition 3.7 that tells us that there exists a $C>0$ independent of m such that:

$$\mathbb{P}_{2g-2}\left[J_{2g-2,m}\right] \lesssim \mathbb{P}_{2g-2}\left[\begin{array}{c} X_{2g-2,1} = X_{2g-2,1} = 0 \\ \text{and } \displaystyle\sum_{i=3}^{8} X_{2g-2,i} \geq Cm \end{array}\right]$$

for $g \to \infty$. If we now apply Theorem 3.3 we get:

$$\mathbb{P}_{2g-2}\left[\begin{array}{c} X_{2g-2,1} = X_{2g-2,1} = 0 \\ \text{and } \displaystyle\sum_{i=3}^{8} X_{2g-2,i} < Cm \end{array}\right] \sim \frac{1}{e^2} \cdot \mathbb{P}_{2g-2}\left[\sum_{i=3}^{8} X_{2g-2,i} \geq Cm\right]$$

for $g \to \infty$. Using Theorem 3.3 again, the limit of this last probability can be made arbitrarily small by increasing m. This means that we have proved that for every $\delta > 0$ there exists an m_δ independent of g such that for all g large enough there exists at least:

$$(1-\delta) \cdot \left|\mathcal{U}^*_{2g-2}\right|$$

vertices of degree $6g-6-m_\delta$ in $\mathcal{MP}(\Sigma_g)$. If we apply Theorem 3.1 then we get the theorem. $\square$

7.3. The genus of the (modular) flip graph

Before we can determine the genus of the modular flip graph, we need to develop counting results for triangulations analogous to those we have for cubic graphs.

7.3.1. Counting results. We need to count triangulations of $\Sigma_{g,1}$. We will approach these triangulations through their oriented dual graphs. It follows from an Euler characteristic argument that in this case the number of vertices of the corresponding graph will need to be $4g-2$. The goal of this section is to derive the triangulation analogues of the results we used in the determination of the genus of the modular pants graph.

We start with the following result (Theorem B of [**Pen92**]):

THEOREM 7.5. [**Pen92**] *We have:*

$$|\{\omega \in \Omega_N;\ S(\omega) \textit{ has } 1 \textit{ puncture}\}| \sim \frac{\sqrt{2}}{3N}\left(\frac{6N}{e}\right)^{3N}$$

for $N \to \infty$.

Recall that the random variables $X_{N,i} : \Omega_N \to \mathbb{N}$ count the number of i-cycles in the corresponding graph. From Theorem 5.1 we get the following:

PROPOSITION 7.6. *Let* $m \in \mathbb{N}$. *There exists a set of mutually independent random variables* $X'_i : \mathbb{N} \to \mathbb{N}$ *for* $i = 1, \ldots, m$ *such that when we restrict to surfaces with* 1 *puncture:*

$$X_{N,i} \to X'_i \textit{ in distribution for } N \to \infty \textit{ and } i = 1, \ldots, m$$

where the limit has to be taken over $N \in 4\mathbb{N}$.

PROOF. This follows from the fact that the random variables are sums of a finite number of converging random variables. □

It also follows from Theorem 4.11 that if a labelled oriented graph H contains no left hand turn cycle then we still have:

$$\mathbb{P}_N\left[H \subset \Gamma \middle|\, S \text{ has 1 punture}\right] \leq C N^{-|E(H)|}$$

where $C > 0$ does not depend on H or N. This means that all the arguments in the proof of Proposition 3.7 still work in the case where we restrict to surfaces with 1 puncture. So we obtain:

PROPOSITION 7.7. *Let* $n, k \in \mathbb{N}$. *There exists a* $C > 0$ *such that the probability:*

$$\mathbb{P}_N\left[\begin{array}{c}\exists\ \mathrm{id} \neq \pi_1, \ldots, \pi_n \in \mathrm{S}_N \textit{ such that } \mathrm{ED}_{\pi_i}(\Gamma) \leq k \\ \textit{and } a(\pi_i) \geq k-1\ \forall i = 1, \ldots, n \textit{ and } \Gamma \\ \textit{has } < Cn \textit{ circuits of length } \leq 2k\end{array} \middle| \, S \textit{ has } 1 \textit{ punture}\right]$$

tends to 0 *as* $N \to \infty$.

In particular, the analogous theorem to Theorem 3.2 also holds in this setting: asymptotically almost surely 1-vertex triangulations have no automorphisms.

This means that we also have the following:

THEOREM 7.8. [**Pen92**]. *Let $I_{g,1}$ denote the number of isomorphism classes of triangulations of $\Sigma_{g,1}$. We have:*

$$I_{g,1} \sim \frac{2}{3\sqrt{\pi}\,(4g-2)^{3/2}} \left(\frac{12g-6}{e}\right)^{2g-1}$$

for $g \to \infty$.

PROOF. We can run the same proof as for Theorem 3.18. The difference however is that we do not need to worry about 1- or 2-circuits: there will not be any 1-circuits and because of the orientation 2-circuits no longer add a factor 2. Furthermore, asymptotically there are no automorphisms. So we get:

$$I_{g,1} \sim \frac{1}{3^{4g-2}(4g-2)!} \cdot |\{\omega \in \Omega_{4g-2};\ S(\omega) \text{ has 1 puncture}\}|$$

Filling in Theorem 7.5 and using Stirling's approximation gives the result. □

7.3.2. The genus. For the full flip graph we have the same result as for the curve graph and the pants graph:

THEOREM 7.9. *Let $g \geq 2$ and $n > 0$. We have:*

$$\gamma(\mathcal{F}\,(\Sigma_{g,n})) = \infty$$

PROOF. We will proof this by embedding a graph with vertex set $\mathbb{Z}^2 \times \{0,1\}$ and edges:

$$x \sim y \;\Leftrightarrow\; |x-y| = 1$$

We first consider the triangulation T_0 of the cylinder given in the first image of Figure 7.4 below:

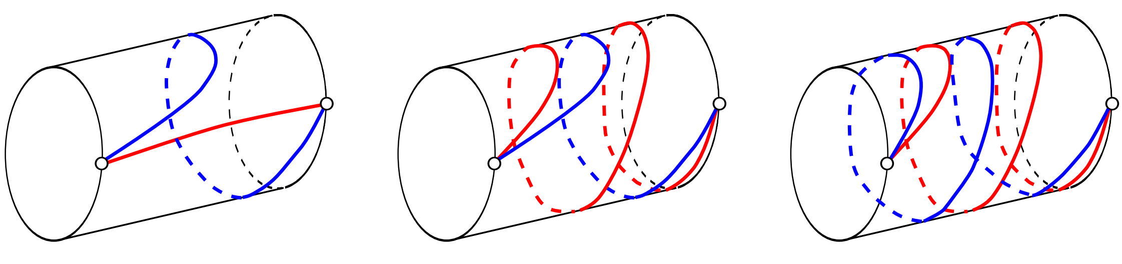


FIGURE 7.4. T_0, F_rT_1 and T_1.

Let F_r denote the flip in the red curve and F_b the flip in the blue curve. We now define the sequence of triangulations: $\{T_i\}_{i\in\mathbb{Z}}$ inductively by:

$$T_{i+1} = F_bF_rT_i \text{ and } T_{i-1} = F_rF_bT_i$$

This means that given a triangulation of $\Sigma_{g,1}$ that contains a cylinder, we obtain a copy of $\mathbb{Z}$ in the flip graph.

We choose a triangulation T of $\Sigma_{g,1}$ that contains two copies of T_0, with red and blue arcs r_1, r_2 and b_1, b_2 respectively and a quadrilateral. Such a triangulation T exists because of our condition on g. The quadrilateral has flip graph K_2. This means that we get a set of vertices which we can label by $\{(T_i, T_j, k)\}_{(i,j,k)\in\mathbb{Z}^2\times\{0,1\}}$. These induce the graph above, which has infinite genus (this follows from Proposition 1.15). □

For the modular flip graph we will focus on 1-vertex triangulations. We have the following theorem:

THEOREM 7.10. *We have:*

$$\gamma\left(\mathcal{MF}\left(\Sigma_{g,1}\right)\right) \sim_{c_1,c_2} \frac{1}{(4g-2)^{3/2}}\left(\frac{12g-6}{e}\right)^{2g}$$

for $g\to\infty$, where $c_1 = \frac{e}{18\sqrt{\pi}}$ and $c_2 = \frac{e}{6\sqrt{\pi}}$.

PROOF. We will use the counting results on random triangulated surfaces from above. The proof is analogous to the proof for the modular pants graph. The main thing we need to control is the degree of the vertices of $\mathcal{MF}\left(\Sigma_{g,1}\right)$.

For the upper bound we use the fact that every triangulation $T \in \mathcal{MF}\left(\Sigma_{g,1}\right)$ contains $6g-3$ arcs, all of which can be flipped. This means that $\deg(T) \leq 6g-3$. Combining this with Theorem 7.5 and Proposition 1.15 gives the upper bound.

For the lower bound we again study the ratio of degree $\geq 6g-3-m$ vertices for $m\in\mathbb{N}$. In this case we do not need to worry about loops in the dual graph, because a 1-vertex triangulation cannot contain such loops. We define the sets:

$$\mathcal{O}_g = \{\text{Triangulations of } \Sigma_{g,1} \text{ with triangles labeled } \{1,\ldots,4g-2\}\}$$

$$\mathcal{W}_g = \{\text{Isomorphism classes of triangulations of } \Sigma_{g,1}\}$$

So $\mathcal{W}_{4g-2}$ is the vertex set of $\mathcal{MF}\left(\Sigma_{g,1}\right)$ and we again have forgetful maps:

$$\{\omega\in\Omega_{4g-2};\ S(\omega)\simeq\Sigma_{g,1}\} \xrightarrow{\pi_1} \mathcal{G}_{2g-2} \xrightarrow{\pi_2} \mathcal{U}_{2g-2}$$

where '$\simeq$' denotes homeomorphism. We also define the set:

$$\mathcal{W}_g^* = \{T\in\mathcal{W}_g;\ \text{The dual graph to } T \text{ carries no non-trivial automorphism}\}$$

Note that because in this case 2-circuits no longer add a factor 2 (like in the proof of Theorem 7.8), we no longer need to add conditions on 2 circuits in the definition of $\mathcal{W}_g^*$. Again we obtain:

$$\frac{|A|}{|\mathcal{W}_g^*|} \geq \mathbb{P}_{4g-2}\left[\pi_1^{-1}\circ\pi_2^{-1}(A)\middle|\, S\simeq\Sigma_{g,1}\right] \text{ for all } A\subset\mathcal{W}_g^*$$

Applying the same reasoning as in the proof for $\mathcal{MP}(\Sigma_g)$ (but now using Proposition 7.7), we obtain that there exists a constant $C > 0$ independent of m such that:

$$\mathbb{P}_{2g-1}\left[\deg(T) \geq 6g-3-m \middle| S \simeq \Sigma_{g,1}\right] \geq 1 - \mathbb{P}_{2g-1}\left[\sum_{i=2}^{8} X_{2g-1,i} \geq Cm \middle| S \simeq \Sigma_{g,1}\right]$$

where we have abused notation slightly, by identifying $\pi_1^{-1} \circ \pi_2^{-1}(\mathcal{W}_g^*)$ with $\mathcal{W}_g^*$. The probability on the right hand side can be made arbitrarily small by increasing m. This, combined with Theorem 7.8 and Proposition 1.15 implies the lower bound. $\square$

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