Mémoire d'habilitation

Extremal and random hyperbolic geometry

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Contents

1 Introduction

In this thesis, we will mainly discuss two types of related questions. The first of these concerns extremal problems in hyperbolic geometry. An example of such a question is: *what is the closed hyperbolic surface of given genus of minimal diameter?* On the one hand, these questions are often the simplest questions that we can ask about the geometric invariants of interest, but on the other hand there are only very few cases in which we are able to answer them.

The second type of questions is probabilistic in nature and often concerns growth and asymptotics in hyperbolic geometry. For example, we will study the geometry of random hyperbolic manifolds. An example of such a question is: *how many short closed geodesics does a random hyperbolic surface of large area have?* Besides making the notion of a "typical" manifold rigorous, random manifolds can also be useful for extremal problems: sometimes it's easier to prove that a random manifold has extremal properties with high probability than providing an explicit construction. This idea is called the probabilistic method and we will see several examples of it.

We have tried to write this text as one coherent story. We will draw connections to work that has been done by others, in hyperbolic geometry, but also in Euclidean geometry and graph theory. Of course, the space is too limited to give proper overviews of all related work, so a selection has been made. We also mention open questions and conjectures throughout the text. Some of these are questions that naturally came up during some of the work we will discuss, others are well known conjectures in the field. In the last chapter, we discuss directions for future research.

Before all this, we will use the rest of this introduction to recall, very briefly, some definitions from hyperbolic geometry.

1.1 Hyperbolic manifolds

To ensure that the readers and the author are thinking of the same thing, we start with a definition:

Definition 1.1. A hyperbolic manifold is a complete Riemannian manifold with constant sectional curvature equal to -1.

We can replace the condition on the curvature by the condition that the manifold is locally isometric to the hyperbolic space $\mathbb{H}^d$ of the right dimension d. Every such manifold is isometric to a quotient $\Gamma\backslash\mathbb{H}^d$, where $\Gamma < \mathrm{Isom}(\mathbb{H}^d)$ is a discrete, torsion-free subgroup of the isometry group $\mathrm{Isom}(\mathbb{H}^d)$ of $\mathbb{H}^d$. This isometry group can be identified with $\mathrm{O}(d,1)^\circ$ and the subgroup of isometries that preserve the orientation $\mathrm{Isom}^+(\mathbb{H}^d) \simeq \mathrm{SO}(d,1)^\circ$. In dimension 2 and 3, we have two accidental isomorphisms:

$$\mathrm{Isom}^+(\mathbb{H}^2) \simeq \mathrm{PSL}(2,\mathbb{R}) \quad \text{and} \quad \mathrm{Isom}^+(\mathbb{H}^3) \simeq \mathrm{PSL}(2,\mathbb{C}).$$

In this thesis we will mainly consider orientable hyperbolic manifolds of finite volume. Most of the time, our hyperbolic manifolds will be 2- or 3-dimensional, but hyperbolic manifolds exist in all dimensions $d \geqslant 2$. In dimension 2, we can even deform a hyperbolic metric on a given surface and we will write $\mathcal{M}_{g,n}$ for the moduli space of hyperbolic surfaces of genus g and with n cusps and $\mathcal{M}_g = \mathcal{M}_{g,0}$. The Mostow–Prasad rigidity theorem [Mos68, Pra73] implies that in dimension greater than two there are no non-trivial deformations. In dimension 3, we still know many constructions of hyperbolic manifolds of finite volume. We can even say, without causing too much controversy, that a "typical" closed 3-dimensional manifold admits a hyperbolic metric. In dimensions above 3, hyperbolic manifolds of finite volume are much rarer. One way to construct them is using arithmetic subgroups of $\mathrm{Isom}^+(\mathbb{H}^d)$. The question whether there are also non-arithmetic manifolds was open for a long time and was resolved by Gromov–Piatetski-Shapiro [GPS88]. However, we still know very few constructions of such manifolds.

For more details on hyperbolic geometry, we refer the reader for instance to: [Bus10, BP92, MR03, Mar22].

2 Flat tori and graphs

In this chapter, we discuss the problems that we will study in this thesis, but in two different contexts, namely flat tori and regular graphs. We do this because there are many analogies with hyperbolic geometry and a lot of ideas that we will use. In the section on graphs (Section 2.2) we will also briefly discuss our work with Maxime Fortier Bourque on kissing numbers of regular graphs [FBP22].

2.1 Packings, coverings and flat tori

Several problems that this thesis will discuss are hyperbolic analogues of very classical (and in general very open) problems in Euclidean geometry. For some motivation and also because we will be using ideas from this area in hyperbolic geometry later on, we will discuss Euclidean packings and coverings in this section. The selection we make is surely very biased and moreover the space is too small to give a serious overview of the theory. We refer to the specialized literature for more details. The book [CS93] is a classic reference, even if there has been a lot of progress since its appearance, the survey articles [PZ04, Oes19] provide overviews of more recent progress.

For us, a **packing in** $\mathbb{R}^n$ is a collection $\mathcal{A} = \{B(a_i, r)\}_{i\in\mathcal{I}}$ of disjoint open balls $B(a_i, r) \subset \mathbb{R}^n$ with center a_i and fixed radius $r > 0$. A **covering of** $\mathbb{R}^n$ is a collection $\mathcal{C} = \{B_c(c_i, r)\}_{i\in\mathcal{I}}$ of closed balls $B_f(c_i, r) \subset \mathbb{R}^n$ with center c_i and radius $r > 0$ such that

$$\bigcup_{i\in\mathcal{I}} B_c(c_i, r) = \mathbb{R}^n.$$

All the questions we will be asking are invariant under homothety, so as soon as the radius of all the balls is the same, we are free to choose it as we like. The most interesting case for us is the case where the centers form a lattice in $\mathbb{R}^n$.

2.1.1 Three invariants

We are going to define three invariants: the **density** and the **kissing number** of a packing and the **thickness**[1] of a covering. We will start with the density: the density of a packing $\mathcal{A} = \{B(a_i, r)\}_{i\in\mathcal{I}}$ is the proportion of the volume that is taken up by the balls in $\mathcal{A}$. Formally:

$$\delta(\mathcal{A}) := \limsup_{R\to\infty} \frac{\mathrm{vol}\left(\left(\bigcup_{i\in\mathcal{I}} B(a_i, r)\right) \cap B(0, R)\right)}{\mathrm{vol}(B(0, R))}.$$

The kissing number is the maximal number of balls that is tangent to a given one in $\mathcal{A}$:

$$\mathrm{Kiss}(\mathcal{A}) := \max_{i\in\mathcal{I}} \left|\{j \in \mathcal{I}\setminus\{i\};\ B_c(a_i, r) \cap B_c(a_j, r) \neq \varnothing\}\right|.$$

The thickness of a covering $\mathcal{C}\{B_c(c_i, r)\}_{i\in\mathcal{I}}$ is the average (with respect to $x \in \mathbb{R}^n$) of the number of balls containing x:

$$\theta(\mathcal{C}) := \liminf_{R\to\infty} \frac{\sum_{i\in\mathcal{I}:\ B_c(c_i,r)\subset B(0,R)} \mathrm{vol}(B_c(c_i, r))}{\mathrm{vol}(B(0, R))}.$$

There is a vast body of literature on these three invariants. The central questions are:

Question 1

Given $n \in \mathbb{N}$, what are:

$$\sup_{\mathcal{A}} \delta(A), \quad \max_{\mathcal{A}} \mathrm{Kiss}(\mathcal{A}) \quad \text{and} \quad \inf_{\mathcal{C}} \theta(\mathcal{C})$$

where $\mathcal{A}$ varies among packings in $\mathbb{R}^n$ and $\mathcal{C}$ varies among coverings of $\mathbb{R}^n$?

2.1.2 Lattices and flat tori

We have already alluded to the fact that there are a packing and a covering associated to each lattice $\Lambda < \mathbb{R}^n$. For the packing, we use the **injectivity radius**:

$$r(\Lambda) = \max\{s > 0;\ B(v, s) \cap B(w, s) = \varnothing \text{ for all } v \neq w \in \Lambda\}$$

[1] This is sometimes also called the density of a covering.

and for a covering we use the **covering radius** of Λ:

$$R(\Lambda) := \min\left\{ s > 0;\ \bigcup_{v\in\Lambda} B_f(v,s) = \mathbb{R}^n \right\}.$$

Given a lattice $\Lambda < \mathbb{R}^n$, the quotient $\mathbb{R}^n/\Lambda$ with the metric induced by the Euclidean metric of $\mathbb{R}^n$ is a flat torus. Moreover, the Killing–Hopf theorem implies that every flat torus is of this form.

In the case of lattices, the three invariants defined above can also be understood in terms of the geometry of the associated flat torus. Indeed, we have

$$\delta(\Lambda) = \frac{\omega_n}{2^n}\frac{\mathrm{sys}(\mathbb{R}^n/\Lambda)^n}{\mathrm{vol}(\mathbb{R}^n/\Lambda)}$$

where $\omega_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)}$ is the volume of a ball of radius 1 in $\mathbb{R}^n$ and $\mathrm{sys}(\mathbb{R}^n/\Lambda)$ is the **systole** of $\mathbb{R}^n/\Lambda$: the minimum length of a closed geodesic in $\mathbb{R}^n/\Lambda$. Equivalently, $\mathrm{sys}(\mathbb{R}^n/\Lambda)$ is the minimal Euclidean norm of a non-zero vector in Λ. The (scale invariant) quantity $\frac{\mathrm{sys}(\mathbb{R}^n/\Lambda)^n}{\mathrm{vol}(\mathbb{R}^n/\Lambda)}$ is called the **systolic ratio** of $\mathbb{R}^n/\Lambda$.

We also observe that $\mathrm{Kiss}(\Lambda)$ is the number of non-zero vectors in Λ of minimal norm, or equivalently: the number of free homotopy classes of oriented geodesics that realize systole of $\mathbb{R}^n/\Lambda$.

The systole and kissing number also admit a spectral interpretation. To this end, we define the **Laplacian** $\Delta : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ by

$$\Big(\Delta f\Big)(x) := -\Big(\mathrm{div}\circ \mathrm{grad} f\Big)(x) = -\Big(\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} f\Big)(x), \quad f\in C^\infty(\mathbb{R}^n),\ x\in\mathbb{R}^n.$$

The Laplacian commutes with Euclidean isometries of $\mathbb{R}^n$ and thus descends to a differential operator $\Delta : C^\infty(\mathbb{R}^n/\Lambda) \to C^\infty(\mathbb{R}^n/\Lambda)$. The eigenfunctions ($L^2$-normalized with respect to the Lebesgue measure) for this operator are the functions $\varphi_w : \mathbb{R}^n/\Lambda \to \mathbb{R}$ defined by

$$\varphi_w(x) = \frac{1}{\sqrt{\mathrm{vol}(\mathbb{R}^n/\Lambda)}}\, e^{2\pi i\cdot\langle x,w\rangle}, \quad x\in\Lambda, w\in\Lambda^\star,$$

where $\langle\cdot,\cdot\rangle : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ is the standard inner product and $\Lambda^\star$ is the **dual lattice** of Λ, defined by

$$\Lambda^\star := \{w\in\mathbb{R}^n;\ \langle w,v\rangle\in\mathbb{Z},\ \forall v\in\Lambda\}.$$

The eigenfunctions $(\varphi_w)_{w\in\Lambda^\star}$ form an orthonormal basis of $L^2(\mathbb{R}^n/\Lambda)$ and the eigenvalue associated to $w \in \Lambda^\star$ is

$$4\pi \left|\left|w\right|\right|^2 .$$

So, the smallest non-zero eigenvalue of the Laplacian on $\mathbb{R}^n/\Lambda$ is $4\pi\,\mathrm{sys}(\mathbb{R}^n/\Lambda^\star)^2$ and $\mathrm{Kiss}(\Lambda^\star)$ is its multiplicity: the dimension of the associated eigenspace.

Finally, the thickness of Λ can be written as

$$\theta(\Lambda) = \omega_n \frac{R(\Lambda)^n}{\mathrm{vol}(\mathbb{R}^n/\Lambda)}$$

Moreover, the covering radius $R(\Lambda)$ coincides with the diameter

$$\mathrm{diam}(\mathbb{R}^n/\Lambda) := \max\left\{\mathrm{d}(x,y);\ x,y \in \mathbb{R}^n/\Lambda\right\},$$

where $\mathrm{d}(x,y)$ denotes the distance between x and y. Indeed, $\mathbb{R}^n/\Lambda$ is a homogeneous space, so its diameter is realized at every point. That is:

$$\mathrm{diam}(\mathbb{R}^n/\Lambda) = \max\left\{\mathrm{d}([0],y);\ y \in \mathbb{R}^n/\Lambda\right\} = R(\Lambda).$$

The conclusion of all of this is that, if we restrict to lattices, Question 1 is equivalent to:

Question 2

Let $n \in \mathbb{N}$, what are:

$$\max_\Lambda \mathrm{sys}(\mathbb{R}^n/\Lambda), \quad \max_\Lambda \mathrm{Kiss}(\Lambda) \quad \text{and} \quad \min_\Lambda \mathrm{diam}(\mathbb{R}^n/\Lambda),$$

where Λ varies among lattices of covolume 1 in $\mathbb{R}^n$?

The moduli space of lattices of covolume 1 in $\mathbb{R}^n$ can be identified with $\mathrm{SL}(n,\mathbb{Z})\backslash\mathrm{SL}(n,\mathbb{R})$. Moreover, the fact that we can replace the supremum and the infimum of Question 1 with a maximum and a minimum is a consequence of Mahler's compactness theorem [Mah46]. Using the fact that duality induces an involution on $\mathrm{SL}(n,\mathbb{Z})\backslash\mathrm{SL}(n,\mathbb{R})$, the maximum systole and the maximum kissing number can be replaced by the maximum of the first eigenvalue and the maximum multiplicity respectively.

2.1.3 A bit of history

As mentioned before, the three questions discussed above have a long history, even if there are still very few dimensions in which their answers are known.

Among all packings, the maximal density is known in dimension 1–3, in dimension 8 and in dimension 24. In dimension 1, the question is trivial and from the classical results of Thue [Thu10] and Fejes Tóth [Fej43] we know that the densest packing in dimension 2 is the hexagonal lattice. The results in dimensions 3, 8 and 24 are consequences of the spectacular progress of the last 30 years. In 1998 Hales–Ferguson [Hal05] demonstrated that the densest 3-dimensional packing is the face-centered cubic lattice, as claimed by Kepler in 1610. Viazovska [Via17] proved that the densest 8-dimensional packing is the E_8 lattice and in dimension 24 it is the Leech lattice, as proved by Cohn–Kumar–Miller–Radchenko–Viazovska [CKM+17]. The last two results use the linear programming method of Cohn–Elkies [CE03], which we will briefly describe in the next section. The idea behind this bound has its analogue in hyperbolic geometry, which we will use in Chapter 3. Among lattices, the maximum density is known in dimension 1–8 [CS93, Table 1.1, p.12] and in dimension 24 [CK09]. The best bound when the dimension $n \to \infty$ is that of Kabatjanskiĭ–Levenšteĭn [KL78] and can be reproduced in using linear programming [CZ14].

Another point of view on density is that of systolic geometry (see [Kat07] for an introduction). There, we try to maximize the systolic ratio among all the Riemannian metrics on our torus. Apart from the trivial case of dimension 1, the answer to this question is only known in dimension 2, due to Loewner who never published his result: the hexagonal torus maximizes the systolic ratio among all Riemannian tori two -dimensional. The hexagonal torus also maximizes the first non-zero eigenvalue of the Laplacian among all Riemannian metrics of the same area [Nad96]. The spectral version of the problem has no meaning in higher dimensions: on any closed smooth manifold of dimension at least 3 there exist Riemannian metrics whose volume is 1, but whose first eigenvalue is arbitrarily large [CD94].

The kissing number has been studied a lot as well (see for example [CS93, PZ04]). The global maximizers are known in dimension 1–4, 8 and 24. The problem is trivial in dimension 1 and an exercise in dimension 2. The 3-dimensional version was the subject of a famous discussion between Isaac Newton and David Gregory: Newton thought it was 12, the kissing number of the face-centered cubic lattice, and Gregory thought there was space for a thirteenth ball. One reason that this is not a simple question is that the configuration of the face-centered cubic lattice is not rigid: we can move the balls while keeping the kissing

number equal to 12. In the end, Newton was right and this was proved by Schütte–van der Waerden [SvdW53] after many attempts by many mathematicians. The maximal kissing numbers in dimension 8 and 24 are those of the E_8 lattice and the Leech lattice respectively, as proved by Odlyzko–Sloane and independently Levenšteĭn [OS79, Lev79]. Their proof is based on the linear programming method of Delsarte-Goethals-Seidel [DGS77]. Much more recently, the maximal kissing number in dimension 4 was determined by Musin [Mus08].

Finally, the maximal thickness of a covering of $\mathbb{R}^n$ is only known in dimensions 1 and 2 and is realized by the hexagonal lattice in dimension 2 [Ker39]. Among lattices, the best coverings are known in dimensions 1–5 [Bam54, DR63, RB75, RB78] The best asymptotic bound known, as $n \to \infty$, is due to Rogers [Rog59] and states that there exists a constant $c > 0$ such that

$$\min\{\theta(\Lambda);\ \Lambda < \mathbb{R}^n \text{ a lattice of covolume } 1\} \leqslant n \cdot \log(n+1)^c$$

for all $n \geqslant 1$.

2.1.4 Linear programming

The results by Cohn–Kumar, Viazovska and Cohn–Kumar–Miller–Radchenko–Viazovska on the densest packings in dimensions 8 and 24 all start with a bound proved by Cohn–Elkies [CE03] and Gorbachev [Gor00] that we will briefly discuss now, mainly because we will treat a hyperbolic version of it in the next chapter.

We will state the Cohn–Elkies–Gorbachev theorem for lattices, because that is the case that corresponds best to hyperbolic manifolds. However, we emphasize that the bound can be adapted to general general sphere packings and the proof of that version is only slightly longer. In the theorem, $\widehat{f} : \mathbb{R}^n \to \mathbb{R}$ will denote the Fourier transform of $f : \mathbb{R}^n \to \mathbb{R}$, given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) \exp(-i\langle x, \xi\rangle) dx, \quad \xi \in \mathbb{R}^n$$

where $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the standard inner product and integration is performed with respect to the Lebesgue measure. We will call $f : \mathbb{R}^n \to \mathbb{R}$ **admissible** if there exist constants $C, \delta > 0$ such that

$$|f(x)| \leqslant \frac{C}{(1+||x||)^{n+\delta}} \quad \text{and} \quad \left|\widehat{f}(\xi)\right| \leqslant \frac{C}{(1+||\xi||)^{n+\delta}}$$

for all $x, \xi \in \mathbb{R}^n$. Cohn–Elkies and Gorbachev proved:

Theorem 2.1 (Cohn–Elkies–Gorbachev)**.** *Suppose there exists a non-zero admissible function $f : \mathbb{R}^n \to \mathbb{R}$ such that*

(a) $f(x) \leqslant 0$ *when* $||x|| \geqslant R$,

(b) $\widehat{f}(\xi) \geqslant 0$ *for all* $\xi \in \mathbb{R}^n$ *and*

(c) $f(0) < \widehat{f}(0)$.

Then $\mathrm{sys}(\mathbb{R}^n/\Lambda) \leqslant R$ *for all lattices* $\lambda < \mathbb{R}^n$ *of covolume* 1.

Proof. Suppose $\Lambda < \mathbb{R}^n$ is a lattice of covolume 1 and whose shortest non-zero lattice vector has norm more than R. Then by the Poisson summation formula and properties (a) and (b) of f:

$$f(0) \geqslant \sum_{v \in \Lambda} f(v) = \sum_{w \in \Lambda^\star} \widehat{f}(w) \geqslant \widehat{f}(0).$$

This contradicts property (c) of f. □

The first time one sees this proof, it seems amazing that anything can come out of this bound: in the displayed string of inequalities, we throw away infinitely many terms in two steps. Nonetheless Cohn and Elkies, by making a computer search for good test functions, already observed that their bound came very close to the systole of the E_8 lattice and the Leech lattice in dimensions 8 and 24 respectively. This started the search for two "magical functions" that saturate the bounds in these dimensions, which were found by Viazovska and Cohn–Kumar–Miller–Radchenko–Viazovska. We refer to [Via17, CKM+17, Oes19] for more on this.

2.2 Regular graphs

The second parallel we would like to draw is with graph theory. Indeed, just like for flat tori, almost all the invariants that we are going to study for hyperbolic manifolds have their analogues in graph theory. Moreover, even if in graph theory the state of the art is more advanced, most of the analogous questions are also open and form the basis of a very active area of research. Like in the last section, we do not at all claim to provide an overview of the state of the art in graph theory, but will make a selection based on analogies with hyperbolic geometry.

The graphs that we will discuss are all regular: for $k \in \mathbb{N}$, a **k-regular graph** is a graph whose vertices all have degree k. Because 0-, 1- and 2-regular graphs are somewhat special, we will assume that $k \geqslant 3$ in what follows.

One reason for the analogy between regular graphs and hyperbolic surfaces is that the $(p+1)$-regular tree can be identified with the coset space $\mathrm{PGL}(2, \mathbb{Q}_p)/\mathrm{PGL}(2, \mathbb{Z}_p)$ (see for example [Ser77, Chapter 2]). Given a torsion-free lattice $\Gamma < \mathrm{PGL}(2, \mathbb{Q}_p)$, the quotient $\Gamma \backslash \mathrm{PGL}(2, \mathbb{Q}_p)/\mathrm{PGL}(2, \mathbb{Z}_p)$ is a finite $(p+1)$-regular graph. So, like hyperbolic surfaces, we can describe graphs using 2×2 matrices. This is not just a superficial connection, we will see that certain techniques (especially those related to arithmetic lattices) carry over from one side to the other.

2.2.1 The girth

Just like for flat tori and manifolds, we can speak about the systole of a graph. In graph theory this is usually called the **girth** of the graph: the minimal length of a cycle in the graph. If G is a graph, we write $\mu(G)$ for its girth. The simple question we are going to ask ourselves is: given $k, n \in \mathbb{N}$, what is the maximum possible girth of a k-regular graph on n vertices?

We start with an elementary upper bound:

Lemma 2.2 (Moore bound)**.** *Let G be a k-regular graph of girth μ, then the number of vertices of G is at least*

$$1 + k \sum_{j=0}^{(\mu-3)/2} (k-1)^j \quad \textit{or} \quad 2 \sum_{j=0}^{(\mu-2)/2} (k-1)^j$$

depending on whether μ is odd or even respectively.

This bound implies that the girth of a k-regular graph on n vertices is bounded by $2\log_{k-1}(n) + O(1)$ as $n \to \infty$. The proof is based on the volume growth of balls in the k-regular tree. Moreover, it is the same as that of the analogous bound in hyperbolic geometry (Lemma 3.1 below).

A graph that saturates the Moore bound is called a **Moore graph**. It turns out there are very few of them. For instance, Moore graphs of odd girth have almost been classified. There are two infinite families of Moore graphs of odd girth (cycles and complete graphs) and two sporadic examples (the Petersson graph and the Hoffman–Singleton graph)

[BI73, Dam73] (see also [EJ08]). The classification is not complete because it is not known whether there exists a Moore graph of degree 57 and girth 5 or not. By definition, if this graph exists, it would have 3250 vertices. More generally, we know that the girth of a Moore graph of degree at least 3 cannot exceed 12.

The natural question that remains, and that also corresponds best to the questions that we are going to ask in hyperbolic geometry, is what happens when we fix the degree k and we make the number of vertices tend towards infinity. Even the question of whether the limit

$$\lim_{n\to\infty} \frac{\max\{\mu(G);\ G \text{ a } k\text{-regular graph on } n \text{ vertices}\}}{\log_{k-1}(n)}$$

exists and if so, what its value is, is open. The first constructions of regular graphs whose girth grows logarithmically as a function of the number of vertices is due to Erdős–Sachs [ES63]. They provided an algorithm that produces infinite sequences $(G_n)_n$ of k-regular graphs on $2n$ vertices with girth $\mu(G_n) \geqslant (1+o(1))\cdot\log(2n)$, as $n \to \infty$. Improving on this construction is a well known and difficult problem that has plenty of history [Mar82, Imr84, Wei84b, LPS88, Mar88, Mor94, Big98, MWW04, Dah14, LS21]. The best upper bound on the asymptotic girth ratio above is still the bound given by the Moore bound. The best lower bound comes from the Ramanujan graphs of Margulis [Mar88] and Lubotzky–Phillips–Sarnak [LPS88] and their generalizations by Morgenstern [Mor94] and states that when the degree equals $k = p^m + 1$ with p an odd prime, then there exists a sequence of k-regular graphs $(G_{n_j})_j$ on n_j vertices such that $n_j \to \infty$ and

$$\lim_{j\to\infty} \frac{\mu(G_{n_j})}{\log_{k-1}(n_j)} = \frac{4}{3}.$$

The upper bound in the above is due to Biggs-Boshier [BB90]. The graphs involved are in fact Cayley graphs of $\mathrm{PGL}(2, \mathbb{Z}/q\mathbb{Z})$ of degree $p^m + 1$.

2.2.2 The kissing number

In analogy with flat tori, we can also define a **kissing number** of a finite graph: the number of distinct oriented cycles in the graph that realize its girth. Together with Maxime Fortier Bourque [FBP22], we proved that Moore graphs can be characterized by their kissing number:

Theorem 2.3. *If G is a connected graph of maximal degree k and girth g on n vertices, then their kissing number satisfies*

$$\mathrm{Kiss}(G) \leqslant \frac{nk(k-1)^{\lfloor g/2 \rfloor}}{g}$$

with equality if and only if G is a Moore graph.

Our proof is elementary and is heavily inspired by Parlier's proof of a similar inequality in hyperbolic geometry [Par13]. For regular graphs, our bound improves on earlier inequalities due to Teo–Koh [TK92], Azarija–Klavžar [AK15].

Using the Moore bound, we can remove the dependence on girth from this theorem and obtain for instance that, a k-regular graph on n vertices has kissing number $\mathrm{Kiss}(G) \ll n^2/\log(n)$. Like the Moore bound, it is unclear whether this is sharp. It is easy, for instance using cyclic covers, to construct sequences of regular graphs of any degree $k \geqslant 3$ whose kissing number grows linearly as a function of their number of vertices.

Together with Maxime Fortier Bourque [FBP22], we showed that the Ramanujan graphs of Lubotzky–Phillips–Sarnak can be used to obtain examples of super-linear growth:

Theorem 2.4. *For every prime number $p \equiv 1 \mod 4$ there is a subsequence $(X^{p,q_k})_k$ of the $(p+1)$-regular graphs $X^{p,q}$ of Lubotzky–Phillips–Sarnak such that*

$$\lim_{k\to\infty} \frac{\log(\mathrm{Kiss}(X^{p,q_k}))}{\log(n_k)} = \frac{4}{3}$$

where n_k is the number of vertices of X^{p,q_k}.

Like Lubotzky–Phillips–Sarnak, we use bounds on the number of integral solutions to certain quadratic equations to analyze their geodesics, but in a different regime.

2.2.3 The spectral gap

For graphs, unlike for tori but just like for hyperbolic manifolds, the spectral questions are distinct from the geometric questions. The question on the spectral gap that we will discuss here can be thought of as a question on connectivity. Indeed, the main reason Lubotzky–Phillips–Sarnak [LPS88] and Margulis [Mar88] introduced their graphs, was to provide examples of explicit optimal expander graphs. An expander sequence is a sequence of graphs that are both sparse (their degree is bounded) and highly connected. For an

overview of the subject, we refer to [HLW06]. Here we will just briefly discuss some of what is known about Ramanujan graphs.

There are multiple ways to measure the connectivity of a graph G. In this section we will use the spectral gap of its adjacency matrix A_G (or equivalently, of its graph Laplacian). If G is a connected k-regular graph on n vertices then its adjacency matrix has real eigenvalues:

$$\lambda_1(G) = k > \lambda_2(G) \geqslant \lambda_3(G) \geqslant \ldots \geqslant \lambda_n(G) \geqslant -k$$

and G is **bipartite** if and only if $\lambda_n(G) = -k$. Recall that $G = (V, E)$ is bipartite if we can write $V = V_1 \sqcup V_2$ and there are no edges interior to V_1, nor to V_2.

We set

$$\lambda(G) = \max\left\{|\lambda_i(G)|\,;\, \lambda_i(G) \neq \pm k\right\}.$$

This number can be seen as a measure of connectivity of G for instance because it shows up in the mixing rate of a random walk on G and also because it relates to the Cheeger constant of G (that we will briefly discuss in Section 6.3). The general rule is that the smaller $\lambda(G)$ is, the more connected G is.

So, the natural extremal question is how small $\lambda(G)$ can be. The Alon-Boppana bound [Alo86] states that for a k-regular graph G,

$$\lambda(G) \geqslant 2\sqrt{k-1} - \varepsilon(G)$$

where the error term $\varepsilon(G) \to 0$ as the number of vertices of G tends to infinity. The significance of the number $2\sqrt{k-1}$ is that the spectrum of the adjacency matrix of the infinite k-regular tree is contained in $[-2\sqrt{k-1}, 2\sqrt{k-1}]$. A sequence $(G_n)_n$ of k-regular graphs on n vertices is called an **expander sequence** if $\lambda(G_n)$ is uniformly bounded from above. A k-regular graph G is called **Ramanujan** if $\lambda(G) \leqslant 2\sqrt{k-1}$.

The construction of Margulis and Lubotzky–Phillips–Sarnak yields Ramanujan graphs of degree p^m+1 for any odd prime p. They can be made bipartite, in which case they're Cayley graphs of $\mathrm{PGL}(2, \mathbb{Z}/q\mathbb{Z})$, or not, in which case they're Cayley graphs of $\mathrm{PSL}(2, \mathbb{Z}/q\mathbb{Z})$.

Besides having very large girth and an asymptotically optimal spectral gap, they also have very small but not optimal diameter [LPS88, Sar19].

More recently, using a completely different argument, Marcus–Spielman–Srivastava [MSS15] proved that there exist infinite sequences of bipartite Ramanujan graphs of every

fixed degree. The non-bipartite case is currently still open.

2.2.4 Random graphs

The final subject we discuss before we get to hyperbolic geometry is that of random regular graphs. Also here, we will just highlight some aspects that have analogues in hyperbolic geometry and refer to the specialized literature (for example [Bol01, JŁR00, Wor99]) for a more complete overview.

The configuration model. An important model for a random k-regular graph is the **configuration model**: we take n (an even number if k is odd) vertices with k half-edges emanating from each of them and uniformly randomly pair up the half-edges. Every pairing is allowed, so in particular, there is a chance the result is disconnected. However, this probability is of the order $O(n^{-1})$ as $n \to \infty$ [Bol81, Wor81a]. We will denote the resulting random regular graph by $G_{n,k}$.

One result of which we will see many analogues in hyperbolic geometry is a classical result by Bollobás [Bol80] and Wormald [Wor81b], who proved that the number of short cycles in $G_{n,k}$ converges to a Poisson distributed random variable as $n \to \infty$. For sharper versions of this result, see [MWW04, Joh15].

The graphs $G_{n,k}$ are also highly connected. The first example of this is their diameter. Before making this precise, we note that the Moore bound (Lemma 2.2) in the case of odd girth is equivalent to:

Lemma 2.5 (Moore bound for the diameter)**.** *Let G be a k-regular graph of diameter d, then G has at most*

$$1 + k \sum_{j=0}^{d-1} (k-1)^j$$

vertices.

In particular, if $(G_n)_n$ is a sequence of k-regular graphs on n vertices and $k \geqslant 3$, then

$$\operatorname{diam}(G_n) \geqslant (1 + o(1)) \cdot \log_{k-1}(n), \quad \text{as } n \to \infty.$$

We will provide a proof of a hyperbolic version of this bound, that uses the exact same idea, in Section 6.2. Bollobás–Fernandez-de-la-Vega [BFdlV82] proved that random regular

graphs saturate this bound. That is,

$$\frac{\operatorname{diam}(G_{n,k})}{\log_{k-1}(n)} \longrightarrow 1$$

in probability as $n \to \infty$.

Finally, $G_{n,k}$ is also nearly Ramanujan, this was a conjecture by Alon [Alo86] and was proved by Friedman [Fri08]. Concretely, for all $\varepsilon > 0$,

$$\mathbb{P}\left(\lambda(G_{n,k}) \leqslant 2\sqrt{k-1} + \varepsilon\right) \longrightarrow 1,$$

as $n \to \infty$. A shorter proof and generalizations of this theorem to random finite degree covers of graphs were found in [Bor20, BC19, BC23].

Random Cayley graphs. Another model of random regular graphs that has been studied is that of random Cayley graphs. That is, we take a sequence of finite groups $(Q_n)_n$ and let $S_n \subset Q_n$ be a random subset. This yields a random graph

$$G_n = \operatorname{Cay}(Q_n, S_n \cup S_n^{-1}),$$

the Cayley graph for Q_n with edges determined by $S_n \cup S_n^{-1}$, where $S_n^{-1} = \{s^{-1};\, s \in S\}$. G_n is connected if and only if S_n is a generating set for Q_n, which is typically the case for the sequences of groups that are considered.

Random Cayley graphs were first studied for sequences of random generating sets whose size grows with the size of the group [AR93, Pak99, LR04, LS04b]. We will consider the case in which $S_n \subset Q_n$ is a uniformly random subset of size k. The case that is best understood is when the sequence of groups is $(\mathbb{G}(\mathbb{F}_q))_q$, where $\mathbb{G}$ is a simple group of fixed Lie type and fixed rank over $\mathbb{F}_q$.

In [GHS+09], Gamburd–Hoory–Shahshahani–Shalev–Virág proved that, with high probability as $q \to \infty$, the girth of a random $2k$-regular Cayley graph G_q of $\mathbb{G}(\mathbb{F}_q)$ satifies

$$\mu\Big(G_q\Big) \geqslant \left(\frac{1}{\dim(\mathbb{G})} + o(1)\right) \cdot \log_{2k-1}(|\mathbb{G}(\mathbb{F}_q)|).$$

Combined with work by Bourgain–Gamburd [BG08], this also implies these graphs are expanders in the case of $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$. This was generalized to all finite simple groups of Lie type by Breuillard–Green–Guralnick–Tao [BGGT15].

For bounds in the case of random Cayley graphs of symmetric groups and finite simple groups of Lie type of growing rank over a fixed finite field, we refer to [Ebe17, LS19, EJ22].

3 Extremal problems I

The goal of this chapter is to present a first selection of results on extremal problems in hyperbolic geometry. A big role in this chapter will be played by our joint project with Maxime Fortier Bourque [BP22, BP23b, BP23c]. In this project we use the Selberg trace formula to prove bounds on spectral and geometric invariants of hyperbolic manifolds, in a way that is analogous to that of Cohn–Elkies–Gorbachev [Gor00, CE03] (see Section 2.1.4). With this method we were able to prove bounds on various invariants with a uniform proof strategy, whereas the best bounds before used very varied methods.

This chapter will also discuss constructions of hyperbolic surfaces of large systole, based on joint work with Alexander Walker [PW18] and on [Pet18].

3.1 The systole

Perhaps the simplest geometric invariant of a hyperbolic manifold M is its **systole** $\mathrm{sys}(M)$: the minimal length of a closed geodesic on M. Equivalently, this is the minimum length of a closed essential curve[1]. Figure 3.1 shows an example.

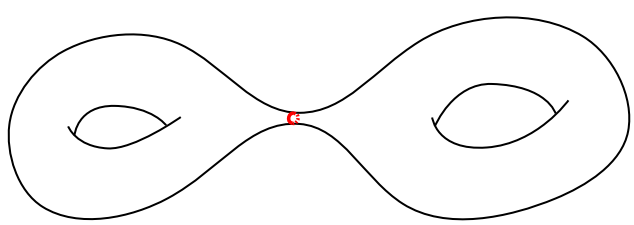

Figure 3.1: A curve that realizes the systole.

It is not difficult to see that there are hyperbolic surfaces of finite (and bounded) area

[1]Essential means that the curve is not homotopic to a point and that there does not exist a family of curves, freely homotopic to the given curve, which leave any compact subset of M.

and arbitrarily small systole. In dimension 3, Dehn surgery techniques – in particular Thurston's convergence theorem (see for example [BP92, Chapter E] or [Mar22, Chapter 15]) – imply the analogous result. In higher dimension, the fact that there exists only a finite number of isometry classes of hyperbolic manifolds of volume $\leqslant v$ and dimension d for all $v > 0$ and $d \geqslant 4$ [Wan72] implies that the volume of a sequence of manifolds whose systole tends towards 0, necessarily tends towards infinity. However, the existence of hyperbolic manifolds of finite volume and arbitrarily small systole is known for any dimension from the work of Agol and Belolipetsky–Thomson [Ago06, BT11].

The simple question that remains is:

Question 3

Given $d \geqslant 2$ and $v > 0$, what is

$$\max \left\{ \mathrm{sys}(M); \quad \begin{array}{c} M \text{ a hyperbolic } d\text{-manifold} \\ \text{of volume } \leqslant v \end{array} \right\} \quad ?$$

And which hyperbolic manifold realizes it?

The fact that this quantity is indeed a maximum and not just a supremum is not trivial. In dimension 2 it is a theorem of Mumford [Mum71], in dimension 3 it is a consequence of the work of Jørgensen–Thurston [Thu02, Theorem 5.12.1] and in dimension at least four Wang's theorem mentioned above implies it. In dimension 2, the function $\mathrm{sys} : \mathcal{M}_g \to (0, \infty)$ is a topological Morse function [Akr03], so in theory it could be used to understand the topology of $\mathcal{M}_g$. However, this currently seems out of reach, even counting the number of local maxima is a very hard problem [FBR22].

Like for all the geometric invariants that we will consider, there is a classical upper bound on systole and the real question is whether this bound is close to being optimal. This classical bound is the analogue of the Moore bound (see Section 2.2.1) and states:

Lemma 3.1. *For every $d \geqslant 2$ there exists a constant $c_d > 0$ such that*

$$\mathrm{sys}(M) \leqslant \frac{2}{d-1} \log \Big(\mathrm{vol}(M) \Big) + c_d,$$

for every closed hyperbolic d-manifold M.

Proof. Take any point $p \in M$. The open ball $B(p, \mathrm{sys}(M)/2)$ of radius $\mathrm{sys}(M)/2$ and center p is isometric to an open ball in $\mathbb{H}^n$ of the same radius.

Indeed, if not, there would be two distinct geodesic segments between p and another point $q \in B(p, \mathrm{sys}(M)/2)$, both shorter than $\mathrm{sys}(M)/2$. The loop that these segments form is strictly less than $\mathrm{sys}(M)$ and can't be contractible (if it were contractible, we could lift it to a geodesic bigon in $\mathbb{H}^d$ and such bigons don't exists). This contradicts the fact that the systole is realized by the shortest closed loop in M.

Now take an open ball $B(\widehat{p}, \mathrm{sys}(M)/2)$ of radius $\mathrm{sys}(M)/2$ around $\widehat{p} \in \mathbb{H}^d$. The observation above implies that

$$\mathrm{vol}(\mathbb{S}^{d-1}) \cdot \int_0^{\mathrm{sys}(M)/2} \sinh^{d-1}(t)dt = \mathrm{vol}(B(\widehat{p}, \mathrm{sys}(M)/2)) \leqslant \mathrm{vol}(M).$$

This implies the lemma. □

If we look closely at the proof above, we observe that in reality, the lemma above is a bound *maximal* injectivity radius. The fact that such a bound implies a bound on the systole is due to the fact that on a negatively curved manifold, the systole is twice the *minimal* injectivity radius.

3.1.1 The maximal systole

For closed hyperbolic manifolds of dimension more than two, Lemma 3.1 is the best known upper bound. In dimension 2, better bounds are known. In [Bav96], Bavard proved that the maximum injectivity radius $\mathrm{injrad}(M)$ of a closed hyperbolic surface M of genus g satisfies:

$$\mathrm{injrad}(M) \leqslant \mathrm{arccosh}\left(\frac{1}{2\sin(\pi/(12g-6))}\right) \overset{g\to\infty}{=} 2\log(g) + 2.680353\ldots + o(1).$$

Moreover, as a bound on the radius of maximal injectivity radius, this bound is sharp: there exist hyperbolic surfaces of all genera $g \geqslant 2$ for which the inequality is an equality. On the other hand, Bavard already noted in his paper that the bound that we derive on the systole cannot be sharp, because of the fact that there exists only a finite number of points in a closed hyperbolic surface at which the injectivity radius is maximal. However, Bavard's bound is better than that of Lemma 3.1, even if it is still based on the maximal injectivity radius.

Recently, with Maxime Fortier Bourque [BP23c], we found the first bound that is better than the bound coming from the maximal injectivity radius. We proved:

Theorem 3.2. *There exists a genus $g_0 \geqslant 2$ such that*

$$\mathrm{sys}(X) < 2\log(g) + 2.409.$$

for every oriented closed hyperbolic surface X of genus $g \geqslant g_0$.

Our proof uses a hyperbolic version of the Cohn–Elkies–Gorbachev method from Section 2.1.4. The proof of the Cohn–Elkies–Gorbachev bound is based on the Poisson summation formula. The analogue of this formula in hyperbolic geometry is the **Selberg trace formula**:

The Selberg trace formula

Let X be a closed, oriented and connected hyperbolic surface of genus g. Let $0 = \lambda_0(X) < \lambda_1(X) \leqslant \dots$ denote the eigenvalues of the Laplacian on X and $\mathcal{C}(X)$ the set of closed oriented geodesics on X. Then

$$\sum_{j=0}^{\infty} \widehat{f}\left(\sqrt{\lambda_j(X) - \tfrac{1}{4}}\right) = 2(g-1)\int_0^{\infty} \widehat{f}(x) x \tanh(\pi x)\, dx + \frac{1}{\sqrt{2\pi}} \sum_{\gamma \in \mathcal{C}(X)} \frac{\Lambda(\gamma) f(\ell(\gamma))}{2\sinh(\ell(\gamma)/2)}$$

for every even function $f : \mathbb{R} \to \mathbb{C}$ for which there exists an $\varepsilon > 0$ such that

- the Fourier transform $\widehat{f}(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\cdot\xi} dx$ is holomorphic on the strip $\left\{z \in \mathbb{C};\ |\mathrm{Im}(z)| < \frac{1}{2} + \varepsilon\right\}$ and
- $\left|\widehat{f}(\xi)\right| = O\left((1 + |\xi|)^{-2-\varepsilon}\right)$ uniformly on $\left\{z \in \mathbb{C};\ |\mathrm{Im}(z)| < \frac{1}{2} + \varepsilon\right\}$.

In this formula, $\ell(\gamma)$ denotes the length of $\gamma \in \mathcal{C}(X)$ and $\Lambda(\gamma)$ the primitive length of γ: the length of the unique primitive geodesic γ_0 for which there exists a $k \geqslant 1$ with $\gamma_0^k = \gamma$.

A function $f : \mathbb{R} \to \mathbb{C}$ that satisfies the conditions above is called **admissible**. We also observe that, if f is admissible, $\widehat{f}$ is even and the expression on the right in the trace formula is therefore well defined: we can choose the square root we want. Selberg proved this formula in [Sel56]. Expositions can be found in [Bus10, Iwa02, Ber11].

Theorem 3.2 is based on the following bound (a hyperbolic analogue of Theorem 2.1):

Theorem 3.3. *Let $g \geqslant 2$. Let f be a non-constant admissible function for which there exists a constant $R > 0$ such that:*

- $f(x) \leqslant 0$ *if* $x \geqslant R$;
- $\widehat{f}(\xi) \geqslant 0$ *for all* $\xi \in \mathbb{R} \cup i\left[-\frac{1}{2}, \frac{1}{2}\right]$;
- $\widehat{f}(i/2) \geqslant 2(g-1)\int_0^\infty \widehat{f}(x) x \tanh(\pi x)\, dx$.

Then $\mathrm{sys}(X) \leqslant R$ *for every* $X \in \mathcal{M}_g$.

Proof. Suppose there exists a hyperbolic surface $X \in \mathcal{M}_g$ such that $\mathrm{sys}(X) > R$. Then $\mathrm{sys}(Y) > R$ for every $Y \in \mathcal{M}_g$ sufficiently close to X. This in particular implies that $f(\ell(\gamma)) \leqslant 0$ for every $\gamma \in \mathcal{C}(Y)$. Moreover, $\widehat{f}\left(\sqrt{\lambda_j(Y) - \frac{1}{4}}\right)) \geqslant 0$ for every $j \geqslant 0$. So the trace formula gives us that

$$\begin{aligned}
\widehat{f}(i/2) &\leqslant \sum_{j=0}^{\infty} \widehat{f}\left(\sqrt{\lambda_j(Y) - \tfrac{1}{4}}\right) \\
&= 2(g-1)\int_0^\infty \widehat{f}(x) x \tanh(\pi x)\, dx + \frac{1}{\sqrt{2\pi}} \sum_{\gamma \in \mathcal{C}(Y)} \frac{\Lambda(\gamma) f(\ell(\gamma))}{2\sinh(\ell(\gamma)/2)} \\
&\leqslant 2(g-1)\int_0^\infty \widehat{f}(x) x \tanh(\pi x)\, dx \\
&\leqslant \widehat{f}(i/2)
\end{aligned}$$

for every Y sufficiently close to X. Then,

$$\widehat{f}\left(\sqrt{\lambda_j(Y) - \tfrac{1}{4}}\right) = 0 \quad \text{pour tout } j \geqslant 1.$$

Seeing how $\widehat{f}$ is holomorphic on a strip (and not constantly equal to zero), the roots of $\widehat{f}$ are isolated. This, combined with the fact that the eigenvalues depend continuously on the metric (see for example [BU83]), implies that all $\lambda_j(Y)$ are constant on an open neighborhood of $X \in \mathcal{M}_g$. This is a contradiction because there are no continuous isospectral deformations of a hyperbolic surface [Gel63]. □

Another way of formulating Theorem 3.3 is to say that if we define, for $R > 0$,

$$g_{\min}(R) := \min\left\{ g \geqslant 2; \quad \begin{array}{c} \text{there exists } X \in \mathcal{M}_g \text{ such that} \\ \mathrm{sys}(X) > R \end{array} \right\}$$

and

$$\mathrm{G}(R) := \sup \left\{ 1 + \frac{f(i/2)}{2\int_0^\infty \widehat{f}(x) x \tanh(\pi x) dx}; \quad \begin{array}{c} f \text{ admissible, non-constant,} \\ f(x) \leqslant 0 \text{ if } x \geqslant R \text{ and} \\ \widehat{f}(\xi) \geqslant 0 \text{ for all } \xi \in \mathbb{R} \cup i\left[-\frac{1}{2}, \frac{1}{2}\right] \end{array} \right\},$$

then

$$g_{\min}(R) \geqslant \mathrm{G}(R)$$

for every $R > 0$.

To prove the Theorem 3.2, we found a series of explicit functions that give rise to very good bounds. We don't have a proof that the functions we found are (near) optimal (asymptotically, as $g \to \infty$). Nonetheless, for example based on the numerical results presented below, we think that they are not far from optimal for Theorem 3.3. We observe that the multiplicative constant in front of the logarithm is equal to 2 in all three limits above (Lemma 3.1, Bavard's bound and Theorem 3.2). The best known examples are of systole $\sim \frac{4}{3}\log(\text{genus})$ (see Section 3.1.2 below), so it is not even clear that the multiplicative constant is optimal. On the other hand, we think that our method cannot improve this multiplicative constant:

Question 4

Let $R(g)$ the optimal bound we can prove on the systole of a closed hyperbolic surface of genus g with Theorem 3.3. that is, $R(g) := \inf\{R > 0;\ \mathrm{G}(R) > g\}$. Is it true that

$$R(g) = 2\log(g) + O(1) \quad \text{as } g \to \infty \quad ?$$

Moreover, what are the optimal functions?

Theorem 3.3 also lends itself to numerical optimization. With Maxime Fortier Bourque, we had our computers search, with `SageMath`, for optimal functions for Theorem 3.3. A posteriori, we used the functions found to prove rigorous numerical bounds on systole in genus 2–20. The result is displayed in Figure 3.2. For the exact numbers we found, we refer to [BP23c]. Our method improves the bound in all genera, except in genus 2, where the maximizer is known: the Bolza surface [Jen81].

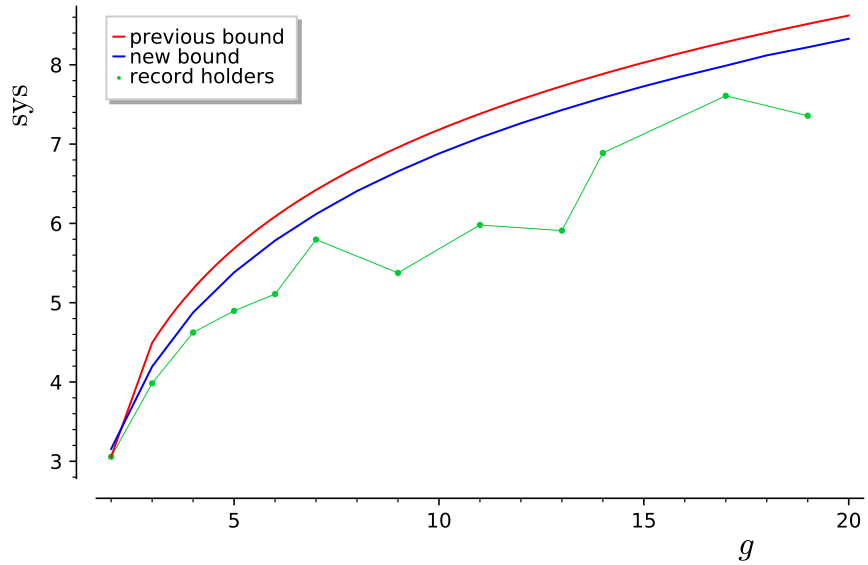

Figure 3.2: The bound on systole in genus 2–20.

3.1.2 Arithmetic surfaces with large systoles

Up until now, we have only considered upper bounds on the maximal possible systole. To find lower bounds, we need examples of manifolds of large systole. In dimension 2, the best known examples are those found by Brooks [Bro88] and Buser–Sarnak [BS94]. Their proof uses an arithmetic construction and was generalized by Katz–Schaps–Vishne [KSV07] in dimensions 2 and 3 and by Murillo [Mur19] in higher dimensions. Brooks and Buser–Sarnak prove the existence of sequences $(X_k)_k$ of closed hyperbolic surfaces of genus g_k, such that $g_k \to \infty$ and

$$\mathrm{sys}(X_k) \geqslant \frac{4}{3}\log(g_k) + O(1) \quad \text{as } k \to \infty.$$

To illustrate the idea, we will give the proof by Brooks and Buser–Sarnak in the non-compact case. The proof is the same as in the compact case, except that the compact case uses congruence coverings of a closed hyperbolic surface coming from a quaternion algebra instead of $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2$. The graph-theoretic analogue of this proof is also behind the fact that the girth of the Ramanujan graphs of Lubotzky–Phillips–Sarnak (see Section 2.2.3) is logarithmic.

Let

$$\Gamma(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{Z}); \quad \begin{matrix} a \equiv d \equiv 1 \mod N, \\ b \equiv c \equiv 0 \mod N \end{matrix} \right\}$$

denote the principal congruence subgroup of level N of $\mathrm{PSL}(2,\mathbb{Z})$.

Lemma 3.4 (Brooks, Buser–Sarnak)**.** *The systole of $X(N) = \Gamma(N)\backslash\mathbb{H}^2$ satisfies*

$$\mathrm{sys}(X(N)) \geqslant 2 \operatorname{arccosh}\left(\frac{N^2-2}{2}\right).$$

Proof. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(N)$ be a hyperbolic element. Then there exist $x_0, x_1, x_2, x_3 \in \mathbb{N}$ such that

$$a = 1 + x_0 N,\ b = x_2 N,\ c = x_3 N \text{ and } d = 1 + x_1 N.$$

Moreover, using the determinant, we get

$$0 = ad - bc - 1 = (x_0 + x_1)N + (x_0 x_1 - x_2 x_3)N^2.$$

So, N divides $x_0 + x_1$. Given that the element isn't parabolic, it's not possible that $x_0 + x_1 = 0$, hence

$$\left| \mathrm{tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \right| \geqslant N^2 - 2.$$

which implies the lemma. □

To compare the systole with the area of $X(N)$, we note that

$$\mathrm{area}(X(N)) = \frac{\pi}{6} N^3 \prod_{p|N \text{ prime}} 1 - \frac{1}{p^2} \quad \text{for } N \geqslant 3$$

This quantity depends on the number of divisors of N, but the quantity

$$\mathrm{area}(X(N))/N^3$$

is uniformly bounded from both sides. In particular,

$$2 \operatorname{arccosh}\left(\frac{N^2-2}{2}\right) = \frac{4}{3} \log(\mathrm{area}(X(N))) + O(1) \quad \text{as } N \to \infty.$$

One way to obtain closed surfaces of large systole from this construction is to compactify

them using the method of Brooks [Bro99]. This gives the same multiplicative constant ($\frac{4}{3}$) in front of the logarithm. It is known that the multiplicative constant $\frac{4}{3}$ is optimal for principal congruence coverings of arithmetic surfaces [Mak13]. Moreover, it is still the state of the art, that is to say, at the time of writing, we know no more than:

$$\limsup_{g\to\infty} \frac{\max\{\mathrm{sys}(X);\, X \in \mathcal{M}_g\}}{\log(g)} \in \left[\frac{4}{3}, 2\right].$$

Schmutz-Schaller has conjectured that $\frac{4}{3}$ is the actual value of this limit supremum [SS98].

3.1.3 A combinatorial construction

In addition to arithmetic constructions, we currently know two other constructions of closed hyperbolic surfaces of logarithmic systole. The aim of this section and the next is to present these two constructions.

Both constructions are very strongly inspired by ideas from graph theory. The first is the result of joint work with Alexander Walker [PW18] and is a hyperbolic version of the construction of Erdős–Sachs [ES63] mentioned in Section 2.2.1. The construction is based on gluing together ideal hyperbolic triangles along their sides. This is similar to the model of random surfaces of Brooks–Makover [BM04], except that we glue carefully instead of randomly.

Since the side lengths of ideal triangles are infinite, there is an infinite number of isometries that can be used for the gluing. We will use **shear** 0 gluings. One way to measure this shear is to draw the two orthogonal projections onto the side we glue of the two vertices which are *not* incident to that side. The signed hyperbolic distance (using the orientation of the surface) between these two points is the shear. So, in our gluings we require the two orthogonal projections to coincide.

The goal is now to find a gluing such that there are no short geodesics on the resulting surface. With Alexander Walker, we found sequences of gluings $(S_N)_N$ of $2N$ triangles, such that their systoles are at least

$$\log(N) - \log\log(N) - O(1) = \log(\mathrm{area}(S_N)) - \log\log(\mathrm{area}(S_N)) - O(1) \tag{3.1.1}$$

as $N \to \infty$. Our construction also comes with some flexibility. We can for instance create such a sequence in which the systole has multiplicity K for any fixed $K \geqslant 1$.

Before we describe the construction, we begin with some observations about the geometry and topology of a triangulated surface. Since it is a surface with cusps, its fundamental group is free. In fact, it's isomorphic to the fundamental group of the dual graph – the graph we obtain if we add a vertex to each triangle and we draw edges between the vertices for each side the corresponding triangles share. The orientation of the surface also induces an orientation of the dual graph: a cyclic order on the set of edges incident to each vertex. Given an essential closed curve[2] γ in the surface, the length of the unique geodesic in the free homotopy class of γ – which we denote $\ell(\gamma)$ – can be calculated using the orientation of the graph. Indeed, we trace the reduced circuit homotopic to γ and each time we cross a vertex we write down an L or R, depending on whether the circuit turns left or right with respect to the orientation. This gives us a word w in L and R. If we now replace the letters L and R by the matrices

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

then w becomes a matrix and

$$\ell(\gamma) = 2 \operatorname{arccosh}\left(\frac{\operatorname{tr}(w(\gamma))}{2}\right)$$

There is some ambiguity in the definition of $w(\gamma)$: if we start at a different vertex or read in the opposite direction, then $w(\gamma)$ changes. However, its trace does not and hence the expression on the right above is well-defined.

To construct our surfaces with large systoles, we need to find a triangulated surfaces whose dual graph has no circuits that carry words of small trace. If we want to obtain surfaces which satisfy the bound of the equation (3.1.1), we need to construct surfaces which do not carry closed curves of trace $< k$ and which contain $\approx k^2 \log(k)$ triangles.

To do this, we start with a gluing of $\approx k^2 \log(k)$ triangles such that each triangle is incident to exactly two other triangles, as in Figure 3.3. We choose this initial gluing in such a way that the surface does not carry curves of trace $< k$. One way to ensure this is to start with a single cycle of $2N \approx k^2 \log(k)$ triangles that carries a word of the form $(LR)^N$.

Now, we need to find a gluing of the boundary of this surface that does not create any circuits of trace $< k$. It is in this part of the proof that the ideas of Erdős et Sachs

[2] Not homotopic to a point nor to a cusp.

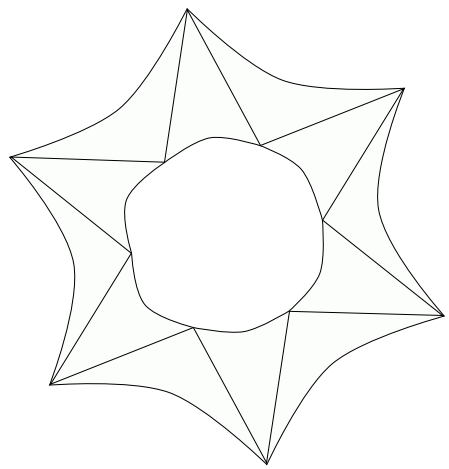

Figure 3.3: A cycle of triangles.

intervene. The idea is to perform an iterative construction. During every step, we perform one of the following two operations:

- If there is still a pair of sides that can be glued together without creating short circuits, we do so (if there are several, we make an arbitrary choice),
- if not, but the surface still has boundary, there must be two sides of two triangles (say c_1 and c_2) on the boundary that are too "close" to glue together. On the other hand, the fact that our surface contains enough triangles ($\approx k^2 \log(k)$) guarantees that there exists a pair of triangles sufficiently "far" from c_1 and c_2, whose sides c_3 and c_4 (which were on the boundary of the initial surface) were glued together earlier in the process. We open up this gluing and we glue c_3 to c_1 and c_2 to c_4.

Since the number of glued sides increases in each step, there is no boundary left at the end and we obtain an surface S_N without boundary whose systole satisfies the equation (3.1.1). The approximation that makes the second operation above work is [PW18]:

Proposition 3.5. *The positive semigroup generated by L and R is*

$$\langle L, R\rangle_+ = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z});\ a,b,c,d \geqslant 0 \right\} =: \mathrm{SL}(2,\mathbb{N}).$$

Moreover,

$$\{A \in \mathrm{SL}(2,\mathbb{N});\ 3 \leqslant \mathrm{tr}\,(A) \leqslant k\} = O\left(k^2 \log(k)\right)$$

as $k \to \infty$.

For more details, we refer to [PW18]. Like in the previous section, the results of Brooks [Bro99] can be used to compactify the surfaces we obtain, which gives rise to sequences of closed surfaces (X_k) whose genus tends to infinity and whose systole is at least $(1 + o(1)) \cdot \log(\text{genus}(X_k))$.

3.1.4 Long systolic pants decompositions

The second construction produces a hyperbolic surface of logarithmic systole such that the geodesics that realize the systole form a pants decomposition of the surface. Here, a **pants decomposition** is a collection of simple closed curves such that the complement of these curves is a collection of **pairs of pants**: three-holed spheres. A hyperbolic metric with a totally geodesic boundary on a pair of pants is determined entirely by the boundary lengths. So one way to specify a hyperbolic metric on a closed surface, is to first choose a pants decomposition and then specify two parameters per curve in the decomposition: a length and a **twist** which describes the gluing of the two pants along the curve (see Figure 3.4 for an example).

A surface such that the geodesics that realize systole form a pants decomposition is not difficult to obtain: we take our favorite pants decomposition and pinch the curves to a sufficiently small length. The collar lemma [Kee74] (see also [Bus10, Chapter 4]) implies that the curves in the pants decomposition are the shortest in the surface. What is more difficult is to do this without creating surfaces with very small systoles. The first such construction is due to Buser [Bus78]. He finds a sequence of surfaces $(X_g)_g$ of genus g whose systole is $\approx \sqrt{\log(g)}$. In [Pet18], this result was improved as follows:

Theorem 3.6. *There exists a sequence of closed orientable hyperbolic surfaces $(X_g)_{g\geqslant 2}$ such that X_g has genus g and*

$$\text{sys}(X_g) \geqslant \frac{4}{7}\log(g) + O(1) \quad \textit{as } g \to \infty.$$

The idea of the proof is to consider the following set:

$$\mathcal{S}_a = \left\{ \begin{array}{l} \text{Closed hyperbolic surfaces of systole } a\text{, that admit a pants decomposition} \\ \text{- such that all the lengths in the decomposition are } a \\ \text{- and all the twists are } a/4 \end{array} \right\}.$$

Afterwards, the proof has two steps. The first step is to show that $\mathcal{S}_a$ is not empty.

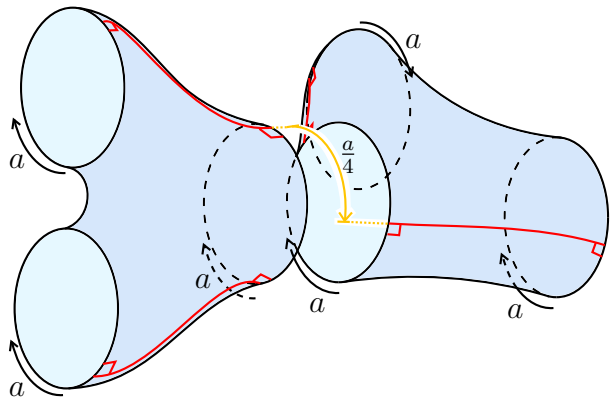

Figure 3.4: A gluing of two hyperbolic pairs of pants of twist $a/4$. Here the twist is the hyperbolic distance between the two points on the curve where the orthogonal geodesics connecting the curve along which we glue with the other boundary components intersect the curve.

The idea is that twisting by $a/4$ forces curves that are not in the pants decomposition to be long. In particular, the only way to have a short curve in a surface with a pants decomposition as in the definition of $\mathcal{S}_a$ is to have a short cycle in the graph dual to the pants decomposition. So if we have a surface with a pants decomposition all of whose lengths are a and all of whose twists are $a/4$ and whose dual graph is of a sufficiently large girth, then this surface is in $\mathcal{S}_a$.

The second step is to define

$$g_{\min}(a) = \min \{\text{genus}(X);\ X \in \mathcal{S}_a\}.$$

Given that $\mathcal{S}_a$ isn't empty, $g_{\min}(a)$ is finite. The goal now is to find an upper bound on $g_{\min}(a)$. To do so we use the diameter of the surface $X_{\min}(a)$ realizing $g_{\min}(a)$. If this diameter is very large compared to a, then $X_{\min}(a)$ contains two pairs of pants P_1 and P_2 that are very far away from each other. Because of this, we can remove P_1 and P_2 and glue the resulting boundary components together without creating any curves of length below a. So we obtain a surface whose systole is still a, but whose genus is below that of $X_{\min}(a)$, thus contradicting minimality.

In conclusion, we obtain a bound on the diameter of $X_{\min}(a)$ in terms of its systole. The diameter of a hyperbolic surface is at least logarithmic as a function of its genus (see Section 6.2 for a detailed discussion), so we obtain a lower bound on the systole in terms of the genus, which is the bound in the theorem above (see [Pet18] for more details).

3.2 The kissing number

There are variants of the method behind our bounds on the systole with Maxime Fortier Bourque (see Section 3.1.1) that can be used to prove bounds on other geometric and spectral invariants of hyperbolic manifolds as well. The idea is always the same: we use the Selberg trace formula with a function that satisfies certain linear conditions to find a bound on the quantity that interests us. Afterwards we optimize the bound on the set of functions which satisfy the conditions.

The second invariant for which we found a new bound is the **kissing number** of a hyperbolic surface: the number of oriented geodesics that realize the systole of the surface. This terminology was invented by Schmutz-Schaller [Sch96a, Sch96b, SS97] in analogy with the number of contacts of a Euclidean lattice. The basic question is:

Question 5

Given $d \geqslant 2$ and $v > 0$, what is

$$\max \left\{ \mathrm{Kiss}(M); \quad \begin{array}{c} M \text{ a hyperbolic } d\text{-manifold} \\ \text{of volume } \leqslant v \end{array} \right\} \quad ?$$

And which hyperbolic manifold realizes it?

The bound that we find is [BP23c]:

Theorem 3.7. *There exists a $g_0 \geqslant 2$ such that*

$$\mathrm{Kiss}(X) < \frac{4.873 \cdot g^2}{\log(g) + 1.2045}.$$

for every closed oriented hyperbolic surface X of genus $g \geqslant g_0$.

This improves a bound of Parlier [Par13], which is also of the form $g^2/\log(g)$, but with a multiplicative constant of 200. Our method is completely different: Parlier's proof is a geometric argument and does not use the Selberg trace formula.

The best known examples are again given by principal congruence coverings of arithmetic surfaces: Schmutz-Schaller [SS97] proved that there exist sequences of congruence coverings $(X_k)_k$ of genus g_k of an arithmetic surface compact such that $g_k \to \infty$ and for all

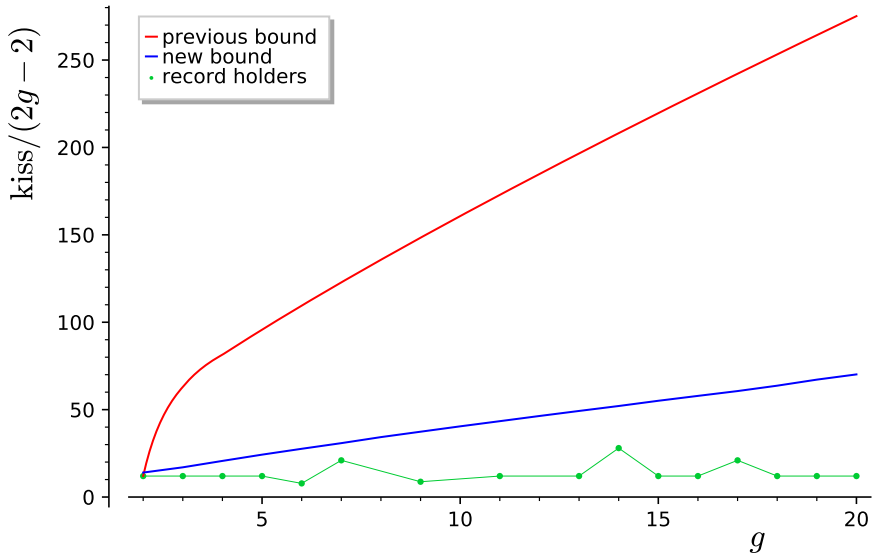

Figure 3.5: The bound on the kissing number in genus 2–20.

$\varepsilon > 0$, the kissing number satisfies

$$\mathrm{Kiss}(X_k) \gg g_k^{4/3-\varepsilon} \quad \text{as } k \to \infty.$$

Schmutz–Schaller [SS97, SS98] conjectured that the exponent $\frac{4}{3}$ is optimal. In higher dimension, constructions of hyperbolic manifolds whose kissing number grows faster than linearly as a function of their volume were found by Dória–Murillo and Dória–Freire–Murillo [DM21, DFM23]. With Maxime Fortier Bourque, we have also proved an analogous bound to Theorem 3.7 for higher dimensional closed hyperbolic manifolds [BP22].

Like for the systole, our method can also be used to prove bounds for surfaces of small area. The result can be found in Figure 3.5. The bound that we obtain is the best known bound, except in the case of genus 2, where the maximum possible kissing number is known: it is 24 [Sch94] and it is achieved by the Bolza surface [Jen81].

3.3 The first eigenvalue

The other three invariants for which we found bounds with Maxime Fortier Bourque are spectral invariants. We consider the Laplacian

$$\Delta_X := -\mathrm{div} \circ \mathrm{grad} : C^\infty(X) \to C^\infty(X)$$

acting on functions on a closed orientable hyperbolic surface X. The eigenvalues of this operator (the same eigenvalues that appear in the Selberg trace formula above) form a discrete spectrum in $[0, \infty)$.

These eigenvalues contain a lot of geometric information about the underlying surface. For example, a classical theorem due to Huber [Hub59, Hub60, Hub61] states that the spectra of the Laplacians of two closed hyperbolic surfaces are the same if and only their length spectra – the multisets containing the lengths of all closed geodesics – are. moreover, the first eigenvalue $\lambda_1(X)$ can be seen as a measure of connectivity of the surface, for instance because of the Cheeger–Buser inequality [Che70, Bus82], that we will discuss in detail in Section 6.3, and also because $\lambda_1(X)$ influences the mixing rate of the geodesic flow of the surface: the larger it is, the faster the flow mixes (see for instance [Mat13] for an effective version).

It has been known for a long time, due to results by Huber and Cheng [Hub74, Che75b] that, if the genus of a hyperbolic surface X is large, then $\lambda_1(X)$ can't be much larger than $\frac{1}{4}$. The significance of $\frac{1}{4}$ here is that it's the spectral gap of the hyperbolic plane. This is the hyperbolic analogue of the Alon–Boppana bound mentioned in Section 2.2.3. The best upper bound that was known before our work with Maxime Fortier Bourque combines the result of Cheng with work of Gage [Gag80] and Bavard [Bav96] and states:

$$\lambda_1(X) \leqslant \frac{1}{4} + \left(\frac{2\pi}{\log(g-1)}\right)^2.$$

With Maxime Fortier Bourque, we proved:

Theorem 3.8. *There exists $g_0 \geqslant 2$ such that*

$$\lambda_1(X) < \frac{1}{4} + \left(\frac{\pi}{\log(g) + 0.7436}\right)^2.$$

for every hyperbolic surface X of genus $g \geqslant g_0$.

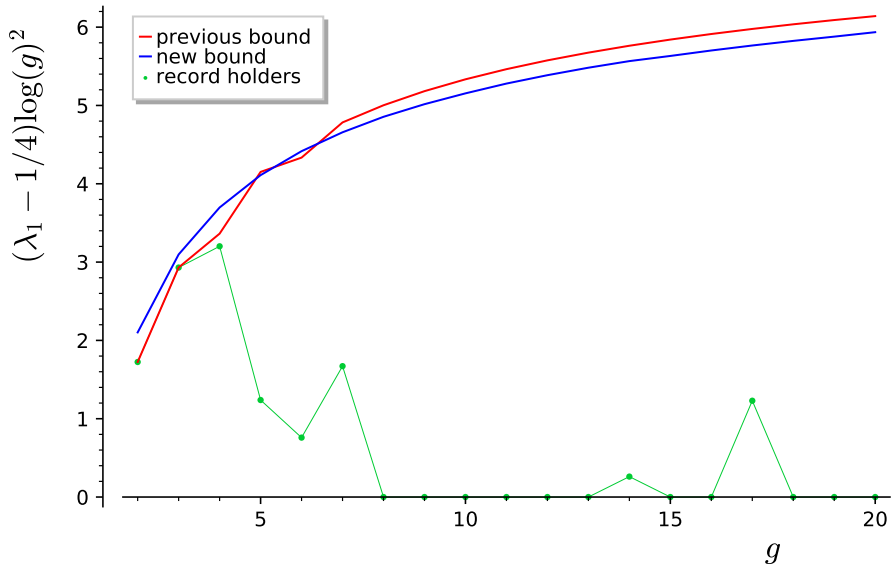

Figure 3.6: The bound on the first eigenvalue in genus 2–20.

Our method improves the error term by a factor of 4. This method is different from that of Cheng and moreover, thanks to the existence of hyperbolic surfaces of small diameter (see Section 6.2), it is not possible to obtain our bound with Cheng's method.

The bounds that we obtain in small genus can be found in Figure 3.6. There has been a lot of recent progress the maximum of the first eigenvalue. The best known upper bounds in genus 2 and 3 are due to work by Bonifacio [Bon22] and Kravchuk–Mazac–Pal [KMP21] and were found using a method called *conformal bootstrap*. These bounds are very close to the first eigenvalues of the Bolza surface and the Klein quartic respectively, so it's natural to conjecture that these two surfaces maximize the first eigenvalue in their moduli spaces. Somewhat curiously, the conformal bootstrap yields a decidedly sharper bound than ours in genus 2 and 3, yet our bound is sharper in higher genus. In neither case, this seems to be due to not running the numerical optimizer long enough. Finally, in genus 4 and 6, a classical bound by Yang–Yau [YY80] is still the best known bound.

The question of whether there exist sequences of closed surfaces whose genus tends to infinity and whose first eigenvalue tends to $\frac{1}{4}$ has been open for a long time. Recently, there has been a spectacular breakthrough due to Hide–Magee [HM23b, Hid23]: such a sequence exists. Hide–Magee construct the sequence as compactifications of random covers of a sphere with three cusps. Their proof makes essential use of the work of Bordenave–Collins [BC19, BC23] on the asymptotic freedom of random permutations and the related

question of spectral gaps in random coverings of graphs. However, the following question remains open:

Question 6

Does there exist a hyperbolic surface $X \in \mathcal{M}_g$ such that

$$\lambda_1(X) \geqslant \frac{1}{4} \text{ or even } \lambda_1(X) > \frac{1}{4}$$

for every $g \geqslant 2$? Such surfaces are sometimes called Ramanujan (or strict Ramanujan) surfaces, in analogy with Ramanujan graphs.

Selberg's conjecture [Sel56] and its generalizations [BLS92] assert that congruence lattices give examples of Ramanujan surfaces (and higher-dimensional Ramanujan manifolds). The best known bounds towards these conjectures are due to Kim–Sarnak [Kim03] in dimension 2 and to Bergeron–Clozel [BC13] for higher dimensional hyperbolic manifolds.

3.4 The multiplicity of the first eigenvalue

Another invariant that we can access with our methods is the multiplicity $m_1(X)$ of the first eigenvalue: the dimension of the eigenspace associated to $\lambda_1(X)$.

The study of this invariant has a long history as well. For the case of surfaces, Colin-de-Verdière (see [CdV86, p.269] and [CdV87, p.601]) has conjectured that, if Σ is a closed surface,

$$\max\{m_1(\Sigma, g);\ g \text{ a Riemannian metric on } \Sigma\} = \mathrm{chr}(\Sigma) - 1,$$

where $\mathrm{chr}(\Sigma)$ is the chromatic number of Σ: the maximal possible number of vertices of a complete graph that embeds in Σ. A theorem by Ringel–Youngs [RY68] states that (if Σ is not the Klein bottle):

$$\mathrm{chr}(\Sigma) = \left\lfloor \frac{1}{2}\left(7 + \sqrt{49 - 24\chi(\Sigma)}\right) \right\rfloor,$$

where $\chi(\Sigma)$ denotes the Euler characteristic of Σ. Colin-de-Verdière also proved that in higher dimension, the situation is very different: on any closed manifold of dimension at least 3 and for any $N \geqslant 1$, one can find a Riemannian metric such that the multiplicity of its first eigenvalue is N [CdV87].

Going back to the case of surfaces, Colin-de-Verdière's conjecture has been proven for the sphere by Cheng [Che76], for the torus and the projective plane by Besson [Bes80] and for the Klein bottle by Colin-de-Verdière and Nadirashvili [CdV87, Nad87]. Colbois–Colin-de-Verdière [CCdV88] have found examples of sequences of closed hyperbolic surfaces whose first eigenvalue multiplicity grows like the square root of their genus, but at a slower rate than predicted by Colin-de-Verdière's conjecture. The best known upper bound on the multiplicity of the first eigenvalue among all Riemannian metrics is due to Sévennec [Sév02], who uses a topological argument based on Courant's nodal domain theorem [Cou23] and topological methods "à la Borsuk–Ulam" to prove that the supremum above can be bounded by $5 - \chi(\Sigma)$, which comes down to $2g + 3$ for a closed orientable surface of genus g.

With our methods we obtain a slight improvement of Sévennec's result in the case of hyperbolic surfaces:

Theorem 3.9. *There exists a $g_0 \geqslant 2$ such that*

$$m_1(X) \leqslant 2g - 1.$$

for every closed hyperbolic surface X of genus $g \geqslant g_0$.

In fact, our method works less well for multiplicity than for the other invariants. The problem is that with the Selberg trace formula, we are only able to prove bounds on the number of eigenvalues in certain intervals. On the other hand, it is known that in any moduli space $\mathcal{M}_g$ and for any $\varepsilon > 0$, we can find hyperbolic surfaces with as many eigenvalues in the interval $(\frac{1}{4}, \frac{1}{4} + \varepsilon)$ as we want [Bus10, p.219]. In other words, our bound on the multiplicity necessarily depends on $\lambda_1(X)$ and it explodes when $\lambda_1(X) \to \frac{1}{4}$. Instead of the trace formula, we use a bound by Otal [Ota08] for surfaces X such $\lambda_1(X) \leqslant \frac{1}{4}$, which states that in this case, $m_1(X) \leqslant 2g - 3$. Afterwards, we use the topological methods of Otal and Sévennec to prove that there exists an explicit $a_g > 0$ such that, $m_1(X) \leqslant 2g - 1$, if $\lambda_1(X) \in (\frac{1}{4}, \frac{1}{4} + a_g]$. For the remaining interval (between $\frac{1}{4} + a_g$ and the upper bound in Theorem 3.8), we use the trace formula. On this small interval, the bound we find is in fact sublinear as a function of g.

A general sublinear bound, under the condition that the systole of the surface is not too small, has recently been found by Letrouit–Machado [LM23]. Their proof is strongly inspired by that of the analogous result for regular graphs [JTY+21].

The bounds that we found in genus 2–20 can be found in Figure 3.7. Our bound is sharp in genus 3. In [BP23b] we proved:

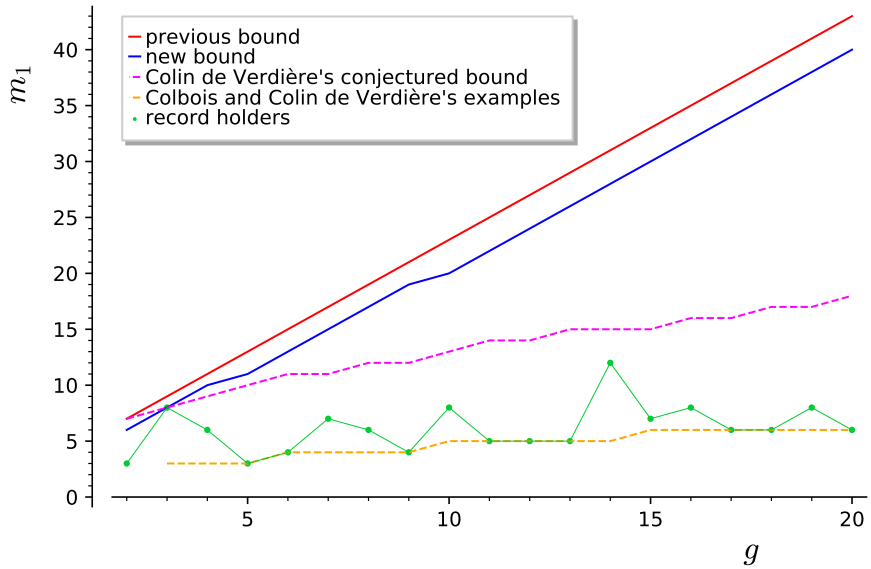

Figure 3.7: The bound on the multiplicity of the first eigenvalue in genus 2–20.

Theorem 3.10. *We have:*

$$\max\{m_1(X);\ X \in \mathcal{M}_3\} = 8$$

and this is realized by the Klein quartic.

The quickest way to define the Klein quartic is to say that it's the unique closed orientable hyperbolic surface of genus 3 whose group of orientation preserving isometries has order 168 (which is maximal in genus 3, by the Hurwitz bound). For more information on the Klein quartic, we refer the reader to [Lev99], a whole book dedicated to this surface.

To prove the fact that the first eigenvalue multiplicity of the Klein quartic is 8, we use the fact that the Klein quartic has so many symmetries. This is a combination of our methods based on the trace formula and the approach that Jenni [Jen81] and Cook [Coo18] used to prove that the multiplicity of the first eigenvalue multiplicity of the Bolza surface and the Klein quartic are 3 and at least 6 respectively.

Concretely, the group of isometries of the Klein quartic is $\mathrm{PGL}(2, \mathbb{Z}/7\mathbb{Z})$ (and the subgroup of orientation preserving isometries is $\mathrm{PSL}(2, \mathbb{Z}/7\mathbb{Z})$) and the action of this group commutes with the Laplacian. This implies that the eigenspaces of the Laplacian decompose into representations of $\mathrm{PGL}(2, \mathbb{Z}/7\mathbb{Z})$. This group has two irreducible representations of dimension 1, three of dimension 6, two of dimension 7 and two of dimension 8. The

goal is therefore on the one hand to rule out representations of dimensions lower than 8 and on the other hand to prove that the multiplicity is at most 8. We first prove, using the spectrum of the quotient of the Klein quartic by $\mathrm{PGL}(2, \mathbb{Z}/7\mathbb{Z})$, that representations of dimension 1 do not appear in the first eigenspace. Then we prove, with the trace formula, that in a certain interval which contains the first eigenvalue of the Klein quartic, there are more than 7 eigenvalues and less than 12. The only positive integer combination of 6, 7 and 8 between 7 and 12 being 8, the multiplicity of the first eigenvalue of the Klein quartic has to be 8.

What is striking is that our method works for genus 3, but not for genus 2. The problem is that the upper bound we prove (6) is larger than the largest known multiplicity (3, for the Bolza surface). Nor do we know whether the Klein quartic is the unique surface of genus 3 of multiplicity 8.

Question 7

- Is it true that
$$\max\{m_1(X);\, X \in \mathcal{M}_2\} = 3?$$
And if so, is the Bolza surface the unique closed orientable surface of genus 2 and first eigenvalue multiplicity 3?

- Is the Klein quartic the unique closed orientable surface of genus 3 and first eigenvalue multiplicity 8?

In upcoming work with Maxime Fortier Bourque, Émile Gruda-Médiavilla and Mathieu Pineauilt, we will prove that Colin-de-Verdière's conjecture is false: there are two counterexamples, one of genus 10 and one of genus 17. The graph in Figure 3.7 will soon get an update as well.

3.5 The number of small eigenvalues

The last invariant that we studied with Maxime Fortier Bourque is the number of small eigenvalues:

$$\mathrm{N}_{\mathrm{small}}(X) = \max\left\{k;\ \lambda_k(X) \leqslant \frac{1}{4}\right\} + 1,$$

where we've added 1 because we're calling the trivial eigenvalue $\lambda_0(X)$. This invariant is of interest for example because small eigenvalues give rise to exponential error terms in the asymptotic count of the number of short geodesics on X (see for example [Bus10, Section 9.6]).

Buser [Bus77] and Otal–Rosas [OR09] proved that:

$$\max\left\{\mathrm{N}_{\mathrm{small}}(X);\ X \in \mathcal{M}_g\right\} = 2g-2 \quad \text{pour tout } g \geqslant 2.$$

The lower bound proved by Buser uses explicit examples of hyperbolic surfaces. The systoles of these examples are very small, so it is a natural question to ask whether this is necessary. The first result which implies that this is indeed the case is due to Huber [Hub76] and says that

$$\mathrm{N}_{\mathrm{small}}(X) < \min\left\{\frac{3\pi^2(g-1)}{8\ (\log(\cosh(\mathrm{sys}(X)/4)))^3}, \frac{g-1}{2\ \log(\cosh(\mathrm{sys}(X)/4))}\right\}.$$

Huber's method is similar to ours, except that he chooses another type of special functions: Huber uses Legendre functions and we use functions constructed with Bessel functions. With our functions, we obtain a small improvement on Huber's result:

Theorem 3.11. *If X is a closed oriented hyperbolic surface of genus $g \geqslant 2$, then*

$$\mathrm{N}_{\mathrm{small}}(X) < \min\left(\frac{24\pi^2(g-1)}{\mathrm{sys}(X)^3}, \frac{16(g-1)}{\mathrm{sys}(X)^2}\right).$$

For the proof that our bound is really strictly smaller than (but still asymptotic to) Huber's bound, we refer to [BP23c, Section 10.2].

Like for the other invariants, in small genus, numerical optimization gives sharper results than the test functions we constructed by hand (see Figure 3.8). For example, we prove that $N_{\mathrm{small}}(X) < 2g-2$ as soon as

$$\mathrm{sys}(X) \geqslant 2.317.$$

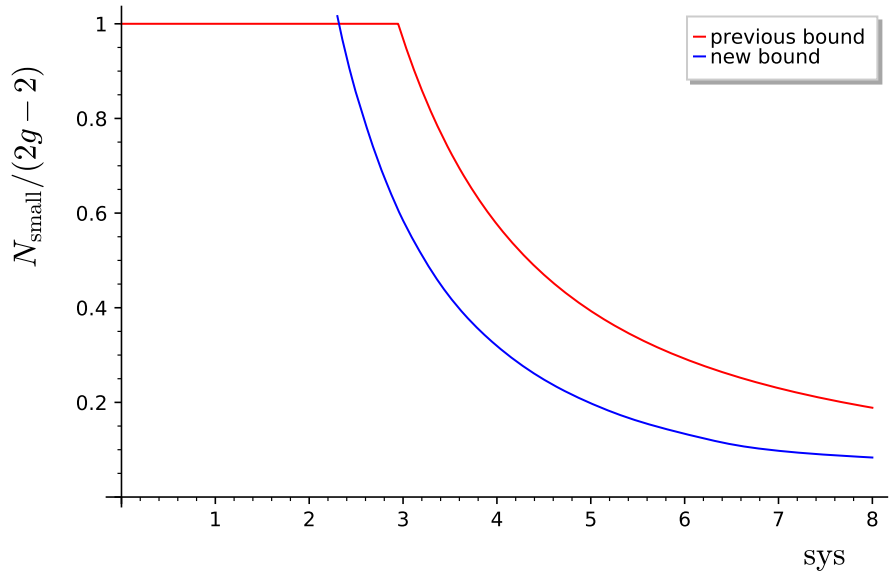

Figure 3.8: The bound on $\mathrm{N}_{\text{small}}/(g-1)$ when the systole is small.

This improves on Huber's earlier mentioned results that imply that $N_{\text{small}}(X) < 2g-2$ when $\text{sys}(X) > 4\,\text{arccosh}(e^{1/4}) = 2.9476\ldots$ (and Jammes's 3.46 that was found using different methods [Jam21]).

4 Counting

A large part of this thesis will be taken up by random manifolds. Before we get to these, the very first question – at least for discrete models of random manifolds – is that of counting: in order to calculate probabilities with respect to the uniform measure on a finite set, one needs to know the number of elements in the set (or at least have a good approximation). Often, this question is already very hard.

A classical example is counting triangulations of manifolds. In the 1960s, Tutte [Tut63] developed a way to enumerate triangulations of 2-dimensional disks and since then, there has been an immense amount of work on counting triangulations of surfaces (we will discuss a very small portion of it in the next chapter). Even if a lot is known, many very basic questions remain open. In dimension 3 and above, we know much less, even the asymptotic number of triangulations of the sphere is out of reach of current methods: the best known bounds are very far apart (see for instance [BZ11, DJ95, AB20, CP21]).

In this chapter we will discuss our work on subgroup growth with Hyungryul Baik and Jean Raimbault [BPR20, BPR19] and with Stefan Friedl, JungHwan Park, Jean Raimbault and Arunima Ray [FPP+21] and our paper with Patricia Cahn and Federica Fanoni on counting mapping class group orbits of curves on surfaces [CFP18],

4.1 Subgroup growth

In this section, we will discuss subgroup growth: the study of the number of finite index subgroups of a given group. Concretely, given a finitely generated group Γ, we define

$$s_n(\Gamma) = |\{H < \Gamma;\, [\Gamma : H] = n\}|\,.$$

For a finitely generated group, this quantity is always finite. The basic question now is how $s_n(\Gamma)$ depends on n and which algebraic and geometric properties of Γ one can deduce

from the information contained in this sequence. Before we'll get to our joint work with Hyungryul Baik and Jean Raimbault on the subgroup growth of Artin and Coxeter groups, we will start with some context. We will only give a quick introduction to the subject of subgroup growth and we make a selection which is very biased by applications to random manifolds. For a more complete introduction we refer to the book by Lubotzky–Segal [LS03].

Subgroup growth is relevant for the study of random manifolds, because a random finite index subgroup of the fundamental group $\pi_1(M)$ of a manifold M gives rise to a random finite degree cover of M. This model has recently been very successful in dimension 2 [MP23, MNP22, HM23b, Nau22] (see Section 5.1.2) and we'll also treat a 3-dimensional example in Section 5.2.2.

4.1.1 Invariant random subgroups

One way to look at these random finite index subgroups is through the lens of invariant random subgroups. For a finitely generated group Γ, $\mathrm{Sub}(\Gamma)$ will denote the **Chabauty space** of subgroups of Γ (see for instance [Gel18] for an introduction).

We will be interested in random index n subgroups of such a group Γ. Given $n \in \mathbb{N}$, we will write

$$\mathcal{S}_n(\Gamma) = \{H < \Gamma;\ [\Gamma : H] = n\} \subset \mathrm{Sub}(\Gamma),$$

so that $s_n(\Gamma) = |\mathcal{S}_n(\Gamma)|$. Studying a random index n subgroup of Γ comes down to understanding the measure μ_n on $\mathrm{Sub}(\Gamma)$, defined by

$$\mu_n = \frac{1}{s_n(\Gamma)} \sum_{H \in \mathcal{S}_n(\Gamma)} \delta_H$$

where δ_H denotes the Dirac mass on $H \in \mathrm{Sub}(\Gamma)$.

μ_n is an example of what is called an **invariant random subgroup** (IRS) of Γ – i.e. a Borel probability measure on $\mathrm{Sub}(\Gamma)$ that is invariant under conjugation by Γ. We will write $\mathrm{IRS}(\Gamma)$ for the space of IRS's of Γ endowed with the weak-* topology. This space has been first studied under this name in [AGV14] and [Bow14] and under a different name in [Ver12].

4.1.2 Estimating the subgroup growth of a group

Most estimates $s_n(\Gamma)$ start with the following lemma. In this lemma, $\mathfrak{S}_n$ denotes the symmetric group on n letters.

Lemma 4.1. *Let Γ be a group. Then*

$$s_n(\Gamma) = \frac{t_n(\Gamma)}{(n-1)!},$$

where

$$t_n(\Gamma) = |\{\varphi \in \mathrm{Hom}(\Gamma, \mathfrak{S}_n);\ \varphi(\Gamma) \text{ acts transitively on } \{1, \dots, n\}\}|$$

Proof. If $H < \Gamma$ is such that $[\Gamma : H] = n$, then Γ acts transitively on the set of cosets

$$\Gamma/H = \{H, g_2H, \dots, g_nH\}$$

The induced permutation action on the indices of the g_i is an action on a set of n elements. In doing this, we have made an arbitrary identification of the last $n-1$ cosets. So, in reality, H gives rise to $(n-1)!$ distinct actions.

In the other direction, given a transitive action of Γ on $\{1, \dots, n\}$, the stabilizer $\mathrm{Stab}_\Gamma(\{1\}) < \Gamma$ is a subgroup of index n. Moreover, if we conjugate the action by an element of $\mathfrak{S}_{n-1}$ acting on $\{2, \dots, n\}$ this doesn't affect the stabilizer. □

For an infinite group Γ, it is rare that one can calculate an explicit formula for $s_n(\Gamma)$, and even if this is possible, the formula is often so complicated that it is difficult to extract information from it. So typically, we rather look for the asymptotic behavior of $s_n(\Gamma)$ as $n \to \infty$. The first example of a class of infinite finitely generated groups for which such an asymptotic expression has been calculated is the class of free groups. First, we observe that the question is trivial for $\mathbb{Z}$: $s_n(\mathbb{Z}) = 1$ for all $n \in \mathbb{N}$. For non-abelian free groups, a result by Dixon [Dix69] implies that, if $r \geqslant 2$ the probability that a homomorphism $\varphi \in \mathrm{Hom}(F_r, \mathfrak{S}_n)$ is transitive tends towards 1 when $n \to \infty$. Here F_r denotes the free group of rank r. Combined with Lemma 4.1, this implies that

$$s_n(F_r) \sim n \cdot (n!)^{r-1} \quad \text{as } n \to \infty$$

for all $r \geqslant 2$. This result was generalized to lattices in $\mathrm{PSL}(2, \mathbb{R})$ by Müller–Schlage-Puchta [MP02, MSP07] and Liebeck–Shalev [LS04a]. Like in the case of free groups, the factorial

growth of $s_n(\Gamma)$ is linked to the area of the associated hyperbolic surface, i.e.

$$\lim_{n\to\infty} \frac{\log(s_n(\Gamma))}{n\log(n)} = \frac{\text{area}(\Gamma\backslash\mathbb{H}^2)}{2\pi} = -\chi(\Gamma),$$

where $\chi(\Gamma)$ denotes the Euler characteristic of Γ. These results form the starting point for earlier mentioned results on the geometry and spectra of random covers of hyperbolic surfaces.

For lattices in $\text{Isom}(\mathbb{H}^d)$ for $d \geqslant 3$, a case that is of obvious interest to us, much less is known. There is no such lattice for which we know an asymptotic equivalent for the sequence $(s_n(\Gamma))_n$. The fact that the lattices in $\text{Isom}(\mathbb{H}^d)$ are large – they have finite index subgroups which surject onto F_2 –, which was proven by Agol [Ago13], using work by Kahn–Markovic [KM12], Wise [Wis21], Bergeron–Wise [BW12] and many others, does imply that regularized counting function

$$s_{\leqslant n}(\Gamma) = \sum_{k=1}^{n} s_k(\Gamma)$$

grows factorially fast as a function of n. On the other hand, Agol's theorem is not effective, so we currently have no good estimates on $\lim_{n\to\infty} \frac{\log(s_n(\Gamma))}{n\log(n)}$ in general. There is one example of a lattice in $\text{Isom}(\mathbb{H}^3)$ for which we can determine this factorial growth rate. We will discuss this in Section 4.1.4.

Also the following question is open:

Question 8

Let $d \geqslant 3$ and let $\Gamma < \text{Isom}(\mathbb{H}^d)$ be a lattice. Can one deduce the volume $\text{vol}(\Gamma\backslash\mathbb{H}^d)$ from the sequence $(s_n(\Gamma))_n$?

If the answer to this question is "yes", then the relation is more complicated than in dimension 2: using the fact that there exist closed hyperbolic manifolds of dimension 3 of arbitrarily large volume but whose fundamental group can be generated with 5 elements (i.e. there is an epimorphism of F_5 onto these fundamental groups), we can prove that there exist hyperbolic manifolds of arbitrarily large volume whose factorial subgroup growth rate is bounded by 4. Indeed, if we have an epimorphism $\Lambda \to \Gamma$, then $s_n(\Gamma) \leqslant s_n(\Lambda)$ for all n. To find examples of hyperbolic manifolds with the desired property, we can for example take a sequence of cyclic coverings of a hyperbolic manifold which fibers on the circle.

A second observation is that the sequence $(s_n(\Gamma))_n$ is a profinite invariant of Γ. In other

words, if two groups Γ and Λ have the same profinite completion (see for example [RZ10]), then $s_n(\Gamma) = s_n(\Lambda)$ for all $n \in \mathbb{N}$. In particular, if the answer is "yes", this also gives a positive answer to the open question of whether the volume of a hyperbolic manifold is a profinite invariant. This question has for instance been raised in [BF20, Question 3.18] and [Liu23] and also connects to more general questions on profinite rigidity of 3-manifold groups, see [Rei15, Rei18] for an introduction and [BMRS20] for examples of profinitely rigid Kleinian groups.

Finally, we note that the situation for lattices in Lie groups of higher rank, in which case arithmetic methods can be used, is wildly different from that in rank one (see for instance [Lub95, LN04, GLP04]).

4.1.3 Right-angled Artin groups

Together with Hyungryul Baik and Jean Raimbault [BPR20, BPR19], we've studied the subgroup growth of right-angled Artin and Coxeter groups. We'll start with the former. Given a finite graph $G = (V, E)$, the associated right-angled Artin group is

$$\Gamma_{\mathrm{A}}(G) = \langle v \in V \mid [v, w] \text{ for all } \{v, w\} \in E \rangle.$$

These groups interpolate between free groups and free abelian groups. Indeed, if G is a complete graph, then $\Gamma_{\mathrm{A}}(G)$ is a free abelian group and if G has no edges, then $\Gamma_{\mathrm{A}}(G)$ is a free group. They show up in hyperbolic geometry for instance because Agol proved that every cocompact lattice in $\mathrm{PSL}(2, \mathbb{C})$ virtually embeds in one of them [Ago13].

For us, they're a good testing ground because they are large groups, but are given by relatively manageable relations. Moreover, they are also a first step towards right-angled Coxeter groups, that appear as lattices in the isometry groups of some hyperbolic spaces.

What we could access with Hyungryul Baik and Jean Raimbault is the factorial subgroup growth rate of these groups:

Theorem 4.2. *let $G = (V, E)$ be a finite graph, then*

$$\lim_{n \to \infty} \frac{\log(s_n(\Gamma_{\mathrm{A}}(G)))}{n \log(n)} = \alpha(G) - 1,$$

where $\alpha(G)$ denotes the independence number of G: the maximal size of a subset $W \subset V$ such that there are no edges between any of the pairs of vertices in W.

The independence number $\alpha(G)$ also has a geometric interpretation. Indeed, $\Gamma_{\mathrm{A}}(G)$ is the fundamental group of a cube complex called the Salvetti complex. The number $\alpha(G)$ counts the maximal number of disjoint hyperplanes in this complex (see for instance [Vog15, Example 5.2]).

The proofs of the upper and lower bounds in the theorem above are distinct. In fact, the lower bound follows from the results on free groups mentioned above. Indeed, if $G = (V, E)$ is a finite graph and $W \subset V$ and independent set – a set of vertices that have no edges between them – then we obtain a surjection

$$\Gamma_{\mathrm{A}}(G) \longrightarrow F_{|W|},$$

by mapping the generators of $\Gamma_{\mathrm{A}}(G)$ corresponding to the elements of W to distinct generators of the free group $F_{|W|}$. This implies that $s_n(\Gamma_{\mathrm{A}}(G)) \geqslant s_n(F_{|W|})$ for all $n \in \mathbb{N}$ and thus proves the lower bound.

The bulk of the proof of the theorem goes into showing that, at the factorial level, this simple lower bound is sharp. Morally, this comes from the fact that typically the centralizers of elements in a symmetric group are very small compared to the size of the symmetric group itself. Our proof uses an induction on the number of vertices of G and yields a bound of the form

$$s_n(\Gamma_{\mathrm{A}}(G)) \leqslant C^{n \log\log(n)} (n!)^{\alpha(G)-1} \text{ for all } n \in \mathbb{N},$$

with a constant $C > 0$ that depends on the graph G. For details, we refer to [BPR20]. Here, we will include the proof in a special case, which is much quicker, leads to a sharper upper bound, and uses a very nice classical theorem from graph theory, due to Kőnig. It is however not very representative of the general case.

Proof of Theorem 4.2 in the case of bipartite graphs. We assume that $G = (V, E)$ is bipartite. That is, we assume that we can write $V = V_1 \sqcup V_2$ and there are no edges interior to V_1 or V_2.

We will need two more notions from graph theory. First of all, a **matching** of G is a set $M \subset E$ of edges such that no pair of edges in M shares an endpoint. A matching is called **maximal** if it has the maximal number of edges among all matchings in the graph. We will write $\mu(G)$ for the number of edges in such a maximal matching.

Secondly, a **vertex cover** of G is a set of vertices $W \subset V$ such that W contains at least

one of the endpoints of every edge in E. A vertex cover is called **minimal** if it contains the minimum number of vertices among all vertex covers of the graph. We will write $\nu(G)$ for the number of vertices in a minimal vertex cover of G.

Kőnig's theorem states that, if G is bipartite:

$$\mu(G) = \nu(G).$$

Moreover, observe that for any graph G,

$$\nu(G) + \alpha(G) = |V|,$$

because every vertex cover is complementary to an independent set and vice versa. So, Kőnig's theorem implies that, when G is bipartite, we can find a matching $M \subset E$ that contains

$$|V| - \alpha(G)$$

edges. Such a matching M is incident to $2(|V| - \alpha(G))$ vertices and hence there are $2\alpha(G) - |V|$ vertices not incident to M. This means that there is a surjection

$$(\mathbb{Z}^2)^{*(|V|-\alpha(G))} * F_{2\alpha(G)-|V|} \longrightarrow \Gamma_{\mathrm{A}}(G),$$

where "$*$" denotes the free product. Writing $h_n(\Gamma) = |\mathrm{Hom}(\Gamma, \mathfrak{S}_n)|$ for any group Γ, the surjection above implies that

$$h_n(\Gamma_{\mathrm{A}}(G)) \leqslant h_n\left((\mathbb{Z}^2)^{*(|V|-\alpha(G))} * F_{2\alpha(G)-|V|}\right) = p(n)^{|V|-\alpha(G)} \cdot (n!)^{\alpha(G)},$$

where $p(n)$ denotes the number of partitions of the natural number n. Together with Lemma 4.1, this implies the theorem in the case of bipartite graphs. □

4.1.4 Right-angled Coxeter groups

The case of right-angled Coxeter groups is already a lot more subtle than that of right-angled Artin groups. First, we remind the reader that the only thing that changes is that we now assume the generators to be involutions. That is, given a finite graph $G = (V, E)$, the associated right-angled Coxeter group is

$$\Gamma_{\mathrm{C}}(G) = \langle v \in V \mid v^2 \text{ for all } v \in V \text{ and } [v, w] \text{ for all } \{v, w\} \in E \rangle.$$

An example from hyperbolic geometry of such a group is the group generated by the reflections in the faces of a regular ideal right-angled octahedron[1] in $\mathbb{H}^3$. The graph defining this group is the 1-skeleton of the cube. In general, it is known that right-angled Coxeter groups can appear as lattices in $\mathrm{Isom}(\mathbb{H}^d)$ only if $d \leqslant 14$, examples of such lattices are known in dimensions 2–8 [PV05].

Just like in the case of right-angled Artin groups, right-angled Coxeter groups admit surjections on groups that are easier to understand. We could take independent sets of vertices of G again and obtain surjections on free products of multiple copies of $\mathbb{Z}/2\mathbb{Z}$. It however turns out we can do better. To this end, we remind the reader that a **clique** in $G = (V, E)$ is a complete subgraph of G. Given a clique C in G, we define its weight as

$$w(C) = 1 - 2^{|C|},$$

where $|C|$ denotes the number of vertices in C. A system $\mathcal{C}$ of independent cliques in G is a collection of disjoint cliques in G such that G contains no edges between any pair of these cliques. The total weight of such a system is

$$w(\mathcal{C}) = \sum_{C \in \mathcal{C}} w(C).$$

We now define the following graph invariant

$$\gamma(G) = \max \left\{ w(\mathcal{C});\, \mathcal{C} \text{ a system of independent cliques in } G \right\}.$$

This invariant plays the same role as the independence number did for right-angled Artin groups. Indeed, given a system $\mathcal{C}$ of independent cliques in G, we obtain a surjection

$$\Gamma_{\mathrm{C}}(G) \longrightarrow \mathop{\ast}_{C \in \mathcal{C}} (\mathbb{Z}/2\mathbb{Z})^{|C|}.$$

It follows from work by Müller [Mül97, Mül96] that

$$\liminf_{n \to \infty} \frac{\log\left(s_n(\Gamma_{\mathrm{C}}(G))\right)}{n \log(n)} \geqslant \gamma(G) - 1.$$

The crux here is that the exponents of the copies of $\mathbb{Z}/2\mathbb{Z}$ in the free product above have an influence on the factorial growth rate. This is why cliques, instead of just isolated vertices, show up. From a combinatorial point of view, this also makes the problem a lot

[1] It turns out that this list of adjectives is long enough to determine a unique polytope up to isometry.

less tractable. Nonetheless, with Hyungryul Baik and Jean Raimbault, we conjecture that, at the factorial level, the lower bound we obtain is still sharp:

Conjecture

Let $G = (V, E)$ be a finite non-complete graph, then

$$\lim_{n\to\infty} \frac{\log\left(s_n(\Gamma_{\mathrm{C}}(G))\right)}{n\log(n)} = \gamma(G) - 1.$$

The reason we exclude complete graphs is that the corresponding groups are finite.

One of our reasons for conjecturing the above is that we proved it holds for a large class of groups:

Theorem 4.3. *Let $G = (V, E)$ be a finite graph for which there exists a (possibly empty) set of vertices $\{v_1, \dots, v_k\} \subset V$ such that*

- *The 1-neighborhood of v_i is a tree for all $i = 1, \dots, k$,*
- *these 1-neighborhoods are disjoint*
- *and the graph*
$$G - \{v_1, \dots, v_k\}$$
is a tree.

Then the conjecture above holds for $\Gamma_{\mathrm{C}}(G)$.

We don't have a heuristic reason for the conditions above, they are the conditions that come out of our proof. We can actually push the methods slightly further and obtain a larger class of graphs. This class is defined properly in [BPR20]. Since we're not expecting that class to be optimal either, we've chosen to slightly weaken the theorem here for ease of exposition. Furthermore, there are again classes of graphs, trees for instance, for which the proof simplifies considerably. We include one example here, the only lattice in $\mathrm{Isom}(\mathbb{H}^3)$ for which we know the factorial subgroup growth rate

Proposition 4.4. *Let G be the 1-skeleton of the cube, so that $\Gamma_{\mathrm{O}} := \Gamma_{\mathrm{C}}(G)$ is the reflection group of the regular right-angled octahedron in $\mathbb{H}^3$. Then the conjecture above is true for Γ_{O}, that is*

$$\lim_{n\to\infty} \frac{\log(s_n(\Gamma_{\mathrm{O}}))}{n\log(n)} = \gamma(G) - 1 = 1.$$

Proof. Given our observations before, we only need to prove the upper bound. Writing S for the 1-skeleton of the square, we have a surjection

$$\Gamma_{\mathrm{C}}(S) * \Gamma_{\mathrm{C}}(S) \longrightarrow \Gamma_{\mathrm{C}}(G).$$

Indeed, we can decompose the vertex set V of G into two sets of 4 vertices, each corresponding to a face of the cube. Since the right-angled Coxeter group corresponding to a disjoint union of two graphs is the free product of the two corresponding right-angled Coxeter groups, the group corresponding to the union of the two squares on these two sets 4 of vertices is $\Gamma_{\mathrm{C}}(S) * \Gamma_{\mathrm{C}}(S)$. The map $\Gamma_{\mathrm{C}}(S) * \Gamma_{\mathrm{C}}(S) \longrightarrow \Gamma_{\mathrm{C}}(G)$ induced by sending the generators corresponding to the vertices to themselves is a well-defined surjection.

Now, $\Gamma_{\mathrm{C}}(S) = D_\infty \times D_\infty$, where $D_\infty \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ is the infinite dihedral group. In particular, $\Gamma_{\mathrm{C}}(S)$ is virtually abelian. As such, it satisfies $h_n(\Gamma_{\mathrm{C}}(S)) \leqslant C^n n!$ for some $C > 0$ (see for instance [LS03, Chapter 1]. This, combined with Lemma 4.1, implies that

$$s_n(\Gamma_{\mathrm{O}}) \leqslant \frac{h_n(\Gamma_{\mathrm{O}})}{(n-1)!} \leqslant \frac{h_n(\Gamma_{\mathrm{C}}(S) * \Gamma_{\mathrm{C}}(S))}{(n-1)!} = \frac{h_n(\Gamma_{\mathrm{C}}(S)) \cdot h_n(\Gamma_{\mathrm{C}}(S))}{(n-1)!} \leqslant n \cdot C^{2n} \cdot n!,$$

which proves our claim. □

For more details on Theorem 4.3 and its proof, which in part relies on the results on Fuchsian groups we mentioned in the previous section, we refer to [BPR20].

Finally, there is one very specific class of right-angled Coxeter groups for which we obtain much better asymptotic results [BPR19]. These are virtually cyclic right-angled Coxeter groups and their free products. It turns out that every virtually cyclic right-angled Coxeter group is of the form

$$\Gamma_r := (\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})^r$$

for some $r \geqslant 0$.

The main advantage of these groups is that we can write down explicit generating functions for the sequences that interest us. Like in some of the proofs above, we start with the sequence

$$h_n(\Gamma_r) = |\mathrm{Hom}(\Gamma_r, \mathfrak{S}_n)|, \quad n \in \mathbb{N},$$

which is in fact equivalent information to the sequence $s_n(\Gamma_r)$ (see [LS03, Corollary 1.1.4]).

Let us write

$$G_r(x) = \sum_{n=0}^{\infty} \frac{h_n(\Gamma_r)}{n!}\, x^n.$$

for the exponential generating function of the sequence $(h_n(\Gamma_r))_n$. We have:

Theorem 4.5. *Let $r \in \mathbb{N}$. Then*

$$G_r(x) = \prod_{j=0}^{r} \left(\left(1 - x^{2^{j+1}}\right)^{-s_{2^j}/2} \exp\left(-2^j\, s_{2^j} + \frac{2^j\, s_{2^j}}{1 - x^{2^j}}\right)\right)$$

where

$$s_{2^j} = s_{2^j}((\mathbb{Z}/2\mathbb{Z})^r) = \frac{\prod_{l=0}^{j-1}(2^r - 2^l)}{\prod_{l=0}^{j-1}(2^j - 2^l)}.$$

The fact that this function is so explicit allows us to apply methods from analytic combinatorics (see [FS09] for a standard reference) in order to prove asymptotic results for its coefficients. The particular methods we use are due to Hayman [Hay56]. Similar methods have a long history in counting homomorphisms of finite groups into symmetric groups (see for instance [CHM51, MW56, MW57, Vol86, Wil86, Mül97]).

Examples of results we obtain are:

Theorem 4.6. *(a) Let $r \in \mathbb{N}$. Then*

$$s_n(\Gamma_r) = n \left(1 + \sum_{\substack{0<j\leqslant r \\ s.t.\ 2^j|n}} 2^j\, s_{2^j}\right) + \sum_{\substack{0\leqslant j\leqslant r \\ s.t.\ 2^{j+1}|n}} 2^j\, s_{2^j},$$

for all $n \in \mathbb{N}$.

(b) Let $m \in \mathbb{N}_{\geqslant 2}$, $r_1, \ldots, r_m \in \mathbb{N}$ and

$$\Gamma = *_{l=1}^{m} \Gamma_{r_l}.$$

Then there exist explicit constants $A_\Gamma, B_\Gamma > 0$ and $C_\Gamma \in \mathbb{Q}$ such that

$$s_n(\Gamma) \sim A_\Gamma\, n^{1+C_\Gamma}\, \exp(B_\Gamma \sqrt{n})\, n!^{m-1}.$$

as $n \to \infty$.

The constants in the theorem above are explicit, but their definitions (especially that of A_Γ) are lengthy. So for these, we refer to [BPR19]. We do note that they behave well

with respect to free products, that is:

$$A_\Gamma = \prod_{l=1}^{m} A_{r_l}, \quad B_\Gamma = \sum_{l=1}^{m} B_{r_l} \quad \text{and} \quad C_\Gamma = \sum_{l=1}^{m} C_{r_l},$$

where $A_r = A_{\Gamma_r}$, $B_r = B_{\Gamma_r}$ and $C_r = C_{\Gamma_r}$. As to their values, we will content ourselves with a table with some values:

	Exact values			Numerical values	
r	A_r	B_r	C_r	A_r	B_r
0	$\frac{1}{\sqrt{8\,\pi}} \cdot \exp\left(-\frac{1}{2}\right)$	2	$-\frac{1}{2}$	$0.1210\ldots$	2
1	$\frac{1}{\sqrt[4]{2048\,\pi^2}} \cdot \exp\left(-\frac{7}{4}\right)$	$2\sqrt{2}$	$-\frac{1}{4}$	$0.01457\ldots$	$2.8284\ldots$
2	$\frac{1}{320\,\sqrt{\pi}} \cdot \exp\left(-\frac{63}{8}\right)$	$2\sqrt{5}$	$\frac{1}{2}$	$6.7020 \cdot 10^{-7}$	$4.4721\ldots$
3	$\frac{1}{68719476736\sqrt{\pi}} \exp\left(-\frac{671}{16}\right)$	8	$\frac{13}{4}$	$5.0248 \cdot 10^{-30}$	8

Table 4.1: The first four values of A_r, B_r and C_r.

4.2 Covers by distinct manifolds

Another natural question related to finite index subgroup is how many *non-isomorphic* subgroups of the same index a given group has. For instance, if Γ is a free group or the fundamental group of a closed orientable surface and $\Lambda < \Gamma$ is a finite index subgroup, then the index alone is enough to determine isomorphism type of Λ. Indeed, it's again a free group or the fundamental group of a closed surface respectively. Moreover, which free group or surface group it is can be determined through an euler characteristic computation. In other words, even if free groups and fundamental groups of closed surfaces (of genus at least 2) have many finite index subgroups, they don't have many pairwise non-isomorphic subgroups of the same index.

Unlike the case of free groups and surface groups, in general, we are not able to prove counting results. Instead, we will just ask whether groups have the property that all their subgroups of the same index are isomorphic. Even if surfaces and graphs play a very prominent role in this text, this particular property should be the exception rather

than the rule. As such, together with Stefan Friedl, JungHwan Park, Jean Raimbault and Arunima Ray [FPP+21], we dubbed groups that have this property **exceptional**:

Definition 4.7. Let Γ be a group. If for all $d \geqslant 1$, for all pairs of subgroups of $\Lambda_1, \Lambda_2 < \Gamma$, both of index d, Λ_1 and Λ_2 are isomorphic as groups, we call Γ exceptional.

Our main motivation came from compact 3-manifolds, in which case the fundamental group often determines the manifold up to homeomorphism. Indeed, if M and N are closed, orientable, aspherical 3-manifolds and $\pi_1(M) \simeq \pi_1(N)$, then M and N are homeomorphic. This follows by combining Perelman's geometrization theorem [Per02, Per03a, Per03b], Mostow's rigidity theorem [Mos68], and work of Waldhausen [Wal68] and Scott [Sco83] (see for instance [AFW15, Theorem 2.1.2])

So, it also makes sense to speak of exceptional manifolds:

Definition 4.8. Let M be manifold. If for all $d \geqslant 1$, for all pairs of covers of $N_1 \to M$ and $N_2 \to M$, both of degree d, N_1 and N_2 are homeomorphic as manifolds, we call M exceptional.

We proved:

Theorem 4.9. *Let M be a compact 3-manifold with empty or toroidal boundary. Then M is exceptional if and only if it is homeomorphic to one of the following manifolds:*

1. $k \cdot \left(\mathbb{S}^1 \times \mathbb{S}^2\right)$ *for* $k \geqslant 1$,
2. $\mathbb{S}^1 \tilde{\times} \mathbb{S}^2$,
3. $\mathbb{S}^1 \times \mathbb{D}^2$,
4. $\mathbb{T}^2 \times I$,
5. $\mathbb{T}^3$,
6. *all spherical manifolds except those with fundamental group* $P_{48} \times \mathbb{Z}/p\mathbb{Z}$ *with* $\gcd(p, 3) = 1$ *and* p *odd, or* $Q_{8n} \times \mathbb{Z}/q$ *with* $\gcd(q, n) = 1$, q *odd, and* $n \geqslant 2$.

In the theorem above, $\mathbb{S}^n$ denotes the n-sphere, $\mathbb{D}^n$ denotes the n-dimensional disk, $\mathbb{T}^n$ denotes the n-torus, I denotes the unit interval $[0, 1]$ and $k \cdot M$ denotes the k-fold connected sum of the manifold M . The finite groups listed in item (6) are defined in [FPP+21, Section 6].

Our proof uses a divide and conquer strategy, we use Kneser's prime decomposition theorem and Perelman's geometrization theorem to divide our manifold into various geometric classes that we then treat separately. For the full proof, we refer to the paper, here we'll give a proof for the hyperbolic case:

Proof that closed hyperbolic 3*-manifolds are not exceptional.* Let M be a closed hyperbolic manifold. By Agol's theorem [Ago13], M admits a cover of finite degree $N \to M$ such that the first Betti number $b_1(N) > 0$.

This means that $\pi_1(N)$ admits a surjection $\varphi : \pi_1(N) \to \mathbb{Z}$ and in particular N admits a sequence of cyclic covers $Y_n^{(1)} \to N$ of degree n. The systoles of this sequence of hyperbolic manifolds are uniformly bounded. Indeed, the subgroup $K = \ker(\varphi)$ necessarily contains hyperbolic elements. Their translation length is an upper bound for the systole of $Y_n^{(1)}$ for all n.

On the other hand, N is residually finite by Malcev's theorem [Mal40]. This means that N admits a sequence of finite degree covers $Y_{n_k}^{(2)} \to N$ whose systoles tend to infinity. Since we have a cyclic cover of any degree, we can find a pair of covers

$$Y_n^{(1)}, Y_n^{(2)} \to N \to M$$

whose systoles are different but whose degrees are the same. By Mostow rigidity [Mos68], $Y_n^{(1)}$ and $Y_n^{(2)}$ cannot be homeomorphic as manifolds. □

The use of Agol's theorem in the proof above makes it considerably shorter, but it's in fact not necessary. In our paper we present a proof that avoids it. It still uses systoles but instead of Agol's theorem, we use the fact that we can embed a lattice in $\mathrm{PSL}(2, \mathbb{C})$ in $\mathrm{GL}_d(\mathbb{F})$ for some number field $\mathbb{F}$ (see [MR03, Theorem 3.1.2]). This allows us to apply Nori–Weisfeiler's strong approximation theorem [Wei84a] to find non-isomorphic subgroups of the same finite index. The advantage of that proof is that it generalizes to all semi-simple Lie groups that are not locally isomorphic to $\mathrm{PGL}(2, \mathbb{R})$. Agol's largeness theorem [Ago13] combined with arguments similar to those in [BGLM02, BGLS10] also allow us to prove that, in the case of hyperbolic 3-manifolds, the number of non-isomorphic subgroups of bounded index still grows factorially fast as a function of the index.

4.3 Counting curves

The final counting problem that we will briefly discuss is that of counting closed curves of surfaces. There are essentially two (related) types of questions: geometric and topological questions. In this section we will consider the latter. We will discuss some aspects of the former in Section 5.1.

Here, we will ask for the number of closed curves on a closed surface Σ, considered up to homeomorphism and isotopy, with a bounded number of self intersections. One way to describe this is to consider the **mapping class group**

$$\mathrm{MCG}(\Sigma) = \mathrm{Homeo}^+(\Sigma)/\mathrm{Homeo}_0^+(\Sigma),$$

where $\mathrm{Homeo}^+(\Sigma)$ denotes the group of orientation preserving homeomorphisms $\Sigma \to \Sigma$ and $\mathrm{Homeo}_0^+(\Sigma)$ the normal subgroup consisting of those homeomorphisms that are isotopic to the identity. In this definition, we can replace homeomorphisms by diffeomorphisms and isotopy with homotopy without affecting the resulting mapping class group.

This group acts on the set of free homotopy classes of closed curves. This allows us to define

$$N_k(\Sigma) := \left| \left\{ \begin{array}{c} \text{free homotopy classes of closed curves} \\ \text{on } \Sigma \text{ with } k \text{ self-intersections} \end{array} \right\} \Big/ \mathrm{MCG}(\Sigma) \right|.$$

Here, the number of self-intersections of a free homotopy class of closed curves is the minimum of the self-intersection number of among all curves in the class. One reason to care about $N_k(\Sigma)$ is that it shows up in the count of the number of closed geodesics on a hyperbolic surface with a fixed number of self intersections and of bounded length [Mir08, Mir16, ES16, EPS20, ES22, DGZZ21, DGZZ22, Liu22, VM22].

There are two immediate questions to ask: how does $N_k(\Sigma)$ depend on k, and how does it depend on Σ? We'll start with the former. Aougab–Souto [AS18], improving upon work by Sapir [Sap16], proved that, if Σ_g is a closed orientable surface of genus $g \geqslant 2$, for all $\delta > 0$,

$$\exp\left((\pi\sqrt{2g-2} - \delta) \cdot \sqrt{k}\right) \leqslant N_k(\Sigma_g) \leqslant \exp\left((4\sqrt{2}\sqrt{2g-2} + \delta) \cdot \sqrt{k}\right),$$

for all $k \gg 0$. Aougab–Souto conjecture that their lower bound is sharp:

Conjecture (Aougab–Souto)

Let Σ_g be a closed orientable surface of genus $g \geqslant 2$, then:

$$\lim_{k\to\infty} \frac{\log(N_k(\Sigma))}{\sqrt{k}} = \pi\sqrt{2g-2}.$$

With Patricia Cahn and Federica Fanoni [CFP18], we considered the opposite regime. That is, we fix $k \geqslant 0$ and ask what happens if we vary the surface. We proved:

Theorem 4.10. *We have*

$$N_k(\Sigma_g) \sim C_k \cdot g^{k+1} \quad \text{as } g \to \infty.$$

where $C_k > 0$ is a constant that depends on k only.

Our proof makes heavy use of results of Hass and Scott [HS94, HS99] on properties of curves that are not in minimal position. The constant C_k is a weighted sum over a finite set of ribbon graphs that depends on k. In particular, it can be made explicit for small values of k. For instance, it's well known that $C_0 = \frac{1}{2}$.

5 Random manifolds

Over the past 20 years, enormous progress has been made on random hyperbolic manifolds. Today there are many different models of random hyperbolic surfaces [BM04, GPY11, Mir13, MP23, BCP21b] and random hyperbolic 3-manifolds [DT06, PR22].

The purpose of this chapter is to discuss these models, their geometric and topological properties, and how the contributions of my co-authors and me fit into this story. We will discuss:

- universal properties of random surfaces obtained by gluing polygons together, which is joint work with Thomas Budzinski and Nicolas Curien [BCP19],
- the distribution of the number of short closed geodesics in random surfaces, based on joint work with Christoph Thäle [PT18] and Maryam Mirzakhani [MP19],
- homological torsion of random Heegaard splittings, based on our project with Hyungryul Baik, David Bauer, Ilya Gekhtman, Ursula Hamenstädt, Sebastian Hensel, Thorben Kastenholz and Daniel Valenzuela [BBG$^+$18],
- random covers of torus knot complements, that we worked on with Elizabeth Baker [BP23a],
- and random 3-manifolds with boundary, that we studied with Jean Raimbault [PR22].

5.1 Random surfaces

Random surfaces have been around for a long time both in mathematics and in theoretical physics. A model that has attracted a lot of attention is that of random triangulations (or more general cell decompositions, often called **maps**) of a fixed surface Σ of finite type. Often this surface is the 2-sphere, in which case the cell decompositions are called **planar**

maps. In this model, we pick randomly pick a triangulation Σ with a bounded number of triangles in the set of homeomorphism classes of all such triangulations and ask what the result looks like as the number of triangles tends to infinity. By now, there are many techniques available (see for instance [LGM12, Mie14, Cur23] and references therein) and we have quite a detailed description of the local [AS03] and global [LG13, Mie13, BM22] geometry of these triangulations. Recently these methods have also found applications in hyperbolic geometry [Bud22, BZ23].

5.1.1 Random Belyĭ surfaces

The first model of random hyperbolic surfaces, called random Belyĭ surfaces, was developed by Brooks–Makover [BM04]. Their goal, inspired by random regular graphs, was to study random closed hyperbolic surfaces of large genus (so the opposite regime to the one of random planar maps), and find applications of the probabilistic method, in particular for the spectral gap.

A random Belyĭ surface is obtained by randomly partitioning the sides of n (an even number) ideal triangles into pairs and then gluing them into an oriented surface according to the side pairing (this is analogous to the configuration model in graph theory, see Section 2.2.4), using gluings of shear 0 (see Section 3.1.3). The resulting surface S_n^O is then conformally compactified, thus obtaining a closed random surface S_n^C, equipped with a conformal structure.

If the genus is at least 2 (which turns out to be generic), this conformal class contains a unique hyperbolic metric by the uniformization theorem. One reason that this is an interesting model of random surfaces is that the non-compact surfaces S_n^O are covers of $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}^2$, which implies that the compact surfaces S_n^C are branched covers of the sphere, branched at at most three points. Such surfaces are called Belyĭ surfaces because Belyĭ proved they are exactly the surfaces that can be defined over $\overline{\mathbb{Q}}$ [Bel79] (see [JS96] for a nice survey). This in particular implies that the set of surfaces we sample from, form a dense set in $\mathcal{M}_g$ for all $g \geqslant 0$.

Topology. The first question is what the topology of these surfaces is like. First of all, we didn't specify that the gluing needs to be into a connected surface. However, it's true that

$$\mathbb{P}\left(S_n^O \text{ and } S_n^C \text{ are connected}\right) \to 1 \quad \text{as } n \to \infty$$

which follows directly from the analogous result on random regular graphs due to Bollobás and Wormald [Bol81, Wor81a]. Knowing this, the only remaining question is their genus (or equivalently, the number of cusps of S_n^O). Very precise results on this are known due to Gamburd (when 4 divides n) [Gam06] and Chmutov–Pittel [CP16]. They for instance prove the following central limit theorem for genus of these surfaces:

$$\frac{\mathrm{genus}(S_n^C) - 1 - \frac{n}{4} - \log(n)}{\sqrt{\log(n)}} \xrightarrow{\text{distribution}} \mathcal{N}(0,1) \quad \text{as } n \to \infty,$$

where $\mathcal{N}(0,1)$ denotes the standard normal distribution.

They prove this using the fact that a random surface can be modeled using a uniformly random pair of permutations $(\sigma_n, \tau_n) \in K(3^n) \times K(2^{3n/2})$. Here $K(3^n) \subset \mathfrak{S}_{3n}$ denotes the conjugacy class of fixed point free permutations of order 3 and $K(2^{3n/2}) \subset \mathfrak{S}_{3n}$ the conjugacy class of fixed point free involutions. The notation "3^n" and "$2^{n/2}$" is exponential notation for the partitions that describe the cycle types of the permutations in these conjugacy classes. The permutation σ_n encodes the oriented labeling of the sides of the ideal triangles and the permutation τ_n describes the side pairing. Their product $\sigma_n \cdot \tau_n$ describes the cusps of S_n^O, in the sense that each cusp naturally corresponds to a cycle in the disjoint cycle decomposition of $\sigma_n \cdot \tau_n$ and moreover the length of that cycle gives us the "size" of the cusp: the number of corners of ideal triangles incident to it.

Using ideas from random walks on finite groups, in particular the Diaconis–Shahshahani lemma [DS81], Gamburd and Chmutov–Pittel proved that, denoting the distribution of $\sigma_n \cdot \tau_n \in \mathfrak{S}_{3n}$ by $\mathbb{P}_{\sigma_n \cdot \tau_n}$, we have that,

$$\mathrm{d_{TV}}(\mathbb{P}_{\sigma_{4k} \cdot \tau_{4k}}, \mathbb{U}_{4k}^{\mathrm{even}}) \to 0 \quad \text{as } k \to \infty$$

and

$$\mathrm{d_{TV}}(\mathbb{P}_{\sigma_{4k+2} \cdot \tau_{4k+2}}, \mathbb{U}_{4k+2}^{\mathrm{odd}}) \to 0 \quad \text{as } k \to \infty,$$

where $\mathrm{d_{TV}}$ denotes total variational distance, $\mathbb{U}_{4k}^{\mathrm{even}}$ the uniform measure on the set of even permutations (the alternating group) and $\mathbb{U}_{4k+2}^{\mathrm{odd}}$ the uniform measure on the set of odd permutations. This for instance implies the central limit theorem above and also that, when properly normalized, the partition describing the cycle type of $\sigma_n \cdot \tau_n$ converges to a Poisson–Dirichlet distributed random variable as $n \to \infty$ (see for instance [ABT00]). Chmutov–Pittel also generalized these results to random gluings of polygons with varying perimeters.

In [BCP19], with Thomas Budzinski and Nicolas Curien, we proved a weaker version (that we dubbed Poisson–Dirichlet universality) of these results for a slightly larger class of polygon gluings. One of the main points of this is that we did this using different methods, that do in particular not rely on the representation theory of the symmetric group.

Instead, we used dynamical exploration techniques that are well known in the world of planar maps and are often called **peeling techniques** (see [Cur23] for an introduction) and that play an important role at multiple points in this text. The idea is very simple. Formally, our probability space is a finite set of pairings of sides of triangles. In particular, an element in the probability space does not make any reference to the order in which the triangles are glued together. However, we can choose to add an order without influencing the probability of any geometric or topological event. Such an order is called a **peeling algorithm** and is often thought of as the order in which the triangulation is being "discovered".

To show how the method works, we'll present a simple example. This example essentially already appears (without this language) in Brooks and Makover's paper [BM04]:

A peeling algorithm

Input:

- An even number $n \in \mathbb{N}$
- A surface $S^{(0)}$ which is a disconnected union of n oriented triangles, with their sides labeled $1, 2, \ldots, 3n$

Iteration:
At step $t = 1, \ldots, 3n/2$, create the random surface $S^{(t)}$ as follows

1. Let $e_1^{(t)}$ and $e_2^{(t)}$ denote two uniformly random edges on $\partial S^{(t-1)}$
2. Glue $e_1^{(t)}$ and $e_2^{(t)}$ together in such a way that the orientations on both sides match. Call the resulting surface $S^{(t)}$.

This procedure generates a random sequence of surfaces $S^{(0)}, \ldots, S^{(3n/2)}$. The order we have introduced is of no influence on the resulting surface. That is, as a random triangulated surface, the law of $S^{(3n/2)}$ is the same as that of S_n^O. Adding this order, however gives us access to the number of vertices in the triangulation. For instance, if we

write V_n for the number of vertices of $S^{(3n/2)}$, then we have

$$\mathbb{E}(V_n) = \sum_{t=1}^{3n/2} \mathbb{E}(V_n^{(t)}),$$

where $V_n^{(t)}$ denotes the number of vertices that in are in the interior of $S^{(t)}$ but not in the interior of $S^{(t-1)}$. The value of $V_n^{(t)}$ is either 0, 1 or 2. The latter only happens if $e_1^{(t)}$ and $e_2^{(t)}$ together form a boundary component of $S^{(t-1)}$, which is rare. Creating one vertex in the interior can happen in two ways, either $e_1^{(t)}$ and $e_2^{(t)}$ are incident to a common vertex (see Figure 5.1), or both $e_1^{(t)}$ and $e_2^{(t)}$ form a full boundary component of $S^{(t-1)}$. Of these two, the former is generic. As such, we obtain

$$\mathbb{E}(V_n) = o(\log(n)) + \sum_{t=1}^{3n/2} \mathbb{P}\left(\begin{array}{c} e_1^{(t)} \text{ and } e_2^{(t)} \text{ are on a boundary component containing} \\ \text{at least 3 edges and are incident to a common vertex} \end{array}\right),$$

as $n \to \infty$ Also the event that $e_1^{(t)}$ and $e_2^{(t)}$ are on a boundary component containing at least 3 edges is generic. Using this and the fact that after step $t-1$, there are $3n-2t+2$ edges left on the boundary, we obtain

$$\mathbb{E}(V_n) = o(\log(n)) + \sum_{t=1}^{3n/2} \frac{2}{3n-2t+1} = \log(n) + o(\log(n))$$

as $n \to \infty$, thus recovering the first order term in the central limit theorem above. For finer estimates for more general gluings and proofs of the claims of genericity above, we refer to [BCP19].

In that paper, we also observed that other geometric properties of the graph structure of a random polygon decomposition seem to be universal. To formalize this, let $\mathcal{P}_n = \{p_1, \ldots, p_k\}$ be a multiset of perimeters of polygons such that $\sum_i p_i = n$. Moreover let $G_{\mathcal{P}_n}$ denote the random graph obtained from a random gluing of polygons of the perimeters prescribed by $\mathcal{P}_n$, using the configuration model. The structure of this graph is not entirely random: there is a parity constraint similar to the constraint showing up in the results of Gamburd and Chmutov–Pittel discussed above. Indeed, an Euler characteristic computation shows that the parity of the number of vertices of $G_{\mathcal{P}_n}$ is the same as the parity of $n+k$. The question is whether this is essentially the only constraint:

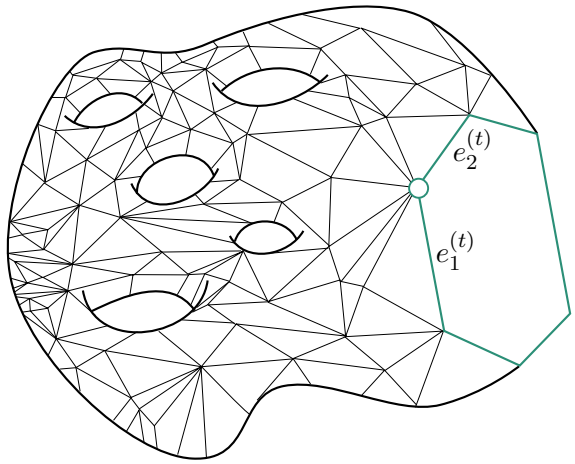

Figure 5.1: Closing off a vertex of $S^{(t)}$

Question 9

Let $\mathbb{G}_n$ be the random graph structure of a uniform random labeled map on n edges, and denote $\mathbb{G}_n^{\text{odd}}$ (respectively $\mathbb{G}_n^{\text{even}}$) be the random graph $\mathbb{G}_n$ conditioned respectively on having an odd (respectively even) number of vertices.
Suppose moreover that $(\mathcal{P}_n)_n$ is a sequence of multisets of perimeters such that $\sum_{p\in\mathcal{P}_n} p = n$ and

$$\lim_{n\to\infty} \frac{|\{p_i \in \mathcal{P}_n;\, p_i = 1\}|}{\sqrt{n}} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{|\{p_i \in \mathcal{P}_n;\, p_i = 2\}|}{n} = 0$$

Then is it true that

$$\mathrm{d}_{\mathrm{TV}}(G_{\mathcal{P}_n}, \mathbb{G}_n^{\epsilon_n}) \to 0 \quad \text{as } n \to \infty \quad ?$$

Here $\epsilon_n \in \{\text{even}, \text{odd}\}$ denotes the parity of $|\mathcal{P}_n| + n$.

The conditions on the number of monogons and bigons in this question are necessary to make the earlier mentioned peeling based proofs work. They for instance play a role in the proofs of the genericity claims before.

Geometry. We also know a lot about the geometry of random Belyĭ surfaces. Brooks and Makover already proved that there exist constants $c_1, c_2, c_3, c_4 > 0$ such that, with high

probability as $n \to \infty$ [1],

$$\mathrm{sys}(S_n^C) \geqslant c_1, \quad \lambda_1(S_n^C) \geqslant c_2, \quad h(S_n^C) \geqslant c_3 \quad \text{and} \quad \mathrm{diam}(S_n^C) \leqslant c_4 \log(\mathrm{genus}(S_n^C)).$$

Here, $h(S_n^C)$ denotes the Cheeger constant of S_n^C, defined as

$$h(S_n^C) = \inf \left\{ \frac{\ell(\partial Y)}{\mathrm{area}(Y)}; \begin{array}{c} Y \subset S_n^C \text{ a submanifold with smooth} \\ \text{boundary and } \mathrm{area}(Y) \leqslant \mathrm{area}(S_n^C) \end{array} \right\},$$

which measures how hard it is to cut S_n^C in two. We will discuss this quantity and its interest at length in Section 6.3. $\mathrm{diam}(S_n^C)$ denotes the diameter of S_n^C. By now, sharper bounds are known on this diameter [BCP21a], which we will briefly discuss in Section 6.2. All these bounds suggest that random Belyĭ surfaces are good expanders. Whether they're near Ramanujan surfaces is currently still open, but it seems quite likely they are, given that the combinatorics and geometry of the model are very close to that of the random covers of Hide–Magee [HM23b]:

Conjecture

For every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\lambda_1(S_n^C) > \frac{1}{4} - \varepsilon\right) = 1.$$

The method of proof of the bounds by Brooks and Makover is to first understand the geometry of the non-compact surfaces S_n^O, using a combination of results on random graphs and hyperbolic geometry and then prove that the given bounds persist (up to manageable errors) in the compactification. This last step is based on the same earlier work due to Brooks [Bro99] that we have mentioned before. This works for the systole, the spectral gap and the Cheeger constant. However, the diameter of S_n^O is infinite. Instead, the logarithmic upper bound on the diameter of S_n^C follows from the bounds on the other three invariants, again using earlier work by Brooks [Bro92].

Together with Christoph Thäle [PT18], we studied the length spectrum of random Belyĭ surfaces. Let $\mathcal{L}(S_n^C)$ denote the **primitive length spectrum** of S_n^C: the multiset of lengths of primitive closed geodesics on S_n^C. This is a random countable subset of $(0, \infty)$. In order to state our theorem, recall from Section 3.1.3 that the lengths of curves on S_n^O

[1] The phrase "with high probability as $n \to \infty$" means that the probability of the described event tends to 1 as $n \to \infty$

are determined by words in two matrices L and R. To this end, we set

$$W = \{\text{words in } L \text{ and } R\}/\sim$$

where two words w and w' are equivalent if (as strings in L and R) if w' can be obtained from w through a combination of the following two operations

- cyclic permutation of the letters,
- reading the word backwards and interchanging L and R.

This equivalence captures the ambiguity in the definition of the word associated to a closed curve mentioned in Section 3.1.3. Given $[w] \in W$, we will write

$$\ell([w]) = 2\text{arccosh}\left(\frac{\text{tr}\,(w)}{2}\right).$$

We proved the following:

Theorem 5.1. *As $n \to \infty$, $\mathcal{L}(S_n^C)$ converges locally in total variational distance*[2] *to a Poisson point process $\mathcal{L}^{\text{BM}}$ of intensity λ^{BM} given by*

$$\lambda^{\text{BM}}(A) = \sum_{\substack{[w]\in W \\ \ell(w)\in A}} \frac{|[w]|}{2\,|w|}, \quad A \subset (0,\infty).$$

Because they play an important role in this text, we remind the reader of the definition of a **Poisson point process**.

Definition 5.2. Let X be a manifold and μ a locally finite Radon measure on X. A Poisson point process on X of intensity μ is a random countable subset $\mathcal{S} \subset X$ such that:

(a) For any compact measurable $A \subset X$, the random variable

$$N_A = |\mathcal{S} \cap A| \sim \text{Poisson}(\mu(A)).$$

(b) If $A_1, \ldots, A_k \subset X$ are disjoint compact subsets, then the random variables $N_{A_1}, \ldots, N_{A_k}$ are an independent family.

[2] See [Kal17, Section 4.4] for a definition.

With Christoph Thäle, we obtain explicit bounds on the total variational distance of the random variables that count the numbers of geodesics of bounded length to the counting variables corresponding to $\mathcal{L}^{\mathrm{BM}}$. This in particular allows us to say something about geodesics whose length grows (slowly) with n. Our theorem strengthens work from the author's PhD thesis [Pet17b, Pet17a] in which the method of moments was used to prove local convergence in distribution. In our work with Christoph Thäle, we use the Chen–Stein method for Poisson approximation instead [Che75a, AGG89, BHJ92].

Our result with Christoph Thäle also implies that the systole of random Belyĭ surfaces is not large (for instance, its expectation is uniformly bounded [Pet17b]). In upcoming work with Mingkun Liu [LP23], we will prove that the construction can be modified, using ideas from graph theory due to Linial–Simkin [LS21], in order to produce random surfaces with logarithmic systoles.

5.1.2 Random covers of finite degree

Another combinatorial model is that of random covers of a fixed surface. That is, one fixes a hyperbolic surface X_1 with a finitely generated fundamental group Γ and takes a uniformly random index n subgroup of Γ, which gives rise to a random cover of degree n of X_1. Due to results by Dixon [Dix69] for free groups and Müller–Schlage-Puchta [MP02, MSP07] and Liebeck–Shalev [LS04a] for surface groups, this model is essentially the same as picking a uniformly random homomorphism

$$\varphi_n \in \mathrm{Hom}(\Gamma, \mathfrak{S}_n)$$

and taking the cover corresponding to the stabilizer $H_n = \mathrm{Stab}_{\varphi_n}(\{1\})$.

If the base surface is a non-compact surface of finite area, and hence Γ is a finitely generated free group, this model is somewhat similar to the model of random Belyĭ surfaces discussed above. Counts of the number of short closed geodesics can be derived from results due to Nica [Nic94] and imply in particular that as an IRS (see Section 4.1.1), H_n converges to the trivial group and the corresponding cover Benjamini–Schramm converges to $\mathbb{H}^2$. As we mentioned in Section 3.3, Hide–Magee [HM23b] proved that these surfaces are near Ramanujan with high probability as $n \to \infty$. A version of the latter result in the case of Schottky groups in $\mathrm{PSL}(2, \mathbb{R})$ is due to Magee–Naud [MN20, MN21].

If the base surface is closed, the fact that its fundamental group is not free makes counting the number of homomorphisms significantly more complicated. Nonetheless, good

asymptotic results for this number are known due to the works by Müller–Schlage-Puchta and Liebeck–Shalev mentioned above.

Also the question of counting the number of short closed geodesics that lift to the cover becomes more complicated, but recently techniques for estimating this were developed by Magee–Naud–Puder [MP23, MNP22], which in particular give rise to a Poisson approximation theorem for the length spectrum of these random covers [PZ22] and again imply convergence of the corresponding IRS and Benjamini–Schramm convergence to the hyperbolic plane.

Like in the Brooks–Makover model (and the Weil–Petersson model that we will discuss below), the results above imply that these random constructions do not give rise to surfaces with large systoles. In the earlier mentioned upcoming work with Mingkun Liu [LP23] we will prove that certain random *regular* covers, much like the random Cayley graphs of Section 2.2.4, do yield to surfaces with logarithmic systoles.

Magee–Naud–Puder also prove that for any $\varepsilon > 0$, with high probability as the degree of the cover tends to infinity, these covers have no new Laplacian eigenvalues[3] in the interval $\left[0, \frac{3}{16} - \varepsilon\right]$. They conjecture that this should hold up to $\frac{1}{4}$:

Conjecture (Magee–Naud–Puder)

Let X be a closed orientable hyperbolic surface and let $X_n \to X$ denote a random cover of degree n of X, as defined above. Moreover, given a closed hyperbolic surface Y, let $\sigma(Y)$ denote the spectrum of its Laplacian. Then for any $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(\sigma(X_n) \cap \left[0, \frac{1}{4} - \varepsilon\right] = \sigma(X) \cap \left[0, \frac{1}{4} - \varepsilon\right]\right) = 1.$$

Finally, Naud determined the asymptotic behavior of the variance of smoothened counting functions of eigenvalues of the twisted Laplace operators [Nau22] and the determinant of these surfaces [Nau23].

5.1.3 The Weil–Petersson measure

The final model of random hyperbolic surface we will discuss in this chapter (we'll introduce another one in Section 6.2) is the model coming from the Weil–Petersson measure on the

[3] New Laplacian eigenvalues on a finite degree cover are eigenvalues that do not correspond to eigenvalues obtained by lifting an eigenfunction from the base.

moduli space $\mathcal{M}_g$ of closed hyperbolic surfaces of genus g. This is the measure coming from the Weil–Petersson volume form, which in turn comes from a Kähler 2-form called the Weil–Petersson form. Random surfaces sampled using this measure were first studied by Mirzakhani [Mir13] and Guth–Parlier–Young [GPY11]: Mirzakhani proved that these random surfaces have similar expansion properties to random Belyĭ surfaces. Guth, Parlier and Young used these surfaces to attack an extremal problem on pants decompositions (see Section 6.1 below for a description of these results).

Together with Maryam Mirzakhani [MP19], we proved a similar Poisson approximation result to Theorem 5.1 for surfaces distributed according to this measure. In the theorem below, X_g will denote a random hyperbolic surface in $\mathcal{M}_g$, distributed according to the Weil–Petersson probabiltiy measure. $\mathcal{L}(X_g)$ will again denote its primitive length spectrum, a random subset of $(0, \infty)$. We proved:

Theorem 5.3. *As $g \to \infty$, $\mathcal{L}(X_g)$ converges locally in distribution*[4] *to a Poisson point process $\mathcal{L}^{\mathrm{WP}}$ of intensity λ^{WP} given by*

$$\lambda^{\mathrm{WP}}(A) = \int_A \frac{\cosh(t) - 1}{t} dt.$$

Our proof is based on the method of moments, combined with Mirzakhani's integration techniques for Weil–Petersson volumes [Mir07a, Mir07b]. Curiously, these same statistics have been observed for certain models of unicellular maps by Janson–Louf [JL22, JL23].

In this section, we will describe some of the ingredients of the proof of the theorem above, starting with an explanation of the Weil–Petersson volume form itself. In the end of this section we'll survey some of the other known results on random surfaces distributed according to the Weil–Petersson measure.

Tecihmüller and moduli spaces. In this text, we won't describe this Kähler structure. Instead, we'll start with Wolpert's results on the associated symplectic form. This will be enough for us, because in the end we're only interested in the volume form. To describe Wolpert's work, we need to first describe **Fenchel–Nielsen coordinates**. These are global coordinates on the **Teichmüller space** of the closed oriented surface Σ_g of genus $g \geqslant 2$:

$$\mathcal{T}_g = \left\{ (X, f); \quad \begin{array}{c} X \text{ an oriented hyperbolic surface and} \\ f : \Sigma_g \to X \text{ an orientation preserving diffeomorphism} \end{array} \right\} \Big/ \sim,$$

[4] See [Kal17, Section 4.1] for a definition.

where $(X_1, f_1) \sim (X_2, f_2)$ if and only if there exists an isometry $m : X_1 \to X_2$ such that the map

$$f_2^{-1} \circ m \circ f_1 : \Sigma_g \to \Sigma_g$$

is homotopic to the identity. The mapping class group $\mathrm{MCG}(\Sigma_g)$ (see Section 4.3 for its defintion) acts on $\mathcal{T}_g$ by

$$[\varphi] \cdot [X, f] = [X, f \circ \varphi^{-1}], \quad [X, f] \in \mathcal{T}_g, \varphi \in \mathrm{MCG}(\Sigma_g)$$

this action is properly discontinuous, but not free, and the quotient is the moduli space $\mathcal{M}_g$.

Given a pants decomposition $\{\alpha_1, \dots \alpha_{3g-3}\}$ of Σ_g, we can define Fenchel–Nielssen coordinates:

$$\left(\ell_{\alpha_1}, \tau_{\alpha_1}, \dots, \ell_{\alpha_{3g-3}}, \tau_{\alpha_{3g-3}}\right) : \mathcal{T}_g \longrightarrow (\mathbb{R}_{>0} \times \mathbb{R})^{3g-3}.$$

The functions ℓ_α measure lengths of curves: $\ell_\alpha([X, f])$ is the length of the unique geodesic on X in the free homotopy class of $f(\alpha)$. The coordinates τ_{α_i} measure the twists along the pants curves (similar to Section 3.1.4). It turns out that the induced map is a homeomorphism and that moreover the coordinate changes (when changing pants decomposition) are analytic.

Wolpert proved that these coordinates are in fact Darboux coordinates for the symplectic form associated to the Weil–Petersson Kähler form [Wol82]. That is, if we write ω_{WP} for the Weil–Petersson symplectic form, then

$$\omega_{\mathrm{WP}} = \sum_{i=1}^{3g-3} d\ell_{\alpha_i} \wedge d\tau_{\alpha_i}.$$

In particular, the associated volume form is just the usual Lebesgue volume form on $(\mathbb{R}_{>0} \times \mathbb{R})^{3g-3}$. Moreover, it descends to a volume form of finite total volume on $\mathcal{M}_g$. This allows us to define a probability measure, simply by normalizing it:

$$\mathbb{P}(A) = \frac{\mathrm{vol}_{\mathrm{WP}}(A)}{\mathrm{vol}_{\mathrm{WP}}(\mathcal{M}_g)}, \quad \forall A \subset \mathcal{M}_g \text{ measurable},$$

where

$$\mathrm{vol}_{\mathrm{WP}}(A) = \int_A \frac{\omega_{\mathrm{WP}}^{\wedge(3g-3)}}{(3g-3)!}, \quad \forall A \subset \mathcal{M}_g \text{ measurable}.$$

Computing Weil–Petersson volumes. Of course, theoretically, Weil–Petersson volumes can be computed by integrating over a suitable fundamental domain for the $\mathrm{MCG}(\Sigma_g)$-action on $\mathcal{T}_g$. Unfortunately, the topology of $\mathcal{M}_g$ is very complicated, so no easy to describe sequences of such fundamental domains are known. There are estimates of Weil–Petersson volumes based on approximate fundamental domains [Pen92, Gru01, ST01], but they are generally not sharp enough to answer the type of questions we are interested in. A notable exception is the paper by Guth–Parlier–Young [GPY11], in which these bounds suffice.

In her thesis [Mir07a, Mir07b], Mirzakhani developed an alternative approach to the calculation of Weil–Petersson volumes. In order to describe this, we need to slightly enlarge our set of moduli spaces. First fix an orientable surface $\Sigma_{g,n}$ of genus g and with n boundary components. Given $L_1, \dots L_n \geqslant 0$, we now define the Teichmüller space $\mathcal{T}_{g,n}(L_1, \dots, L_n)$ as

$$\mathcal{T}_{g,n}(L_1, \dots, L_n) = \left\{ (X, f); \quad \begin{array}{c} X \text{ an oriented hyperbolic surface with totally} \\ \text{geodeisc boundary and } f : \Sigma_{g,n} \to X \text{ an} \\ \text{orientation preserving diffeomorphism} \end{array} \right\} \Big/ \sim,$$

By convention, when we say "totally geodesic boundary component of length 0" we mean "cusp". The notion of equivalence is the same as before, except that we now also require the maps $f_2^{-1} \circ m \circ f_1 : \Sigma_{g,n} \to \Sigma_{g,n}$ to preserve the boundary components of $\Sigma_{g,n}$ setwise. Likewise, the mapping class group $\mathrm{MCG}(\Sigma_{g,n})$ is the group of isotopy classes of diffeomorphisms $\Sigma_{g,n} \to \Sigma_{g,n}$ that preserve the boundary components setwise. The corresponding moduli space is

$$\mathcal{M}_{g,n}(L_1, \dots, L_n) = \mathcal{T}_{g,n} / \mathrm{MCG}(\Sigma_{g,n}).$$

By work of Goldman [Gol84] and Wolpert [Wol82], this moduli space carries also carries a symplectic form that admits the same expression in terms of Fenchel–Nielsen coordinates. In particular, we can speak of its Weil–Petersson volume.

In order to find functions we can integrate, Mirzakhani's idea is to introduce a class of functions $\mathcal{M}_g \to \mathbb{R}$ that she calls **geometric functions**. These are constructed as follows:

Definition 5.4. Let $\Gamma = (\gamma_1, \dots, \gamma_k)$ be a sequence of pairwise disjoint essential free homotopy classes of closed curves on $\Sigma_{g,n}$. Moreover, let $F : \mathbb{R}^k_{\geqslant 0} \to \mathbb{R}$. Then the geometric function $F^\Gamma : \mathcal{M}_{g,n}(L_1, \dots, L_n) \to \mathbb{R}$ is defined by

$$F^\Gamma(X) = \sum_{(\alpha_1, \dots, \alpha_k) \in \mathrm{MCG}(\Sigma_{g,n}) \cdot (\gamma_1, \dots, \gamma_k)} F(\ell_{\alpha_1}(X), \dots, \ell_{\alpha_k}(X))$$

For example, if γ is a simple non-separating closed curve on Σ_g and $F = \chi_{[0,L]}$ is the characteristic function of the interval $[0, L]$, then the geometric function $F^\gamma : \mathcal{M}_g \to \mathbb{N}$ is given by

$$F^\gamma(X) = \left| \left\{ \begin{array}{c} \text{simple non-separating closed geodesics} \\ \text{of length at most } L \text{ on } X \end{array} \right\} \right|.$$

To state Mirzakhani's integration formula, we will, given a sequence of pairwise disjoint essential free homotopy classes of closed curves $\Gamma = (\gamma_1, \dots, \gamma_k)$ on $\Sigma_{g,n}$, $L \in \mathbb{R}^n_{\geqslant 0}$ and $x \in \mathbb{R}^k_{\geqslant 0}$, use the notation $V_{g,n}(\Gamma, L, x)$ for the Weil–Petersson volume of the moduli space of hyperbolic metrics on the surface with boundary $\overline{\Sigma_{g,n} - (\gamma_1 \cup \cdots \cup \gamma_k)}$, whose boundary lengths are $L_1, \dots, L_n, x_1, x_1, x_2, x_2, \dots, x_k, x_k$.

In [Mir07a], Mirzakhani proved the following formula:

Theorem 5.5 (Mirzakhani integration formula)**.** *Let $\Gamma = (\gamma_1, \dots, \gamma_k)$ be a sequence of pairwise disjoint essential free homotopy classes of closed curves on $\Sigma_{g,n}$ and $F : \mathbb{R}^k_{\geqslant 0} \to \mathbb{R}$. Then*

$$\int_{\mathcal{M}_{g,n}(L)} F^\Gamma(X)\, d\,\mathrm{vol}_{\mathrm{WP}}(X) = C_\Gamma \cdot \int_{\mathbb{R}^k_{\geqslant 0}} F(x_1, \dots x_k) \cdot V_{g,n}(\Gamma, L, x) \cdot x_1 \cdots x_k \, dx_1 \cdots dx_k,$$

where $C_\Gamma > 0$ is a computable constant that depends on Γ only.

The constant C_Γ in the theorem above is computable, but it's also very easy to get it wrong. For a formula for it, we refer to [Wri20, Theorem 4.1]. Here, we just note that if $\Gamma = (\gamma_1, \dots, \gamma_k)$ is such that $\Sigma_g - (\gamma_1 \cup \cdots \cup \gamma_k)$ is connected (this determines a unique $\mathrm{MCG}(\Sigma_g)$-orbit), then $C_\Gamma = 2^{-k}$.

Mirzakhani used this formula, combined with the beautiful idea that the McShane–Mirzakhani identity turns constant functions into geometric functions, to derive a recurrence for Weil–Petersson volumes of moduli spaces of surfaces with boundary. These recurrences are however quite complicated, so it is a lot work to extract good asymptotic bounds on Weil–Petersson volumes of moduli spaces of surfaces of large genus (that are crucial for applications to random surfaces) from it. Nonetheless, Mirzakhani and Zograf [MZ15] proved that, for any $n, K \geqslant 0$ fixed

$$\mathrm{vol}_{\mathrm{WP}}(\mathcal{M}_{g,n}) = c_{\mathrm{WP}} \cdot \frac{(2g-3+n)!(4\pi^2)^{2g-3+n}}{\sqrt{g}} \left(1 + \sum_{k=1}^{K} \frac{a_{n,k}}{g^k} + O\left(\frac{1}{g^{K+1}} \right) \right) \quad \text{as } g \to \infty,$$

where $c_{\mathrm{WP}} > 0$ is a universal constant and the constants $a_{n,k}$ are effectively computable.

Based on numerical data, Zograf [Zog20] conjectured:

Conjecture (Zograf)

The multiplicative constant in the asymptotic expansion for Weil–Petersson volumes satisfies:

$$c_{\mathrm{WP}} = \frac{1}{\sqrt{\pi}}.$$

The current best known asymptotic expansions on the ratios

$$\mathrm{vol}_{\mathrm{WP}}(\mathcal{M}_{g,n}(L_1, \ldots, L_n))/\,\mathrm{vol}_{\mathrm{WP}}(\mathcal{M}_{g,n})$$

are due to Anantharaman–Monk [AM22] and improve on estimates by Mirzakhani [MP19].

The length spectrum. The proof of the Poisson approximation theorem for the length spectrum (Theorem 5.3) uses the method of moments. First of all, we will write $N_{[a,b]} : \mathcal{M}_g \to \mathbb{N}$ for the random variable that counts the number of closed geodesics whose length lies in $[a, b]$. The goal is then to estimate the factorial moments

$$\mathbb{E}\left(\prod_{i=1}^{k}\Big(N_{[a_i,b_i]}(X_g)\Big)_{m_i}\right) = \frac{1}{\mathrm{vol}_{\mathrm{WP}}(\mathcal{M}_g)}\int_{\mathcal{M}_g}\prod_{i=1}^{k}\Big(N_{[a_i,b_i]}(X)\Big)_{m_i}\,d\,\mathrm{vol}_{\mathrm{WP}}(X),$$

where

$$\Big(N_{[a,b]}(X)\Big)_m := N_{[a,b]}(X)\cdot\Big(N_{[a,b]}(X)-1\Big)\cdots\Big(N_{[a,b]}(X)-m+1\Big).$$

In particular, we need to show that, for any collection of disjoint intervals $[a_1, b_1], \ldots, [a_k, b_k]$ and any $m_1, \ldots, m_k \in \mathbb{N}$,

$$\mathbb{E}\left(\prod_{i=1}^{k}\Big(N_{[a_i,b_i]}(X_g)\Big)_{m_i}\right) \longrightarrow \prod_{i=1}^{k}\lambda_{[a_i,b_i]}^{m_i},$$

as $g \to \infty$.

To do so, we first observe that the random variable $\prod_{i=1}^{k}(N_{[a_i,b_i]})_{m_i} : \mathcal{M}_g \to \mathbb{N}$ counts ordered tuples of geodesics: the number of tuples of $m_1 + \ldots + m_k$ geodesics such that the lengths of m_1 of these geodesics lie in $[a_1, b_1]$, the lengths of m_2 of these geodesics lie in $[a_2, b_2]$, et cetera.

Next, we split this counting function into two counting functions. The first of these counts only tuples of simple geodesics that pairwise don't intersect each other and moreover

leave the surface connected once removed. The second function counts the remaining tuples.

The first function is a geometric function, so we can develop its expectation using Theorem 5.5. The resulting expression can then be estimated using the earlier mentioned asymptotic results on (ratios of) Weil–Petersson volumes. The resulting limit is exactly the expression above.

The second part of the proof is to show that the expectation of the function that counts tuples that do intersect or separate is negligible when $g \to \infty$. To do that, we first argue that short tuples with intersections give rise to small subsurfaces with a short boundary. Indeed, if we take a regular neighborhood of such a tuple, then we obtain a subsurface with boundary, the length of which is comparable to the length of the curve. It then remains to prove that it's rare that a surface has such small subsurfaces with a short boundary. This again uses the asymptotic results mentioned above.

Other results on the geometry and spectrum of X_g. During the last couple of years there has been an explosion of work on random surfaces of large genus distributed according to the Weil–Petersson measure.

The geometry of closed geodesics on these surfaces has been further investigated in [NWX23, WX22a, HSWX23, MT22]. For example, in [WX22a], Wu and Xue showed that the phenomenon that "most short geodesics are simple" that we alluded to in the previous section actually persists for geodesics on X_g of length up to $o(\sqrt{g})$. This confirmed a conjecture by Lipnowski–Wright [LW23]. They also showed that geodesics whose length is significantly more than $\sqrt{g}$ tend not to be simple. More recently, in [DS23], Dozier–Sapir showed that for this last bit, a uniform spectral gap and injectivity alone are enough and randomness is not necessary.

Wu–Xue [WX22b] and Lipnowski–Wright [LW23] independently showed that for any $\varepsilon > 0$, as $g \to \infty$, $\lambda_1(X_g) > \frac{3}{16} - \varepsilon$. Hide [Hid22] showed explicit lower bounds for random surfaces with $o(\sqrt{g})$ cusps. More recently, Anantharaman–Monk [AM23] proved that in fact $\lambda_1(X_g) > \frac{2}{9} - \varepsilon$ with high probability as $g \to \infty$. Moreover, they have announced that in future work, they will be able to push this up to $\frac{1}{4} - \varepsilon$. Further spectral statistics were determined in [Mon22, Nau23, Rud23, RW23] and delocalization results on the eigenfunctions of the Laplacian on X_g were proved in [LMS17, Tho21, GLMST21].

Finally, we note that the geometry and spectrum of random surfaces of bounded genus with many cusps is wildly different. We refer to [SW23, HT22, Bud22, BZ23] for more

information.

5.2 Random 3-manifolds

The first two models of random surfaces discussed above – random Belyĭ surfaces and random finite degree covers – have natural analogues in dimension 3. Because of Mostow rigidity, the model based on the Weil–Petersson measure does not.

However, both models we get are currently intractable. The analogue to random Belyĭ surfaces would be 3-manifolds obtained by randomly gluing a finite number of tetrahedra together along their faces. The problem is that the probability that the resulting complex is a manifold tends to zero as the number of tetrahedra grows [DT06, Proposition 2.8]. One could of course condition on the complex being a manifold. In any event, it's hard to avoid having to count the number of triangulated manifolds and the best bounds we have on these numbers are not precise enough for our purposes (see for instance [CP21]). The analogue of the random cover model: taking a random finite degree cover of a fixed hyperbolic 3-manifold of finite volume has the same issue, namely that we currently don't have good bounds on the subgroup growth of the fundamental groups of these manifolds (see Section 4.1).

In this section, we will discuss three alternative models. First we will describe Dunfield and Thurston's model of random Heegaard splittings. We will in particular describe our results on homological torsion with Hyungryul Baik, David Bauer, Ilya Gekhtman, Ursula Hamenstädt, Sebastian Hensel, Thorben Kastenholz and Daniel Valenzuela [BBG+18]. After this, we will describe a class of non-hyperbolic 3-manifolds for which we do have sufficiently sharp bounds on subgroup growth to carry out the study of random finite degree covers, namely torus knot complements. This is joint work with Elizabeth Baker [BP23a]. The third and final model we'll discuss is one that we studied with Jean Raimbault [PR22] and gives rise to random hyperbolic manifolds with boundary.

5.2.1 Random Heegaard splittings

Defintions. Dunfield and Thurston's solution to the problem above was to consider random Heegaard splittings. These are defined as follows. A (3-dimensional) **handle body** is a 3-manifold with boundary obtained by gluing a finite number of 1-handles (copies of $[0,1] \times \mathbb{D}$, where $\mathbb{D}$ denotes the 1-dimensional disk) to a 3-ball $\mathbb{B}$ (along $\{0\} \times \mathbb{D}$ and

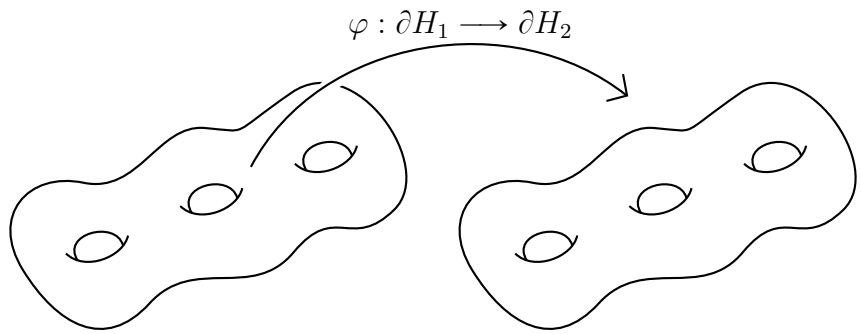

Figure 5.2: A Heegaard splitting

$\{1\} \times \mathbb{D}$). Alternatively, these can be thought of as the closed regular neighborhood of a finite graph in $\mathbb{R}^3$. Given a handle body H, its boundary ∂H is a closed surface, the **genus** of H is the genus of ∂H.

If H_1 and H_2 are handle bodies and $\varphi : \partial H_1 \to \partial H_2$ is an orientation reversing diffeomorphism, then

$$M_\varphi = (H_1 \sqcup H_2) \,/\, (x \in \partial H_1 \sim \varphi(x) \in \partial H_2)$$

is a closed orientable 3-manifold (see Figure 5.2 for a schematic picture). The decomposition of M_φ is what is called a **Heegaard splitting**. The genus of the Heegaard splitting is the genus of the handle bodies involved.

Moise proved that every closed 3-manifold can be triangulated [Moi52]. It follows from this that every closed 3-manifold admits a Heegaard splitting. In fact, it admits infinitely many non-isomorphic Heegaard splittings. The minimal genus among these splitting is called the **Heegaard genus** of the manifold.

An important observation for us is that if $\varphi_1 : \partial H_1 \to \partial H_2$ and $\varphi_2 : \partial H_1 \to \partial H_2$ are isotopic, then M_{φ_1} and M_{φ_2} are diffeomorphic. In other words, M_φ is determined by the image of φ in the mapping class group[5] $\mathrm{MCG}(\partial H)$. The mapping class group $\mathrm{MCG}(\Sigma_g)$ is a finitely generated group, so we obtain a notion of random manifolds by performing a random walk on it. That is, we fix some probability measure μ on $\mathrm{MCG}(\Sigma_g)$ of finite support[6] and let φ_n denote the random mapping class distributed according to the n-fold convolution μ^{*n}. We will assume that the support of μ generates a non-elementary

[5]Technically we've identified two sets of mapping classes here: φ was supposed to be orientation reversing rather than orientation preserving and go between two distinct (but diffeomorphic) surfaces.

[6]This condition is stricter than typically necessary, usually a finite first moment type condition suffices. However not to get into unnecessary technical difficulties, we'll stick to finite support.

subgroup of $\mathrm{MCG}(\Sigma_g)$. The associated **random Heegaard splitting** is the manifold M_{φ_n}.

On a side note, we mention there is a second way to construct a manifold out of φ_n, namely we can consider the associated mapping torus. This model has also been studied and it turns out that geometrically the resulting manifolds are very similar to random Heegaard splittings (see for instance [Via21]).

Geometry and topology. Dunfield and Thurston introduced random Heegaard splittings in [DT06] to study the virtual Haken conjecture (which at that point had not yet been proved). They ended up proving a negative result: among abelian covers of a random Heegaard splitting M_{φ_n} of genus 2, Haken covers are rare. They also conjectured that M_{φ_n} should be hyperbolic with high probability as $n \to \infty$, which was later proved by Maher [Mah10]. By Mostow rigidity, the resulting metric is determined by the topology (so φ_n) alone, which allows one to ask what its geometry is like.

Viaggi proved that the volume of M_{φ_n} satisfies a law of large numbers: it grows linearly in n, at a rate determined uniquely by the measure μ driving the random walk. Feller–Sisto–Viaggi [FSV22] proved bounds on their injectivity radius and diameter and found a new proof of the fact that they're hyperbolic that avoids the use of the Perelman's hyperbolization theorem. Finally, Hamenstädt–Viaggi [HV22] proved that the first non-zero eigenvalue of their Laplacian decays quadratically in their volume (see [BGH20] for the case of random mapping tori). In particular, random Heegaard splittings have very different spectral properties from random graphs and random surfaces.

The drawback of both random Heegaard splittings and random mapping tori is that in the end, we're not sampling from all manifolds after all. We once and for all fix the genus of the handle body and the randomness only comes from the mapping class. In particular, we fix an a priori upper bound on the Heegaard genus of the manifolds we consider. This in turn also implies a global upper bound on the injectivity radius of our manifolds [Whi02], i.e. even at uniformly bounded scales, the geometry of M_{φ_n} is different from $\mathbb{H}^3$ at every point. This for instance means that, unlike random regular graphs and all the models of random surfaces we have discussed above, the manifolds M_{φ_n} do not Benjamini–Schramm converge to $\mathbb{H}^3$ (see [ABB+17] for definitions).

Homological torsion. Nonetheless, random Heegaard splittings can be used for different purposes. They for instance form a good testing ground for conjectures that are still out of reach for all manifolds.

Together with Hyungryul Baik, David Bauer, Ilya Gekhtman, Ursula Hamenstädt, Sebastian Hensel, Thorben Kastenholz and Daniel Valenzuela [BBG+18], we studied the growth of torsion in homology in towers of covers of random 3-manifolds. Before we state our results, we start with some context.

Given a closed 3-manifold M, let $H_1(M;\mathbb{Z})_{\text{tors}}$ denote the torsion subgroup of the integral first homology group $H_1(M;\mathbb{Z})$. Lück conjectured the following[7] (see for instance [Lüc13, Conjecture 1.12(2)] and also [BV13, Conjecture 1.3] for a restricted version):

Conjecture (Lück)

If $M = \Gamma\backslash\mathbb{H}^3$ is a closed hyperbolic 3-manifold and $(\Gamma_i \lhd \Gamma)_{i=1}^{\infty}$ is a sequence of finite index normal subgroups such that

$$\bigcap_{i=1}^{\infty} \Gamma_i = \{e\},$$

then

$$\lim_{i\to\infty} \frac{\log\left(|H_1(\Gamma_i\backslash\mathbb{H}^3;\mathbb{Z})_{\text{tors}}|\right)}{[\Gamma:\Gamma_i]} = \frac{\text{vol}(M)}{6\pi}.$$

The reason that the constant $\frac{1}{6\pi}$ shows up in this conjecture relates to ℓ^2-torsion of hyperbolic manifolds. The idea is that the normalized homological torsion of the covers should approximate the ℓ^2-torsion of the manifold, in a similar way to how normalized Betti numbers $b_k(\Gamma_i\backslash\mathbb{H}^3;\mathbb{Z})$ / $[\Gamma:\Gamma_i]$ approximate the ℓ^2-Betti numbers of M, by Lück's approximation theorem [Lüc94]. It's also known that [Kam18], if true, the conjecture above implies the conjecture we briefly discussed in Section 4.1: the volume of a hyperbolic 3-manifold is a profinite invariant.

Lê [Lê18] proved that the limit in the conjecture is always bounded from above by $\text{vol}(M)/6\pi$. On the other hand, currently not a single example of a hyperbolic 3-manifold and a sequence of normal subgroups with trivial intersection that exhibits exponential torsion growth (let alone at the predicted rate) is known. The best evidence available is a result by Sun [Sun15] proving that, given a closed hyperbolic manifold M, we can make any finite abelian group appear as a summand in $H_1(\widehat{M};\mathbb{Z})$ for some regular cover $\widehat{M} \to M$ of finite degree.

On the other hand, if we relax the conditions, much more is known. First of all,

[7] Lück's conjecture is stated in a much more general setting, but we will stick to hyperbolic 3-manifolds here.

Liu [Liu19] proved that every closed hyperbolic 3-manifold $M = \Gamma\backslash\mathbb{H}^3$ admits a nested sequence of subgroups $(\Gamma_i < \Gamma)_{i=1}^{\infty}$ with trivial intersection such that the torsion in their first homology grows exponentially. However, these subgroups are not normal and the corresponding covers do not Benjamini–Schramm converge to $\mathbb{H}^3$. In particular, there is no reason to expect that the exponential growth rate equals the one in the conjecture.

The first examples of exponential torsion growth in general were given by sequences of cyclic (or more generally, abelian) covers. So, this comes down to letting go of the condition that the intersection of the subgroups is trivial. In this case, the exponential growth rate of the torsion in the homology of the covers is determined by the Mahler measure of the Alexander polynomial, namely the torsion in the sequence grows exponentially if and only if the multiplicative Mahler measure of the Alexander polynomial is different from 1 [Ril90, GAnS91, SW02b, SW02a, Rai12, Le14]. Note that in order for a manifold to have an infinite sequence of abelian covers, its first Betti number needs to be positive.

Our paper [BBG+18] studies exponential torsion growth for random Heegaard splittings. We study two problems: the growth of torsion in homology as a function of the step length of the random walk and the existence of a sequence of cyclic covers with exponential torsion growth. We prove:

Theorem 5.6. *(a) Suppose the support of the measure μ generates $\mathrm{MCG}(\Sigma_g)$, then there exists a constant $\alpha_\mu > 0$ such that*

$$\frac{\log\left(|H_1(M_{\varphi_n};\mathbb{Z})_{\mathrm{tors}}|\right)}{n} \longrightarrow \alpha_\mu$$

in probability as $n \to \infty$.

(b) Suppose the support of the measure μ generates $\mathcal{I}(\Sigma_g)$ then M_{φ_n} admits a sequence of cyclic covers with exponential torsion growth with high probability as $n \to \infty$.

In item (b) above, the group $\mathcal{I}(\Sigma_g)$ is the **Torelli subgroup** of $\mathrm{MCG}(\Sigma_g)$, defined by

$$\mathcal{I}(\Sigma_g) \coloneqq \{\varphi \in \mathrm{MCG}(\Sigma_g);\ \varphi_* = \mathrm{Id} : H_1(\Sigma_g;\mathbb{Z}) \to H_1(\Sigma_g;\mathbb{Z})\}\,,$$

the subgroup of mapping classes that act trivially on homology. Randomly walking on this group instead of the full mapping class group ensures that the first Betti number of M_{φ_n} is positive, in fact, a Mayer–Vietoris argument shows that it's at least g. On the other hand, if we randomly walk on the full mapping class group, the probability that the first Betti number is positive is exponentially small [Kow08, Proposition 7.19]. Item (a)

answers a question by Kowalski, who had already proved that homological torsion grows super polynomially fast [Kow08, Section 7.7].

The main idea behind our proofs is to translate these questions into questions about random walks on linear groups. For item (a), we first observe that, by a Mayer–Vietoris argument

$$H_1(M_\varphi;\mathbb{Z}) = H_1(\Sigma_g;\mathbb{Z}) \,/\, \langle L, \varphi_* L\rangle, \tag{5.2.1}$$

where $\varphi_* : H_1(\Sigma_g;\mathbb{Z}) \to H_1(\Sigma_g;\mathbb{Z})$ is the map φ induces on homology and L is a Lagrangian in $H_1(\Sigma_g;\mathbb{Z})$ given by

$$L = \ker\Big(i_* : H_1(\Sigma_g;\mathbb{Z}) \to H_1(H;\mathbb{Z})\Big),$$

where $i : \Sigma_g \to H$ is identification of Σ_g with the boundary ∂H of our handle body H.

The action homology gives rise to a linear representation to a symplectic group

$$\mathrm{MCG}(\Sigma_g) \longrightarrow \mathrm{Sp}(2g,\mathbb{Z}).$$

So our random walk descends to a random walk $((\varphi_n)_*)_n$ on $\mathrm{Sp}(2g,\mathbb{Z})$. Looking at (5.2.1), asking for a positive first Betti number is asking that the vector spaces $(\varphi_n)_*(L\otimes\mathbb{R})$ and $L\otimes\mathbb{R}$ intersect non-trivially, which turns out to be non-generic. The torsion in $H_1(M_{\varphi_n};\mathbb{Z})$ can be computed as the determinant of a block in the matrix $(\varphi_n)_*$, which grows exponentially. Both of these are results from random walks on linear groups [GM89, BQ14, BQ16].

To prove item (b), in which case the symplectic representation doesn't "see" anything, we use work of Looijenga [Loo97]. Looijenga shows that a linear representations of the mapping class group on the homology of abelian covers do not annihilate the Torelli group. To control torsion growth, we use the earlier mentioned results that relate this to the Mahler measure of the Alexander polynomial.

5.2.2 Random covers of torus knot complements

We have already mentioned before that very little is known about the subgroup growth of lattices in $\mathrm{PSL}(2,\mathbb{C})$. However, there are (non-hyperbolic) 3-manifolds for which something can be said, namely Seifert fibered manifolds. The subgroup growth of orientable circle bundles over surfaces was determined by Liskovets and Mednykh [LM00] and the subgroup growth of Euclidean manifolds can be derived from general results on the subgroup growth of virtually abelian groups [dSMS99, Sul16].

Together with Elizabeth Baker [BP23a], We studied groups of the form

$$\Gamma_{p_1,\dots,p_m} = \langle x_1, \dots x_m |\ x_1^{p_1} = x_2^{p_2} = \cdots = x_m^{p_m} \rangle.$$

When $\gcd(p,q) = 1$ and $p, q \geqslant 2$ then $\Gamma_{p,q}$ is the fundamental group of a (p,q)-torus knot complement. More generally, $\Gamma_{p_1,\dots,p_m}$ is a central extension of the form

$$1 \longrightarrow \mathbb{Z} \longrightarrow \Gamma_{p_1,\dots,p_m} \overset{\Phi_{p_1,\dots,p_m}}{\longrightarrow} C_{p_1} * \cdots * C_{p_m} \longrightarrow 1, \tag{5.2.2}$$

where C_p denotes the finite cyclic group of order p and $\Phi_{p_1,\dots,p_m}$ is the map that sends the generator x_j to a generator of C_{p_j}.

We will consider the case in which

$$-m + 1 + \sum_{i=1}^{m} \frac{1}{p_i} < 0.$$

This includes all torus knot groups. In this setting $\Gamma_{p_1,\dots,p_m}$ appears as a non-uniform lattice in $\mathrm{PSL}(2,\mathbb{R}) \times \mathbb{R}$ (see for instance [Eck04, Proposition 7.2]). $\Phi_{p_1,\dots,p_m}(\Gamma_{p_1,\dots,p_m})$ is then the projection onto $\mathrm{PSL}(2,\mathbb{R})$ of $\Gamma_{p_1,\dots,p_m}$ and is a Fuchsian group that acts on the hyperbolic plane $\mathbb{H}^2$. The displayed expression above is the orbifold Euler characteristic of $\Phi_{p_1,\dots,p_m}(\Gamma_{p_1,\dots,p_m})\backslash\mathbb{H}^2$.

Subgroup growth. Recall that $s_n(\Gamma_{p_1,\dots,p_m})$ denotes the number of subgroups of index n in $\Gamma_{p_1,\dots,p_m}$. Because $\Gamma_{p_1,\dots,p_m}$ is a central extension of a free product of finite groups by $\mathbb{Z}$, a standard argument, combined with results due to Müller [Mül96], Volynets [Vol86] and Wilf [Wil86] on the subgroup growth on free products of finite groups proves the following:

Theorem 5.7. *Let $p_1, \dots, p_m \in \mathbb{N}_{>1}$ such that $\sum_{j=1}^{m} \frac{1}{p_j} < m - 1$. Then it holds that*

$$s_n(\Gamma_{p_1,\dots,p_m}) \sim A_{p_1,\dots,p_m} \cdot n^{-1/2} \cdot \exp\left(\sum_{i=1}^{m} \sum_{\substack{0<j<p_i \\ s.t.\ j|p_i}} \frac{n^{j/p_i}}{j} \right) \cdot \left(\frac{n}{e}\right)^{n\cdot\left(m-1-\sum_{i=1}^{m}\frac{1}{p_i}\right)}$$

as $n \to \infty$, where

$$A_{p_1\dots p_m} = \sqrt{2\pi} \exp\left(- \sum_{i:\ p_i\ even} \frac{1}{2p_i} \right) \prod_{i=1}^{m} p_i^{-1/2}.$$

Random subgroups and covers. The asymptotic equivalent in Theorem 5.7 allows us to study a uniformly random subgroup

$$H_n < \Gamma_{p_1,\dots,p_m}.$$

This is an example of an Invariant Random Subgroup (IRS) – i.e. a conjugation invariant Borel measure on the Chabauty space of subgroups of $\Gamma_{p_1,\dots,p_m}$ (see Section 4.1.1).

In order to make topological statements, we will also fix a classifying space $X_{p_1,\dots,p_m}$ for $\Gamma_{p_1,\dots,p_m}$. For instance, if $p, q \geqslant 2$ and $\gcd(p,q) = 1$ we can take the corresponding torus knot complement. More generally, since $\Gamma_{p_1,\dots,p_m}$ appears as a torsion-free lattice in $\mathrm{PSL}(2,\mathbb{R}) \times \mathbb{R}$, we may take the manifold $\Gamma\backslash(\mathbb{H}^2 \times \mathbb{R})$. H_n gives rise to a random degree n cover of $X_{p_1,\dots,p_m}$.

First of all, we determined the limit of the random subgroup H_n:

Theorem 5.8. *As $n \to \infty$, H_n converges to*

$$L_{p_1,\dots,p_m} := \ker(\Phi_{p_1,\dots,p_m}) \simeq \mathbb{Z}.$$

as an IRS.

For a (p,q) torus knot, $L_{p,q}$ is the subgroup generated by the longitude. Using results by Elek [Ele10, Lemma 6.1] and Lück [Lüc94], this implies the following corollary on the growth of the Betti numbers of these random subgroups:

Corollary 5.9. *We have that*

$$\lim_{n\to\infty} \frac{b_k(H_n;\mathbb{R})}{n} = \begin{cases} m - 1 - \sum_{i=1}^{m} \frac{1}{p_i} & \text{if } k = 1,2 \\ 0 & \text{otherwise.} \end{cases}$$

in probability.

Finally, we studied the problem of curve counting. To this end, given a conjugacy class $K \subset \Gamma_{p_1,\dots,p_m}$, we will write

$$Z_K(H_n) = \left|\{\gamma H_n \in \Gamma_{p_1,\dots,p_m}/H_n;\ g \cdot \gamma H_n = \gamma H_n\}\right|,$$

where $g \in K$ is any element. In topological terms, K corresponds to a free homotopy class of loops in $X_{p_1,\dots,p_m}$ and $Z_K(H_n)$ is the number of closed lifts of that loop to the cover of

$X_{p_1,\dots,p_m}$ corresponding to H_n. We obtained the following Poisson approximation for these random variables:

Theorem 5.10. *Let $p_1,\dots,p_m \in \mathbb{N}_{>1}$ be such that $\sum_{j=1}^m \frac{1}{p_j} < m-1$. If for all $g \in K_i$, for all $i = 1,\dots,r$, the image $\Phi_{p_1,\dots,p_m}(g)$ is either trivial or of infinite order, then, as $n \to \infty$, the random variables $Z_{K_i}(H_n)$, $i = 1,\dots,r$ are asymptotically independent. Moreover,*

- *if $K_i \subset L_{p_1,\dots,p_m}$ then*
$$\lim_{n\to\infty} \mathbb{P}\left(Z_{K_i}(H_n) = n\right) = 1$$
- *and if $K_i \not\subset L_{p_1,\dots,p_m}$ is the conjugacy class of the kth power of a primitive element g_0 then $Z_{K_i}(H_n)$ converges in distribution to a random variable $Z^\infty_{K_i}$.*
  - *If $\Phi_{p_1,\dots,p_m}(g_0)$ is not a product of two elements of order two,*
$$Z^\infty_{K_i} \sim \sum_{d|k} d \cdot X_{1/d},$$
where $X_{1/d} \sim \mathrm{Poisson}(1/d)$ and $X_1,\dots,X_{1/k}$ are independent.
  - *If $\Phi_{p_1,\dots,p_m}(g_0)$ is a product of two elements of order two,*
$$Z^\infty_{K_i} \sim \sum_{d|k} 2d \cdot X^{d,1}_{1/2d} + \sum_{d|k,\ d\ even} d \cdot X^{d,2}_{1/2} + d \cdot X^{d,3}_{1/2} + \sum_{d|k,\ d\ odd} d \cdot X^{d,4}_1$$
where $X^{d,1}_{1/2d} \sim \mathrm{Poisson}(1/2d)$, $X^{d,2}_{1/2}, X^{d,3}_{1/2} \sim \mathrm{Poisson}(1/2)$, $X^{d,4}_1 \sim \mathrm{Poisson}(1)$ and all these variables are independent.

The fact that products of two involutions play a special role in this theorem was first observed by Doron Puder and Tomer Zimhoni, who noticed an error in a previous version of our paper. They also generalized some of the Poisson approximations we proved on the way to the one stated above [PZ22].

For the conjugacy classes that were left out of the previous statement –those that project to finite order elements in the free product of cyclic groups– we proved:

Theorem 5.11. *Let $p_1,\dots,p_m \in \mathbb{N}_{>1}$ be such that $\sum_{j=1}^m \frac{1}{p_j} < m-1$. If the images of the elements of K_i under $\Phi_{p_1,\dots,p_m}$ have order $k_i \in \mathbb{N}$ for $i = 1,\dots,r$, then the vector of random variables*
$$\left(\frac{Z_{K_1}(H_n) - n^{1/k_1} - \varepsilon_1 \cdot n^{1/2k_1}}{\sqrt{p_{j_1}/k_1} \cdot n^{1/2k_1}}, \dots, \frac{Z_{K_r}(H_n) - n^{1/k_r} - \varepsilon_r \cdot n^{1/2k_r}}{\sqrt{p_{j_r}/k_r} \cdot n^{1/2k_r}}\right)$$

converges in distribution to a $\mathcal{N}(0,1)^{\otimes r}$*-distributed random variable as* $n \to \infty$*. Here* $p_{j_i} \in \mathbb{N}$ *is such that* $\Phi_{p_1,\dots,p_m}(K_i)$ *is the conjugacy class of* $x_{j_i}^{l_i}$*, for* $i = 1, \dots, r$*. Finally* ε_i *equals* 1 *if* p_{j_i}/k_i *is even and* 0 *otherwise.*

The idea behind the proofs of our results on random subgroups is to first prove the analogous results for random index n subgroups of $C_{p_1} * \cdots * C_{p_m}$ and then use the fact that most index n subgroups of $\Gamma_{p_1,\dots,p_m}$ come from index n subgroups of $C_{p_1} * \cdots * C_{p_m}$ to upgrade these into results about $\Gamma_{p_1,\dots,p_m}$.

First, we consider the problem of counting the number of fixed points of an element $g \in C_{p_1} * \cdots * C_{p_m}$ under a random homomorphism $C_{p_1} * \cdots * C_{p_m} \to \mathfrak{S}_n$. There are two cases to consider:

- Müller–Schlage-Puchta [MSP04, MSP10] proved the central limit theorem we need for finite order elements.
- If g is of infinite order, its number of fixed points can be approximated by a sum of multiples of Poisson-distributed random variables. Like in Section 5.1.3, we prove this by estimating the factorial moments of the random variables that count the fixed points of g.

To turn these results into statements about $\Gamma_{p_1,\dots,p_m}$ instead of $C_{p_1} * \cdots * C_{p_m}$, we use the fact that asymptotically most actions factor through the projection $\Phi_{p_1,\dots,p_m}$ (this is essentially Theorem 5.7). To promote them to statements on subgroups, we use Lemma 4.1.

Finally, the fact that a conjugacy class $K \subset \Gamma_{p_1,\dots,p_m}$ typically has very few lifts to H_n if it does not lie in $L_{p_1,\dots,p_m}$ and typically has n lifts if it does (this is essentially the content of Theorems 5.10 and 5.11), implies that H_n converges to $L_{p_1,\dots,p_m}$ (Theorem 5.8).

5.2.3 Random 3-manifolds with boundary

The final model of random 3-manifolds we will discuss is a modification of the model of random gluings of tetrahedra. We noted above that the probability that a gluing of n tetrahedra along their faces is a manifold tends to 0 as $n \to \infty$. However, the only points in the resulting complex that do not have neighborhoods that are homeomophic to an open set in $\mathbb{R}^3$ are the vertices in the complex.

As such, one obtains a random manifold with boundary by truncating the tetrahedra at their vertices and gluing the resulting polytopes (see Figure 5.3) at random along their

hexagonal faces.

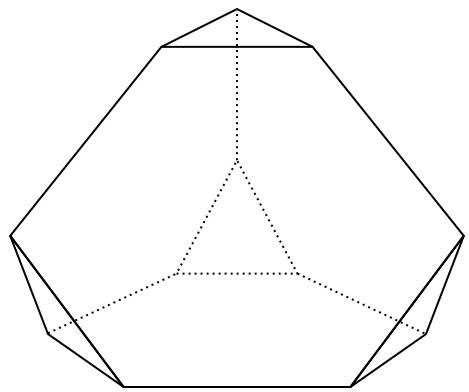

Figure 5.3: A truncated tetrahedron. M_n is built by randomly gluing n copies of this polyhedron together along their hexagonal faces.

It can be derived from Moise's theorem [Moi52] that all compact 3-manifolds with boundary admit a decomposition into a finite number of copies of this polytope. The question of studying this model has been evoked before (see for instance [DHM15, Question 6.2]) but we aren't aware of any prior work on it.

Together with Jean Raimbault [PR22], we investigated the geometry and topology of these random manifolds. Formally, we let M_n denote a random gluing of n truncated tetrahedra along their hexagonal faces, obtained by using the configuration model to pair the faces (conditioned on not having loops and multiple edges in the dual graph) and then by gluing each pair of matched faces with one of the three possible orientation reversing simplicial gluings.

Results. The topological properties of the random manifolds we obtain are as follows:

Theorem 5.12 (Topology)**.** *(a) We have*

$$\lim_{n\to\infty} \mathbb{P}\Big(M_n \textit{ is connected and has a single boundary component}\Big) = 1$$

(b) The genus $g(\partial M_n)$ of the boundary of M_n satisfies

$$\lim_{n\to+\infty} \mathbb{P}\Big(n - \theta(n) \leqslant g(\partial M_n) \leqslant n+1\Big) = 1,$$

for any function $\theta : \mathbb{N} \to \mathbb{R}$ that grows super-logarithmically[8]*.*

[8] By this we mean that $\lim_{n\to\infty} \frac{\theta(n)}{\log(n)} = +\infty$.

(c) *Let $\mathcal{D}\, M_n$ denotes the double of M_n along its boundary and $g(\mathcal{D}\, M_n)$ its Heegaard genus. Then*

$$\lim_{n\to+\infty} \mathbb{P}\Big(n - \theta(n) \leqslant g(\mathcal{D}\, M_n) \leqslant n + \theta(n)\Big) = 1,$$

for any function $\theta : \mathbb{N} \to \mathbb{R}$ that grows super-logarithmically.

(d) *There exists C such that the Betti numbers $b_1(M_n)$ and $b_1(M_n, \partial M_n)$ satisfy*

$$\lim_{n\to+\infty} \mathbb{P}\Big(b_1(M_n, \partial M_n) \leqslant \theta(n)\Big) = 1, \ \lim_{n\to+\infty} \mathbb{P}\Big(\ |b_1(M_n) - n| \leqslant \theta(n)\Big) = 1$$

for any function $\theta : \mathbb{N} \to \mathbb{R}$ that grows super-logarithmically.

Our main reason for studying this model is that we hoped that it would give rise to hyperbolic manifolds, which turns out to be the case. Note that it follows from Mostow rigidity that if M_n carries a hyperbolic metric with totally geodesic boundary, then this metric is unique up to isometry. As such, one can also ask for the geometric properties of this metric. We prove:

Theorem 5.13 (Geometry)**.** *We have*

$$\lim_{n\to+\infty} \mathbb{P}\Big(M_n \text{ carries a hyperbolic metric with totally geodesic boundary}\Big) = 1.$$

This metric has the following properties:

(a) *The hyperbolic volume* $\mathrm{vol}(M_n)$ *of M_n satisfies:*

$$\mathrm{vol}(M_n) \sim n \cdot v_O \quad \text{as } n \to \infty$$

in probability. Here v_O denotes the volume of the regular right-angled ideal hyperbolic octahedron.

(b) *There exists a constant $c_\lambda > 0$ so that the first discrete Laplacian eigenvalue $\lambda_1(M_n)$ of M_n satisfies*

$$\lim_{n\to+\infty} \mathbb{P}\Big(\lambda_1(M_n) > c_\lambda\Big) = 1.$$

(c) *There exists a constant $c_d > 0$ such that the diameter* $\mathrm{diam}(M_n)$ *of M_n satisfies:*

$$\lim_{n\to+\infty} \mathbb{P}\Big(\ \mathrm{diam}(M_n) < c_d \log(\mathrm{vol}(M_n))\Big) = 1$$

(d) There exists a constant $c_s > 0$ such that the systole $\mathrm{sys}(M_n)$ *of M_n satisfies:*

$$\lim_{n\to+\infty} \mathbb{P}\Big(\mathrm{sys}(M_n) > c_s\Big) = 1$$

(e) For every $\varepsilon > 0$,

$$\lim_{n\to+\infty} \mathbb{P}\left(\frac{1-\varepsilon}{4n} < \mathrm{sys}(\mathcal{D}\, M_n) < \frac{1}{n^{1-\varepsilon}}\right) = 1.$$

The same holds for the minimal length among arcs in M_n that are homotopically non-trivial relative to ∂M_n.

Finally, it turns out that the Benjamini–Schramm limit of M_n can be identified with a tree of regular right-angled ideal octahedra (the same polytope we discussed in Section 4.1.4) pointed at a uniform random point (this makes sense since this manifold has a cofinite group of isometries).

Methods. In big lines, our proofs consist of two steps. The goal of the first of these is to understand the combinatorics of the cell decomposition of M_n. Most of this part of the proof is spent on the combinatorics of the interior edges. There are two basic questions: how many interior edges are there, and how many truncated tetrahedra are incident to them. To this end, we let $E(M_n)$ denote the number of interior edges and $E_k(M_n)$ the number of interior edges that have k tetrahedra incident to them. In the latter variable, tetrahedra are counted with multiplicity. That is, if an edge appears multiple times in the boundary of a given tetrahedron, this tetrahedron adds to its "length" each time. As such, we have

$$\sum_{k=1}^{6n} k \cdot E_k(M_n) = 6n.$$

To understand these numbers, we use peeling techniques similar to those in Section 5.1.1. We for instance prove that, uniformly for all $k = o(\sqrt{n})$,

$$\mathbb{E}\Big(E(M_n)\Big) = \frac{1}{2}\log(n) + O(1) \quad \text{and} \quad \mathbb{E}\Big(E_k(M_n)\Big) = \frac{1}{2k}(1+o(1)) \quad \text{as } n \to \infty.$$

These and the other similar bounds we find are sufficient to prove Theorem 5.12.

The bounds we obtain are less sharp than the analogous bounds for random Belyĭ surfaces (see Section 5.1.1). For instance the following question is open:

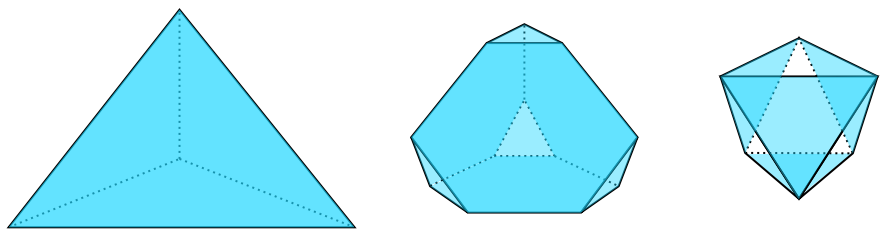

Figure 5.4: Three polytopes: a tetrahedron, a truncated tetrahedron and an octahedron

Question 10

Does the random normalized partition

$$1^{E_{6n}(M_n)}\left(\frac{6n-1}{6n}\right)^{E_{6n-1}(M_n)}\cdots\left(\frac{1}{6n}\right)^{E_1(M_n)}$$

converge to a Poisson–Dirichlet distributed random variable as $n \to \infty$?

The hyperbolization of M_n is based on the observation that if we contract the interior edges of the cell decomposition, and remove the resulting points, we obtain a non-compact manifold X_n that is decomposed into octahedra (see Figure 5.4).

The other way around, M_n can be obtained from X_n by a Dehn filling (with cylinders). The reason that this is useful is that X_n admits a complete hyperbolic metric of finite volume with totally geodesic boundary. Indeed, we can equip the octahedra in the decomposition with the metric of a regular right-angled ideal octahedron and perform the gluings with isometries.

There is a lot of work, starting with Thurston's Dehn filling theorem (see for instance [BP92, Chapter E] or [Mar22, Chapter 15]), on the question of when a Dehn filling of a hyperbolic manifold admits a hyperbolic metric and how close this metric is to the original metric. We use Andreev's theorem [RHD07] and recent work by Futer–Purcell–Schleimer [FPS22] on Dehn fillings. This, combined with our combinatorial bounds, implies the manifolds M_n are hyperbolizable and moreover allows us to estimate their geometric properties.

Further questions and results. More recently, the author's graduate student Anna Roig Sanchis [RS23] proved a similar Poisson approximation theorem to Theorems 5.1 and 5.3 for the primitive length spectrum $\mathcal{L}(M_n)$ of the random 3-manifolds M_n:

Theorem 5.14 (Roig Sanchis)**.** *As $n \to \infty$, $\mathcal{L}(M_n)$ converges locally in distribution to a*

Poisson point process $\mathcal{L}^{3\mathrm{D}}$ *of computable intensity* $\lambda^{3\mathrm{D}}$.

There are still plenty of elementary geometric and topological questions that remain unanswered by the results above. For example:

Question 11

(a) Is $H_1(M_n;\mathbb{Z})_{\mathrm{tors}}$ trivial with high probability as $n \to \infty$? Or a weaker variant: does

$$\frac{\log\left(|H_1(M_n;\mathbb{Z})_{\mathrm{tors}}|\right)}{n} \longrightarrow 0$$

in probability?

(b) Is $b_1(M_n;\partial M_n) = 0$ with high probability as $n \to \infty$?

(c) Do the spectral gaps of M_n and $\mathcal{D}\,M_n$ converge in probability as $n \to \infty$? And if so what are these limits?

6 Extremal problems II

This penultimate chapter will be dedicated to application of the probabilistic method – i.e. using probability theory to attack extremal problems – in hyperbolic geometry.

This is especially effective in the study of connectivity properties of hyperbolic manifolds. For example, during the last three years, a lot of progress has been made on questions on the connectivity of closed hyperbolic surfaces of large area. The three most common ways to measure how connected a hyperbolic surface is, are through its diameter, its Cheeger constant and its spectral gap. Some of what we have learned in the last three years is:

- With Thomas Budzinski and Nicolas Curien, we used a random construction to prove that the minimal possible diameter of a hyperbolic surface of genus g is asymptotic to $\log(g)$ [BCP21b]. Due to the volume entropy of the hyperbolic plane, this is the smallest one can hope for.
- As we mentioned in Section 3.3, using compactifications of random covers of the thrice punctured sphere, Hide and Magee [HM23b] showed that the maximal possible spectral gap of a closed hyperbolic surface of genus g tends to that of the hyperbolic plane (which is $\frac{1}{4}$) as $g \to \infty$, thus resolving a longstanding conjecture (see for instance [BBD88]).
- Contrary to the previous two, it turns out that the maximal possible Cheeger constant of a hyperbolic surface of large area is strictly smaller than that of the hyperbolic plane. With Thomas Budzinski and Nicolas Curien [BCP22], we recently proved that the Cheeger constant of a hyperbolic surface of large area can't be much larger than $\frac{2}{\pi} \approx 0.63\ldots$ (the Cheeger constant of the hyperbolic plane is 1), confirming a conjecture by Lipnowski–Wright [Wri20].

The goal of this chapter is to describe our joint work with Thomas Budzinski and

Nicolas Curien on diameters and Cheeger constants.

6.1 The Bers constant and related problems

But before that, we describe what to the author's best knowledge is the first application of probabilistic methods to extremal problems in hyperbolic geometry: the paper on pants decompositions by Guth–Parlier–Young [GPY11].

Here the **Bers length** of a hyperbolic surface is the minimum over all pants decompositions of that surface of the maximal length among the curves in the pants decomposition. The **Bers constant** B_g in genus g is the maximum over $\mathcal{M}_g$ of the Bers length and plays an important role in hyperbolic geometry of surfaces and 3-manifolds. The value of the Bers constant is known only in genus 2 [Gen11]. Buser [Bus10, Chapter 5] provided examples of surfaces whose Bers length is $\gg \sqrt{g}$ and the current best known upper bound on B_g is due to Parlier [Par23] and is:

$$B_g \leqslant 4\pi(g-1).$$

Earlier linear bounds were proved in [Bus10, Chapter 5] and [Par14]. Buser conjectured his construction is essentially optimal:

Conjecture (Buser)

The Bers constant satisfies

$$B_g = O(\sqrt{g}) \quad \text{as } g \to \infty.$$

Guth, Parlier and Young studied the related problem of **total pants length**. From the earlier mentioned bounds on the Bers constant, it follows that every hyperbolic surface admits a pants decomposition of total length $\leqslant 12\pi(g-1)^2$. It's however not at all clear that this is sharp. Using the probabilistic method, they proved a lower bound. Concretely, they proved that for any $\varepsilon > 0$,

$$\mathbb{P}\left(\begin{array}{c} X \in \mathcal{M}_g \text{ admits a pants decomposition} \\ \text{of total length } \leqslant g^{7/6-\varepsilon} \end{array}\right) \longrightarrow 0 \text{ as } g \to \infty,$$

where the probability is taken with respect to the Weil–Petersson measure (see Section

5.1.3). It in particular follows that

$$\liminf_{g\to\infty} \frac{\log\left(\sup_{X\in\mathcal{M}_g}\left\{\mathrm{MTPL}(X)\right\}\right)}{\log(g)} \in \left[\frac{7}{6}, 2\right],$$

where

$$\mathrm{MTPL}(X) := \min\left\{\sum_i \ell_{\alpha_i}(X);\ (\alpha_i)_i \text{ a pants decomposition of } X\right\}$$

denotes the minimal total pants length of X. The bounds above currently still are the best bounds on the maximal minimal total pants length. Moreover, no explicit constructions of sequences of surfaces whose minimal total pants length grows faster than linearly in their genus is known at the moment.

6.2 The diameter

Together with Thomas Budzinski and Nicolas Curien, we studied at hyperbolic surfaces of small diameter. Let us first prove an elementary lower bound (similar to Lemma 3.1) that works in any dimension:

Lemma 6.1. *Let $d \geqslant 2$. There exists a constant $c_d > 0$ such that*

$$\mathrm{diam}(M) \geqslant \frac{1}{d-1}\log(\mathrm{vol}(M)) - c_d$$

for all closed hyperbolic d-manifolds M.

Proof. Again denote the R-ball around a point $p \in M$ by $B(p,R)$. By definition of the diameter, we have

$$B(p, \mathrm{diam}(M)) = M$$

for any $p \in M$. Now the volume of $B(p, \mathrm{diam}(M))$ is smaller than the volume of a ball of the same radius in $\mathbb{H}^n$. So, using the formula of the volume of a ball in $\mathbb{H}^d$ again, we get

$$\mathrm{vol}(M) = \mathrm{vol}(B(p, \mathrm{diam}(M))) \leqslant \mathrm{vol}(\mathbb{S}^{n-1}) \cdot \int_0^{\mathrm{diam}(M)} \sinh^{n-1}(t)dt,$$

which implies the lemma. □

The only dimension in which a better lower bound is known is dimension 2. Bavard [Bav96] proved that the diameter of a closed hyperbolic surface M of genus g satisfies

$$\operatorname{diam}(M) \geqslant \operatorname{arccosh}\left(\frac{1}{\sqrt{3}\tan\left(\pi/(12g-6)\right)}\right).$$

This improves on the lemma above by an additive constant.

Given the bounds above, we say that manifolds with small diameter are manifolds with logarithmic diameter. All the models of random manifolds from the previous chapter, except random Heegaard splittings, yield manifolds with small diameter. However, except in the case of random Belyĭ surfaces, as we will discuss below, it's not clear how close their diameters come to the known lower bounds.

In [BCP21b] we proved that, in dimension 2, the bound from Lemma 6.1 is asymptotically sharp:

Theorem 6.2. *The minimal diameter among closed hyperbolic surfaces of genus g satisfies*

$$\lim_{g\to\infty} \frac{\min\left\{\operatorname{diam}(X);\ X \in \mathcal{M}_g\right\}}{\log(g)} = 1.$$

6.2.1 A model for random surfaces

The proof of this theorem is based on a model of random surfaces that we haven't mentioned yet. First of all, we will write P_a for the hyperbolic pair of pants all of whose boundary components have length $a > 0$. The random surface $S_{g,a}$ will be the the surface obtained by randomly gluing $2g-2$ copies of P_a together along their boundary components with gluings of twist 0, again using the configuration model (Figure 6.1 shows an example).

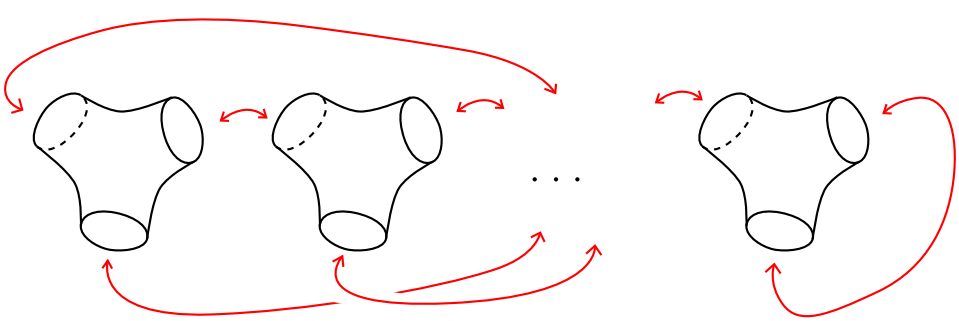

Figure 6.1: A random gluing of pairs of pants.

What we really prove with Thomas Budzinski and Nicolas Curien is that for every $\varepsilon > 0$ there exists a boundary length $a > 0$ such that

$$\lim_{g\to\infty} \mathbb{P}\Big(\operatorname{diam}(S_{g,a}) \leqslant (1+\varepsilon)\log(g)\Big) = 1. \tag{6.2.1}$$

6.2.2 Ingredients of the proof

There are two basic inputs, a geometric one and a probabilistic one.

Geometric input. The main geometric input comes from counting problems in the hyperbolic plane. Given a discrete group $\Gamma < \operatorname{Isom}(\mathbb{H}^2)$ and $x \in \mathbb{H}^2$, one can ask for the growth, as a function of R, of the function

$$N_R(\Gamma, x) = |\Gamma \cdot x \cap B(x, R)|,$$

where $B(x, R)$ denotes the disk of radius R around x. There exists a lot of literature on this problem (see for instance [GN12, Section 1.3] for an overview). We will be interested be interested in the orbit growth of the group Γ_a: the group generated by the reflections in the three non-consecutive sides of length $a/2$ of H_a – a right-angled hexagon in $\mathbb{H}^2$ with three non-consecutive sides of length $a/2$ (Figure 6.2). Combining results by Patterson [Pat88]

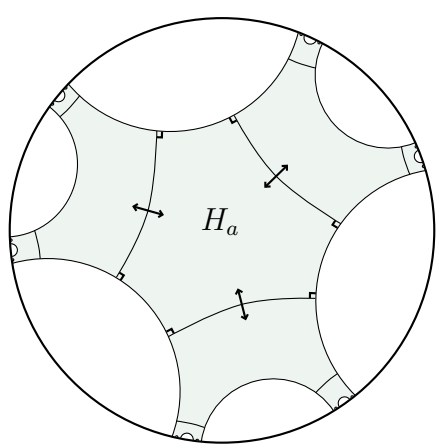

Figure 6.2: The orbit of H_a under Γ_a.

and McMullen [McM98], we obtain that for every $a > 0$ there exist constants $C_a > 0$ and $\delta_a \in (0, 1)$ such that

$$N_R(\Gamma_a, x) \sim C_a \cdot e^{\delta_a R}, \quad \text{as } R \to \infty.$$

Moreover, $\delta_a \to 1$ as $a \to \infty$.

The pants tree. We will apply these counting results to a hyperbolic surface of infinite area we will call the pants tree T_a. This surface is formed by gluing countable many copies of P_a according to the pattern of a trivalent tree. Figure 6.3 shows an example. We use

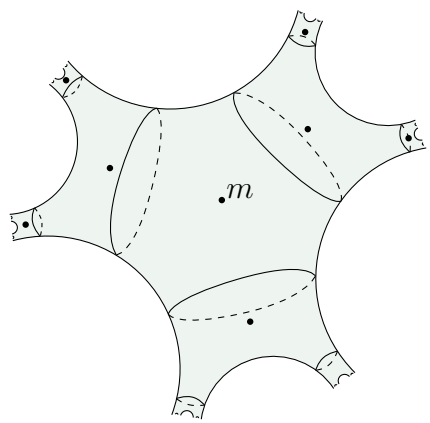

Figure 6.3: A pants tree

the orbit counting results from above as follows. Fix a point $m \in P_a$. For convenience, we fix this to be a midpoint of one of the copies of H_a that P_a is made out of. We will now fix one of the copies of P_a in T_a and denote its midpoint by m_0. Moreover $N_a(R)$ will denote the number of midpoints in T_a at distance at most R from m_0.. Essentially because T_a consists of two copies of the tiling in Figure 6.2, it follows from the orbit counting results above that

$$N_a(R) \sim C_a e^{\delta_a R}, \quad \text{as } R \to \infty. \tag{6.2.2}$$

Probabilistic input. First of all observe that there exists a constant D_a, depending on a only, such that

$$\operatorname{diam}(S_{g,a}) \leqslant \max \left\{ \mathrm{d}(m, m'); \begin{array}{c} m \text{ and } m' \text{ midpoints of} \\ \text{copies of } P_a \text{ in } S_{g,a} \end{array} \right\} + D_a$$

so it will be sufficient to control the maximal distance between midpoints on $S_{g,a}$. For argument's sake, we will imagine $S_{g,a}$ has the geometry of the pants tree T_a around every midpoint – this is not quite true, but using peeling techniques we prove that it's close enough to true for the rest of the proof to work. This part of the argument is similar to the proof by Bollobás–Fernandez-de-la-Vega [BFdlV82] of the fact that for a random k-regular G_n on n vertices $\operatorname{diam}(G_n)/\log_{k-1}(n) \to 1$ in probability.

Let us sketch the rest of the argument. Pick any pair of midpoints m, m' of $S_{g,a}$. Because the geometry is like that of T_a, the number of midpoints at distance at most R from m (resp. m') is (by (6.2.2)) $N_a(R) \sim C_a e^{\delta_a R}$. Furthermore let $R > 0$ be such that

$$N_a(R) > g^{1/2+\varepsilon},$$

This happens when

$$R \approx \frac{1/2+\varepsilon}{\delta_a} \log\left(\frac{g}{C_a}\right).$$

Now, given a pair of boundary components of copies of P_a, the probability that they are *not* glued together is roughly $\frac{6g-8}{6g-7} = 1 - \frac{1}{6g-7}$. Falsely imagining these probabilities for different pairs are all independent we get that the probability that none of the pants at distance $\leqslant R$ from m are glued to a pair of pants at distance $\leqslant R$ from m' is

$$\leqslant \left(\left(1 - \frac{1}{6g-7}\right)^{g^{1/2+\varepsilon}}\right)^{g^{1/2+\varepsilon}} = o(g^{-2}) \quad \text{as } g \to \infty$$

summing this over the $\approx g^2$ pairs of midpoints, we see that

$$\mathbb{P}_g\left(\begin{array}{c}\text{There is a path of length } \leqslant 2R \\ \text{between every pair of midpoints}\end{array}\right) \xrightarrow{g\to\infty} 1.$$

So we get for all $\varepsilon > 0$:

$$\mathbb{P}_g\left(\operatorname{diam}(S_{g,a}) \leqslant \frac{1+2\varepsilon}{\delta_a} \log\left(\frac{g}{C_a}\right) + D_a\right) \xrightarrow{g\to\infty} 1.$$

Since $\delta_a \to 1$ as $a \to \infty$, which proves (6.2.1) and hence Theorem 6.2.

6.2.3 Further remarks and questions

The random surfaces $S_{g,a}$ are somewhat of a funny model for random surfaces. They for instance all come with an order two symmetry, so they aren't very suitable for studying the geometry of a "typical" hyperbolic surface of large genus.

In a companion paper together with Thomas Budzinski and Nicolas Curien [BCP21a], we proved that random Belyĭ surfaces miss the optimal diameter by a factor 2. The fact that their diameters are at least $2\log(\text{genus})$ follows from the fact that they have multiple

large embedded disks, which in turn follows from Gamburd and Chmutov-Pittel's Poisson–Dirichlet approximation theorems (see Section 5.1.1). To prove that the diameter is at most $(2 + o(1)) \log(\text{genus})$, we use peeling techniques again. We furthermore conjectured that if one conditions on surfaces whose triangulation has only one vertex, the diameter should be asymptotic to $\log(\text{genus})$.

It would be interesting two know the behavior of the diameter for the other models of random manifolds we described in Sections 5.1 and 5.2.3:

Question 12

(a) Let X_g and Y_n denote a Weil–Petersson random surface of genus g and a random cover of degree n of a closed hyperbolic surface Y_1 respectively. Do

$$\frac{\operatorname{diam}(X_g)}{\log(g)} \quad \text{and} \quad \frac{\operatorname{diam}(Y_n)}{\log(n)}$$

converge in probability (as g and n tend to infinity respectively) and if so, what are these limits?

(b) Let $\mathcal{D}\, M_n$ denote the double of a random 3-manifold with boundary built out of n truncated tetrahedra. Does

$$\frac{\operatorname{diam}(M_n)}{\log(\operatorname{vol}(\mathcal{D}\, M_n))}$$

converge in probability as $n \to \infty$?

We also note that a uniform lower bound on the spectral gap (like those of Magee–Naud–Puder [MNP22] and Anantharaman–Monk [AM23]) and the systole imply a logarithmic upper bound on the diameter [Mag20]. However, this is not enough to get the optimal multiplicative constant, even if the surfaces are Ramanujan. In particular, the fact that random Belyĭ surfaces have diameter $\sim 2\log(\text{genus})$ does not imply they are not Ramanujan.

6.3 The Cheeger constant

The last connectivity problem we will discuss in this chapter is that of the **Cheeger constant**, that we already briefly discussed in Section 5.1.1. Given a Riemannian d-

manifold M, its Cheeger constant is

$$h(M) = \inf \left\{ \frac{\mathrm{vol}_{d-1}(\partial A)}{\mathrm{vol}_d(A)}; \quad \begin{array}{c} A \subset M \text{ compact with smooth} \\ \text{boundary and } \mathrm{vol}_d(A) \leqslant \mathrm{vol}_d(M)/2 \end{array} \right\}.$$

It measures how hard it is to cut off a piece of the manifold. It's called the Cheeger constant, because Cheeger [Che70] proved that

$$\lambda_1(M) \geqslant \frac{h(M)^2}{4}$$

for any complete Riemannian manifold M. In fact, Cheeger stated this inequality only for compact manifolds. His proof however works for manifolds of infinite volume as well. In that case constant functions are not in $L^2(M)$ and $\lambda_1(M)$ should be interpreted as the infimum of the spectrum of the Laplacian

$$\lambda_1(M) = \inf \left\{ \frac{\int_M ||\nabla f||^2 \, d\,\mathrm{vol}_M}{||f||_2^2}; \; f \in C^\infty(M) \cap L^2(M) \right\}.$$

Buser [Bus82] proved a converse to Cheeger's inequality, assuming bounds on the Ricci curvature. Moreover, Brooks proved that if a sequence of closed hyperbolic manifolds has uniformly bounded Cheeger constant and systole, their diameter is automatically logarithmic as a function of their volume [Bro92]. However, just like for the spectral gap, the resulting bound does not yield the optimal multiplicative constant.

The Cheeger constant of $\mathbb{H}^d$ equals

$$h(\mathbb{H}^d) = d - 1.$$

It is not attained by a smooth submanifold but can be approximated by balls of growing radius. In particular, Cheeger's inequality is sharp for $\mathbb{H}^d$: $\lambda_1(\mathbb{H}^d) = h(\mathbb{H}^d)^2/4$.

The natural extremal question that arises is what the maximal possible Cheeger constant of a closed hyperbolic manifold of large volume is. It already follows from Theorem 3.8 and similar bounds in higher dimension [Che75b] combined with Cheeger's inequality that, if the volume is large, the Cheeger constant cannot be significantly larger than that of the hyperbolic space of the same dimension. There is also a more direct proof of this:

Lemma 6.3. *For any $d \geqslant 2$, there exists a constant $c_d > 0$ such that*

$$h(M) \leqslant d - 1 + \frac{c_d}{\log(\mathrm{vol}(M))}$$

for any closed hyperbolic d-manifold M.

Proof. It follows from the Kazhdan–Margulis theorem [KM68] that there exists a constant $r_d > 0$ such that every closed hyperbolic d-manifold contains an embedded ball of radius r_d (see [Mar89, Yam82, Fan15] for effective versions).

Suppose $x \in M$ is the center of such a ball $B_{r_d}(x)$. On the other hand, for any $t > 0$ such that $\mathrm{vol}(B_t(x)) \leqslant \mathrm{vol}(M)/2$ ($t \lesssim \frac{1}{d-1}\log(\mathrm{vol}(M))$ suffices),

$$\frac{d}{dt}\mathrm{vol}(B_t(x)) \geqslant h(M) \cdot \mathrm{vol}(B_t(x)).$$

Combining these two, we obtain

$$\begin{aligned}\mathrm{vol}(B_{r_d}(x)) \cdot \exp\Big(h(M) \cdot (t - r_d)\Big) &\leqslant \mathrm{vol}(B_t(x)) \\ &\leqslant \mathrm{vol}(B_t(\widetilde{x})) = \mathrm{vol}(\mathbb{S}^{d-1}) \cdot \int_0^t \sinh^{d-1}(s)ds,\end{aligned}$$

where $\widetilde{x} \in \mathbb{H}^d$ is any point. This implies the lemma. □

6.3.1 Results

Again, the question is how sharp this bound is. As opposed to the case of the spectral gap and the diameter, the natural conjecture is that this is not sharp. This is the case for the analogous problem for regular graphs [Bol88, BZ02, Alo97] and has been conjectured for surfaces for instance by Lipnowski–Wright [Wri20]. Together with Thomas Budzinski and Nicolas Curien [BCP22], we proved this conjecture for surfaces:

Theorem 6.4. *We have*

$$\limsup_{g\to\infty} \sup\{h(X);\ X \in \mathcal{M}_g\} \leqslant \frac{2}{\pi} \approx 0.63\ldots.$$

The main point of this result is that there is a gap between the Cheeger constant of $\mathbb{H}^2$ and that of a closed orientable hyperbolic surface of large genus. There is no reason

to expect that the value we get is optimal. It is however optimal for our method of proof. The theorem also implies that Cheeger's inequality cannot be used to prove that hyperbolic surfaces of large genus are Ramanujan.

Similar gap results were already known for principal congruence covers of $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}^2$, random covers of the Bolza surface and random Belyĭ surfaces [BZ02, SW21, SW22].

6.3.2 Method of proof

Our proof is inspired by Bollobás's proof in the case of regular graphs, so we will briefly discuss this first. If $G = (V, E)$ is a graph, then its Cheeger constant[1] is

$$h(G) = \inf\left\{\frac{|E(A, V-A)|}{|A|};\ A \subset V \text{ finite and } |A| \leqslant |V|/2\right\},$$

where $E(A, V-A)$ denotes the set of edges connecting A to $V-E$. There is an analogous theory for the Cheeger constant in graph theory, we refer to [HLW06] for an introduction.

We will denote the infinite k-regular tree by $\mathbb{T}_k$. Its Cheeger constant equals $h(\mathbb{T}_k) = k-2$. There is an analogous elementary bound to Lemma 6.3 that states that the Cheeger constant of a finite k-regular graph on a large number of vertices cannot be significantly larger than $k-2$.

Bollobás's [Bol88] idea is that we can do a lot better with a elementary probabilistic argument, that we will paraphrase next. Indeed, let $G = (V, E)$ be a finite k-regular graph. We will write $n = |V|$. Now randomly color each vertex black or white, each with probability $\frac{1}{2}$ independently from each other. Let B and W denote the sets of black and white vertices respectively. We observe that $|B|$ and $|W|$ are binomial variables and apply Chebyshev's inequality to them to obtain:

$$\mathbb{P}\left(\left||B| - \frac{n}{2}\right| \geqslant n^{\frac{1}{2}+\delta}\right) = \mathbb{P}\left(\left||W| - \frac{n}{2}\right| \geqslant n^{\frac{1}{2}+\delta}\right) \leqslant \frac{1}{4}\, n^{-2\delta}.$$

Moreover, since an edge is part of $E(B, W)$ if and only if it runs between vertices of different colors, $\mathbb{E}\left(|E(B,W)|\right) = k \cdot n/4$. So, by Markov's inequality

$$\mathbb{P}\left(|E(B,W)| \geqslant (1+\delta)\cdot\frac{k\cdot n}{4}\right) \leqslant \frac{1}{1+\delta}.$$

[1] There are multiple options for the isoperimetric constant of a graph. For more information, we refer to [HLW06]

As a result

$$\mathbb{P}\left(\frac{|E(B,W)|}{\min\{|B|\,,|W|\}} \leqslant \frac{k}{2}\cdot\frac{n}{n-2\,n^{\frac{1}{2}+\delta}}\right) \geqslant 1-\frac{1}{4}\,n^{-2\delta}-\frac{1}{1+\delta},$$

thus yielding an upper bound of $\frac{k}{2}+o(1)$ (as $|V|\to\infty$) on the Cheeger constant, which for large degree is better by a factor of almost 2. Alon [Alo97] later sharpened this and obtained an improvement of the trivial bound in all degrees.

Looking at this proof with hyperbolic surfaces in mind, what we need to be able to do is randomly selecting half the hyperbolic surface X. With Thomas Budzinski and Nicolas Curien, we do this by using a **Poisson–Voronoi tessellation**. This is a random cell decomposition of the surface that is obtained as follows. First of all, we consider a Poisson point process $\mathcal{S}$ (see Definition 5.2) whose intensity is of the form $\lambda\cdot\mu_{\text{area}}$, where μ_{area} denotes the hyperbolic area measure on X and $\lambda>0$. The associated cell decomposition of X is the **Voronoi decomposition** corresponding to $\mathcal{S}$. The 2-dimensional cells of this decomposition are given by

$$C(s)=\{x\in X;\,\mathrm{d}(x,s)\leqslant \mathrm{d}(x,s') \text{ for all } s'\in\mathcal{S}\setminus\{s\}\}\,,\quad s\in\mathcal{S}$$

The 1-skeleton of the decomposition consists of points on X that are equidistant to multiple elements of $\mathcal{S}$.

So, to randomly split our closed hyperbolic surface X into two sets, we equip it a Poisson–Voronoi tessellation and color each top dimensional cell black or white, both with probability $\frac{1}{2}$, independently of all other cells. We call the resulting random subsets B_λ and W_λ respectively. We then prove a concentration result for $\min\{\text{area}(B_\lambda),\text{area}(W_\lambda)\}$ that is similar to the one for graphs we proved above. Most of the work goes into proving that

$$\limsup_{\lambda\to 0}\sup_{g\to\infty}\sup_{X\in\mathcal{M}_g}\frac{1}{\text{area}(X)}\,\mathbb{E}\left(\ell(\partial B_\lambda)\right)\leqslant\frac{1}{\pi}.$$

In fact, using results by Isokawa [Iso00] together with the existence of sequences of surfaces whose genus and systole tend to infinity, one can show that the above is an equality (which is why one can't do better than $\frac{2}{\pi}$ with our proof). Once we have these two bounds, the proof proceeds in the same way as in the graph case.

Finally, there is a particular cell decomposition of the hyperbolic plane that naturally arises from our proof. Indeed, in our proof we let the parameter λ tend to 0, which comes down to decreasing the number of points per unit of area. The quantity $\frac{1}{\pi}$ that shows

up can be thought of as $\frac{1}{2}$ times the ratio between the boundary length and the area of a typical cell of a limiting tessellation that we dubbed the **pointless Poisson–Voronoi tessellation** of $\mathbb{H}^2$. In this tessellation, the points defining the centers of the cells of this tessellation have converged out of the boundary. The cells themselves however remain visible from any base point in $\mathbb{H}^2$. The properties of this tessellation have recently been further studied for hyperbolic spaces in [DCE+23] and for symmetric spaces of higher rank in [FMW23, Mel23].

6.3.3 Further remarks and questions

Poisson–Voronoi tessellations are a classical object in stochastic geometry, see for instance [Iso00, BS01, HM22, HM23a, OT23, HOOT23] for versions in hyperbolic geometry and [CCE21] in a more general Riemannian geometric setting.

Given what's known for hyperbolic surfaces and graphs, it's natural to conjecture:

Conjecture

For every $d \geqslant 3$, there exists a number $\varepsilon_d > 0$ such that

$$h(M) < d - 1 - \varepsilon_d$$

for all closed hyperbolic d-manifolds of sufficiently large volume.

In fact, there is no reason that random decompositions based on Poisson–Voronoi tessellations don't work in higher dimension. What's less clear is whether the bound they yield gets better in higher dimension. For instance, is it asymptotic to $d/2$ as $d \to \infty$?

The distribution of Cheeger constants of random surfaces is also an interesting question:

Question 13

Is there some $h > 0$ such that $h(X) \longrightarrow h$ in probability as $\mathrm{area}(X) \to \infty$, for any of the models of random surfaces in Section 5.1 ?

If such a number exists, it would be a natural candidate for the maximal possible Cheeger constant of a hyperbolic surface of large genus.

7 Future directions

In the chapters above, we have tried to convince the reader that simple extremal questions connect to many different aspects of hyperbolic geometry. Moreover, in hyperbolic geometry, just like in Euclidean geometry and graph theory, many of the simplest questions – of which we have listed some throughout the text – are still open. In this last chapter, we discuss some future directions of research, based on some of the questions above and other questions like them.

Systoles. In Section 3.1, we discussed the maximal possible systole of a hyperbolic manifold of bounded volume. By now, we have plenty of constructions of sequences of closed hyperbolic surfaces whose systoles grow logarithmically as a function of their genus – which is the fastest possible rate. There are arithmetic constructions and more combinatorial constructions, both of which we discussed above. Moreover, in upcoming work with Mingkun Liu [LP23], we will present two random constructions, both inspired by ideas from graph theory [LS21, GHS+09].

In higher dimensions, currently only arithmetic examples are known. It would be interesting to have other constructions. An example of a question is wherer a variant of the construction from Section 3.1.3 can be made to work for 3-manifolds built out of regular right-angled octahedra.

A related, but probably much harder, question is what the systole, and more generally the length spectrum of a random finite degree subgroup of the corresponding reflection group behaves like. This would require deriving much sharper bounds on the subgroup growth of that reflection group than those we found with Hyungryul Baik and Jean Raimbault (see Section 4.1.4). Already proving that a random finite degree cover Benjamini–Schramm converges to $\mathbb{H}^3$ would be very interesting. If these random orbifolds behave like their 2-dimensional counterparts, then one would expect that their systoles do not tend to infinity. In order to obtain manifolds of large systoles, it would also be interesting study

random normal subgroups.

Linear programming. A more manageable project is to generalize our linear programming bounds with Maxime Fortier Bourque, that we discussed in Chapter 3, to (potentially non-compact) hyperbolic orbifolds of finite volume. The only thing that needs to be done is deal with the extra terms that show up in the trace formula. It would for instance be interesting to see how the bound one obtains compares with the growth in the sequences of non-compact manifolds with super-linear kissing number found by Dória–Murillo and Dória–Freire–Murillo [DM21, DFM23] and the bound by Fanoni–Parlier [FP15] in dimension 2.

A much more ambitious (and speculative) question on our linear programming bounds is whether, like in Euclidean geometry, there exist magical functions in some special cases. The bounds we have presented in this text don't make this seem very hopeful: our bounds for hyperbolic surfaces are much further away from the best examples than the Cohn–Elkies bounds on sphere packings were. However, based on recent numerics by Émile Gruda-Médiavilla and Mathieu Pineauilt, it seems like in genus 10 and 17, the bounds might be closer.

Expansion. Above, we have discussed three measures of connectivity of hyperbolic manifolds: their diameter, their Cheeger constant and their spectral gap. We have recently learned a lot about these invariants in dimension 2. Indeed, there exist sequences of closed hyperbolic surfaces whose diameter [BCP21b] and spectral gap [HM23b] saturate the classical bounds on these quantities. Moreover, the Cheeger constant behaves differently: a closed hyperbolic surface of large genus has a smaller Cheeger constant than the hyperbolic plane [BCP22] (see Chapter 6).

It would be nice to know what the actual maximum of the Cheeger constant is in large genus. Moreover, we have already mentioned in Section 6.3 that the Cheeger constants of random hyperbolic surfaces of large genus are not yet fully understood. Also for graphs, the analogous questions are still open.

In higher dimensions we know much less. It doesn't seem unlikely that our method with Thomas Budzinski and Nicolas Curien will allow us to prove a similar gap result on the Cheeger constant to the 2-dimensional case. The fact that higher dimensional hyperbolic manifolds are less flexible than surfaces (i.e. they satisfy the Mostow–Prasad rigidity theorem) and that as a result it is much harder to come up with good models of random manifolds, also makes the questions on the diameter and spectral gap harder

to approach. One model that does show expansion properties is the model of random 3-manifolds with boundary and their (closed) doubles that we studied with Jean Raimbault [PR22]. It would be very interesting get better estimates on their diameter and spectral gap. In fact, there are reasons to believe that their diameter misses the optimal rate by a factor 4 and that their spectral gap is related to that of the Apollonian group (for which very good numerical approximations exist).

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