Extremal problems and probabilistic methods in hyperbolic geometry

Mini course in the

Virtual Seminar on
Geometry and Topology

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LECTURE 1

A very brief reminder on hyperbolic geometry

In this lecture we introduce the main actors of this mini course: hyperbolic manifolds. We will state many facts without proof. For a more complete treatment, we refer to [**BP92, Rat06**].

1.1. Hyperbolic manifolds

There are multiple equivalent definitions of what a hyperbolic manifold is. The shortest of these is perhaps the following:

DEFINITION 1.1.1. A *hyperbolic n-manifold* is an n-dimensional Riemannian manifold, whose metric is complete and has constant sectional curvature equal to -1.

The first example of such a manifold is hyperbolic n-space $\mathbb{H}^n$. The Killing–Hopf theorem states that there is a unique (up to isometry) simply connected hyperbolic n-manifold, so we may define $\mathbb{H}^n$ to be that manifold. A more concrete way of thinking of $\mathbb{H}^n$ is by specifying what is called a *model*: a concrete simply connected complete Riemannian manifold with a metric of constant sectional curvature -1. There are various models that are useful for various purposes. We mention:

- the *ball model*:
$$\mathbb{D}^n = \left\{ x \in \mathbb{R}^n \;:\; \sum_{i=1}^n x_i^2 < 1 \right\}, \quad ds^2 = 4\,\frac{dx_1^2 + \ldots + dx_n^2}{\left(1 - \sum_{i=1}^n x_i^2\right)^2},$$
- the *upper half space model*:
$$\mathbb{U}^n = \{\, x \in \mathbb{R}^n \;:\; x_n > 0 \,\}, \quad ds^2 = \frac{dx_1^2 + \ldots + dx_n^2}{x_n^2},$$
- the *hyperboloid model*:
$$\mathbb{L}^n = \left\{\, x \in \mathbb{R}^{n+1} \;:\; \langle x, x\rangle_{n,1} = -1,\; x_0 > 0 \,\right\},$$
where $\langle x, y\rangle_{n,1} = -x_0y_0 + x_1y_1 + \ldots + x_ny_n$, with the metric given by the restriction of $\langle \cdot, \cdot \rangle_{(n,1)}$ to
$$T_x\,\mathbb{L}^n = \left\{\, y \in \mathbb{R}^{n+1} \;:\; \langle x, y\rangle_{n,1} = 0 \,\right\}.$$

We leave it to the reader to check that all these metrics indeed have constant sectional curvature -1.

Note that, using $\mathbb{H}^n$, we can alternatively define hyperbolic n-manifolds as

- complete Riemannian n-manifolds that are locally isometric to $\mathbb{H}^n$,
- or manifolds whose charts map to $\mathbb{H}^n$ and such that all chart transitions are restrictions of isometries of $\mathbb{H}^n$,

- or manifolds of the form $\Gamma\backslash\mathbb{H}^n$, where Γ is a discrete torsion-free group of isometries of $\mathbb{H}^n$.

1.2. Isometries

The isometry group $\mathrm{Isom}(\mathbb{H}^n)$ and orientation preserving isometry group $\mathrm{Isom}^+(\mathbb{H}^n)$ satisfy

$$\mathrm{Isom}(\mathbb{H}^n) \simeq \mathrm{PO}(1,n) := \left\{ A \in \mathrm{Mat}_{n+1}(\mathbb{R}) \ : \ \begin{array}{c} \langle x, y\rangle_{n,1} = \langle Ax, Ay\rangle_{n,1} \ \forall x, y \in \mathbb{R}^{n+1} \\ A \cdot \mathbb{L}^n = \mathbb{L}^n \end{array} \right\},$$

where $\mathrm{Mat}_n(\mathbb{R})$ denotes the set of $n \times n$ matrices with coefficients in $\mathbb{R}$, and

$$\mathrm{Isom}^+(\mathbb{H}^n) \simeq \mathrm{PSO}(1,n) := \{ A \in \mathrm{PO}(1,n) \ : \ \det(A) = 1 \}.$$

The fact that the groups $\mathrm{PO}(1,n)$ and $\mathrm{PSO}(1,n)$ act by isometries on $\mathbb{H}^n$ can be seen directly from the hyperboloid model $\mathbb{L}^n$. The proof of the fact that there are no other isometries can be found in [**Rat06**, Chapter 3] or [**BP92**, Chapter A].

In low dimensions there are two accidental isomorphisms

$$\mathrm{Isom}^+(\mathbb{H}^2) \simeq \mathrm{PSL}(2,\mathbb{R}) := \{ A \in \mathrm{Mat}_2(\mathbb{R}) \ : \ \det(A) = 1 \} \big/ \{\pm \mathrm{Id}_2\},$$

where Id_k denotes the $k \times k$ identity matrix, and

$$\mathrm{Isom}^+(\mathbb{H}^3) \simeq \mathrm{PSL}(2,\mathbb{C}) := \{ A \in \mathrm{Mat}_2(\mathbb{C}) \ : \ \det(A) = 1 \} \big/ \{\pm \mathrm{Id}_2\}.$$

The action of $\mathrm{PSL}(2,\mathbb{R})$ by isometries is that on $\mathbb{U}^2$ by linear fractional transformations – i.e. we see the upper hald plane as a subset of $\mathbb{C}$: $\mathbb{U}^2 = \{ z \in \mathbb{C} \ : \ \mathrm{Im}(z) > 0 \}$ and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az+b}{cz+d}$$

for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{R})$, $z \in \mathbb{U}^2$.

The action of $\mathrm{PSL}(2,\mathbb{C})$ on $\mathbb{H}^3$ is harder to describe. First of all, $\mathrm{PSL}(2,\mathbb{C})$ acts on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by linear fractional transformations – i.e.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az+b}{cz+d}$$

for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{C})$, $z \in \widehat{\mathbb{C}}$. It turns out that every such map φ can be written as the composition of two inversions in circles $C_1, C_2 \subset \widehat{\mathbb{C}}$. If we see $\widehat{\mathbb{C}}$ as the boundary of $\mathbb{U}^3$, then these circles define two hemispheres H_1 and H_2 in $\mathbb{U}^3$. φ now acts on $\mathbb{U}^3$ by the composition of the inversions (reflections) in H_1 and H_2. For the details, we refer to [**Bea95**, Section 3.3].

1.3. Hyperbolic surfaces

Before we ask questions about hyperbolic manifolds, we need some examples of them. We start with surfaces.

One way to construct hyperbolic surfaces is using pairs of pants. In what follows, we sketch how this works. For details, see for instance [**Bus10**, Section 1.7].

For ease of drawing, we will work in the disk model $\mathbb{D}^2$. In this model, geodesics are exactly straight diagonals through the center of $\mathbb{D}^2$ and half-circles orthogonal to $\partial\,\mathbb{D}^2$. A *right-angled hexagon* $H \subset \mathbb{D}^2$ is a compact, simply connected set whose boundary is geodesic, except at exactly six points, at which the geodesic segments coming from the right and left meet at right angles. Figure 1 shows an example.

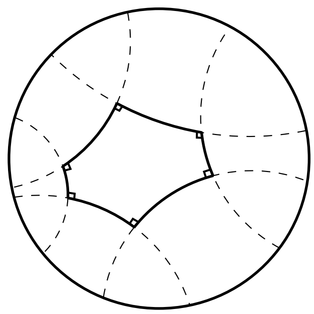

FIGURE 1. A right angled hexagon in $\mathbb{D}^2$

It turns out that, given three numbers $a, b, c > 0$, there exists a unique (up to isometry) right angled hexagon with three non-consecutive sides of lengts a, b and c.

Given two such hexagons, with the same side lengths, we can use three isometries to glue them along three non-consecutive sides of the same lengths, from which we obtain a hyperbolic pair of pants: a 2-sphere with three boundary components equipped with a hyperbolic metric. Figure 2 illustrates this gluing procedure.

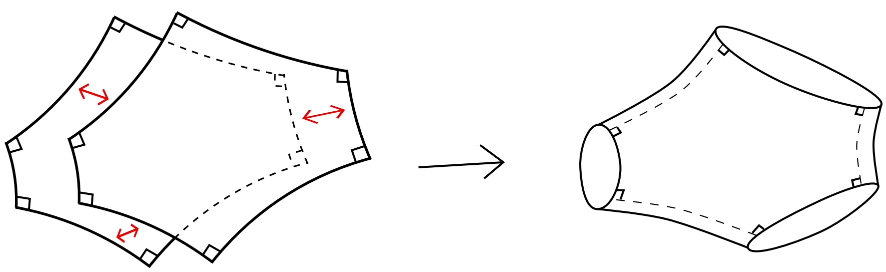

FIGURE 2. Gluing two right-angled hexagons into a pair of pants

It turns out that the hyperbolic metric on a pair of pants is determined (up to isometry) by the lengths of its three boundary components.

Finally, given two copies P_1 and P_2 of the same hyperbolic pair of pants, we may glue them together using three isometries between their boundary components. The result is a genus 2 surface. Figure 3 shows how this gluing works.

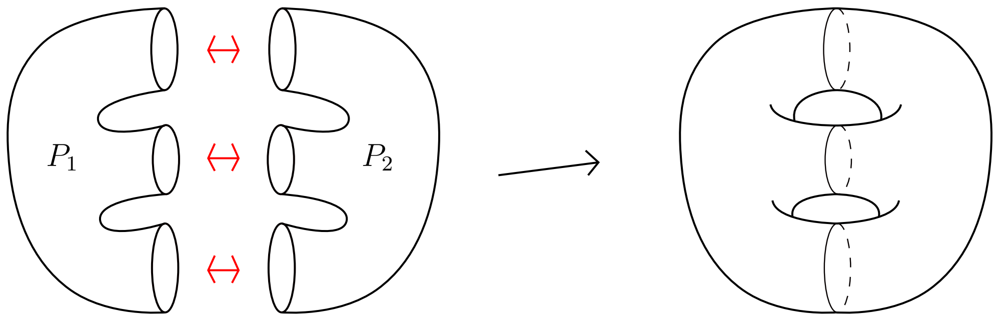

FIGURE 3. Gluing two pair of pants into a surface of genus 2

Note that we have multiple choices to make in this gluing process. First of all, we have to choose three boundary lengths for our pairs of pants. On top of that, we have to choose the three isometries between the pairs of boundary components. These lengths are naturally parametrized by $\mathbb{R}_+^3$. The three isometries can be parametrized by three copies of the circle. We will however choose to create some multiplicity and parameterize these by $\mathbb{R}$. So all in all, we get a space

$$\mathcal{T}_2 = \mathbb{R}_+^3 \times \mathbb{R}^3$$

of hyperbolic surfaces of genus 2 – called *Teichmüller space* of surfaces of genus 2. It turns out that this space contains a copy of *every* isometry type of hyperbolic surface of genus 2. In fact, it contains many copies of each isometry type (not just because of the multiplicity we introduced in the second half of the coordinates). The quotient in which all isometric pairs of surfaces is identified is called the *Moduli space* $\mathcal{M}_2$ of hyperbolic surfaces of genus 2.

Of course, a similar construction works for any genus. An Euler characteristic computation tells us that we need $2g-2$ pairs of pants for a closed surface of genus g. This gives us a Teichmüller space

$$\mathcal{T}_g = \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$$

and a moduli space $\mathcal{M}_g$. The parameters we used to construct $\mathcal{T}_g$ are called *Fenchel-Nielsen coordinates*, the first $3g-3$ are called the *length coordinates* and the last $3g-3$ coordinates are called the *twist coordinates*.

REMARK 1.3.1. The reason we pick the twists in $\mathbb{R}$ and not in the circle is that we then have a homeomorphism

$$\mathcal{T}_g \to \mathcal{T}(S) = \left\{ (X,f) \; : \; \begin{array}{c} X \text{ a Riemann surface} \\ f : S \to X \text{ a homeomorphism} \end{array} \right\} \Big/ \sim$$

where S is a Riemann surface of genus g and $(X,f) \sim (Y,g)$ if and only if there exists an holomorphism $\varphi : X \to Y$ such that $g^{-1} \circ \varphi \circ f : S \to S$ is isotopic to the identity. The topology on $\mathcal{T}(S)$ is induced by quasiconformal maps: (X,f) and (Y,g) are close if and only if there exists a map $h : X \to Y$ such that $g^{-1} \circ h \circ f : S \to S$ is "close" to a conformal map. See for instance [**Hub06, IT92**] for proper definitions.

1.4. 3-manifolds

The situation in higher dimensions is wildly different than that of surfaces. One of the main reasons for this is the following theorem:

THEOREM 1.4.1 (Mostow–Prasad rigidity theorem). *Let $n \geq 3$ and let M and N be hyperbolic n-manifolds of finite volume. If $\pi_1(M) \simeq \pi_1(N)$ then M and N are isometric.*

For a proof, see for instance [**BP92, Rat06**]. This theorem in particular implies that there are no interesting deformation spaces of hyperbolic structures of finite volume on a fixed smooth manifold of dimension more than two. Together with the fact that the fundamental group of a hyperbolic manifold is finitely presented, it also implies there are only countably many hyperbolic manifolds of finite volume (up to isometry).

Like Teichmüller theory, hyperbolic 3-manifolds is a vast subject and there is no way to do justice to it in a short introduction. So instead of trying to, we will just (have to) content ourselves with some examples of hyperbolic 3-manifolds.

One good source of hyperbolic 3-manifolds is knot complements. A *knot* in the 3-sphere $\mathbb{S}^3$ is a smooth embedding $K : \mathbb{S}^1 \to \mathbb{S}^3$. Figure 4 shows an example: the figure eight knot.

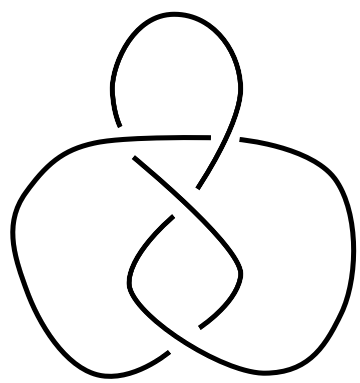

FIGURE 4. The figure eight knot.

Riley [**Ril75a, Ril75b**] discovered that the complement of the figure eight knot in $\mathbb{S}^3$ admits the structure of a hyperbolic 3-manifold of finite volume. In fact, Thurston later on determined exactly which knot complements (many of them) admit hyperbolic structures [**Thu82**].

This, again using work by Thurston, also leads to many examples of closed hyperbolic 3-manifolds, via the process of Dehn filling. Very briefly, every knot complement is homotopic to a 3-manifold M with one boundary component, homeomorphic to a 2-torus. If we now fix a solid torus T and a diffeomorphism $f : \partial T \to \partial M$, we obtain a closed 3-manifold M_f. Thurston proved that, if M itself admits a hyperbolic structure, then so does M_f for "most" choices of f [**Thu78**] (see also [**BP92**, Chapter E]).

1.5. Higher dimensions

In dimension higher than 3, hyperbolic manifolds are much harder to come by. There are still countably infinitely many closed (or of finite volume) hyperbolic n-manifolds for all $n \geq 4$, but they are much less well understood. One big difference with lower dimensional manifold is Wang's theorem, which states:

THEOREM 1.5.1 ([**Wan72**]). *The number of hyperbolic n-manifolds of volume $\leq v$ (up to isometry) is finite.*

First of all there are *arithmetic* hyperbolic manifolds. These are manifolds whose fundamental group is an arithmetic subgroup of $\mathrm{Isom}^+(\mathbb{H}^n)$. We will not go into the (lengthy) definition of what an arithmetic group is, but very roughly, they come from taking the integral points in an algebraic group. Prototypical examples are $\mathrm{PSL}(2,\mathbb{Z}) < \mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{Z}[i]) < \mathrm{PSL}(2,\mathbb{C})$ and their finite index subgroups. These two examples give rise to 2- and 3- orbifolds and manifolds (via their subgroups), but arithmetic groups exist in all dimensions.

Not every hyperbolic n-manifold is arithmetic. In dimension 4 and above this is a celebrated theorem due to Gromov – Piatetski-Shapiro [**GPS88**]. Their proof is constructive. It goes by taking two arithmetic hyperbolic n-manifolds M_1 and M_2 that both contain an embedded copy of a fixed hyperbolic $(n-1)$-manifold, cutting the n-manifolds along these $(n-1)$-manifolds and gluing the resulting blocks together along their boundary. It turns out that if M_1 and M_2 are not *commensurable* (i.e. do not have a common finite degree cover), then the result will be non-arithmetic. Several similar cut-and-paste constructions are by now known [**BT11, Rai13, GL14**].

LECTURE 2

Extremal problems

Now that we know everything about hyperbolic manifolds, it's time to define the invariants this course will be about: their systole, kissing number, diameter, spectral gap and Cheeger constant. The main question we will discuss is: what are the extremal values of these invariants and what do the manifolds that realize these values look like? Many of these questions make sense, and are indeed interesting, in a broader context (there is often an obvious generalization to manifolds of non-positive curvature and analogues to all our questions have also been studied for finite regular graphs), but we will stick to hyperbolic manifolds.

2.1. The systole

The first of these is the systole of a hyperbolic manifold. Given a rectifiable curve α on a Riemannian manifold, we will denote its length by $\ell(\alpha)$.

DEFINITION 2.1.1. Let M be a hyperbolic manifold that is not simply connected. Then the *systole* of M is

$$\operatorname{sys}(M) = \inf \{ \, \ell(\gamma) \, : \, \gamma \text{ a closed geodesic on } M \, \} .$$

If $\pi_1(M)$ is finitely generated, which is the case for most of the manifolds we will study in what follows, then the systole is realized by some closed geodesic in M. We will use the word "systole" for this geodesic as well.

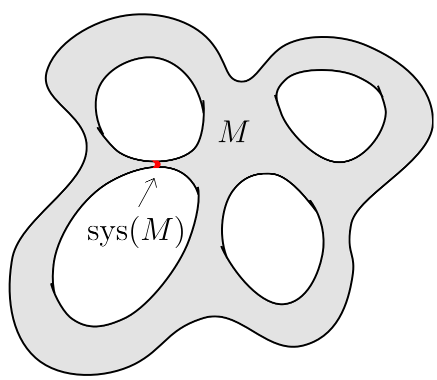

FIGURE 1. The systole of a manifold

We also note the classical fact that (because of the strictly negative curvature of M), there is a unique closed geodesic in each non-trivial non-peripheral[1] free homotopy

[1] A peripheral non-null homotopic closed curve is a curve whose homotopy class contains no geodesic. These are curves that can be homotoped into a cusp of M. If $M = \Gamma\backslash\mathbb{H}^n$ then the conjugacy

class of curves in M and this geodesic minimizes the length in the homotopy class. As such, we can equivalently define the systole to be the minimal length of a closed, homotopically non-trivial non-peripheral curve in M.

We have now arrived at our first question. We will denote the volume of a hyperbolic manifold M by $\mathrm{vol}(M)$.

QUESTION 1. Fix $n \geq 2$. How does

$$\mathrm{S}_n(v) := \max\{\,\mathrm{sys}(M) \,:\, M \text{ a hyperbolic } n-\text{manifold of } \mathrm{vol}(M) \leq v\,\}$$

grow as a function of v?

This question is open, even when $n = 2$. Knowing the function on the nose is to much to ask for, so what we are really asking for is the asymptotic behavior of S_n as $v \to \infty$. Let us discuss what is known about this. We start with an easy upper bound in the closed case:

LEMMA 2.1.2. *Let $n \geq 2$. There exists a constant $c_n > 0$ such that for all closed hyperbolic n-manifolds M:*

$$\mathrm{sys}(M) \leq \frac{2}{n-1}\log(\mathrm{vol}(M)) + c_n.$$

PROOF. Pick any point $p \in M$. Now the open ball $B(p, \mathrm{sys}(M)/2)$ is isometric to an open ball in $\mathbb{H}^n$. Indeed, if not, then there would be two distinct geodesics segments between p and another point $q \in B(p, \mathrm{sys}(M)/2)$, both of length less than $\mathrm{sys}(M)/2$ (Figure 2 shows the situation). The loop formed by these two segments has length stricly less than $\mathrm{sys}(M)$ and is not contractible (if it were contractible then it would lift to a geodesic bigon in $\mathbb{H}^n$, these don't exist). This is in contradiction with the fact that the systole is realized by the shortest closed curve on M.

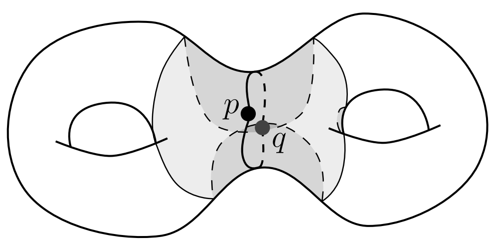

FIGURE 2. Two short segments that form a short loop

Now let $B(\widetilde{p}, \mathrm{sys}(M)/2)$ denote an open ball of radius $\mathrm{sys}(M)/2$ in $\mathbb{H}^n$. Our observation above implies that

$$\mathrm{vol}(B(\widetilde{p}, \mathrm{sys}(M)/2)) \leq \mathrm{vol}(M).$$

The former can be computed explicitly (see [**Rat06**, Exercise 3.4.6]) and satisfies

$$\mathrm{vol}(B(\widetilde{p}, \mathrm{sys}(M)/2)) = \mathrm{vol}(\mathbb{S}^{n-1}) \cdot \int_0^{\mathrm{sys}(M)/2} \sinh^{n-1}(t)dt,$$

class in Γ corresponding to such a curve consists of parabolic isometries of $\mathbb{H}^n$. M can only contain such curves if it is non-compact.

where $\mathbb{S}^k$ denotes the k-sphere in $\mathbb{R}^n$ with its round metric. This readily implies the lemma. $\square$

What is somewhat remarkable is that in general, no sharper upper bound than the Lemma above is known. The only dimension in which something better is known is dimension two, which is due to Bavard [**Bav96**]. He proved that the systole of a closed orientable hyperbolic surface M of genus g satisfies

$$\mathrm{sys}(M) \leq 2\mathrm{arccosh}\left(\frac{1}{2\sin(\pi/(12g-6))}\right).$$

We have [2]

$$2\mathrm{arccosh}\left(\frac{1}{2\sin(\pi/(12g-6))}\right) \sim 2\log(g)$$

as $g \to \infty$. The Gauss–Bonnet theorem implies that the area of a closed orientable hyperbolic surface of genus g equals $4\pi(g-1)$, so asymptotically, even Bavard's bound is still not better than the lemma above.

The only genus for which the closed hyperbolic surface of maximal systole is known is genus 2. Jenni [**Jen84**] proved that the Bolza surface has maximal systole among hyperbplic surfaces of genus 2.

Buser and Sarnak [**BS94**] proved that there exist sequences of closed orientable hyperbolic surfaces $(X_k)_k$ of genus g_k such that $g_k \to \infty$ as $k \to \infty$ and

$$\mathrm{sys}(X_k) \geq \frac{4}{3}\log(g_k) + O(1)$$

as $k \to \infty$. These surfaces are congruence covers of certain closed arithmetic surfaces. This was later generalized by Katz–Schapps–Vishne [**KSV07**] to a larger class of arithmetic surfaces and 3-manifolds and by Murillo [**Mur19**] who, again using arithmetic methods, proved the existence of hyperbolic n-manifolds M of arbitrarily large volume and systole

$$\mathrm{sys}(M) \geq \frac{8}{n(n+1)}\log(\mathrm{vol}(M)) + O_n(1).$$

Some non-arithmetic constructions of surfaces with logarithmic systoles are also known [**PW18, Pet18**]. In summary, it is known that $S_n(v)$ grows logarithmically as a function of v, but the rate is not known.

In the case of (not necessarily closed) hyperbolic manifolds of finite volume, less is known. The proof of Lemma 2.1.2 falls apart because the curve we find might be homotopic into a cusp. For surfaces with cusps, bounds have been proved by Schmutz [**Sch94**] and Fanoni–Parlier [**FP15**]. Schmutz also proved that principal congruence covers of $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}^2$ maximize the systole among hyperbolic surfaces of the same signature. In higher dimensions, no upper bounds seem to be known.

In dimension 2, one can also ask for *local* maxima for the systole as a function on $\mathcal{M}_g$. Many local maxima have been found by Schmutz [**Sch93**], Hamenstädt [**Ham01**] and Fortier Bourque–Rafi [**FBR20**]. One reason that these local maxima is interesting is that Akrout [**Akr03**] proved that the systole is a topological Morse function on $\mathcal{M}_g$. So in theory, they could be used to understand the (very complicated) topology of $\mathcal{M}_g$.

[2] We will write that $f(x) \sim g(x)$ as $x \to \infty$ to indicate that $\lim_{x\to\infty} f(x)/g(x) = 1$.

REMARK 2.1.3. As a side note, we mention that in dimensions $n = 2, 3$ we have

$$\inf \{ \operatorname{sys}(M) \, : \, M \text{ a closed hyperbolic } n\text{-manifold with } \operatorname{vol}(M) \leq v \} = 0$$

as soon as v is large enough. In dimension 2 explicit surfaces of a fixed genus and arbitrarily small systole can easily be constructed using the pants decompositions from Section 1.3. In dimension 3, it follows from Thurston's Dehn Surgery theorem mentioned in Section 1.4. This theorem implies that, when we take more and more complicated Dehn fillings of a fixed knot complement, the resulting sequence of closed hyperbolic manifolds converges to the hyperbolic knot complement. As such, these manifolds are of bounded volume but their systole tends to zero.

In dimension at least four, the situation is slightly different. Wang's theorem (Theorem 1.5.1) implies that the infimum above is taken over a finite set of manifolds and hence is not zero. It is known, due to work by Agol [**Ago06**] and Belolipetsky–Thomson [**BT11**] that it tends to zero as $v \to \infty$.

2.2. The kissing number

The second, related, invariant we will look into is the *kissing number* of M:

DEFINITION 2.2.1. The *kissing number* $\operatorname{Kiss}(M)$ of M is the number of pairwise non-homotopic closed geodesics realizing the systole.

The reason for this terminology comes from flat tori. A flat torus is a manifold of the form $\Lambda \backslash \mathbb{R}^n$, where $\Lambda < \mathbb{R}^n$ is a lattice (a discrete subgroup isomorphic to $\mathbb{Z}^n$). The systole of $\Lambda \backslash \mathbb{R}^n$ is twice the maximal radius R such that all the open balls of radius R in $\mathbb{R}^n$ around the points in Λ are pairwise disjoint. The kissing number of $\Lambda \backslash \mathbb{R}^n$ is the number of balls in the resulting packing that is tangent to (read: that kiss) any given ball (see Figure 3 for an example). On a side note, the growth of the largest possible

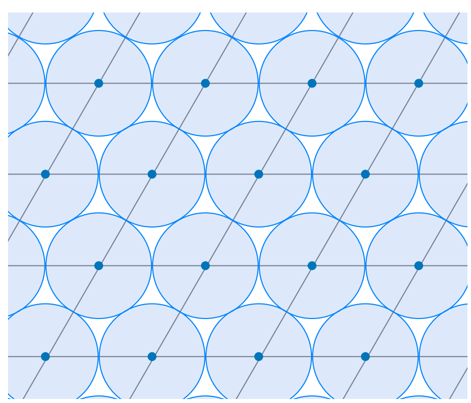

FIGURE 3. The hexagonal lattice: the two-dimensional lattice of maximal kissing number

kissing number of a packing in $\mathbb{R}^n$ as a function of n is known to be exponential, but the rate is also still an open question.

For a hyperbolic manifold $M = \Gamma\backslash\mathbb{H}^n$, $\mathrm{Kiss}(M)$ does not necessarily equal the kissing number of a packing of $\mathbb{H}^n$ with centers in $\Gamma \cdot x_0$ for some $x_0 \in \mathbb{H}^n$, but the name is kept in analogy. The natural extremal problem associated to kissing numbers is:

QUESTION 2. Fix $n \geq 2$. How does

$$\mathrm{K}_n(v) := \max \{ \mathrm{Kiss}(M) \ : \ M \text{ a hyperbolic } n-\text{manifold of } \mathrm{vol}(M) \leq v \}$$

grow as a function of v?

Also this question is wide open in general. The currently best known general upper bound in the closed case is due to Parlier (in dimension 2) and Fortier Bourque–Petri:

THEOREM 2.2.2 ([**Par13**], [**FBP19**]). *Let $n \geq 2$. There exists a constant $A_n > 0$ such that for all closed hyperbolic n-manifolds M,*

$$\mathrm{Kiss}(M) \leq A_n \ \mathrm{vol}(M) \ \frac{\exp\left(\frac{n-1}{2} \, \mathrm{sys}(M)\right)}{\mathrm{sys}(M)}.$$

The proofs of this theorem, which we will not get into in these lectures, are different. Parlier's proof is based on a geometric argument, whereas the proof by Fortier Bourque–Petri is based on the Selberg trace formula, a formula that makes a link between the lengths of geodesics on M and the eigenvalues of the Laplace operator on M. Note that the theorem implies that if we want to find sequences of manifolds whose kissing numbers grow faster than $\mathrm{vol}(M)^{1+\alpha}$ for some $\alpha > 0$, these manifolds need to have logarithmic systoles.

A similar bound for surfaces of finite area is due to Fanoni–Parlier [**FP15**]. The case of higher dimensional non-compact hyperbolic manifolds of finite volume is still open.

Using a quantatative versions of the Margulis lemma together with Lemma 2.1.2, we also obtain a bound that depends on volume alone.

COROLLARY 2.2.3 ([**Par13**],[**FBP19**]). *For every $n \geq 2$, there exists a constant $A_n' > 0$ such that*

$$\mathrm{Kiss}(M) \leq A_n' \, \frac{\mathrm{vol}(M)^2}{\log(1 + \mathrm{vol}(M))}$$

for every closed hyperbolic n-manifold M.

PROOF. For manifolds with small systole, a stronger inequality of the form

$$\mathrm{Kiss}(M) \leq A_n'' \, \mathrm{vol}(M) \, \mathrm{sys}(M)^{\lfloor \frac{n-2}{2} \rfloor / \lfloor \frac{n+1}{2} \rfloor}$$

follows from estimates on the volume of Margulis tubes around short geodesics due to Keen [**Kee74**] in dimension 2 and Buser [**Bus80**] in higher dimensions.

So we may assume that $\mathrm{sys}(M) \geq \varepsilon_n$ for some ε_n. The function $x \mapsto \exp\left(\frac{n-1}{2} \cdot x\right)/x$ is increasing for x large enough. So, Lemma 2.1.2 implies the bound. □

As is the case for systoles, congruence covers of arithmetic manifolds seem to provide good examples of manifolds with high kissing numbers. Schmutz [**SS97**] used these to prove the existence of (both closed and non-compact) sequences hyperbolic surfaces $(X_k)_k$ with

$$\mathrm{Kiss}(X_k) \geq \mathrm{area}(X_k)^{4/3-\varepsilon} \quad \text{and} \quad \mathrm{area}(X_k) \overset{k\to\infty}{\longrightarrow} \infty$$

for every $\varepsilon > 0$. Very recently, Dória–Murillo [**DM20**] proved the existence of a sequence $(M_k)_k$ of non-compact hyperbolic 3 manifolds of finite volume with

$$\mathrm{Kiss}(M_k) \geq C \cdot \frac{\mathrm{vol}(M_k)^{31/27}}{\log(1+\mathrm{vol}(M_k))}$$

for some constant $C > 0$.

2.3. The diameter

The other three invariants we will talk about all relate to the "connectedness" of hyperbolic manifolds. The first of these is the diameter:

$$\mathrm{diam}(M) = \sup\{\, \mathrm{d}(x,y) \,:\, x,y \in M \,\}, \tag{1}$$

where $\mathrm{d} : M \times M \to [0,\infty)$ denotes the distance function on M. Of course, this is only an interesting invariant for *closed* hyperbolic manifolds.

The question we will ask is:

Question 3. Let $n \geq 2$. How does

$$\mathrm{D}_n(v) := \min\{\, \mathrm{diam}(M) \,:\, M \text{ a closed hyperbolic } n\text{-manifold with } \mathrm{vol}(M) \geq v \,\}$$

grow as a function of v ?

It by now won't surprise the reader that also this question is open in all generality. Let us first prove an elementary lower bound (similar in spirit to Lemma 2.1.2):

Lemma 2.3.1. *Let $n \geq 2$. There exists a constant $c_n > 0$ such that*

$$\mathrm{diam}(M) \geq \frac{1}{n-1}\log(\mathrm{vol}(M)) - c_n$$

for all closed hyperbolic n-manifolds M.

Proof. Again denote the R-ball around a point $p \in M$ by $B(p,R)$. By definition of the diameter, we have

$$B(p, \mathrm{diam}(M)) = M$$

for any $p \in M$. Now the volume of $B(p, \mathrm{diam}(M))$ is smaller than the volume of a ball of the same radius in $\mathbb{H}^n$. So, using the formula of the volume of a ball in $\mathbb{H}^n$ again, we get

$$\mathrm{vol}(M) = \mathrm{vol}(B(p,\mathrm{diam}(M))) \leq \mathrm{vol}(\mathbb{S}^{n-1}) \cdot \int_0^{\mathrm{diam}(M)} \sinh^{n-1}(t)dt,$$

which implies the lemma. □

The only dimension in which a better lower bound is known is dimension two. Bavard [**Bav96**] proved that the diameter of a closed hyperbolic surface M of genus g satisfies

$$\mathrm{diam}(M) \geq \mathrm{arccosh}\left(\frac{1}{\sqrt{3}\tan\left(\pi/(12g-6)\right)}\right),$$

again, as $g \to \infty$, $\mathrm{arccosh}\left(\frac{1}{\sqrt{3}\tan(\pi/(12g-6))}\right) \sim \log(g)$, which means that asymptotically it gives the same bound as the lemma above.

So, just like manifolds with large systole are manifolds with logarithmic systole, manifolds with small diameter are manifolds with logarithmic diameter.

It turns out that, due to recent work by Budzinski–Curien–Petri, in dimension 2, the bound from Lemma 2.3.1 can asymptotically be saturated:

THEOREM 2.3.2 ([**BCP20**]). *The minimal diameter among hyperbolic surfaces of area $\geq a$ satisfies:*

$$\lim_{a\to\infty} \frac{\mathrm{D}_2(a)}{\log(a)} = 1.$$

The proof of this theorem is based on a random construction of hyperbolic surfaces. We will discuss how this works in the next two lectures.

In higher dimensions, the asymptotic behavior of $\mathrm{D}_n(v)$ is less well understood. It can be derived from spectral properties of certain arithmetic manifolds (see for instance [**Clo03, BC13**] for these properties and Section 2.5 for the connection) that for every $n \geq 2$, there exists a constant C_n such that

$$\limsup_{v\to\infty} \frac{\mathrm{D}_n(v)}{\log(v)} \leq C_n,$$

but a sharp statement like Theorem 2.3.2 is not known.

Finally, we mention the opposite problem: looking for manifolds with large diameter. In dimensions two and three, this is not an intersting problem: there are closed manifolds of bounded volume and arbitrarily large diameter. In dimension two, these can easily be constructed using a pants decomposition with pairs of pants with very short boundary geodesics. In dimension three, this follows from Thurston's work on Dehn fillings (see Section 1.4). In higher dimensions, the situation is very different, Burger–Schroeder [**BS87**] proved that for every $n \geq 4$, there exists a constant $A_n > 0$ such that

$$\mathrm{diam}(M) \leq A_n \cdot \mathrm{vol}(M)$$

for all closed hyperbolic n-manifolds M.

2.4. The Cheeger constant

Another measure of the connectivity of a manifold is its Cheeger constant. Intuitively, this measures how hard it is to cut a large piece off of the manifold:

DEFINITION 2.4.1. Let $n \geq 2$ and let M be a hyperbolic n-manifold of finite volume. The *Cheeger constant* of M is

$$h(M) = \inf\left\{ \frac{\mathrm{vol}_{n-1}(\partial N)}{\mathrm{vol}_n(N)} \;:\; \begin{array}{c} N \subset M \text{ a smooth submanifold} \\ \text{with } 0 < \mathrm{vol}_n(N) \leq \mathrm{vol}_n(M)/2 \end{array} \right\}.$$

Figure 4 shows an example of a manifold with small Cheeger constant.

If one is interested in "highly connected" hyperbolic manifolds, the natural question is:

QUESTION 4. Fix $n \geq 2$. What is

$$\mathrm{H}_n(v) = \max\left\{ h(M) \;:\; M \text{ a hyperbolic } n\text{-manifold with } \mathrm{vol}(M) \geq v \right\}.$$

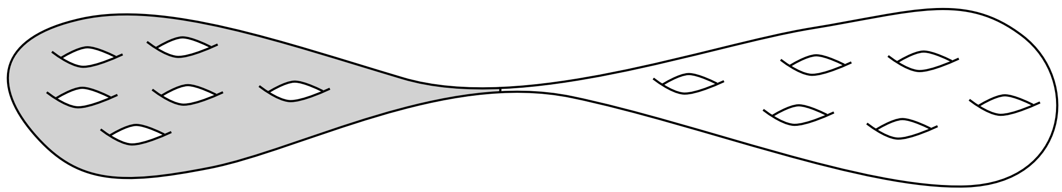

FIGURE 4. A manifold with small Cheeger constant: the shaded area can be cut off using a very small cut.

Combining work by Cheeger [**Che70**] and Cheng [**Che75**] we obtain that there exists a constant $c_n > 0$ (see Section 2.5) such that

$$h(M)^2 \leq (n-1)^2 + \frac{c_n}{\mathrm{diam}(M)^2}$$

for all closed hyperbolic n-manifolds M. In particular (using Lemma 2.3.1)

$$\limsup_{v\to\infty} \mathrm{H}_n(v) \leq n-1.$$

On the other hand, it is known that

$$\liminf_{v\to\infty} \mathrm{H}_n(v) > 0,$$

this can for instance be seen from arithmetic constructions [**Clo03**]. For example, in dimension two, the best known lower bound follows by combining work due to Kim–Sarnak [**Kim03**], Brooks [**Bro99**] and Buser [**Bus82**] and is:

$$\liminf_{a\to\infty} H_2(a) \geq \frac{-32+\sqrt{\frac{6923}{2}}}{160} = 0.1677\ldots.$$

However, whether $\lim_{v\to\infty} \mathrm{H}_n(v)$ exists and what its value should be, is open, even in dimension two.

Finally, a manifold that has a large Cheeger constant also has a small diameter (unless it has very small systole). Brooks proved:

THEOREM 2.4.2 ([**Bro92**]). *Let M be a closed hyperbolic n-manifold M. Then*

$$\mathrm{diam}(M) \leq \frac{2}{h(M)} \log\left(\frac{\mathrm{vol}(M)}{2\,\mathrm{vol}(B(r/2))}\right) + r,$$

where $B(r/2)$ denotes any ball of radius $r/2$ in $\mathbb{H}^n$ and $r = \min\{\mathrm{sys}(M), 1\}$.

PROOF. Fix $x, y \in M$ and let $V(x,t)$ denote the volume of a ball of radius t around x. When $t \leq \mathrm{sys}(M)/2$, $V(x,t) = \mathrm{vol}(B(t))$, using the same argument we saw in the proof of Lemma 2.1.2.

Moreover, when t is small enough such that $V(x,t) \leq \mathrm{vol}(M)/2$,

$$\frac{d}{dt}V(x,t) = \mathrm{vol}_{n-1}(\partial V(x,t)) \geq h(M)\cdot V(x,t)$$

Set $r = \min\{\mathrm{sys}(M)/2, 1/2\}$. Combining the two observations above, we get

$$V(x,t) \geq e^{h(M)\cdot(t-r)}V(x,r).$$

This holds up until $V(x,t) = \mathrm{vol}(M)/2$ which hence happens before

$$t_0 = \frac{1}{h(M)} \log\left(\frac{\mathrm{vol}(M)}{2V(x,r)}\right) + r.$$

Now the balls of volume $\mathrm{vol}(M)/2$ around x and y intersect, so

$$\mathrm{diam}(M) \leq 2t_0.$$

□

2.5. The spectral gap

The final (related) invariant we will consider is the spectral gap of a hyperbolic manifold. To this end, let us denote the Laplacian operator on functions on M by

$$\Delta = -\mathrm{div} \circ \mathrm{grad} : C^\infty(M) \to C^\infty(M).$$

We will mostly restrict to closed manifolds in this section (see for instance [**Iwa95**] for the non-compact case). We start with the spectral theorem (See for instance [**Bus10**, Chapter 7] or [**Ber16**, Chapter 3] for a proof):

THEOREM 2.5.1 (Spectral theorem). *Let M be a closed connected hyperbolic n-manifold. The eigenvalue problem*

$$\Delta\varphi = \lambda\varphi$$

has a complete orthonormal system of C^∞-eigenfunctions $\varphi_0, \varphi_1, \ldots$ in $L^2(M)$ with corresponding eigenvalues $\lambda_0, \lambda_1, \ldots$ that satisfy

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \nearrow \infty.$$

The first non-zero Laplacian eigenvalue of M, that we will denote by $\lambda_1(M)$ is also a measure of connectivity. In what follows we will explain why. Cheng [**Che75**] proved that there exists a constant

$$\lambda_1(M) \leq \frac{(n-1)^2}{4} + \frac{c_n}{\mathrm{diam}(M)^2}.$$

It is open whether this bound is asymptotically tight:

QUESTION 5. Define

$$\Lambda_n(v) = \sup\left\{ \lambda_1(M) \ : \ M \text{ a closed hyperbolic } n\text{-manifold with } \mathrm{vol}(M) \geq v \right\}.$$

Does it hold that

$$\lim_{v\to\infty} \Lambda_n(v) = \frac{(n-1)^2}{4} \quad ?$$

This question of course connects to the Selberg conjecture [**Sel65**] (and its generalizations [**BLS92**]), which states that

$$\lambda_1(\Gamma\backslash\mathbb{H}^2) \geq \frac{1}{4}$$

for every principal congruence subgroup $\Gamma < \mathrm{PSL}(2,\mathbb{Z})$. These surfaces are not closed, however using work by Brooks and Brooks–Makover [**Bro99, BM01**], a proof of the Selberg conjecture would imply

$$\lim_{a\to\infty} \Lambda_2(a) = \frac{1}{4}.$$

Selberg's conjecture is still open. Selberg himself proved that $\lambda_1(\Gamma\backslash\mathbb{H}^2) \geq \frac{3}{16}$ for principal congruence subgroups Γ. This has since been improved by various authors. The current record is due to Kim–Sarnak [**Kim03**]

$$\lambda_1(\Gamma\backslash\mathbb{H}^2) \geq \frac{975}{4096} \approx 0.238\ldots,$$

which, using the same work by Brooks and Brooks–Makover [**Bro99, BM01**], that

$$\liminf_{a\to\infty} \Lambda_2(a) \geq \frac{975}{4096}.$$

Bergeron–Clozel [**BC13**] proved that

$$\liminf_{v\to\infty} \Lambda_n(v) \geq n-2.$$

One of the reasons that $\lambda_1(M)$ is a measure of connectivity is:

THEOREM 2.5.2 (Cheeger–Buser inequalities, [**Che70**],[**Bus82**]). *Let M be a closed hyperbolic n-manifold. Then*

$$\frac{h(M)^2}{4} \quad\leq\quad \lambda_1(M) \quad\leq\quad 2(n-1)h(M) + 10h(M)^2.$$

So, if we have a sequence of closed hyperbolic n-manifolds $(M_k)_k$, then $\lambda_1(M) \stackrel{k\to\infty}{\longrightarrow} 0$ if and only if $h(M_k) \stackrel{k\to\infty}{\longrightarrow} 0$. Moreover, by combining the theorem above with Theorem 2.4.2, we obtain that a uniformly bounded spectral gap also implies a logarithmic diameter.

In fact, quantatively, a better bound on $\mathrm{diam}(M)$ in terms of $\lambda_1(M)$ and $\mathrm{vol}(M)$ can be obtained by a direct argument. This argument is written up in dimension two by Magee [**Mag20**] and one obtains that if M is a closed hyperbolic surface with $\lambda_1(M) \geq (1-\delta^2)/4$ and $\mathrm{sys}(M) > \varepsilon$ then

$$\mathrm{diam}(M) \leq \frac{2}{1-\delta}\log(\mathrm{area}(M)) + \frac{4}{1-\delta}\log\log(\mathrm{area}(M)) + C_{\varepsilon.\delta},$$

where $C_{\varepsilon.\delta} > 0$ is a constant depending on ε and δ alone.

LECTURE 3

The minimal diameter of a hyperbolic surface

Now that we've treated some of the context, it's time to prove something. The goal of this lecture is to determine the asymptotic behavior of the minimal diameter among hyperbolic surfaces of genus g, i.e. to prove the following theorem we mentioned in the previous lecture:

THEOREM 2.3.2. *The minimal diameter among hyperbolic surfaces of area $\geq a$ satisfies:*

$$\lim_{a\to\infty} \frac{\mathrm{D}_2(a)}{\log(a)} = 1.$$

Note that because of Lemma 2.3.1, we only need to prove that

$$\limsup_{a\to\infty} \frac{\mathrm{D}_2(a)}{\log(a)} \leq 1.$$

3.1. A model for random surfaces

As we mentioned, we will prove this theorem using a random construction. So, let us introduce this construction first.

First of all, recall from our discussion from Section 1.3, that given $a > 0$, there exists a unique (up to isometry) hyperbolic pair of pants P_a all of whose boundary components have length a. In short, our random surface will be constructed as follows: we take $2g-2$ copies of P_a and a uniformly random matching between the $6g-6$ boundary components of the pants and then glue the pants together according to the matching. For the gluing we will set the twist equal to zero, so that the combinatorics completely determine the gluing.

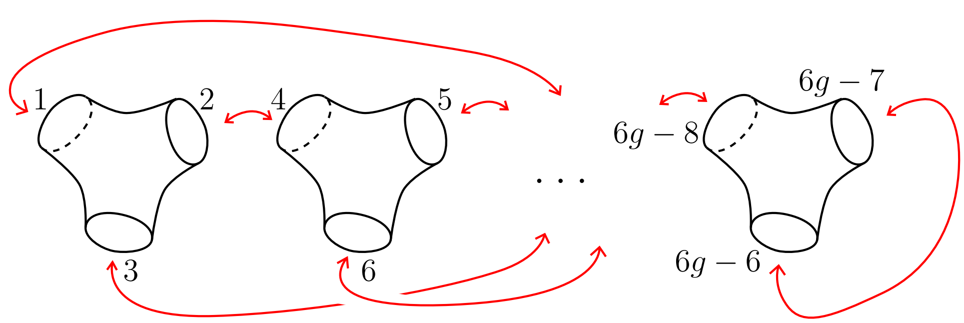

FIGURE 1. A random gluing of pairs of pants.

For the proofs later on, we will formalize this as follows. We will label the boundary components of the first copy of P_a by 1, 2 and 3, those of the second copy of P_a by 4, 5 and 6, et cetera. This means that, given $a > 0$, for every element

$$\omega \in \Omega_g := \Big\{ \text{Partitions of } \{1, \ldots, 6g-6\} \text{ into pairs} \Big\}$$

we obtain a surface

$$S_a(\omega)$$

by gluing the copies of P_a together, with twist 0, according to ω. In order to turn S_a into a random surface, we will use the uniform probability measure $\mathbb{P}_g$ on Ω_g. That is

$$\mathbb{P}_g[A] = \frac{|A|}{|\Omega_g|}, \quad A \subseteq \Omega_g.$$

Now that we have defined out model for random surfaces, we can state what we will really prove:

THEOREM 3.1.1 ([**BCP20**]). *For every $\varepsilon > 0$ there exists a number $a > 0$ such that*

$$\lim_{g\to\infty} \mathbb{P}_g[\mathrm{diam}(S_a) \leq (1+\varepsilon)\log(g)] = 1.$$

We will use the rest of this lecture and the beginning of the next one to prove this theorem. We will start with the necessary hyperbolic geometry and then prove the theorem.

3.2. Orbit counting

The main geometric input comes from counting problems in the hyperbolic plane. Given a discrete group $\Gamma < \mathrm{Isom}(\mathbb{H}^2)$ and $x \in \mathbb{H}^2$, one can ask for the growth, as a function of R, of the function

$$N_R(\Gamma, x) = \left| \Gamma \cdot x \cap B(x, R) \right|,$$

where $B(x, R)$ denotes the disk of radius R around x. There exists a vast body of literature on this problem. We will be interested be interested in the orbit growth of the group Γ_a: the group generated by the reflections in the three non-consecutive sides of length $a/2$ of H_a – a right-angled hexagon in $\mathbb{H}^2$ with three non-consecutive sides of length $a/2$ (Figure 2).

Pattersson and McMullen proved that:

THEOREM 3.2.1 ([**Pat88, McM98**]). *For every $a > 0$ there exist constants $C_a > 0$ and $\delta_a \in (0, 1)$ such that*

$$N_R(\Gamma_a, x) \sim C_a \cdot e^{\delta_a R}, \quad \text{as } R \to \infty.$$

Moreover, $\delta_a \to 1$ as $a \to \infty$.

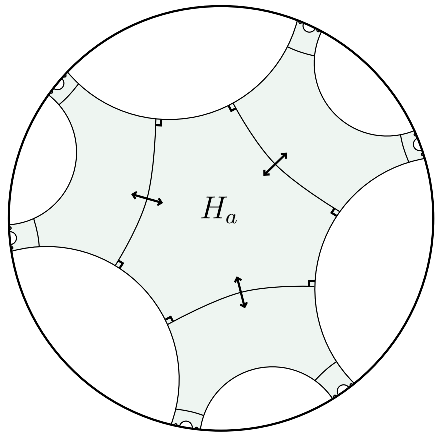

FIGURE 2. The orbit of H_a under Γ_a.

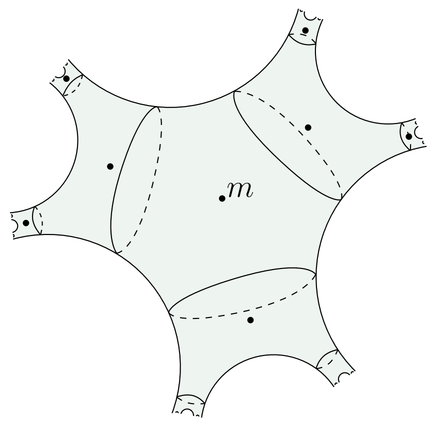

FIGURE 3. A pants tree

3.3. The pants tree

A crucial role in the proof will be played by a hyperbolic surface of infinite area we will call the pants tree T_a. This surface is formed by gluing countable many copies of P_a according to the pattern of a trivalent tree. Figure 3 shows an example.

This surface will be used as a model for the local geometry of our random surfaces S_a.

We will use the orbit counting results from above as follows. Fix a point $m \in P_a$. For convenience, we will fix this to be a midpoint of one of the copies of H_a that P_a is made out of. We will now fix one of the copies of P_a in T_a and denote its midpoint by m_0. Moreover $N_a(R)$ will denote the number of midpoints in T_a at distance at most R from m_0.. Essentially because T_a consists of two copies of the tiling in Figure 2, it follows from Theorem 3.2.1 that

$$N_a(R) \sim C_a e^{\delta_a R}, \quad \text{as } R \to \infty. \tag{2}$$

3.4. A proof sketch

Before we give the actual proof of the theorem above, which we will discuss in the next lecture, let us first discuss the idea of the proof. First of all observe that there exists a constant D_a, depending on a only, such that

$$\mathrm{diam}(S_a(\omega)) \leq \max \left\{ \mathrm{d}(m,m') \ : \ \begin{array}{c} m \text{ and } m' \text{ midpoints of} \\ \text{copies of } P_a \text{ in } S_a(\omega) \end{array} \right\} + D_a$$

so it will be enough to control the maximal distance between midpoints on S_a. Now, for argument's sake, imagine S_a has the geometry of T_a around every midpoint. Now pick any pair of midpoints m, m' of S_a. Because the geometry is like that of T_a, the number of midpoints at distance at most R from m (resp. m') is (by (2)) $N_a(R) \sim C_a e^{\delta_a R}$. Now suppose R is so that

$$N_a(R) > g^{1/2+\varepsilon},$$

This happens when

$$R \approx \frac{1/2+\varepsilon}{\delta_a} \log\left(\frac{g}{C_a}\right).$$

Now, given a pair of boundary components of copies of P_a, the probability that they are *not* glued together is roughly $\frac{6g-8}{6g-7} = 1 - \frac{1}{6g-7}$. Falsely imagining these probabilities for different pairs are all independent we get that the probability that none of the pants at distance $\leq R$ from m are glued to a pair of pants at distance $\leq R$ from m' is

$$\leq \left(\left(1 - \frac{1}{6g-7}\right)^{g^{1/2+\varepsilon}} \right)^{g^{1/2+\varepsilon}} = o(g^{-2}) \quad \text{ as } g \to \infty$$

summing this over the $\approx g^2$ pairs of midpoints, we see that

$$\mathbb{P}_g \left[\begin{array}{c} \text{There is a path of length } \leq 2R \\ \text{between every pair of midpoints} \end{array} \right] \stackrel{g\to\infty}{\longrightarrow} 1.$$

So we get for all $\varepsilon > 0$:

$$\mathbb{P}_g \left[\mathrm{diam}(S_a) \leq \frac{1+2\varepsilon}{\delta_a} \log\left(\frac{g}{C_a}\right) + D_a \right] \stackrel{g\to\infty}{\longrightarrow} 1.$$

Since $\delta_a \to 1$ as $a \to \infty$, this would prove the theorem.

LECTURE 4

More random manifolds

In our final lecture, we will first finish the proof of Theorem 3.1.1 – the asymptotic behavior of the minimal diameter among closed hyperbolic surfaces of genus g. After that we will discuss other models of random surfaces and (briefly) higher dimensional manifolds. We will finish with some open questions. The attentive reader will notice that the text below is significantly longer than that of the other lectures. This is because we will give some more details (especially about random triangulations) than we did during the actual lecture.

4.1. The real proof of Theorem 3.1.1

We now start by presenting the real proof of the following theorem that we discussed in the previous lecture:

THEOREM 3.1.1 ([**BCP20**]). *For every $\varepsilon > 0$ there exists a number $a > 0$ such that*

$$\lim_{g\to\infty} \mathbb{P}_g[\mathrm{diam}(S_a) \leq (1+\varepsilon)\log(g)] = 1.$$

The two main lies in the proof sketch above are the independence we assumed in all the gluing probabilities and the assumption that S_a looks *exaclly* like T_a everywhere.

4.1.1. A peeling algorithm. In order to produce a proper proof, we will use a technique known as *peeling*. The idea is that, without changing the probabilities of geometric event, we may glue our surface together in a specific order.

This goes as follows: fix $i \in \{1,\ldots,2g-2\}$. To this number corresponds a pair of pants out of which we will build S_a. Denote its midpoints by m_i. Our goal is to understand the geometry of the neighborhood of m_i.

Fix some $\varepsilon > 0$. We will now build S_a in $3g-3$ steps (one for each pants curve). So at each step $t \in \{1,\ldots,3g-3\}$, we have a surface $S_a^{(t)}$. $S_a^{(0)}$ is the surface consisting of $2g-2$ disjoint pairs of pants. At step t, we perform the following operation:

- If the component of $S_a^{(t-1)}$ containing m_1 contains fewer than $\lceil g^{1/2+\varepsilon}\rceil$ pairs of pants: glue (with twist 0) the boundary component of $S_a^{(t-1)}$ that is closest to m_i (in the hyperbolic metric) to a boundary component that has not been used yet (picked uniformly at random). Call the result $S_a^{(t)}$.
- If not: pick to boundary components of $S_a^{(t-1)}$ uniformly at random and glue them together with twist 0. Call the result $S_a^{(t)}$.

Note that there are issues with the notion of "closest to m_i":

- There might be multiple boundary components at equal (minimal) distance, in this case we pick one uniformly at random.

- It might happen that $S_a^{(t)}$ has a connected component without boundary that contains m_i. In this case we will say a *disconnection* has happened in the process. In that case we will just pick a component uniformly at random to glue.

Note that the arbitrary order (or *peeling algorithm*) we introduced in the construction of S_a has no influence on the probabilities of any geometric or combinatorial properties of S_a. It will however help us control certain probabilities.

4.1.2. Tree-like neighborhoods. Our first goal now is to show that up to the first $\approx g^{1/2+\varepsilon}$ pants around m_i, the pants tree T_a really is a good approximation for the geometry around m_i. The corresponding neighborhood of m_i is not isometric to a subset of T_a, but the number of defects is low.

To this end, we will say

Definition 4.1.1. A step t in the peeling proces described above is *bad* if during this step, two boundary components of the connected component of $S_a^{(t-1)}$ containing m_i are glued together.

Note that this does *not* include a disconnection event.

We now have:

Lemma 4.1.2. *Let τ denote the time in the peeling process at which we have used at least $g^{1/2}\log(g)$ pairs of pants.*

(a) For every $\varepsilon > 0$, there exists a $K \in \mathbb{N}$, such that

$$\mathbb{P}_g\left[\begin{array}{c}\text{During the first } g^{1/2-\varepsilon} \text{ steps, at}\\ \text{least } K \text{ steps are bad}\end{array}\right] = o(g^{-3}) \quad \text{as } g \to \infty$$

(b) Moreover,

$$\mathbb{P}_g\left[\begin{array}{c}\text{During steps } g^{1/2-\varepsilon} \text{ up to } \tau\\ \text{at least } \log(g)^3 \text{ steps are bad}\end{array}\right] = o(g^{-3}) \quad \text{as } g \to \infty.$$

Proof. We start with item (a). First note that the boundary of the connected component of $S_a^{(t)}$ containing m_i has at most $3+t$ components. As such, the probability that set t is bad is at most

$$\mathbb{P}_g[\text{Step } t \text{ is bad}] \leq \frac{2+t}{6g-7-2t} \leq \text{cst} \cdot \frac{1}{g^{1/2+\varepsilon}}$$

for $t \leq g^{1/2-\varepsilon}$. So the probability that K steps are bad among the first $g^{1/2-\varepsilon}$ is at most

$$\sum_{t_1<t_2<\ldots<t_K\leq\leq g^{1/2-\varepsilon}} \mathbb{P}_g[\text{Steps } t_1,\ldots,t_K \text{ are bad}] \leq \text{cst} \cdot \left(g^{1/2-\varepsilon}\right)^K \left(\frac{1}{g^{1/2+\varepsilon}}\right)^K$$
$$\leq \text{cst}_K \cdot g^{-2\varepsilon K}$$

so our claim holds for $K > 3/(2\varepsilon)$.

The proof of item (b) is similar. We get that for $g^{1/2-\varepsilon} \leq t \leq \tau$,

$$\mathbb{P}_g[\text{Step } t \text{ is bad}] \leq \text{cst} \cdot \frac{\log(n)}{g^{1/2}}$$

So

$$\mathbb{P}_g\left[\begin{array}{c}\text{During steps } g^{1/2-\varepsilon} \text{ up to } \tau \\ \text{at least } \log(g)^3 \text{ steps are bad}\end{array}\right] \leq \frac{1}{(\log^3(g))!}(g^{1/2}\log(g))^{\log^3(g)}\left(\frac{\log(n)}{g^{1/2}}\right)^{\log^3(g)} = o(g^{-3})$$

as $g \to \infty$. □

So the neighborhood of a fixed point consisting of the closed $\approx g^{1/2}$ pants is indeed tree-like. It's not so hard to see (see [**BCP20**] for a proof) that this implies that exponential growth of volume stays intact:

Lemma 4.1.3. *Fix $\varepsilon > 0$, $a \in (0,\infty)$ and $K \geq 0$. Suppose that during an exploration as above, there are fewer than K bad steps until time $g^{1/2-\varepsilon}$ and less than $\log^3(g)$ bad steps until time τ.*

Then, if g is large enough, either the surface is disconnected, or the maximal distance R reached in S_a from the midpoint of the pair of pants where the exploration started satisfies

$$R \leq \frac{1}{2}\left(\frac{1}{\delta_a} + \varepsilon\right)\log g.$$

4.1.3. Finishing the proof. Above we've seen that with probability $1 - o(g^{-3})$ the neighborhood N_i consisting of the closest $g^{1/2}\log(g)$ pants along a midpoint m_i is tree-like (assuming there is no disconnection).

Now we need that such neighborhoods connect. First, assuming there is no disconnection, observe that N_i has $\geq \text{cst}\cdot g^{1/2}\log(g)$ boundary components. So the probability that, in an exploration of τ steps near m_j, we don't connect to N_i is bounded by

$$\mathbb{P}_g\left[\begin{array}{c}\text{In the first } \tau \text{ steps of} \\ \text{the peeling process around } m_j \\ \text{we don't connect to } N_i\end{array}\right] \leq \left(1 - \text{cst}\frac{\log(g)}{g^{1/2}}\right)^{g^{1/2}\log(g)} = o(g^{-3})$$

as $g \to \infty$.

So we get that

$$\mathbb{P}_g\left[\begin{array}{c}\text{There exist midpoints } m_i \text{ and } m_j \text{ that have } \mathrm{d}(m_i,m_j) \geq 2R \\ \text{and } S_a \text{ is connected}\end{array}\right] \leq \sum_{i,j}\mathbb{P}_g\left[\begin{array}{c}\mathrm{d}(m_i,m_j) \geq 2R \\ \text{and } S_a \text{ is connected}\end{array}\right] = o(g^{-1})$$

where R is as in Lemma 4.1.3.

Finally, we use that

$$\mathbb{P}_g[S_a \text{ is disconnected}] = o(g^{-1})$$

as $g \to \infty$ (this was first proved by Bollobás and Wormald [**Bol81**], [**Wor81**]), to conclude the proof.

4.2. Other models for random surfaces

There are plenty of other models around for random hyperbolic surfaces. In this section we will discuss some of them and what is known about them. We will give more details than were provided in the original lecture.

4.2.1. The Brooks–Makover model. We start with a model for random surfaces introduced by Brooks–Makover [**BM04**]. The idea of this model is to take an even number of ideal hyperbolic triangles, glue these along their sides randomly into an oriented non-compact hyperbolic surface and conformally compactify the result.

4.2.1.1. *Formal topological set up.* Formally, the model is based on the *configuration model* of random 3-regular graphs on $2N$ vertices and is very close to the model we used for the minimal diameter of a hyperbolic sruface. Or probability space will be the set

$$\Omega_N := \Big\{\text{Partitions of } \{1,\ldots,6N\} \text{ into pairs}\Big\}$$

equipped with the uniform probability measure $\mathbb{P}_N$.

This $\omega \in \Omega_N$ can be turned into a closed oriented surface $S(\omega)$ as follows. Take $2N$ triangles (2-simplices) $\Delta_1,\ldots,\Delta_{2N}$, and label the sides of the first triangle with the labels 1, 2 and 3, those of the second 4, 5 and 6 and so forth (see the figure below).

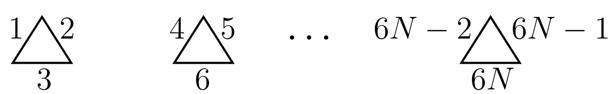

FIGURE 1. $2N$ labeled triangles.

Each of these triangles naturally comes with an orientation (induced by the cyclic order of the labels on the sides). For each pair of labels $c = \{i,j\} \in C$ fix an orientation reversing simplicial map φ_c between the corresponding sides. We set

$$S(C) = \bigsqcup_{i=1}^{2N} \Delta_i / \sim$$

where the equivalence relation is given by the collection of maps $\{\varphi_c\}_{c\in C}$.

Figure 2 gives some examples for $N = 1$.

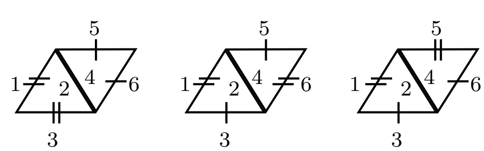

FIGURE 2. The surfaces corresponding to the configurations $\{\{1,3\},\{2,4\},\{5,6\}\}$, $\{\{1,6\},\{2,4\},\{3,5\}\}$ and $\{\{1,5\},\{2,4\},\{3,6\}\}$: a sphere, a torus and a sphere respectively.

4.2.1.2. *The resulting topology.* Before we properly discuss how to turn the surfaces $S(\omega)$ into hyperbolic surfaces, we discuss their topology.

The first question is of course whether $S(\omega)$ is connected. It turns out that typically it is, for exactly the same reason that our random pants decompositions above were typically connected: the connectivity of random trivalent graphs, due to Bollobás and Wormald:

Theorem 4.2.1 ([**Bol81**], [**Wor81**]). *We have*

$$\lim_{N\to\infty} \mathbb{P}_N[S \textit{ is connected}] = 1.$$

Given that $S(\omega)$ is a closed orientable surface, the only remaining topological question is what its genus is $g(S(\omega))$. A simple Euler characteristic computation gives that for $\omega \in \Omega_N$ (if $S(\omega)$ is connected):

$$g(S(\omega)) = \frac{N+2-V(\omega)}{2},$$

where V is the number of vertices of the triangulation S comes with. Good estimates on the asymptotic behavior of V as $N \to \infty$, have been worked out by Gamburd and Chmutov–Pittel. They proved the following theorem, in which the *total variational distance* between two random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega' \to \mathbb{R}$ is defined as

$$\mathrm{d_{TV}}\,(X,Y) := \sup\left\{\, |\mathbb{P}(X\in A) - \mathbb{P}(Y\in A)| \;:\; A\subseteq \mathbb{R} \text{ measurable}\,\right\}.$$

Theorem 4.2.2 ([**Gam06**],[**CP16**]). *Let $V_N : \Omega_N \to \mathbb{N}$ be the random variable that counts the number of vertices of the triangulation on S. Moreover, let $\mathcal{N}_N$ a random variable that is normally distributed with mean $\log(N)$ and standard deviation $\sqrt{\log(N)}$. Then*

$$\mathrm{d_{TV}}\,(V_N, \mathcal{N}_N) \to 0$$

as $N \to \infty$.

For us the most important consequence of this theorem is that the genus of our random surfaces is roughly $N/2$ (up to only a logarithmic error). In particular, this model gives us random surfaces of large genus.

Proof sketch of Theorem 4.2.2. The idea of proof is based on a different description of our probability space Ω_N: we will parametrize random surfaces with pairs of permutations instead of with configurations.

This goes as follows. First of all the orientation (the cyclic order of the labelled sides) of the triangles we start with can be captured in a permutation. This permutation consists of a product of three-cycles, one corresponding to each triangle. Figure 3 shows the idea:

This leads to a permutation

$$\sigma = (1\;2\;3)(4\;5\;6)\cdots(6N-2\;\;6N-1\;\;6N) \in \mathfrak{S}_{6N}\,.$$

Likewise, the configuration $\omega \in \Omega_N$ itself can also be recorded in a permutation. We simply write down a two-cycle $(a_i\;b_i)$ for each pair $\{a_i,b_i\} \in \omega$ and concatenate all these (disjoint) two-cycles. This leads to another permutation

$$\tau = (a_1\;b_1)(a_2\;b_2)\cdots(a_{3N}\;b_{3N}) \in \mathfrak{S}_{6N}$$

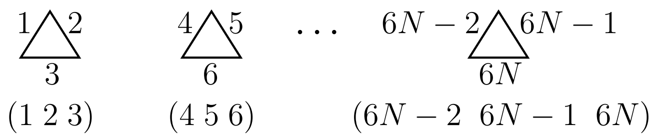

FIGURE 3. $2N$ labeled triangles.

The cycle type of τ (the fact that τ has exactly $3N$ two-cycles and no cycles of any other length) determines a conjugacy class in the symmetric group $\mathfrak{S}_{6N}$. As such, Ω_N can be identified with a conjugacy class, which we shall denote by $K\left(2^{3N}\right) \subset \mathfrak{S}_{6N}$.

Let us denote the conjugacy class of σ by $K\left(3^{2N}\right) \subset \mathfrak{S}_{6N}$. In our model for random surfaces, σ is fixed. We could of course also randomly pick it in $K\left(3^{2N}\right)$. This would just come down to a random relabeling of the triangles and as such wouldn't change the probabilities of any graph theoretic or topological property. This leads to a probability space

$$\Omega'_N = K\left(3^{2N}\right) \times K\left(2^{3N}\right)$$

endowed with the uniform probability measure. Let us denote the surface corresponding to $(\sigma, \tau) \in \Omega''_N$ by $S(\sigma, \tau)$ and the corresponding triangulation by $\mathcal{T}(\sigma, \tau)$.

So far, this description using the symmetric group might sound a little artificial. The crux however is that the number of vertices of the triangulation $\mathcal{T}$ of S (the only part of the Euler characteristic that does not come directly from the set up) can be expressed in terms of a permutation. Indeed, we claim that the number of vertices of $\mathcal{T}$ is exactly the number of cycles in a disjoint cycle decomposition of the permutation

$$\sigma\tau \in \mathfrak{S}_{6N}\,.$$

To see this, note that the permutation $\sigma\tau$ describes 'traversing the side between two triangles and then turning left'. Indeed, if σ is applied to a label l, then the label that comes out is exactly the label to the left of it on the same triangle. Likewise, if we apply τ, we obtain the label on the opposite side of the edge l represents. As such, the cycles in $\sigma\tau$ correspond one to one to "cycles" of triangles around a fixed vertex (see Figure 4).

Now what Gamburd, using work on random walks on finite groups due to Diaconis–Shahshahani [**DS81**], proved is that, as $N \to \infty$ over even numbers, the total variational distance between the random permutation $\sigma\tau$ and a permutation π chosen uniformly at random in the alternating group $\mathfrak{A}_{6N}$ tends to zero. Since the cycle statistics of such uniformly random permutations are well understood, the result follows. Chmutov–Pittel completed this by proving that when N is odd, $\sigma\tau$ is asymptotically uniform on the other coset of A_{6N}. □

Another consequence of the equidistribution Gamburd and Chmutov–Pittel prove is that we can say something about the large vertices of a random triangulated surface. $2N$ triangles have a total of $6N$ corners. When we glue these triangles together into a surface, these corneres are partitioned into sets according to the vertices at which they meet. The sizes of these sets (how many triangles meet in each vertex) give a partition

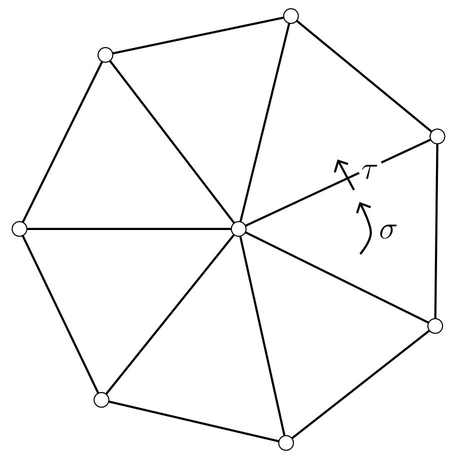

FIGURE 4. The correspondence between cycles in $\sigma\tau$ and vertices

of the number $6N$. If we normalize this partition (by dividing all the sizes by $6N$), we obtain a random variable

$$\pi_{\text{vert}} : \Omega_N \to \mathcal{P}_\infty := \left\{ \lambda \in [0,1]^{\mathbb{N}} \ : \ \sum_{i\in\mathbb{N}} \lambda_i = 1, \ \lambda_1 \geq \lambda_2 \geq \ldots \right\}.$$

It now follows from the equidistribution proved by Gamburd and Chmutov–Pittel that, as $N \to \infty$, π_{vert} converges in distribution to a *Poisson-Dirichlet* distributed random variable

$$\pi_{\text{PD}} : \Omega \to \mathcal{P}_\infty .$$

Such a variable can be described by simple stick breaking process. We start with an interval of length 1, break it in two at a point chosen uniformly at random using the Lebesgue measure, then break the piece on the left in two at a uniformly random point and repeat this ad inifinitum. The partition is now given by the lengths of the resulting pieces of stick, reordered by size so that the image lies in the set $\mathcal{P}_\infty$.

4.2.1.3. *The geometry.* Brooks–Makover [**BM04**] used the topological model described above to obtain a model for random closed hyperbolic surfaces of large genus. As we mentioned above, the idea is as follows:

(1) Glue ideal hyperbolic triangles according to the configuration. The result of this will be a hyperbolic surface with punctures (coming from the missing vertices of ideal triangles).
(2) Compactify the surface, from which we obtain (generically) a closed hyperbolic surface.

Let us elaborate a little bit on how both steps work, starting with step 1. First of all, we will of course glue the ideal hyperbolic triangles together with isometries so that the hyperbolic metric on them descends. However, because their sides have infinite length there is not just one isometry between a pair of sides. Figure 5 illustrates this issue with two gluings.

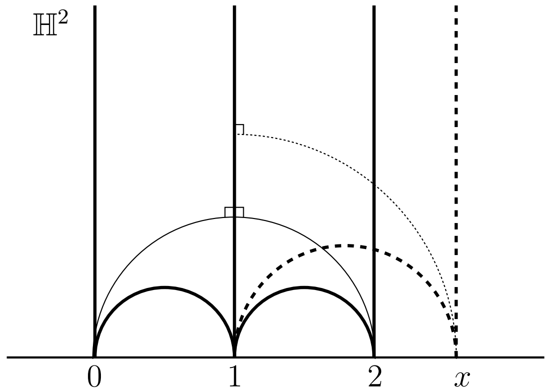

FIGURE 5. Shear.

We need one pair of points that are identified to determine the isometry between a pair of sides entirely. A natural candidate for this pair of points is constructed as follows. On both triangles involved in the gluing, take the vertex that is not part of the side in question and drop the unique perpendicular from the vertex to the side (in the figure above, these are the dotted lines). This defines two points on the side.

In a general gluing, the distance between the two special points will be called the *shear* of the gluing. We will always set the shear to 0 in our gluings. Together with this information, a configuration $\omega \in \Omega_N$ now specifies a hyperbolic surface $S_o(\omega)$ with punctures.

We will only sketch step 2 (for more details see [**Bro99, BM04**]), which if the genus $g(S(\omega))$ of $S_o(\omega)$ is at least 2 produces a closed hyperbolic surface $S_c(\omega)$. It relies on the uniformization theorem. I.e. to every isometry classes of complete hyperbolic metrics on a finite type surface (of negative Euler characteristic) corresponds a unique complex structure up to biholomorphism.

The idea is that around every puncture we can find a region that is isometric / biholomorphic to

$$C_t = \left\{ z \in \mathbb{H}^2 \,:\, \mathrm{Im}(z) > t \right\} / (z \mapsto z+1)$$

for some $t > 0$. This is not so hard to see from our construction. From this region we can find a biholomorphic map to the punctured unit disk

$$\{ z \in \mathbb{C} \,:\, 0 < |z| < 1 \}.$$

Adding the point $z = 0$ in for every region gives us a closed surface with a complex structure on it. As such, the uniformization theorem gives us a hyperbolic metric when the genus is at least 2 (which, given Theorem 4.2.2, is typical). We will denote the hyperbolic surface we obtain by $S_c(\omega)$.

The reason that this leads to an interesting model for random closed hyperbolic surfaces is the following theorem due to Belyĭ [**Bel79**]:

THEOREM 4.2.3. *The inclusion*

$$\left(\bigcup_{N=1}^{\infty} \{ S_c(\omega) \,:\, \omega \in \Omega_N \} \bigcap \mathcal{M}_g \right) \subset \mathcal{M}_g$$

is dense for every $g \geq 2$*.*

Belyĭ's theorem is actually a theorem about when an algebraic curve over the complex numbers can be written as a curve over $\overline{\mathbb{Q}}$. The statement above relies on the identification of these curves with hyperbolic surfaces (see [**JS96**] for more information). It should also be noted that the analogous statement for the surfaces $S_o(C)$ is false: for every pair $(g, n) \in \mathbb{N}^2$ we only obtain finitely many surfaces of genus g with n punctures, whereas a similar construction to that in Section 1.3 shows that there are uncountably many hyperbolic surfaces of genus g and with n punctures.

4.2.1.4. *Geometric properties of random surfaces.* We now connect back up to our extremal problems. Brooks and Makover proved the following theorem:

THEOREM 4.2.4 ([**BM04**]). *There exist positive constants C_1, C_2, C_3 and C_4 such that:*

(a) The first eigenvalue $\lambda_1(S_c(C))$ satisfies

$$\mathbb{P}_N[\lambda_1(S_c) \geq C_1] \to 1.$$

(b) The Cheeger constant $h(S_c(C))$ satisfies

$$\mathbb{P}_N[h(S_c) \geq C_2] \to 1.$$

(c) The shortest geodesic $\mathrm{sys}(S_c(C))$ *satisfies*

$$\mathbb{P}_N[\mathrm{sys}(S_c) \geq C_3] \to 1.$$

(d) The diameter $\mathrm{diam}(S_c(C))$ *satisfies*

$$\mathbb{P}_N[\mathrm{diam}(S_c) \leq C_4 \log(g)] \to 1.$$

PROOF SKETCH. Note that a combination items (b) and (c) together with Theorem 2.4.2 and 2.5.2 imply items (a) and (d). So, let us sketch the proof of these two facts, starting with item (b).

First of all, we will use the fact the dual graph to the triangulation on S – the graph whose vertices are the triangles of the triangulation, which share an edge if the triangles share a side (see Figure 6) – is a random 3-regular graph.

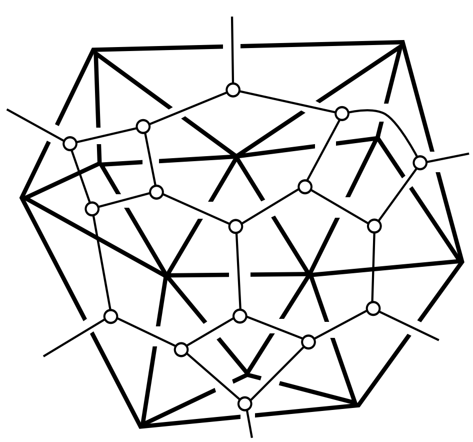

FIGURE 6. The dual graph to a triangulation

The corresponding configuration model for random regular graphs is very well studied. In particular, it is known that the Cheeger constant of these graphs $G = (V, E)$, defined as

$$h(G) = \min \left\{ \frac{|\{\text{edges between } U \text{ and } V \setminus U\}|}{|U|} \; : \; U \subset V, \; |U| \leq |V|/2 \right\}$$

is asymptotically uniformly bounded from below. That is, Bollobás [**Bol88**] proved that there exists a constant $C > 0$ such that

$$\lim_{n \to \infty} \mathbb{P}\left[\text{A random 3-regular graph } G \text{ on } n \text{ vertices satisfies } h(G) \geq C\right] = 1$$

In fact, Bollobás's result is effective, he proved that the result holds for $C = 0.1844\ldots$.

The next step is to turn this into a statement on surfaces. This uses a version of the Brooks–Burger transfer principle [**Bur86, Bro86**], (see also [**Bre14**]). The rough idea is that if the non-compact surface S_o has a subsurface $Y \subset S_o$ of large area and with a short boundary, then the triangles in the triangulation of S_o that have a large enough intersection with Y form a set of vertices in the dual G graph that connect to the rest of the graph with only a small number of edges (the latter comes from Y itself having a small boundary). This in turn would mean that the dual graph has a small Cheeger constant. After this, bounds by Brooks [**Bro99**] can be used to control the change of the Cheeger constant during the compactification process.

To control the systole of S_c, we observe that the geodesic realizing the systole on S_c is a closed curve that is not homotopic to a puncture of S_o. This means its trajectory in the triangulation contains at least one left hand turn and one right hand turn. A bit of hyperbolic trigonometry (using the fact that all the shears are 0) implies that length of the corresponding geodesic on S_o is at least $2 \cdot \text{arccosh}(3/2)$. Now using the same comparison results by Brooks [**Bro99**], this implies item (c). □

4.2.1.5. *Are these surfaces extremal?* Seeing Theorem 4.2.4, the first question is of course, do these surfaces solve any of the extremal problems posed above? In fact, Brooks and Makover introduced their model with this in mind, they were mainly interested in λ_1.

Let us discuss our invariants one by one, starting with the systole. Recall that we would be hoping for surfaces with *logarithmic* systole. Unfortunately, it turns out that, even if the systole of these surfaces does not tend to zero, it also doesn't tend to infinity. For instance, we have the following result, in which we will denote the *expected value* of a random variable $X : \Omega_n \to \mathbb{R}$ with respect to $\mathbb{P}_n$ by

$$\mathbb{E}_N[X] = \sum_{\omega \in \Omega_n} \mathbb{P}_n[\omega] \cdot X(\omega).$$

THEOREM 4.2.5 ([**Pet17**]). *In the Brooks–Makover model we have:*

$$\lim_{N \to \infty} \mathbb{E}_N[\text{sys}(S_c)] = 2.484\ldots.$$

Of course, this doesn't mean that these random surfaces can't be used to attack the question of maximal systole: we could still hope to prove that the probability $\mathbb{P}_N[\text{sys}(S_c) > C \cdot \log(g(S_c))]$ remains positive for a certain $C > 0$, thus establishing the existence of such surfaces. In the case of regular graphs, this has been shown to work for some (not (yet) optimal) $C > 0$ by McKay–Wormald–Wysocka [**MWW04**]. For Brooks–Makover random surface this is known to work up to length $\approx \log\log(g(S_c))$, due to work by Petri–Thäle [**PT18**], whether this can be pushed further is open.

Proof sketch of Theorem 4.2.5. The proof relies on computing the expected systole of the non-compact surface S_o and then using Brooks's work [**Bro99**] to control the expected systole if S_c.

In order to determine the systole of S_o, we study the random variables

$$Z_L : \Omega_N \to \mathbb{N}$$

that count the number of geodesics of length at most L on S_o. It turns out that these random variables converge to Poisson variables (see Theorem 4.2.6 below) with means that depend on L alone[1]. In [**Pet17**] this was done using the method of moments. Better bounds were later proved in [**PT18**] using the Chen–Stein method. Ignoring convergence issues, since

$$\mathbb{P}_n[\mathrm{sys}(S_o) \geq x] = \mathbb{P}[Z_x(S_o) = 0]$$

we can use the Poisson variables we found to write down an expression for the large n limit of $\mathbb{E}_n[\mathrm{sys}(S_c)]$, which gives the $2.484\ldots$ from the theorem. □

Let us discuss the Poisson distribution result mentioned in the proof above. The result states:

Theorem 4.2.6 ([**Pet17, PT18**]). *Fix $x > 0$ Let $Z_x : \Omega_N \to \mathbb{N}$ denote the random variable that counts the number of closed geodesics of length at most x on S_o.*

Then, as $N \to \infty$, Z_x converges in total variational distance to a Poisson-distributed random variable with a mean λ_x, depending on x alone. Moreover, λ_x is explicitly computable for every $x > 0$.

Proof. First of all, counting closed curves needs to be translated to a combinatorial question. First of all note that every closed curve in the surface S_o can be homotoped to a closed cycle in the dual graph to the triangulation of S_o (see Figure 7)

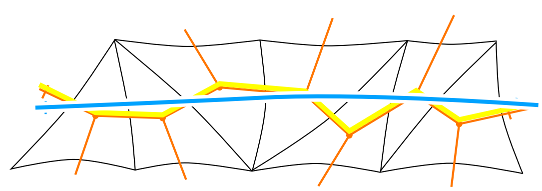

Figure 7. A closed curve and the cycle in the graph, homotopic to it

If this curve does not encircle a cusp on S_o, it is freely homotopic to a unique closed geodesic, minimizing the length in the homotopy class. The length of this geodesic can

[1]Recall that a random variable $X : \Omega \to \mathbb{N}$ is said to be Poisson-distributed with mean $\lambda > 0$ if and only if

$$\mathbb{P}[X = k] = \frac{\lambda^k e^{-\lambda}}{k!}.$$

for all $k \in \mathbb{N}$

easily be computed in terms of the combinatorics of the circuit. Indeed, if we think of $S_o = \Gamma\backslash\mathbb{H}^2$, we need to find an element $g \in \Gamma$ corresponding to our curve. This goes as follows. At every triangle the circuit traverses, it turns either right or left. If we record these turns we get a finite string of L's and R's. We now replace these letters with the matrices

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

So for every closed curve γ we obtain a matrix M_γ by multiplying out the matrices L and R according to the string. The geodesic $\widetilde{\gamma}$ homotopic to γ now has length

$$\ell(\widetilde{\gamma}) = 2 \cdot \operatorname{arccosh}\left(\frac{\operatorname{tr}(M_\gamma)}{2}\right)$$

Note that the word in L and R is only determined up to cyclic permutation: if we start at a different triangle, we get a cyclic permutation. Moreover, if we traverse the curve backwards, we get the word read backwards with L and R interchanged, because $L = R^t$, this corresponds to taking a transpose. Luckily the trace is invariant under cyclic permutations and transposes.

It is not hard to see that the number of words w in L and R with trace $2 < \operatorname{tr}(w) \leq 2\cosh(x/2)$ is finite for all $x > 0$. So, if we want to count all closed geodesics of length $\leq x$, we just need to count the number of appearances of the corresponding words as cycles in the dual graph to the triangulation of S_o. I.e.

$$Z_x = \sum_{w \in W_x} Z_{[w]},$$

where

- $W_x = \{\, w \text{ word in } \{L, R\} \,:\, 2 < \operatorname{tr}(w) \leq 2\cosh(x/2) \,\} / \sim$ and $w \sim w'$ if w' can be obtained from w' by a cyclic permutation and potentially reading it backwards and interchanging L and R.
- and $Z_{[w]}$ counts the number of cycles carrying $[w]$.

The next step is proving the convergence to Poisson variables. We will prove that $Z_{[w]}$ converges to a Poisson variable. This is almost enough to prove the claim about Z_x: sums of *independent* Poisson variables are Poisson variables. We'll comment in the end where the asymptotic independence comes from.

For simplicity of exposition, we will follow the proof from [**Pet17**], which relies on the method of moments. Let us first recall what this is. Given a random variable $X : \Omega \to \mathbb{N}$ and $k \in \mathbb{N}$, we will write

$$(X)_k = X \cdot (X-1) \cdots (X-k+1)$$

The method of moments now states that if we have a sequence $\{X_n\}_n$ of random variables and there exists a $\lambda > 0$ such that

$$\lim_{n\to\infty} \mathbb{E}[(X_n)_k] = \lambda^k \tag{3}$$

for all $k \in \mathbb{N}$, then, as $n \to \infty$, X_n converges in distribution to a Poisson distributed random variable with mean λ (see for instance [**Bol01**] for a proof).

So, the rest of our proof consists of proving (3) holds for our random variables $Z_{[w]} : \Omega_N \to \mathbb{N}$. Let us start with $\mathbb{E}_N[Z_{[w]}]$. We write

$$\mathbb{E}_N[Z_{[w]}] = \sum_{c \in C_{[w],N}} \mathbb{E}_N[\mathbb{1}_c]$$

where

- $C_{[w],N}$ is the set of labeled cycles with labels in $\{1, \ldots, 6N\}$ that carry the word $[w]$.
- $\mathbb{1}_c$ is the $\{0, 1\}$-valued random value that is 1 if and only if c appears in the configuration.

Note that

$$\mathbb{E}_N[\mathbb{1}_c] = \mathbb{P}_N[c \text{ apears in } S_o] = \frac{1}{(6N-1)(6N-3)\ldots(6N-2m+1)}$$

where m is the length of the cycle: the number of gluings of triangles involved in the cycle.

We will now separate or set $C_{[w],N}$ into finitely many sets $C^H_{[w],N}$ of cycles according to the isomorphism type H of the dual graph they represent. So we get

$$\mathbb{E}_N[Z_{[w]}] = \sum_H \left|C^H_{[w],N}\right| \frac{1}{(6N-1)(6N-3)\ldots(6N-2e(H)+1)}, \tag{4}$$

where $e(H)$ denotes the number of edges of H.

Our goal is now to prove that $\mathbb{E}_N[Z_{[w]}]$ is dominated by the term in which H is a a cycle that visits every vertex at most once. Let us write H_0 for this graph and first compute this term. We claim that

$$\left|C^{H_0}_{[w],N}\right| = 3^{|w|} \frac{|[w]|}{2\,|w|} 2N(2N-1)\cdots(2N-|w|+1).$$

In order to count this, we have to count the number of ways to label H_0.

- The factors $2N(2N-1)\cdots(2N-|w|+1)$ come from choosing a labeled triangle for each triangle in H.
- The factor $3^{|w|}$ comes from choosing a labelled side for one outgoing edge of H at each triangle. The label second outgoing edge is determined by w.
- We have to repeat this for every representative of $[w]$, which gives a factor $|[w]|$
- We overcount in the process described above, because it uses a starting point and a direction, so we divide by $2\,|w|$.

The above implies that

$$\left|C^{H_0}_{[w],N}\right| \frac{1}{(6N-1)(6N-3)\ldots(6N-2e(H_0)+1)} \overset{N\to\infty}{\longrightarrow} \frac{|[w]|}{2\,|w|}.$$

For the other graphs in the sum above, we have

$$\left|C^H_{[w],N}\right| \leq (2N)^{v(H)},$$

where $v(H)$ denotes the number of vertices of H. So, for those terms we have

$$\left|C^H_{[w],N}\right| \frac{1}{(6N-1)(6N-3)\dots(6N-2e(H)+1)} = O(N^{v(H)-e(H)}) \quad \text{as } N \to \infty$$

since the Euler characteristic $v(H) - e(H)$ of any connected graph H that is neither a circuit nor a tree is negative, this shows that the other terms in the sum in (4) tend to zero and we have

$$\mathbb{E}[Z_{[w]}] \overset{N\to\infty}{\longrightarrow} \frac{|[w]|}{2\,|w|},$$

which proves (3) for $k = 1$.

For the higher order moments, we observe that $(Z_{[w]})_k$ counts the number k-tuples of distinct $[w]$-carrying cycles. In particular, $\mathbb{E}[(Z_{[w]})_k]$ can be controlled with similar arguments –that we will not work out here – to the above. In order to prove the independence we mentioned earlier, we also need to control moments of the form $\mathbb{E}[(Z_{[w_1]})_k(Z_{[w_2]})_k \cdots (Z_{[w_m]})_k]$, since these random variables again count tuples of cycles, this can be done with similar methods as well. □

Random construction do not seem like a natural source for surfaces with large kissing numbers: Kissing numbers are a measure of symmetry and random objects are not symmetric. So, the next natural question is whether their diameter is extremal. Brooks–Makover (Theorem 4.2.4) already proved it's logarithmic, so the question is whether they (asymptotically) saturate the bound from Lemma 2.3.1. It turns out, from work of Budzinski–Curien–Petri, they miss this bound by a factor two:

Theorem 4.2.7 ([**BCP19**]). *The diameter of Brooks–Makover random surfaces satisfies:*

$$\operatorname{diam}(S_c) \sim 2 \cdot \log(g(S_c)) \quad \text{as } N \to \infty$$

in probability.

Proof sketch. We will not comment on the upper bound, but it uses a different (technically more involved) version of the peeling arguments in Section 4.1.2 below. The lower bound is a direct consequence of the fact that the sizes of the vertices asymptotically follow a Poisson–Dirichlet distribution. Indeed, it is not so hard to see from the stick-breaking description in Section 4.2.1.2 the triangulation will typically contain at least two vertices of linear size (in N). Again using Brooks's work [**Bro99**], it turns out that in the compactification process, there will be two large (of radius $\log(N) - O(1)$) disjoint embedded disks around these vertices. The distance between the midpoint of these disks is at least the sum of their radii, hence at least $2\log(N) - O(1)$. □

4.2.2. Weil–Petersson random surfaces. Another way to define a model for random surfaces is by picking a random point in the moduli space $\mathcal{M}_g$ of closed oriented surfaces of genus g.

For this, we need a measure on $\mathcal{M}_g$. We will use the measure coming from the *Weil–Petersson metric*. It's a theorem of Wolpert [**Wol82**], that on the Teichmüller space $\mathcal{T}_g$, this volume form is

$$d\,\mathrm{vol}_{\mathrm{WP}} = d\ell_1 \wedge d\tau_1 \wedge \cdots \wedge d\ell_{3g-3} \wedge d\tau_{3g-3},$$

where $\ell_1, \ldots, \ell_{3g-3}$ and $\tau_1, \ldots, \tau_{3g-3}$ denote the length and twist coordinates respectively with respect to a fixed pants decomposition. Moreover, this volume form descends to a volume form of finite total volume on $\mathcal{M}_g$. So, we may define a probability measure on $\mathcal{M}_g$ by

$$\mathbb{P}_g^{\mathrm{WP}}[A] = \frac{\mathrm{vol}_{\mathrm{WP}}(A)}{\mathrm{vol}_{\mathrm{WP}}}$$

for all measurable $A \subset \mathcal{M}_g$.

In [**Mir07**], Mirzakhani developed recursive methods to evaluate integrals of functions on $\mathcal{M}_g$. These methods also allow one to study the geometry of a random point in $\mathcal{M}_g$. Mirzakhani proved:

THEOREM 4.2.8 ([**Mir13**]). *We have*

(a) There exist constants $C, \varepsilon_0 > 0$ such that for all $g \geq 2$ and $\varepsilon < \varepsilon_0$

$$\frac{1}{C} \cdot \varepsilon^2 \leq \mathbb{P}_g^{\mathrm{WP}}[\mathrm{sys}(X) < \varepsilon] \leq C \cdot \varepsilon^2.$$

(b) For every $\varepsilon > 0$,

$$\lim_{g\to\infty} \mathbb{P}_g^{\mathrm{WP}}\left[h(X) \geq \frac{\log(2)}{2\pi + \log(2)} - \varepsilon\right] = 1.$$

(c) For every $\varepsilon > 0$,

$$\lim_{g\to\infty} \mathbb{P}_g^{\mathrm{WP}}\left[h(X) \geq \frac{\log^2(2)}{(4\pi + \log(4))^2} - \varepsilon\right] = 1.$$

(d) We have

$$\lim_{g\to\infty} \mathbb{P}_g^{\mathrm{WP}}[\mathrm{diam}(X) \leq 40\log(X)] = 1.$$

Note that $\frac{\log(2)}{2\pi+\log(2)} = 0.099\ldots$ and $\frac{\log^2(2)}{(4\pi+\log(4))^2} = 0.0024\ldots$.

We also observe that, except for item (a), Weil–Petersson random behave in a similar way to the Brooks–Makover model.

Moreover, they also satisfy a similar Poisson approximation theorem, due to Mirzakhani–Petri:

THEOREM 4.2.9 ([**MP19**]). *Fix $x > 0$ and let $N_x : \mathcal{M}_g \to \mathbb{N}$ denote the function that counts the number of closed geodesics of length $\leq x$. Then, as $g \to \infty$, N_x converges in distribution to a Poisson distributed random variable with mean*

$$\lambda_x = \int_0^x \frac{e^t + e^{-t} - 2}{2t} dt.$$

This was again proved with the method of moments, now in combination with Mirzakhani's integration methods for moduli space.

4.2.3. Random covers. Another model of a random surface can be defined as follows. Fix any hyperbolic surface X of finite area. Then X has finitely many covers of degree n for any $n \in \mathbb{N}$. So we get a random surface by picking such a cover uniformly at random.

First of all we note that for this model, it is already quite hard to count the number of elements in the probability space. This was first done by Müller–Schlage-Puchta [**MP02**] in the closd case and later generalized to all Fuchsian groups by Liebeck–Shalev [**LS04**]. Very recently, analogues of Theorems 4.2.4, 4.2.6 and 4.2.9 were proved by Magee–Puder [**MP20**] and Magee–Naud–Puder [**MNP20**] for random covers of a closed hyperbolic surface X. We mention in particular their theorem that if $\lambda_1(X) \geq \frac{3}{16}$ then for every $\varepsilon > 0$,

$$\mathbb{P}\left[\text{A random degree } n \text{ cover } Y \to X \text{ has } \lambda_1(Y) \geq \frac{3}{16} - \varepsilon\right] \overset{n\to\infty}{\longrightarrow} 1.$$

4.3. Random 3-manifolds

We haven't spoken much about higher dimensional random manifolds yet. One reason for this is that fewer models are around.

One model that is well studied in dimension three is that of random Heegard splittings and the related model of random mapping tori. We will not go into these models in this lectures, but mention that they behave very differently from the models of random surfaces we've seen. For instance, we've seen random surfaces always turn out to be highly connected. This is known not to hold for random Heegaard splittings and mapping tori: For both of these models $\lambda_1(M)$ tends to zero as $\mathrm{vol}(M)$ goes up Baik–Gekhtman–Hamenstädt [**BGH20**] and independently Lenzhen–Souto determined bounds on the rate at which this happens for mapping tori and Hamenstädt–Viaggi [**VH19**] determined the rate for random Heegaard splittings.

One of the problems in dimension three and above is that the natural analogue of the model by Brooks–Makover does not work: if you randomly glue n tetrahedra together along their faces, the probability that the result is a manifold tends to zero as $n \to \infty$ (see for instance [**DT06**] for a proof). However, the problem turns out to be only at the vertices of the complex. So, by truncating the tetrahedra, one obtains a model for a random 3-manifold with boundary. The resulting compact manifolds carry hyperbolic metrics of finite volume with totally geodesic boundary, their volume is linear in the number of tetrahedra and their Cheeger constants do not tend to zero. This is the subject of upcoming joint work with Jean Raimbault [**PR20**].

4.4. Questions

Besides the extremal questions we posed in the second lecture, there are also many interesting open questions about random manifolds. We will finish this text by listing a few:

Question 6. Find new models for random (hyperbolic) manifolds. In particular, find models that show similar properties to all the models of random surfaces we've seen above, like high connectivity.

QUESTION 7. What is the geometry of a random degree n cover of a fixed closed hyperbolic manifold? Is it highly connected? Does it Benjamini–Schramm converge to $\mathbb{H}^n$? Currently, even the asymptotic growth rate of the number of degree n covers as $n \to \infty$ are not known in dimension more than two.

QUESTION 8. Does any (or do all) of the models for random surfaces discussed above satisfy

$$\mathbb{P}[\lambda_1(X) \geq \frac{1}{4} - \varepsilon] \overset{\text{area}(X)\to\infty}{\longrightarrow} 1$$

for every $\varepsilon > 0$?

QUESTION 9. By Wang's Theorem (Theorem 1.5.1), the number of hyperbolic n-manifolds of volume $\leq v$ is finite for all $n \geq 4, v > 0$. What is the geometry of such a manifold picked uniformly at random?

Bibliography

[Ago06] Ian Agol. Systoles of hyperbolic 4-manifolds. Preprint, arXiv:math/0612290, 2006.

[Akr03] H. Akrout. Singularités topologiques des systoles généralisées. *Topology*, 42(2):291–308, 2003.

[Bav96] Christophe Bavard. Disques extrémaux et surfaces modulaires. *Ann. Fac. Sci. Toulouse Math. (6)*, 5(2):191–202, 1996.

[BC13] Nicolas Bergeron and Laurent Clozel. Quelques conséquences des travaux d'Arthur pour le spectre et la topologie des variétés hyperboliques. *Invent. Math.*, 192(3):505–532, 2013.

[BCP19] Thomas Budzinski, Nicolas Curien, and Bram Petri. The diameter of random belyi surfaces. Preprint, arXiv: 1910.11809, 2019.

[BCP20] Thomas Budzinski, Nicolas Curien, and Bram Petri. On the minimal diameter of closed hyperbolic surfaces. *Duke Math. J., to appear*, 2020+.

[Bea95] Alan F. Beardon. *The geometry of discrete groups*, volume 91 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. Corrected reprint of the 1983 original.

[Bel79] G. V. Belyĭ. Galois extensions of a maximal cyclotomic field. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(2):267–276, 479, 1979.

[Ber16] Nicolas Bergeron. *The spectrum of hyperbolic surfaces*. Universitext. Springer, Cham; EDP Sciences, Les Ulis, 2016. Appendix C by Valentin Blomer and Farrell Brumley, Translated from the 2011 French original by Brumley [2857626].

[BGH20] Hyungryul Baik, Ilya Gekhtman, and Ursula Hamenstädt. The smallest positive eigenvalue of fibered hyperbolic 3-manifolds. *Proc. Lond. Math. Soc. (3)*, 120(5):704–741, 2020.

[BLS92] M. Burger, J.-S. Li, and P. Sarnak. Ramanujan duals and automorphic spectrum. *Bull. Amer. Math. Soc. (N.S.)*, 26(2):253–257, 1992.

[BM01] Robert Brooks and Eran Makover. Riemann surfaces with large first eigenvalue. *J. Anal. Math.*, 83:243–258, 2001.

[BM04] Robert Brooks and Eran Makover. Random construction of Riemann surfaces. *J. Differential Geom.*, 68(1):121–157, 2004.

[Bol81] Béla Bollobás. Random graphs. In *Combinatorics (Swansea, 1981)*, volume 52 of *London Math. Soc. Lecture Note Ser.*, pages 80–102. Cambridge Univ. Press, Cambridge-New York, 1981.

[Bol88] Béla Bollobás. The isoperimetric number of random regular graphs. *European J. Combin.*, 9(3):241–244, 1988.

[Bol01] Béla Bollobás. *Random graphs*, volume 73 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2001.

[BP92] Riccardo Benedetti and Carlo Petronio. *Lectures on hyperbolic geometry*. Universitext. Springer-Verlag, Berlin, 1992.

[Bre14] Emmanuel Breuillard. Expander graphs, property (τ) and approximate groups. In *Geometric group theory*, volume 21 of *IAS/Park City Math. Ser.*, pages 325–377. Amer. Math. Soc., Providence, RI, 2014.

[Bro86] Robert Brooks. Combinatorial problems in spectral geometry. In *Curvature and topology of Riemannian manifolds (Katata, 1985)*, volume 1201 of *Lecture Notes in Math.*, pages 14–32. Springer, Berlin, 1986.

[Bro92] Robert Brooks. Some relations between spectral geometry and number theory. In *Topology '90 (Columbus, OH, 1990)*, volume 1 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 61–75. de Gruyter, Berlin, 1992.

[Bro99] Robert Brooks. Platonic surfaces. *Comment. Math. Helv.*, 74(1):156–170, 1999.

[BS87] Marc Burger and Viktor Schroeder. Volume, diameter and the first eigenvalue of locally symmetric spaces of rank one. *J. Differential Geom.*, 26(2):273–284, 1987.

[BS94] P. Buser and P. Sarnak. On the period matrix of a Riemann surface of large genus. *Invent. Math.*, 117(1):27–56, 1994. With an appendix by J. H. Conway and N. J. A. Sloane.

[BT11] Mikhail V. Belolipetsky and Scott A. Thomson. Systoles of hyperbolic manifolds. *Algebr. Geom. Topol.*, 11(3):1455–1469, 2011.

[Bur86] Marc Burger. Grandes valeurs propres du laplacien et graphes. In *Séminaire de Théorie Spectrale et Géométrie, No. 4, Année 1985–1986*, pages 95–100. Univ. Grenoble I, Saint-Martin-d'Hères, 1986.

[Bus80] P. Buser. On Cheeger's inequality $\lambda_1 \geq h^2/4$. In *Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979)*, Proc. Sympos. Pure Math., XXXVI, pages 29–77. Amer. Math. Soc., Providence, R.I., 1980.

[Bus82] Peter Buser. A note on the isoperimetric constant. *Ann. Sci. École Norm. Sup. (4)*, 15(2):213–230, 1982.

[Bus10] Peter Buser. *Geometry and spectra of compact Riemann surfaces.* Modern Birkhäuser Classics. Birkhäuser Boston, Ltd., Boston, MA, 2010. Reprint of the 1992 edition.

[Che70] Jeff Cheeger. A lower bound for the smallest eigenvalue of the Laplacian. In *Problems in analysis (Papers dedicated to Salomon Bochner, 1969)*, pages 195–199. 1970.

[Che75] Shiu Yuen Cheng. Eigenvalue comparison theorems and its geometric applications. *Math. Z.*, 143(3):289–297, 1975.

[Clo03] Laurent Clozel. Démonstration de la conjecture τ. *Invent. Math.*, 151(2):297–328, 2003.

[CP16] Sergei Chmutov and Boris Pittel. On a surface formed by randomly gluing together polygonal discs. *Adv. in Appl. Math.*, 73:23–42, 2016.

[DM20] Cayo Dória and Plinio Murillo. Hyperbolic 3-manifolds with large kissing number. Preprint, arXiv:2003.01863, 2020.

[DS81] Persi Diaconis and Mehrdad Shahshahani. Generating a random permutation with random transpositions. *Z. Wahrsch. Verw. Gebiete*, 57(2):159–179, 1981.

[DT06] Nathan M. Dunfield and William P. Thurston. Finite covers of random 3-manifolds. *Invent. Math.*, 166(3):457–521, 2006.

[FBP19] Maxime Fortier Bouque and Bram Petri. Kissing numbers of closed hyperbolic manifolds. Preprint, arXiv: 1905.11083, 2019.

[FBR20] Maxime Fortier Bouque and Kasra Rafi. Local maxima of the systole function, accepted. *Journal of the European Mathematical Society, to appear*, 2020+.

[FP15] Federica Fanoni and Hugo Parlier. Systoles and kissing numbers of finite area hyperbolic surfaces. *Algebr. Geom. Topol.*, 15(6):3409–3433, 2015.

[Gam06] Alex Gamburd. Poisson-Dirichlet distribution for random Belyi surfaces. *Ann. Probab.*, 34(5):1827–1848, 2006.

[GL14] Tsachik Gelander and Arie Levit. Counting commensurability classes of hyperbolic manifolds. *Geom. Funct. Anal.*, 24(5):1431–1447, 2014.

[GPS88] M. Gromov and I. Piatetski-Shapiro. Nonarithmetic groups in Lobachevsky spaces. *Inst. Hautes Études Sci. Publ. Math.*, (66):93–103, 1988.

[Ham01] Ursula Hamenstädt. New examples of maximal surfaces. *Enseign. Math. (2)*, 47(1-2):65–101, 2001.

[Hub06] John Hamal Hubbard. *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1.* Matrix Editions, Ithaca, NY, 2006. Teichmüller theory, With contributions by Adrien Douady, William Dunbar, Roland Roeder, Sylvain Bonnot, David Brown, Allen Hatcher, Chris Hruska and Sudeb Mitra, With forewords by William Thurston and Clifford Earle.

[IT92] Y. Imayoshi and M. Taniguchi. *An introduction to Teichmüller spaces.* Springer-Verlag, Tokyo, 1992. Translated and revised from the Japanese by the authors.

[Iwa95] Henryk Iwaniec. *Introduction to the spectral theory of automorphic forms.* Biblioteca de la Revista Matemática Iberoamericana. [Library of the Revista Matemática Iberoamericana]. Revista Matemática Iberoamericana, Madrid, 1995.

[Jen84] Felix Jenni. Über den ersten Eigenwert des Laplace-Operators auf ausgewählten Beispielen kompakter Riemannscher Flächen. *Comment. Math. Helv.*, 59(2):193–203, 1984.

[JS96] Gareth Jones and David Singerman. Belyĭ functions, hypermaps and Galois groups. *Bull. London Math. Soc.*, 28(6):561–590, 1996.

[Kee74] L. Keen. Collars on Riemann surfaces. pages 263–268. Ann. of Math. Studies, No. 79, 1974.

[Kim03] Henry H. Kim. Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2. *J. Amer. Math. Soc.*, 16(1):139–183, 2003. With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak.

[KSV07] Mikhail G. Katz, Mary Schaps, and Uzi Vishne. Logarithmic growth of systole of arithmetic Riemann surfaces along congruence subgroups. *J. Differential Geom.*, 76(3):399–422, 2007.

[LS04] M. W. Liebeck and A. Shalev. Fuchsian groups, coverings of Riemann surfaces, subgroup growth, random quotients and random walks. *J. Algebra*, 276(2):552–601, 2004.

[Mag20] Michael Magee. Letter to bram petri. available at http://maths.dur.ac.uk/ lcxt26/diameter.pdf, 2020.

[McM98] Curtis T. McMullen. Hausdorff dimension and conformal dynamics. III. Computation of dimension. *Amer. J. Math.*, 120(4):691–721, 1998.

[Mir07] Maryam Mirzakhani. Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces. *Invent. Math.*, 167(1):179–222, 2007.

[Mir13] Maryam Mirzakhani. Growth of Weil-Petersson volumes and random hyperbolic surfaces of large genus. *J. Differential Geom.*, 94(2):267–300, 2013.

[MNP20] Michael Magee, Frédéric Naud, and Doron Puder. A random cover of a compact hyperbolic surface has relative spectral gap $\frac{3}{16} - \varepsilon$. Preprint, arXiv:2003.10911, 2020.

[MP02] T. W. Müller and J. C. Puchta. Character theory of symmetric groups and subgroup growth of surface groups. *J. London Math. Soc. (2)*, 66(3):623–640, 2002.

[MP19] Maryam Mirzakhani and Bram Petri. Lengths of closed geodesics on random surfaces of large genus. *Comment. Math. Helv.*, 94(4):869–889, 2019.

[MP20] Michael Magee and Doron Puder. The asymptotic statistics of random covering surfaces. Preprint, arXiv: 2003.05892, 2020.

[Mur19] Plinio G. P. Murillo. Systole of congruence coverings of arithmetic hyperbolic manifolds. *Groups Geom. Dyn.*, 13(3):1083–1102, 2019. With an appendix by Cayo Dória and Murillo.

[MWW04] Brendan D. McKay, Nicholas C. Wormald, and Beata Wysocka. Short cycles in random regular graphs. *Electron. J. Combin.*, 11(1):Research Paper 66, 12, 2004.

[Par13] Hugo Parlier. Kissing numbers for surfaces. *J. Topol.*, 6(3):777–791, 2013.

[Pat88] S. J. Patterson. On a lattice-point problem in hyperbolic space and related questions in spectral theory. *Ark. Mat.*, 26(1):167–172, 1988.

[Pet17] Bram Petri. Random regular graphs and the systole of a random surface. *J. Topol.*, 10(1):211–267, 2017.

[Pet18] Bram Petri. Hyperbolic surfaces with long systoles that form a pants decomposition. *Proc. Amer. Math. Soc.*, 146(3):1069–1081, 2018.

[PR20] Bram Petri and Jean Raimbault. A model for random three-manifolds. In preparation, 2020.

[PT18] Bram Petri and Christoph Thäle. Poisson approximation of the length spectrum of random surfaces. *Indiana Univ. Math. J.*, 67(3):1115–1141, 2018.

[PW18] Bram Petri and Alexander Walker. Graphs of large girth and surfaces of large systole. *Math. Res. Lett.*, 25(6):1937–1956, 2018.

[Rai13] Jean Raimbault. A note on maximal lattice growth in $\mathrm{SO}(1, n)$. *Int. Math. Res. Not. IMRN*, (16):3722–3731, 2013.

[Rat06] John G. Ratcliffe. *Foundations of hyperbolic manifolds*, volume 149 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2006.

[Ril75a] Robert Riley. Discrete parabolic representations of link groups. *Mathematika*, 22(2):141–150, 1975.

[Ril75b] Robert Riley. A quadratic parabolic group. *Math. Proc. Cambridge Philos. Soc.*, 77:281–288, 1975.

[Sch93] P. Schmutz. Riemann surfaces with shortest geodesic of maximal length. *Geom. Funct. Anal.*, 3(6):564–631, 1993.

[Sch94] Paul Schmutz. Congruence subgroups and maximal Riemann surfaces. *J. Geom. Anal.*, 4(2):207–218, 1994.

[Sel65] Atle Selberg. On the estimation of Fourier coefficients of modular forms. In *Proc. Sympos. Pure Math., Vol. VIII*, pages 1–15. Amer. Math. Soc., Providence, R.I., 1965.

[SS97] P. Schmutz Schaller. Extremal Riemann surfaces with a large number of systoles. In *Extremal Riemann surfaces (San Francisco, CA, 1995)*, volume 201 of *Contemp. Math.*, pages 9–19. Amer. Math. Soc., Providence, RI, 1997.

[Thu78] W. P. Thurston. The geometry and topology of three-manifolds. available at http://msri.org/publications/books/gt3m/, 1978.

[Thu82] William P. Thurston. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc. (N.S.)*, 6(3):357–381, 1982.

[VH19] Gabriele Viaggi and Ursula Hamenstädt. Small eigenvalues of random 3-manifolds. Prerpint, arXiv: 1903.08031, 2019.

[Wan72] Hsien Chung Wang. Topics on totally discontinuous groups. pages 459–487. Pure and Appl. Math., Vol. 8, 1972.

[Wol82] Scott Wolpert. The Fenchel-Nielsen deformation. *Ann. of Math. (2)*, 115(3):501–528, 1982.

[Wor81] Nicholas C. Wormald. The asymptotic connectivity of labelled regular graphs. *J. Combin. Theory Ser. B*, 31(2):156–167, 1981.