Lecture 10

Random triangulated surfaces

10.1 The Euler characteristic

In what follows we will estimate the genus of a random surface. We will follow Brooks and Makover's paper for this [BM04] ([DT06] give a similar proof).

We start with a definition. In this definition, a *triangulation* $\mathcal{T} = (V, E, F)$ of a closed surface S will be the data of a finite set of points $V = \{v_1, \dots . v_k\} \in S$ (called *vertices*), a finite set of arcs $E = \{e_1, \dots, e_l\}$ with endpoints in the vertices (called edges) so that the complement $S \smallsetminus (\cup v_i \cup e_j)$ consists of a collection of disks $F = \{f_1, \dots, f_m\}$ (called *faces*) that all connect to exactly 3 edges.

Note that a triangulation $\mathcal{T}$ here is a slightly more general notion than that of a simplicial complex (it's an example of what Hatcher calls a Δ-complex [Hat02]). Figure 10.1 below gices an example of a triangulation of a torus that is not a simplicial complex.

We also note that the triangulated random surfaces we defined in the previous lecture (unsurprisingly) naturally come with a triangulation.

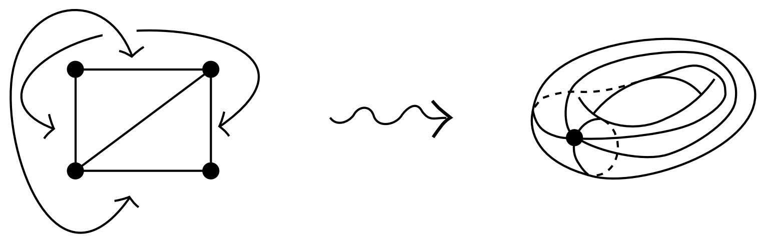

Figure 10.1: A torus with a triangulation

Definition 10.1. S be a surface with a triangulation $\mathcal{T} = (V, E, F)$. The *Euler characteristic* of S is given by

$$\chi(S) = |V| - |E| + |F|.$$

Because $\chi(S)$ can be defined entirely in terms of singular homology (see [Hat02][Theorem 2.4] for details), it is a homotopy invariant. In particular this implies it should only depend on the genus of our surface S. We have

Lemma 10.2. *Let S be a closed connected and oriented surface of genus g. We have*

$$\chi(S) = 2 - 2g.$$

Proof. See Exercise 10.1. □

Just like pants decompositions, triangulations also have dual graphs. The dual graph to a triangulation also comes with a cyclic order of the edges incident to every vertex. Such an order is called an *orientation*. The graph together with this order is sometimes called an oriented graph (*ribbon graph* and *fatgraph* are also terms that appear in the literature).

Definition 10.3. Let S be a closed surface and let $\mathcal{T} = \{V, E, F\}$ be a triangulation of X. The *dual graph* $G_{\mathcal{T}}$ to $\mathcal{T}$ is the graph obtained by setting

- $V(G_{\mathcal{T}}) = F$, the set of faces of $\mathcal{T}$,
- $E(G_{\mathcal{T}}) = E$ the set of edges of $\mathcal{T}$
- an edge in $E(G_{\mathcal{T}})$ is incident to a face if it is a boundary component of that face.

The orientation on $G_{\mathcal{T}}$ is induced by the orientation of S. That is, three edges incident to a face are said to be cyclicly oriented if their orientation agrees with that on the surface.

It is not hard to see that, just like the dual graph to a pants decomposition, the dual graph to a triangulation is 3-regular.

The dual graph to a triangulation can be embedded into the corresponding surface. Figure 10.2 gives an example:

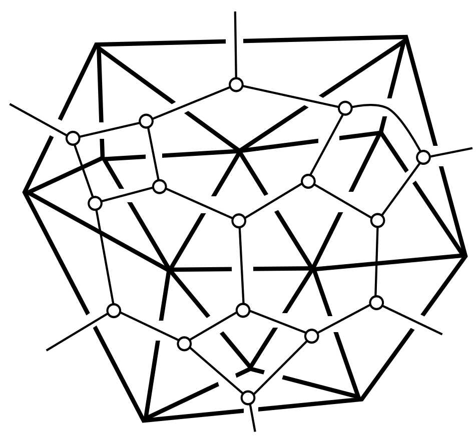

Figure 10.2: A part of a triangulation and its dual graph

Finally we note that if a 3-regular graph is oriented, we can make sense of left hand and right hand turns at a vertex. That is, if a path traverses a vertex in an oriented graph, then we say it turns left if it traverses the vertex in the direction opposite to the cyclic order and right if it traverses the vertex in the direction of the cyclic order. Figure 10.3 gives an example.

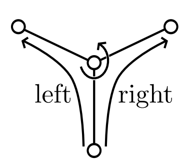

Figure 10.3: A left and a right turn at a vertex. The arrow at the middle vertex indicates the orientation at that vertex.

10.2 The genus of a random surface

Now we return to the random surfaces we associated to configurations. Because these random surfaces come with a triangulation, we can associate a dual graph $G(\omega)$ to each $\omega \in \Omega_N$. Of course, this is exactly the graph associated to ω in the configuration model for random 3-regular graphs. Moreoever the cyclic order at the vertices is by construction the cyclic order of the labels at those vertices.

Our first observation is the following:

Lemma 10.4. *Let $N \in \mathbb{N}$ and $\omega \in \Omega_N$ so that $S(\omega)$ is connected. Moreover, let $L(\omega)$ denote the number of cycles in $G(\omega)$ that consist of left hand turns exclusively. Then the genus $g(\omega)$ of $S(\omega)$ is given by*

$$g(\omega) = 1 + \frac{N}{2} - \frac{L(\omega)}{2}.$$

Proof. ω comes with a triangulation that has $2N$ faces and $3N$ edges (we start with $6N$ edges and pair them). The number of vertices $V(\omega)$ is not immediately clear. As such

$$2 - 2g(\omega) = \chi(S(\omega)) = V(\omega) + 2N - 3N = V(\omega) - N.$$

Hence

$$g(\omega) = 1 + \frac{N}{2} - \frac{V(\omega)}{2}.$$

The crucial observation is now that $V(\omega) = L(\omega)$. Indeed, the dual graph to the triangles around every vertex form a left hand turn cycle and conversely, gluing triangles along a left hand turn cycle leads to a vertex. $\square$

We immediately conclude:

Lemma 10.5. *Let $N \in \mathbb{N}$ and $\omega \in \Omega_N$ so that $S(\omega)$ is connected. Then*

$$g(\omega) \leq \frac{N+1}{2}.$$

Proof. The number of left hand turn cycles (or equivalently the number of vertices) is at least 1. $\square$

We actually claim that on average, the genus of a random surface is quite close to what it maximally can be. That is, it is equal to $N/2$ with a small error term. To make this precise, we have the following theorem due to Brooks and Makover [BM04]:

Theorem 10.6. *Let $N \in \mathbb{N}$. We have*

$$\mathbb{E}_N[L] \leq \frac{3}{2}\log(3N) + 3.$$

Proof. To prove this, we are going to slightly modify our probability space. Let Ω'_N be the set of ordered configurations. That is, Ω'_N contains the same configurations as Ω_N, but we make a distinction between the different order in which the pairs of labels appear in the configuration. As such

$$|\Omega'_N| = (3N)! \cdot |\Omega_N|.$$

The probability measure on Ω'_N is again just the uniform measure.

From the point of view of graphs or surfaces, the order in a configuration of course doesn't make a difference. As such, we might as well compute $\mathbb{E}_N[L]$ using Ω'_N.

The point of working with Ω'_N is that we can now speak of what the i^{th} pair of sides is that is glued together and what the graph looks like after the i^{th} step (for $i = 1, \ldots, 3N$). Figure 10.4 shows an example:

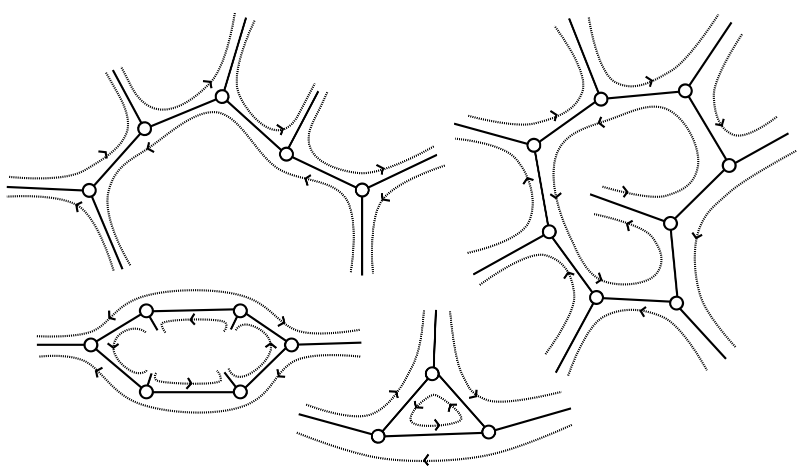

Figure 10.4: What the graph might look like after 21 steps. Half edges are added to each vertex in order to make the total degree 3.

Now let $L_i(\omega)$ be the number of left hand turn cycles created in the i^{th} step in $\omega \in \Omega'_N$. Clearly

$$L(\omega) = \sum_{i=1}^{3N} L_i(\omega)$$

and hence

$$\mathbb{E}_N[L] = \sum_{i=1}^{3N} \mathbb{E}[L_i(\omega)].$$

So, one strategy would be to try to control the distribution of $L_i(\omega)$. This turns out to be difficult as such, but a slight modification will work.

In step i we can create either 0, 1 or 2 extra left hand turn cycles. To see this, note that, when we draw all the left hand turn paths in the graph at before step i (the dotted paths in Figure 10.4), at every half-edge that is yet unpaired, one path starts and one path ends (note that these might be one and the same path, like in the component on the top right in Figure 10.4). As such, we can connect at most two pairs of paths into left hand turn cycles when pairing two edges. To see that two is possible, consider the component on the bottom left in Figure 10.4.

In the picture Figure 10.4 it is impossible to create two left hand turn cycles in the next step with most of the unpaired half edges (there is only one pair of half edges with which we can do this. Actually, with most of the unpaired half edges it is possible to create one left hand turn cycle in exactly two ways (the exceptions being one of unpaired half edges in the top right component and the two unpaired half edges in the bottom left component).

These special cases turn out to be the main issue in the proof. As such, let us give them a name. We will call an unpaired half edge e a *bottleneck* if the left hand turn path ending at it and the left hand turn path starting at it both connect it to the same half edge e' (we have already seen that the case $e = e'$ is possible).

Now, let $B_i(\omega)$ denote the number of bottle necks that are created at step i. If we want to create a left hand turn cycle with a bottle neck, we clearly need to pair it with another bottle neck. Hence

$$L(\omega) \leq \sum_{i=1}^{3N} L_i^*(\omega) + \frac{1}{2} \sum_{i=1}^{3N} B_i(\omega),$$

where $L_i^*(\omega)$ denotes the number of left hand turn cycles that are created without using bottle necks. The inequality here comes from the fact that at some point, a bottle neck may be destroyed again.

Now note that before the i^{th} step, there are $6N - 2i + 2$ half edges left to form the i^{th} pair with. We have the following cases

1. If the first edge that is chosen for the pair is not a bottleneck, then by the arguments above, we have that
$$\mathbb{E}[L_i^*] = \frac{2}{6N - 2i + 1}.$$
By a similar argument, there are at most two half edges with which we may create a bottleneck, hence
$$\mathbb{E}[B_i] \leq \frac{2}{6N - 2i + 1}.$$

2. If the first edge that is chosen is a bottleneck, then
$$\mathbb{E}[L_i^*] = \mathbb{E}[B_i] = 0.$$

So in either case, we have
$$\mathbb{E}[L_i^*] \leq \frac{2}{6N - 2i + 1} \text{ and } \mathbb{E}[B_i] \leq \frac{2}{6N - 2i + 1}.$$
Hence
$$L(\omega) \leq \frac{3}{2} \sum_{i=1}^{3N} \frac{2}{6N - 2i + 1} = \frac{3}{2} \sum_{i=1}^{3N} \frac{1}{(3N - i) + 1/2}.$$
Now we use that
$$\frac{1}{x + 1/2} \leq \log(x + 1) - \log(x)$$
for all $x \geq 1$ and we are done. □

We obtain:

Corollary 10.7. *Let $N \in \mathbb{N}$. We have*
$$\frac{N}{2} - \frac{3}{4}(\log(3N) + 2) \leq \mathbb{E}_N[g] \leq \frac{N}{2} + \frac{1}{2}.$$

Proof. The upper bound is direct from Lemma 10.5. The lower bound comes from putting together Lemma 10.4 and Theorem 10.6. □

10.3 Exercises

Exercise 10.1. In this exercise we prove Lemma 10.2 in two different ways.

1. (a) Describe a way to obtain a closed oriented surface of genus g from a polygon with $4g$ sides.

   (b) Use a triangulation of a $4g$-gon to prove Lemma 10.2.

2. (a) Show that if a closed oriented surface S is the connect sum of two closed oriented surfaces S_1 and S_2 then

$$\chi(S) = \chi(S_1) + \chi(S_2) - 2.$$

   (b) Compute the euler characteristic of the 2-sphere and the torus and use those to prove Lemma 10.2.

Exercise 10.2. Show that for a random surface in the configuration model we have

$$\mathbb{P}_N\left[g \leq \frac{N}{2} - x\right] \leq \frac{3\log(3N) + 6}{4x + 4}$$

Bibliography

[BM04] Robert Brooks and Eran Makover. Random construction of Riemann surfaces. *J. Differential Geom.*, 68(1):121–157, 2004.

[DT06] Nathan M. Dunfield and William P. Thurston. Finite covers of random 3-manifolds. *Invent. Math.*, 166(3):457–521, 2006.

[Hat02] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.