Lecture 4

Random graphs II

4.1 Automorphisms

It turns out that a typical regular graph on a large number of vertices does not have any non-trivial symmetries. This is orginally due to Bollobás [Bol82] and independently McKay and Wormald [MW84]. Exercise 4.3 follows a proof due to Wormald [Wor86].

Theorem 4.1. *Let* $k \in \mathbb{N}_{\geq 3}$*. We have*

$$\lim_{n\to\infty} \mathbb{E}_{n,k}\left[|\mathrm{Aut}(G)|\right] = 1.$$

Proof. See Exercise 4.3. □

4.2 The number of simple graphs

Given $n, k \in \mathbb{N}$ so that $n \cdot k$ is even, let $\mathcal{U}_{n,k}$ denote the set of ismorphism classes of simple k-regular graphs on n vertices. The following count is independently due to Bender and Canfield [BC78], Bollobás [Bol80] and Wormald [Wor78].

Theorem 4.2. *Let $k \in \mathbb{N}_{\geq 3}$. Then:*

$$|\mathcal{U}_{n,k}| \sim \frac{e^{-(k^2-1)/4}(n \cdot k)!!}{(k!)^n \cdot n!}$$

as $n \to \infty$.

Proof. Let $\mathcal{G}^*_{n,k}$ denote the subset of $\mathcal{G}_{n,k}$ consisting of configurations that give rise to a simple graph. We have an obvious map $\mathcal{G}^*_{n,k} \to \mathcal{U}_{n,k}$ that consists of forgetting the labels. This map is far from injective. However, it will turn out the cardinality of the fibers depends only on n, k and the number of automorphisms. As such, Theorem 4.1 tells us that up to a small error, we may assume that this cardinality is constant.

Let us work this idea out. The first thing we will do is add an intermediate step to the map $\mathcal{G}^*_{n,k} \to \mathcal{U}_{n,k}$. Let $\mathcal{V}_{n,k}$ denote the set of k-regular graphs with vertex set $\{1, \ldots, n\}$. We obtain maps

$$\mathcal{G}^*_{n,k} \to \mathcal{V}_{n,k} \to \mathcal{U}_{n,k}$$

by first forgetting the labels of the half-edges and then the labels on the vertices.

First note that the map $\mathcal{G}^*_{n,k} \to \mathcal{V}_{n,k}$ is constant to 1. Indeed, the number of pre-images of an element $G \in \mathcal{V}_{n,k}$ is equal to the number of ways to label the half edges at every vertex (note that this uses that G has no loops and no multiple edges). As such $\mathcal{G}^*_{n,k} \to \mathcal{V}_{n,k}$ is $(k!)^n$ to 1.

We have a natural action of $\mathfrak{S}_n \curvearrowright \mathcal{V}_{n,k}$, where $\mathfrak{S}_n$ denotes the symmetric group on n letters, by permuting the labels of the vertices. Furthermore

$$\mathcal{U}_{n,k} = \mathcal{V}_{n,k} / \mathfrak{S}_n \,.$$

By Burnside's lemma (see Exercise 4.1), we have

$$|\mathcal{U}_{n,k}| = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} |\{G \in \mathcal{V}_{n,k};\ \pi \cdot G = G\}|$$

Regrouping the terms sum, we get

$$\begin{aligned} |\mathcal{U}_{n,k}| &= \frac{1}{n!} \sum_{G \in \mathcal{V}_{n,k}} |\mathrm{Aut}(G)| \\ &= \frac{1}{n!} \sum_{a=1}^{n!} a \cdot |\{G \in \mathcal{V}_{n,k};\ |\mathrm{Aut}(G)| = a\}| \end{aligned}$$

Now we use that the map $\mathcal{G}^*_{n,k} \to \mathcal{V}_{n,k}$ is constant to 1 to obtain

$$|\mathcal{U}_{n,k}| = \frac{1}{(k!)^n \cdot n!} \sum_{a=1}^{n!} a \cdot \left| \left\{ G \in \mathcal{G}^*_{n,k};\ |\mathrm{Aut}(G)| = a \right\} \right|$$

Hence

$$\frac{\left| \left\{ G \in \mathcal{G}^*_{n,k};\ |\mathrm{Aut}(G)| = 1 \right\} \right|}{(k!)^n \cdot n!} \leq |\mathcal{U}_{n,k}|$$

and

$$|\mathcal{U}_{n,k}| \leq \frac{\left|\mathcal{G}^*_{n,k}\right|}{(k!)^n \cdot n!} + \frac{1}{(k!)^n \cdot n!} \sum_{a=2}^{n!} a \cdot \left| \left\{ G \in \mathcal{G}_{n,k};\ |\mathrm{Aut}(G)| = a \right\} \right|.$$

Let us now first work out the lower bound:

$$\frac{\left| \left\{ G \in \mathcal{G}^*_{n,k};\ |\mathrm{Aut}(G)| = 1 \right\} \right|}{(k!)^n \cdot n!} \geq \frac{\left|\mathcal{G}^*_{n,k}\right| - \left| \left\{ G \in \mathcal{G}_{n,k};\ |\mathrm{Aut}(G)| > 1 \right\} \right|}{(k!)^n \cdot n!}.$$

Note that a configuration $C \in \mathcal{G}_{n,k}$ gives rise to a simple graph if and only if $X_{n,k,1}(C) = X_{n,k,2}(C) = 0$. As such Corollary 3.8 implies that

$$\left|\mathcal{G}^*_{n,k}\right| = \mathbb{P}_{n,k}\left[X_{n,k,1}(C) = X_{n,k,2}(C) = 0\right] \cdot |\mathcal{G}_{n,k}| \sim e^{-\lambda_{k,1} - \lambda_{k,2}} \cdot |\mathcal{G}_{n,k}|$$

as $n \to \infty$. On the other hand, Theorem 4.1 implies that

$$\left| \left\{ G \in \mathcal{G}_{n,k};\ |\mathrm{Aut}(G)| > 1 \right\} \right| / |\mathcal{G}_{n,k}| \to 0$$

as $n \to \infty$.

For the upper bound we note that

$$\sum_{a=2}^{n!} a \cdot \frac{\left| \left\{ G \in \mathcal{G}_{n,k};\ |\mathrm{Aut}(G)| = a \right\} \right|}{|\mathcal{G}_{n,k}|} = \mathbb{E}_{n,k}\left[|\mathrm{Aut}(G)|\right] - \mathbb{P}_{n,k}\left[|\mathrm{Aut}(G)| = 1\right].$$

Thus

$$\sum_{a=2}^{n!} a \cdot \frac{\left| \left\{ G \in \mathcal{G}_{n,k};\ |\mathrm{Aut}(G)| = a \right\} \right|}{|\mathcal{G}_{n,k}|} \to 0$$

as $n \to \infty$ by Theorem 4.1.

Putting all of the above together, we see that

$$|\mathcal{U}_{n,k}| \sim e^{-\lambda_{k,1} - \lambda_{k,2}} \cdot \frac{|\mathcal{G}_{n,k}|}{(k!)^n \cdot n!} = \frac{e^{-(k-1)/2 - (k-1)^2/4} (n \cdot k)!!}{(k!)^n \cdot n!} = \frac{e^{-(k^2-1)/4} (n \cdot k)!!}{(k!)^n \cdot n!}.$$

□

4.3 Expansion

This section is mainly based on [HLW06].

4.3.1 Definition

Loosely speaking, an expander graph is a sequence of graphs that is both sparse and well-connected. There are many applications for these sequences of graphs. One of the earliest contexts in which they came up is in a problem from computer science: building a large network of computers in which it's not possible to disconnect a large piece of the network by cutting a small number of cables (well-connectedness) but not connecting too many computers to each other (sparseness).

There are two things to be made precise: sparseness and well-connectedness of a graph. A good candidate for the notion of sparseness is of course a uniform bound on the degree. We will set a stronger condition and assume k-regularity for some fixed k.

The idea of well-connectedness can be made precise with the Cheeger constant. In the following definition, given a graph G and a set of vertices $U \subset V(G)$, we will denote the set of edges that connect U to $V(G) \setminus U$ by ∂U.

Definition 4.3. Let G be a finite connected graph. The *Cheeger constant* or *Expansion ratio* of G is given by:

$$h(G) = \min \left\{ \frac{|\partial U|}{|U|};\ U \subset V(G),\ |U| \leq |V(G)|/2 \right\}.$$

An expander graph will be a sequence of graphs that is both sparse and well-connected:

Definition 4.4. Fix $k \in \mathbb{N}_{\geq 3}$. An *expander graph* is a sequence $(G_n)_{n\in\mathbb{N}}$ of connected k-regular graphs so that

$$|V(G_n)| \to \infty$$

as $n \to \infty$ and there exists a $\varepsilon > 0$ so that

$$h(G_n) > \varepsilon$$

for all $n \in \mathbb{N}$.

4.3.2 Eigenvalues

There exists an equivalent characterisation of expander graphs in terms of eigenvalues. We first need to define what an adjacency matrix is.

Definition 4.5. Given a graph G on the vertex set $\{1, \ldots, n\}$. The adjacency matrix $A(G) \in \mathrm{M}_n(\mathbb{R})$ is given by

$$A(G)_{ij} = m \text{ if and only if } i \text{ and } j \text{ share } m \text{ edges.}$$

Note that $A(G)$ is a self-adjoint matrix and as such has real eigenvalues, let us denote these by $\lambda_1(G) \geq \lambda_2(G) \geq \ldots \geq \lambda_n(G)$. Note that these eigenvalues do not depend on the labelling of the vertices. As such it makes sense to speak of the eigenvalues associated to a graph G.

Lemma 4.6. *Let G be a finite k-regular graph*

(a) $\lambda_1(G) = k$.

(b) G is connected if and only if $\lambda_1(G) > \lambda_2(G)$.

Proof. Exercise 4.2. □

This lemma implies that, given a connected regular graph, the first non-trivial eigenvalue is given by

$$\lambda(G) = \lambda_2(G).$$

4.4 Exercises

Exercise 4.1. (Burnside's lemma) Let G be a finite group and X a finite set so that $G \curvearrowright X$. Prove that

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |\{x \in X;\ g \cdot x = x\}|.$$

Exercise 4.2.

(a) Let G be a graph and $A(G)$ its adjacency matrix. Show that

$$(A^r)_{ij}$$

records the number of walks of length r between vertices i and j.

(b) Prove Lemma 4.6. Hint for part (b) of the lemma: consider the eigenvalues of the matrix

$$\mathrm{Id}_n - \frac{1}{k}A(G),$$

where Id_n denotes the $n \times n$ identity matrix. In particular: show that the eigenfunctions corresponding to the eigenvalue 0 of this matrix are constant on connected components.

Exercise 4.3. Disclaimer: *This exercise is here for completeness, it will not be part of the material for the exam.*

In this exercise we prove Theorem 4.1. It states that for $k \geq 3$:

$$\lim_{n\to\infty} \mathbb{E}_{n,k}\left[|\mathrm{Aut}(G)|\right] = 1.$$

We will go through the proof of [Wor86] step by step.

In what follows, K_n will denote the complete graph (every vertex is connected to every other vertex) on vertex set $\{1, \ldots, n\}$ and $\mathfrak{S}_n$ the symmetric group on n letters. Note that $\mathfrak{S}_n \curvearrowright K_n$.

(a) Given $\sigma \in \mathfrak{S}_n$, let $A(\sigma)$ denote its support and let

$$\mathcal{H}(\sigma) = \left\{ H \text{ graph on } \{1,\ldots,n\};\ \begin{array}{c}\text{every edge of } H \text{ has at least one} \\ \text{end in } A(\sigma),\ \deg_H(v) \leq k \text{ for all} \\ v \in \{1,\ldots,n\} \text{ with equality for} \\ \text{all } v \text{ not fixed by } \sigma \text{ and } \sigma \cdot H = H \end{array} \right\}.$$

Note that $\sigma \in \mathfrak{S}_n$ is and automorphism of $G(C)$ for some $C \in \mathcal{G}_{n,k}$ if and only if $G(C)$ contains one of the graphs in $\mathcal{H}(\sigma)$ as a subgraph.

Given $H \in \mathcal{H}(\sigma))$, let $X_H : \mathcal{G}_{n,k} \to \mathbb{N}$ be the random variable that counts the number of appearances of H in a configuration. Note that H comes with labelled vertices, but not with labelled half edges. Show that there exists a $C > 0$, independent of n so that

$$\mathbb{E}_{n,k}\left[|\mathrm{Aut}(G)|\right] \leq \sum_{\sigma\in\mathfrak{S}_n} \sum_{H\in\mathcal{H}(\sigma))} \mathbb{E}_{n,k}\left[X_H\right] \leq 1 + \sum_{\sigma\in\mathfrak{S}_n\setminus\{\mathrm{Id}\}} \sum_{H\in\mathcal{H}(\sigma))} (C/n)^{e(H)},$$

where $e(H)$ denotes the number of edges in H and $\mathrm{Id} \in \mathfrak{S}_n$ denotes the identity element.

(b) We need to introduce some notation. Let $\sigma \in \mathfrak{S}_n$ and $H \in \mathcal{H}(\sigma)$.

- For $2 \leq i \leq 6$, let $s_i(\sigma)$ denote the number of i-cycles in σ.
- Let $a(\sigma)$ the cardinality of the support $A(\sigma)$ of σ in $\{1, \ldots, n\}$.
- Let $r(H, \sigma)$ denote the number of edges of $H_{A(\sigma)}$, the restriction of the graph H to $A(\sigma)$
- Let $e_1(H, \sigma)$ denote the number of edges of H fixed by σ.
- Let $m(H, \sigma) = r(H, \sigma) - f(H, \sigma)$, where $f(H, \sigma)$ denotes the number of orbits of σ on the set of edges $E(K_n)$ contained in H (equivalently these are the number of orbits of σ on the edges of H that have both of their endpoints in $A(\sigma)$).

Prove the following bounds on the number of choices for a permutation with parameters given:

- The number of subsets $A \subset \{1, \ldots, n\}$ of size of $a(\sigma)$ is bounded from above by
$$\frac{n^{a(\sigma)}}{a(\sigma)!}$$
- Given such a subset $A \subset \{1, \ldots, n\}$, the number of permutations $\sigma \in \mathfrak{S}_n$ with support A and s_i i-cycles for $i = 2, \ldots, 6$ is at most
$$\frac{a(\sigma)!}{\prod_{i=2}^{6} s_i!}.$$

(c) Now prove the following bounds for the number of graphs H:

- Given $\sigma \in \mathfrak{S}_n$ with support $A(\sigma)$ and $s_2(\sigma)$ 2-cycles. The number of choices for the $e_1(H, \sigma)$ edges on $A(\sigma)$ that are fixed by σ is at most
$$2^{a(\sigma)}.$$
- The remaining $r(H) - e_1(H, \sigma)$ edges of H with both their endpoints in A form $f(H, \sigma) - e_1(H, \sigma)$ full orbits of σ (otherwise the graph H would not be fixed by σ). Prove that there are at most
$$\frac{a(\sigma)^{2(r(H.\sigma) - m(H,\sigma) - e_1(H,\sigma))}}{(r(H, \sigma) - m(H, \sigma) - e_1(H, \sigma))!}$$

choices for these orbits.

- Prove that after these edges, there are $k \cdot a(\sigma) - 2 \cdot r(H, \sigma)$ edges in H left.
- Prove that these remaining edges lie in at most $(k \cdot a(\sigma) - 2 \cdot r(H, \sigma))/2$ orbits.
- Prove that there are at most

$$(n-a)^{(k \cdot a(\sigma) - 2 \cdot r(H,\sigma))/2}$$

ways to choose these orbits

(d) Use (b) and (c) to show that the contribution T of all pairs (σ, H) with all the parameters $s_2, \ldots, s_6$, a, r, e_1 and m fixed to the sum in (a) can be bounded by

$$T \leq \frac{B^a \cdot a^{2(r-m-e_1)} \cdot n^{(2-k)a/2}}{(r-m-e_1)! \prod_{i=2}^{6} s_i^{s_i}}$$

where $B > 0$ is some constant independent of n. Hint: use that there exists a constant $B' > 0$ so that

$$B'^p \cdot p^p \leq p! \leq p^p$$

for all $p \in \mathbb{N}$ (This uniform constant $B' > 0$ exists by Stirling's approximation).

(e) Show that this implies that

$$T \leq C \cdot n^{-C \cdot a}$$

(e) Conclude that by summing over all the possible values of the parameters $s_2, \ldots, s_6$, a, r, e_1 and m we get

$$\mathbb{E}_{n,k}\left[|\mathrm{Aut}(G)|\right] = 1 + O\left(n^{-C}\right)$$

as $n \to \infty$.

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