Lecture 5

Random graphs III: Expansion

5.1 Expansion through eigenvalues

The fact that expansion can also be measured using eigenvalues is the content of the following theorem by Dodziuk [Dod84], Alon-Milman [AM85] and Alon [Alo86]:

Theorem 5.1. *Let G be a finite connected k-regular graph, then*

$$\frac{k-\lambda(G)}{2} \leq h(G) \leq \sqrt{2k(k-\lambda(G))}.$$

Proof. We follow the proof from [HLW06, Theorem 4.11] and start with the lower bound on $h(G)$. Let us (arbitrarily) label the vertices of our graph by $\{1,\ldots,n\}$ and let $A = A(G)$ be the adjacency matrix of our graph G. Given vector $f \in \mathbb{R}^n \setminus \{0\}$, the *Rayleigh quotient* of f is given by

$$R_A(f) = \frac{\langle f, Af\rangle}{\langle f, f\rangle},$$

where $\langle\cdot,\cdot\rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the inner product.

Given $S \subset \{1,\ldots,n\}$, we define $f_S \in \mathbb{R}^n$ by

$$(f_S)_i = \begin{cases} n - |S| & \text{if } i \in S \\ -|S| & \text{if } i \notin S. \end{cases}$$

Note that

$$\langle f_S, (1, \ldots, 1)\rangle = 0$$

for all $S \subsetneq \{1, \ldots, n\}$ so that $S \neq \emptyset$. Now write

$$f_S = \sum_{i=2} \langle f_S, g_i\rangle \cdot g_i,$$

where $\{g_i\}_{i=1}^n$ is an orthonormal basis of eigenvectors of A, so that g_i correpsonds to λ_i for $i = 1, \ldots, n$. Note that we may choose $g_1 = \frac{1}{\sqrt{n}} \cdot (1, \ldots, 1)$. This implies that

$$\langle f_S, Af_S\rangle = \sum_{i=2} \langle f_S, g_i\rangle^2 \cdot \lambda_i \leq \lambda(G) \cdot ||f_S||^2 .$$

So we obtain

$$R_A(f_S) \leq \lambda(G).$$

On the other hand, an easy computation gives that

$$||f_S||^2 = n\,|S|\,(n - |S|)$$

and

$$\langle f_S, Af_S\rangle = n \cdot k \cdot |S|\,(n - |S|) - n^2\,|\partial S| .$$

Filling this in in the Rayleigh quotient for a set $S \subset \{1, \ldots, n\}$ so that $|S| \leq n/2$ and $h(G) = |\partial S| / |S|$, we obtain that

$$\lambda(G) \geq \frac{n \cdot k \cdot |S|\,(n - |S|) - n^2\,|\partial S|}{n\,|S|\,(n - |S|)} = k - \frac{n\,|\partial S|}{|S|\,(n - |S|)} \geq k - 2 \cdot h(G),$$

which proves the lower bound.

For the upper bound on $h(G)$ we will use the Laplacian matrix $L \in \mathrm{M}_n(\mathbb{R})$ given by

$$L = k \cdot \mathrm{Id}_n - A,$$

where $\mathrm{Id}_n \in \mathrm{M}_n(\mathbb{R})$ denotes the n-dimensional identity matrix. Note that L has eigenvalues $k - \lambda_i(G)$, corresponding to the same eigenvectors $g_i \in \mathbb{R}^n$ for all $i = 1, \ldots, n$. We will again consider the associated Rayleigh quotients, given by

$$R_L(f) = \frac{\langle f, Lf\rangle}{\langle f, f\rangle},$$

for all $f \in \mathbb{R}^n \setminus \{0\}$.

In what follows, we assume that at most half of the entries of $g_2 \in \mathbb{R}^n$ are positive (we may assume this, because we can replaye g_2 by $-g_2$). Let Define $f \in \mathbb{R}^n$ by

$$f_i = \max\{(g_2)_i, 0\},$$

for $i = 1, \ldots, n$.

We now make two claims:

Claim 1. We have:

$$R_L(f) \leq k - \lambda(G).$$

Claim 2. We have:

$$\frac{h(G)^2}{2k} \leq R_L(f).$$

Note that if we prove these two claims, we prove the theorem.

Proof of Claim 1. Let us write $\mathrm{supp}(f) = V^+ \subset \{1, \ldots, n\}$. For $i \in V^+$ we have:

$$\begin{aligned}
(Lf)_i &= k \cdot f_i - \sum_{j=1}^{n} A_{ij} f_j \\
&= k \cdot (g_2)_i - \sum_{j \in V^+} A_{ij} (g_2)_j \\
&\leq k \cdot (g_2)_i - \sum_{j=1}^{n} A_{ij} (g_2)_j \\
&= (Lg_2)_j \\
&= (k - \lambda(G)) \cdot (g_2)_i.
\end{aligned}$$

As such

$$\langle f, Lf \rangle \ = \ \sum_{i=1}^{n} f_i (Lf)_i \ \leq \ (k - \lambda(G)) \sum_{i \in V^+} (g_2)_i^2,$$

where we used that $f_i = 0$ for $i \notin V^+$ in the second step. This means that

$$\langle f, Lf \rangle \leq (k - \lambda(G)) \cdot ||f||^2,$$

which proves our claim. $\square$

Proof of Claim 2. To prove this claim, we note that

$$\langle f, Lf\rangle = \sum_{i=1}^{n} \left(k \cdot f_i^2 - \sum_{r=1}^{k} \sum_{\substack{j\in\{1,\dots,n\},\ i \text{ and } j \\ \text{share exactly } r \text{ edges}}} r \cdot f_i f_j \right)$$

By reordering the terms above, we can see this sum as a sum over the edges $E(G)$ of G to obtain:

$$\langle f, Lf\rangle = \sum_{e\in E(G)} (f_{v_1(e)} - f_{v_2(e)})^2,$$

where $v_1(e)$ and $v_2(e)$ are the (not necessarily distinct) endpoints of e (in arbitrary order).

Now we assume that the vertices $\{1, \dots, n\}$ are labelled so that $f_1 \geq f_2 \geq \dots \geq f_n$. We have

$$\begin{aligned} h(G) \cdot \|f\|^2 &= h(G) \sum_{i=1}^{n} f_i^2 \\ &= h(G) \cdot \sum_{i\in V^+} (f_i^2 - f_{i+1}^2) \cdot i \end{aligned}$$

The second equality follows from a telescoping argument and the fact that by assumption $f_{i+1} = 0$ for $i = |V^+|$. Set $[i] = \{1, \dots i\}$. By definition of the Cheeger constant, we have that

$$h(G) \leq |\partial[i]| / i.$$

So we obtain:

$$\begin{aligned} h(G) \cdot \|f\|^2 &\leq \sum_{i\in V^+} (f_i^2 - f_{i+1}^2) \cdot |\partial[i]| \\ &= \sum_{i=1}^{n-1} (f_i^2 - f_{i+1}^2) \cdot |\partial[i]| \\ &= \sum_{\substack{e\in E(G) \\ v_1(e)<v_2(e)}} \sum_{i=v_1(e)}^{v_2(e)-1} (f_i^2 - f_{i+1}^2). \end{aligned}$$

With another telescoping argument, we get:

$$\begin{aligned} h(G) \cdot ||f||^2 &\leq \sum_{\substack{e \in E(G) \\ v_1(e) < v_2(e)}} (f_{v_1(e)}^2 - f_{v_2(e)}^2) \\ &= \sum_{\substack{e \in E(G) \\ v_1(e) < v_2(e)}} (f_{v_1(e)} + f_{v_2(e)}) \cdot (f_{v_1(e)} - f_{v_2(e)}) \end{aligned}$$

Now we use the Cauchy-Schwarz inequality, which says that

$$\sum_{\substack{e \in E(G) \\ v_1(e) < v_2(e)}} (f_{v_1(e)} + f_{v_2(e)}) \cdot (f_{v_1(e)} - f_{v_2(e)})$$

$$\leq \sqrt{\sum_{\substack{e \in E(G) \\ v_1(e) < v_2(e)}} (f_{v_1(e)} + f_{v_2(e)})^2} \cdot \sqrt{\sum_{\substack{e \in E(G) \\ v_1(e) < v_2(e)}} (f_{v_1(e)} - f_{v_2(e)})^2}.$$

We have:

$$\sqrt{\sum_{\substack{e \in E(G) \\ v_1(e) < v_2(e)}} (f_{v_1(e)} + f_{v_2(e)})^2} \leq \sqrt{2 \sum_{\substack{e \in E(G) \\ v_1(e) < v_2(e)}} f_{v_1(e)}^2 + f_{v_2(e)}^2} = \sqrt{2k} \cdot ||f|| \,.$$

So, using our earlier observation on $\langle f, Lf \rangle$, we obtain:

$$h(G) \cdot ||f||^2 \leq \sqrt{2k} \cdot ||f|| \cdot \sqrt{\langle f, Lf \rangle},$$

which proves Claim 2. □

Putting Claims 1 and 2 together yields the theorem. □

We already alluded to the following immediate consequence of the theorem above:

Corollary 5.2. *Let $k \geq 3$. A sequence $(G_n)_n$ of k-regular graphs G_n on n vertices is an expander if and only if there exists an $\varepsilon > 0$ so that*

$$k - \lambda(G_n) > \varepsilon$$

for all $n \in \mathbb{N}$ (n even if k is odd).

$k - \lambda(G)$ is often called the *spectral gap* of G, note that it is also the smallest non-zero eigenvalue of the Laplacian

$$L(G) = k \cdot \mathrm{Id}_n - A(G)$$

of the graph G that we used in the proof of the theorem above.

5.2 Existence

We have not yet discussed whether or not expander graphs exist. The first proof of the existence of expanders actually was a random construction and is due to Pinsker [Pin73]. For instance due to work of Margulis [Mar73] and Lubotzky-Phillips-Sarnak [LPS88], there are also explicit examples of sequences of expander graphs.

We will give a probabilistic existence proof. In fact, it is known that random regular graphs are near optimal (their second eigenvalue $\lambda(G)$ is essentially as small as it could possibly be) expanders [Fri08] with probability tending to one. We will follow a shorter proof, due to Broder-Shamir [BS87], with a result that is less strong. Our exposition is based on that in [HLW06, Theorem 7.5].

5.2.1 A different model

We will consider a slightly different model than the configuration model that we have considered so far. This model is called the *permutation model* and works as follows. Given $k \in \mathbb{N}$ and elements $\pi_1, \ldots, \pi_k \in \mathfrak{S}_n$, where $\mathfrak{S}_n$ denotes the symmetric group on n letters, a $2k$-regular graph $G(\pi_1, \ldots, \pi_k) = (V, E, \mathcal{I})$ is obtained by setting

$$V = \{1, \ldots, n\},\ E = \left\{e_{i,\pi_j(i)};\ i = 1, \ldots, n, j = 1, \ldots, k\right\}$$

and

$$\mathcal{I} = \{(e_{i,j}, i),\ (e_{i,j}, j);\ i, j = 1, \ldots, n\}\,.$$

In other words, vertex i is connected to vertex $\pi_j(i)$ for all $j = 1, \ldots, k$. Figure 5.1 gives an example:

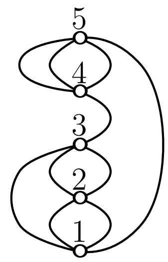

Figure 5.1: The graph corresponding to the permutations $\pi_1 = (1\ 2\ 3)(4\ 5)$ and $\pi_2 = (1\ 2\ 3\ 4\ 5)$.

As such, this model gives as a probability space

$$\Omega_{n,k}^{\mathrm{perm}} = \mathfrak{S}_n^k$$

of $2k$-regular graphs with the usual uniform probability measure $\mathbb{P}_{n,k}^{\mathrm{perm}}$. Furthermore, we have

$$\left|\Omega_{n,k}^{\mathrm{perm}}\right| = (n!)^k.$$

It should be stressed that when we consider $\mathbb{P}_{n,k}^{\mathrm{perm}}$ as a probability measure on the set of isomorphism classes of $2k$-regular graphs, we obtain a different measure than the measure $\mathbb{P}_{n,2k}$ coming from the configuration model.

It does turn out that $\mathbb{P}_{n,k}^{\mathrm{perm}}$, $\mathbb{P}_{n,2k}$ and the uniform measure $\mathbb{P}_{n,2k}^{\mathrm{unif}}$ on the set of isomorphism classes are all contiguous: if $(A_n)_n$ is a sequence of sets of isomorphism classes of $2k$-regular graphs on n vertices then

$$\mathbb{P}_{n,k}^{\mathrm{perm}}[A_n] \to 0 \quad \Leftrightarrow \quad \mathbb{P}_{n,2k}^{\mathrm{unif}}[A_n] \to 0 \quad \Leftrightarrow \quad \mathbb{P}_{n,2k}[A_n] \to 0$$

as $n \to \infty$. In particular, if we can prove that a random graph is an expander with probability tending to 1 in any of these models, we get the same statement for free for the other two models. In the proof of Theorem 4.2 we have essentially already proved the contiguity of $\mathbb{P}_{n,k}^{\mathrm{unif}}$ and the restriction of $\mathbb{P}_{n,k}$ to simple graphs (see Exericise 5.2). We will not prove it for the permutation model in this course and will content ourselves with the statement that graphs coming from the permutation model are expanders. The interested reader is refered to [Wor99, Section 4] for details on contiguity.

5.3 Exercises

Exercise 5.1. Fix $k \geq 3$ and let $(G_n)_n$ be any sequence of k-regular graphs so that G_n has n vertices. Show that

$$\lambda(G_n) \geq \sqrt{k} \cdot (1 - o(1))$$

as $n \to \infty$.

Exercise 5.2. Recall that $\mathcal{U}_{n,k}$ denotes the set of isomorphism classes of simple k-regular on n vertices and that

$$\mathcal{G}^*_{n,k} = \{C \in \mathcal{G}_{n,k};\ G(C) \text{ is simple}\}.$$

Let $\mathbb{P}^{\mathrm{unif}}_{n,k}$ denote the uniform probability measure on $\mathcal{U}_{n,k}$. Furthermore, let $\mathbb{P}^*_{n,k}$ denote the measure on $\mathcal{U}_{n,k}$ obtained from restricting $\mathbb{P}_{n,k}$ to simple graphs in $\mathcal{G}_{n,k}$ and then pushing it forward to $\mathcal{U}_{n,k}$. In other words, if $\pi : \mathcal{G}^*_{n,k} \to \mathcal{U}_{n,k}$ is the map that forgets all labels, then

$$\mathbb{P}^*_{n,k}[A] = \mathbb{P}_{n,k}[\pi^{-1}(A)|\ \mathcal{G}^*_{n,k}]$$

for all $A \subset \mathcal{U}_{n,k}$. Show that $\mathbb{P}^*_{n,k}$ and $\mathbb{P}^{\mathrm{unif}}_{n,k}$ are contiguous.

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