Lecture 7

A crash course the geometry of hyperbolic surfaces

7.1 The hyperbolic plane

Hyperbolic geometry originally developed in the early 19^{th} century to prove that the parallel postulate in Euclidean geometry is independent of the other postulates. From this perspective, the hyperbolic plane can be seen as a geometric object satisfying a collection of axioms very similar to Euclid's axioms for Euclidean geometry, but with the parallel postulate replaced by something else. From a more modern perspective, hyperbolic geometry is the study of manifolds that admit a Riemannian metric of constant curvature -1.

From the classical point of view, any concrete description of the hyperbolic plane is a *model* for two-dimensional hyperbolic geometry, in the same way that $\mathbb{R}^2$ is a model for Euclidean geometry.

Because this is a crash course, we will describe only one model for the hyperbolic plane: the upper half plane model. We note however that other models (like for instance the Klein model, the Poincaré model and the hyperboloid model) do exist. For a more complete reference, we refer to [Bea95, Chapter 7].

Given a smooth manifold M, let TM denote its tangent bundle. Recall that a Riemannian manifold (M, g) is a manifold M equipped with a smooth map

$$g : TM \times TM \to \mathbb{R},$$

called the *Riemannian metric*, so that the restriction $g_p : T_pM \times T_pM \to \mathbb{R}$ is a real inner product.

Definition 7.1. The hyperbolic plane $\mathbb{H}^2$ is the complex domain

$$\mathbb{H}^2 = \{z \in \mathbb{C};\ \Im(z) > 0\}$$

equipped with the Riemannian metric $g_{x+iy} : T_{x+iy}\mathbb{H}^2 \times T_{x+iy}\mathbb{H}^2 \to \mathbb{R}$ given by

$$g_{x+iy}(v, w) = \frac{1}{y^2}\big(dx(v) \cdot dx(w) + dy(v) \cdot dy(w)\big)$$

for all $x \in \mathbb{R}$ and $y \in (0, \infty)$

Because they are convenient, we will almost always work in local coordinates $x = \Re(z)$ and $y = \Im(z)$ for all $z \in \mathbb{H}^2$. We will denote the corresponding tangent vector fields by $\partial/\partial x$ and $\partial/\partial y$ respectively.

Let us first note that even thought distances in $\mathbb{H}^2$ behave very differently than in Euclidean geometry, the angles are the same. Indeed, locally the metric is just a scalar multiple of the usual inner product, so angles are no different.

Example 7.2. Let us compute the hyperbolic length of the straight line segment between $ai \in \mathbb{H}^2$ and $bi \in \mathbb{H}^2$ (denoted $[ai, bi]$) for $0 < a < b \in \mathbb{R}$. We may parameterize this segment by

$$\gamma : [0, 1] \to [ai, bi] \text{ given by } \gamma(t) = (1 - t) \cdot ai + t \cdot bi.$$

We have

$$\frac{d}{dt}\gamma(t) = -a\frac{\partial}{\partial y}_{\gamma(t)} + b\frac{\partial}{\partial y}_{\gamma(t)} = (b - a)\frac{\partial}{\partial y}_{\gamma(t)}.$$

So

$$g\left(\frac{d}{dt}\gamma(t), \frac{d}{dt}\gamma(t)\right) = \frac{(b - a)^2}{(a + t(b - a))^2}.$$

This means that the length of the line segment is given by

$$\begin{aligned}\ell([ai,bi]) &= \int_0^1 \sqrt{g\left(\frac{d}{dt}\gamma(t), \frac{d}{dt}\gamma(t)\right)}dt \\ &= \int_0^1 \frac{b-a}{a+t(b-a)}dt \\ &= [\log(a+t(b-a))]_0^1 \\ &= \log(b/a).\end{aligned}$$

Recall that given a connected Riemannian manifold (M, g), the *distance* between two points $p, q \in M$ is given by

$$\mathrm{d}\,(p,q) = \inf\left\{\ell(\gamma);\ \gamma : [0,1] \to M \text{ smooth, } \gamma(0) = p \text{ and } \gamma(1) = q\right\}.$$

Example 7.3. We claim that for $ai, bi \in \mathbb{H}^2$ with $0 < a < b \in \mathbb{R}$ we have

$$\mathrm{d}\,(ai, bi) = \log(b/a).$$

In Example 7.2 we have already shown that

$$\mathrm{d}\,(ai, bi) \leq \log(b/a),$$

so all we have to do is show the other inequality. Let $\gamma : [0,1] \to \mathbb{H}^2$ be any other smooth path with $\gamma(0) = ai$ and $\gamma(1) = bi$. Write

$$x(t) = \Re(\gamma(t)) \text{ and } y(t) = \Im(\gamma(t)),$$

so $\gamma(t) = x(t) + iy(t)$. We have

$$\begin{aligned}\ell(\gamma) &= \int_0^1 \sqrt{g\left(\frac{d}{dt}\gamma(t), \frac{d}{dt}(\gamma(t)\right)}dt \\ &= \int_0^1 \frac{1}{y(t)}\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}dt,\end{aligned}$$

where $\dot{x}(t) = dx(t)/dt$ and $y\dot{(t)} = dy(t)/dt$. As such

$$\ell(\gamma) \geq \int_0^1 \frac{\dot{y}(t)}{y(t)}dt = \log(b/a),$$

which proves our claim.

Let $\mathrm{Mat}(2,\mathbb{R})$ denote the set of 2×2 real matrices and define the group

$$\mathrm{PSL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2,\mathbb{R});\ ad - bc = 1 \right\} / \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

The group $\mathrm{PSL}(2,\mathbb{R})$ acts on $\mathbb{H}^2$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az+b}{cz+d} \tag{7.1}$$

for all $z \in \mathbb{H}^2$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{R})$. Note that the expression above is well-defined, that is, it does not depend on the representative matrix we choose. In Exercise 7.1 we prove that this actually defines a $\mathrm{PSL}(2,\mathbb{R})$-action on $\mathbb{H}^2$ and that the action is by *isometries*. That is

$$\mathrm{d}\,(Az, Aw) = \mathrm{d}\,(z, w)$$

for all $z, w \in \mathbb{H}^2$ and $A \in \mathrm{PSL}(2,\mathbb{R})$. When acting on $\mathbb{H}^2$, the elements of $\mathrm{PSL}(2,\mathbb{R})$ are called Möbius transformations. We claim (but will not prove) that all orientation preserving isometries of $\mathbb{H}^2$ are Möbius transformations.

Proposition 7.4. *Let $A : \mathbb{H}^2 \to \mathbb{H}^2$ be a smooth map that preserves orientation so that*

$$\mathrm{d}\,(Az, Aw) = \mathrm{d}\,(z, w)$$

for all $z, w \in \mathbb{H}^2$, then A is a Möbius transformation.

A consequence of this is the following:

Proposition 7.5. *Let $z, w \in \mathbb{H}^2$. Then*

$$\mathrm{d}\,(z, w) = \cosh^{-1}\left(1 + \frac{|z-w|^2}{2 \cdot \Im(z) \cdot \Im(w)}\right).$$

Proof. First of all, for z and w on the imaginary axis, this formula restricts to the formula from Example 7.3. As such, our strategy will be to prove that the expression on the right is invariant under Möbius transformations (as well as the expression on the left) and then to show that every pair of elements $z, w \in \mathbb{H}^2$ can be mapped to the imaginary axis by Möbius transformations.

The first fact comes down to checking that

$$\frac{|z-w|^2}{2 \cdot \Im(z) \cdot \Im(w)} = \frac{|Az-Aw|^2}{2 \cdot \Im(Az) \cdot \Im(Aw)}$$

for all $A \in \mathrm{PSL}(2,\mathbb{R})$ and $z, w \in \mathbb{H}^2$. This is a straightforward computation that we leave to the reader.

To show that we can move every pair of points to the imaginary axis with a Möbius transformation, we may assume that not both z and w are on the imaginary axis.

First suppose that z and w lie on a vertical line $\{x = b\}$. In this case the Möbius transformation $z \mapsto z - b$ maps both points to the imaginary axis.

Now suppose that z and w do not lie on a vertical line. Let C be the unique Euclidean circle through z and w that is perpendicular to the real line. Let α be one of the two points on the intersection $C \cap \mathbb{R}$.

$$z \mapsto \frac{-1}{z-\alpha}$$

is a Möbius transformation. We claim that it sends C to a straight line. One way to check this is by parameterization. Indeed, suppose C has center $\beta \in \mathbb{R}$ and suppose $\beta > \alpha$. We can then parameterize

$$C(t) = \beta + e^{2\pi i t}(\beta - \alpha), \; t \in \left(0, \frac{1}{2}\right)$$

It is a straightforward computation to check that

$$\Re\left(\frac{-1}{C(t)-\alpha}\right) = \frac{-1}{2(\beta-\alpha)}.$$

As such, our Möbius transformation sends z and w to two elements that lie on a vertical line and we are done. $\square$

We note that Möbius transformations preserve the set of half circles orthogonal to $\mathbb{R}$ and vertical lines in $\mathbb{H}^2$ (see Exercise 7.2).

Recall that a *geodesic* $\gamma : \mathbb{R} \to \mathbb{H}^2$ is a smooth path so that

$$\mathrm{d}\left(\gamma(t), \gamma(s)\right) = |t-s|$$

for all $t, s \in \mathbb{R}$.

It follows from the proof and the two examples above that:

Proposition 7.6. *The image of a geodesic $\gamma : \mathbb{R} \to \mathbb{H}^2$ is a vertical line or a half circle orthogonal to $\mathbb{R}$. Moreover, every vertical line and half circle orthogonal to the real line can parameterized as a geodesic.*

We will often forget about the parametrization and call the image of a geodesic a geodesic as well. Note that it follows from the proposition above that given any two distinct points $z, w \in \mathbb{H}^2$ there exists a unique geodesic $\gamma \subset \mathbb{H}^2$ so that both $z \in \gamma$ and $w \in \gamma$. Furthermore, it also follows given a point $z \in \mathbb{H}^2$ and a geodesic γ that does not contain it, there is a unique perpendicular from z to γ (a geodesic γ' that intersects γ once perpendicularly and contains z)

The final fact we will need about the hyperbolic plane is:

Proposition 7.7. *Let $z \in \mathbb{H}$ and let $\gamma \subset \mathbb{H}^2$ be a geodesic so that $z \notin \gamma$ then*

$$\mathrm{d}\,(z, \gamma) := \inf \left\{ \mathrm{d}\,(z, w)\,;\; w \in \gamma \right\}$$

is realized by the intersection point of the perpendicular from z to γ.

Proof. This follows from Pythagoras' theorem for hyperbolic triangles. Indeed, given three points in $\mathbb{H}^2$ so that the three geodesics through them form a right angled hyperbolic triangle with sides of length a, b and c (where c is the side opposite the right angle), we have

$$\cosh(a) \cosh(b) = \cosh(c)$$

(see Exercise 7.3). This means in particular that $c > b$.

So, any other point on γ is further away from z than the point w realizing the perpendicular. Because that other point forms a right angled triangle with w and z. □

7.2 Surfaces

A *surface* is a smooth two-dimensional manifold. We call a surface *closed* if it is compact and has no boundary. A surface is said to be of *finite type* if it can be obtained from a closed surface by removing a finite number of points and (smooth) open disks. In what follows, we will always assume our surfaces to be orientable.

Example 7.8. To properly define a manifold, one needs to not only describe the set but also give smooth charts. In what follows we will content ourselves with the sets (Exercise 7.4 completes the picture).

(a) The 2-*sphere* is the surface

$$\mathbb{S}^2 = \left\{(x, y, z) \in \mathbb{R}^3;\ x^2 + y^2 + z^2 = 1\right\}.$$

(b) Let $\mathbb{S}^1$ denote the circle. The 2-*torus* is the surface

$$\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$$

(c) Given two (oriented) surfaces S_1, S_2, their *connected sum* $S_1 \# S_2$ is defined as follows. Take two closed sets $D_1 \subset S_1$ and $D_2 \subset S_2$ that are both diffeomorphic to closed disks, via diffeomorphisms

$$\varphi_i : \left\{(x, y) \in \mathbb{R}^2;\ x^2 + y^2 \leq 1\right\} \to D_i, \quad i = 1, 2,$$

so that φ_1 is orientation preserving and φ_2 is orientation reversing.

Then

$$S_1 \# S_2 = \left(S_1 \smallsetminus \mathring{D}_1 \sqcup S_2 \smallsetminus \mathring{D}_2\right) / \sim$$

where $\mathring{D}_i$ denotes the interior of D_i for $i = 1, 2$ and the equivalence relation $\sim$ is defined by

$$\varphi_1(x, y) \sim \varphi_2(x, y) \quad \text{for all } (x, y) \in \mathbb{R}^2 \text{ with } x^2 + y^2 = 1.$$

The figure below gives an example.

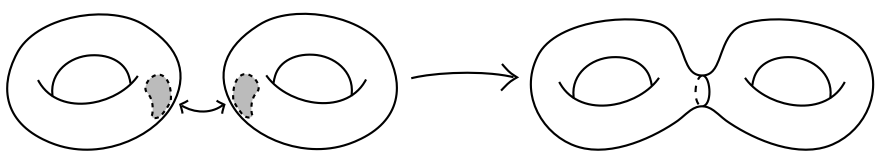

Figure 7.1: A connected sum of two tori.

Like our notation suggests, the manifold $S_1 \# S_2$ is independent (up to diffeomorphism) of the choices we make (the disks and diffeomorphisms φ_i). This is a non-trivial result, the proof of which we will skip. Likewise, we will also not prove that the connected sum of surfaces is an associative operation and that $\mathbb{S}^2 \# S$ is diffeomorphic to S for all surfaces S.

A classical result from the 19^{th} century tells us that the three simple examples above are enough to understand all finite type surfaces up to diffeomorphism.

Theorem 7.9. Classification of closed surfaces *Every closed surface is diffeomorphic to the connected sum of a* 2*-sphere with a finite number of tori.*

Indeed, because the diffeormorphism type of a finite type surface does not depend on where we remove the points and open disks (another claim we will not prove), the theorem above tells us that a finite type surface is (up to diffeomorphism) determined by a triple of positive integers (g, b, n), where

- g is the number of tori in the connected sum and is called the *genus* of the surface.

- b is the number of disks removed and is called the number of *boundary components* of the surface.

- n is the number of points removed and is called the number of *punctures* of the surface.

we will denote the corresponding surface by $\Sigma_{g,b,n}$ and will write $\Sigma_g = \Sigma_{g,0,0}$.

7.3 Hyperbolic surfaces

For this section we will mainly follow [Bus10]. A hyperbolic surface will be a finite type surface equipped with a metric that locally makes it look like $\mathbb{H}^2$.

Because we will want to deal with surfaces with boundary, we need half spaces. Let $\gamma \subset \mathbb{H}^2$ be a geodesic. $\mathbb{H}^2 \setminus \gamma$ consists of two connected components C_1 and C_2. We will call $\mathcal{H}_i = C_i \cup \gamma$ a *closed half space* $(i = 1, 2)$. So for example

$$\left\{z \in \mathbb{H}^2;\ \Re(z) \leq 0\right\}$$

is a closed half space.

We formalize the notion of a hyperbolic surface as follows:

Definition 7.10. A finite type surface S with atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ is called a *hyperbolic surface* if $\varphi_\alpha(U_\alpha) \subset \mathbb{H}^2$ for all $\alpha \in A$ and

1. for each $p \in S$ there exists an $\alpha \in A$ so that $p \in U_\alpha$ and

   - If $p \in \partial S$ then
   $$\varphi_\alpha(U_\alpha) = V \cap \mathcal{H}$$
   for some open set $V \subset \mathbb{H}^2$ and some closed half space $\mathcal{H} \subset \mathbb{H}^2$.
   - If $p \in \mathring{S}$ then $\varphi_\alpha(U_\alpha) \subset \mathbb{H}^2$ is open.

2. For every $\alpha, \beta \in A$ and for each connected component C of $U_\alpha \cap U_\beta$ we can find a Möbius transformation $A : \mathbb{H}^2 \to \mathbb{H}^2$ so that
$$\varphi_\alpha \circ \varphi_\beta^{-1}(z) = A(z)$$
for all $z \in \varphi_\beta(C) \subset \mathbb{H}^2$.

Note that every hyperbolic comes with a metric: every chart is identified with an open set of $\mathbb{H}^2$ which gives us a metric. Because the chart transitions are restrictions of isometries of $\mathbb{H}^2$, this metric does not depend on the choice of chart and hence is well defined.

Definition 7.11. A hyperbolic surface S is called *complete* if the induced metric is complete.

7.4 Exercises

Exercise 7.1. (a) Show that the action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{H}^2$ defined in (7.1) is indeed an action. That is, show that if $A, B \in \mathrm{PSL}(2,\mathbb{R})$ and $z \in \mathbb{H}^2$ then
$$Az \in \mathbb{H}^2 \text{ and } (A \cdot B)z = A(Bz).$$

(b) Recall that if M is a manifold and $f : M \to M$ a diffeomorphism, then we obtain a linear map
$$Df_p : T_pM \to T_{f(p)}M$$

called the differential of f. One way to describe this map is as follows. Given $v \in T_pM$, Take $\gamma : (-1,1) \to M$ so that

$$\gamma(0) = p \text{ and } \frac{d}{dt}\gamma(0) = v$$

and define

$$(Df)v = \frac{d}{dt}(f \circ \gamma)(0).$$

Given $A \in \mathrm{PSL}(2,\mathbb{R})$, show that its derivative DA_z (as a map from $\mathbb{H}^2$ to itself) satisfies

$$g_{Az}(DA_z v, DA_z w) = g_z(v, w)$$

for all $z \in \mathbb{H}^2$ and $v, w \in T_z\mathbb{H}^2$.

(c) Given a smooth path $\gamma : [0,1] \to \mathbb{H}^2$ and $A \in \mathrm{PSL}(2,\mathbb{R})$, we obtain a new smooth path $A \circ \gamma : [0,1] \to \mathbb{H}^2$. Show that

$$\ell(\gamma) = \ell(A \circ \gamma).$$

Conclude that

$$\mathrm{d}\,(Az, Aw) = \mathrm{d}\,(z, w)$$

for all $z, w \in \mathbb{H}^2$ and $A \in \mathrm{PSL}(2,\mathbb{R})$.

Exercise 7.2. Let $C \subset \mathbb{H}^2$ be a half circle orthogonal to $\mathbb{R}$ or a vertical line and let $A : \mathbb{H}^2 \to \mathbb{H}^2$ be a Möbius transformation. Show that $A(C)$ is a vertical line or half circle orthogonal to $\mathbb{R}$.

Hint: consider what a Möbius transformation does to the endpoints (NB: ∞ is a possible endpoint) of half circles orthogonal to $\mathbb{R}$ and vertical lines

Exercise 7.3. *Pythagoras' theorem:* Suppose $x, y, z \in \mathbb{H}^2$ form a right angled triangle (that is, the geodesic between x and y intersects that between y and z perpendicularly) and let

$$a = \mathrm{d}\,(x,y)\,,\ b = \mathrm{d}\,(y,z) \text{ and } c = \mathrm{d}\,(z,x)\,.$$

Prove that

$$\cosh(a) \cdot \cosh(b) = \cosh(c).$$

Hint: just like in the proof of Proposition 7.5 you may assume that the geodesic between y and z is the imaginary axis.

Exercise 7.4. Define an atlas for $\mathbb{S}^2$ and $\mathbb{T}^2$.

Bibliography

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