Lecture 8

Pants decompositions I

8.1 Pairs of pants

Even though Definition 7.10 is a complete definition, it is not very descriptive. In what follows we will describe a concrete cutting and pasting construction for hyperbolic surfaces.

We start with right angled hexagons. Let $\gamma_1, \ldots, \gamma_6 \subset \mathbb{H}^2$ be consistently oriented geodesics so that

$$|\gamma_i \cap \gamma_j| = \begin{cases} 1 & \text{if } |i-j| = 1 \text{ or if } \{i,j\} = \{1,6\} \\ 0 & \text{otherwise.} \end{cases}$$

and the oriented angle at every intersection point is $\pi/2$. Now let $\mathcal{H}_1, \ldots, \mathcal{H}_6$, be half spaces defined by the geodesics $\gamma_1, \ldots, \gamma_6$ so that the intersection $\cap_{i=1}^{6} \mathcal{H}_i$ is non-empty and compact. Then $\cap_{i=1}^{6} \mathcal{H}_i \subset \mathbb{H}^2$ is called a right angled hexagon.

The picture to have in mind is:

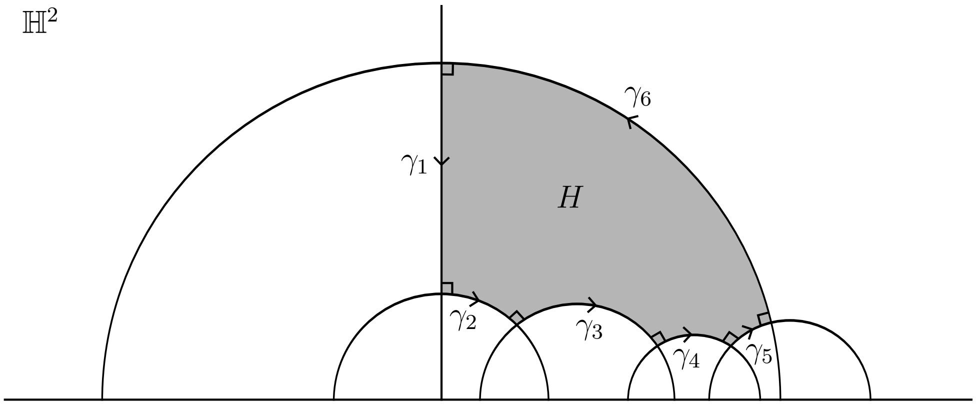

Figure 8.1: A right angled hexagon H.

It turns out that the lengths of three non-consecutive sides determine a right angled hexagon up to isometry.

Proposition 8.1. *Let $a, b, c \in (0, \infty)$. Then there exists a right angled hexagon $H \subset \mathbb{H}^2$ with three non-consecutive sides of length a, b and c respectively. Moreover, if H' is another right angled hexagon with this property, then there exists a Möbius transformation $A : \mathbb{H}^2 \to \mathbb{H}^2$ so that*

$$A(H) = H.$$

Proof. Let us start with the existence. Let γ_{im} denote the positive imaginary axis and set

$$B = \left\{ z \in \mathbb{H}^2;\ \mathrm{d}(z, \gamma_{im}) = c \right\}.$$

B is a one-dimensional submanifold of $\mathbb{H}^2$. Because the map $z \mapsto \lambda z$ is an isometry that preserves γ_{im} for every $\lambda > 0$, it must also preserve B. This means that B is a (straight Euclidean) line.

Now construct the following picture:

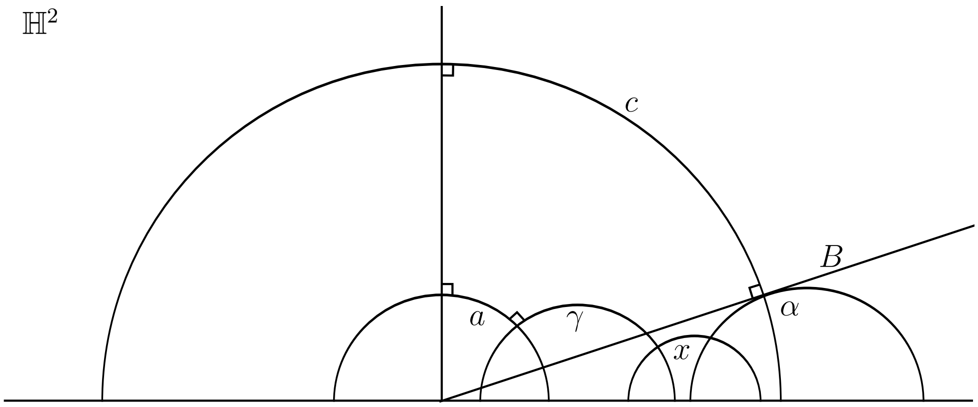

Figure 8.2: Constructing a right angled hexagon $H(a, b, c)$.

That is, we take the geodesic though the point $i \in \mathbb{H}^2$ perpendicular to γ_{im} and at distance a draw a perpendicular geodesic γ. furthermore, for any $p \in B$, we draw the geodesic α that realizes a right angle with the perpendicular from p to γ_{im}. Now let

$$x = \mathrm{d}(\alpha, \gamma) = \inf \{\mathrm{d}(z, w);\ z \in \gamma,\ w \in \alpha\}.$$

Because of Proposition 7.7, x is realized by the common perpendicular to α and γ. By moving p over B, we can realize any positive value for x and hence obtain our hexagon $H(a, b, c)$.

We also obtain uniqueness from the picture above. Indeed, given any right angled hexagon H' with three non-consequtive sides of length a, b and c, apply a Möbius transformation $A : \mathbb{H}^2 \to \mathbb{H}^2$ so that the geodesic segment of length a starts at i and is orthogonal to the imaginary axis. This implies that the geodesic after a gets mapped to the geodesic γ. Furthermore, one of the endpoints of the geodesic segment of length c needs to lie on the line B. We now know that the the geodesic α before that point needs to be tangent to B. Because α and β have a unique common perpendicular. The tangency point of α to B determines the picture entirely. Because the function that assigns the length x of the common perpendicular to the tangency point is injective, we obtain that there is a unique solution. □

One of our main building blocks for hyperbolic surfaces is the following:

Definition 8.2. Let $a, b, c \in (0, \infty)$. A *pair of pants* is a hyperbolic surface that is diffeomorphic to $\Sigma_{0,3,0}$ such that the boundary components have length a, b and c respectively.

Proposition 8.3. *Let $a, b, c \in (0, \infty)$ and let P and P' be pairs of pants with boundary curves of lengths a, b and c. Then there exists an isometry $\varphi : P \to P'$.*

Proof sketch. There exists a unique orthogonal geodesic (this essentially follows from Proposition 7.7, in Proposition 8.6 we will do a similar proof in full) between every pair of boundary components of P.

These three orthogonals decompose P into right-angled hexagons out of which three non-consecutive sides are determined. Proposition 8.1 now tells us that this determines the hexagons up to isometry and this implies that P is also determined up to isometry. □

Note that it also follows from the proof sketch above that the unique perpendiculars cut each boundary curve on P into two geodesic segments of equal length. Moreover, we obtain a standard parmeterization of the boundary pair of pants.

If P is a pair of pants and $\delta \subset \partial P$ is one of its boundary components, let us write $\ell(\delta)$ for the length of δ. Recall that an isometry between Riemannian manifolds M and N is a diffeomorphism $\varphi : M \to N$ so that

$$\mathrm{d}_M(x, y) = \mathrm{d}_N(\varphi(x), \varphi(y))$$

for all $x, y \in M$.

Example 8.4. Given two pairs of pants P_1 with boundary components δ_1, δ_2 and δ_3 and P_2 with boundary components γ_1, γ_2 and γ_3 so that

$$\ell(\delta_1) = \ell(\gamma_1),$$

we can choose an orientation reversing isometry $\varphi : \delta_1 \to \gamma_1$ and from that obtain a hyperbolic surface

$$S = P_1 \sqcup P_2 / \sim,$$

where $\varphi(x) \sim x$ for all $x \in \delta_1$. Note that S is diffeomorphic to $\Sigma_{0,4,0}$.

8.2 Simple closed curves

Given a manifold M, recall that two embeddings $\gamma_1, \gamma_2 : \mathbb{S}^1 \to M$ are called *freely homotopic*, if there exists a continuous map:

$$H : \mathbb{S}^1 \times [0,1] \to X$$

so that

$$H(t,0) = \gamma_1(t) \text{ and } H(t,1) = \gamma_2(t)$$

for all $t \in \mathbb{S}^1$. The difference between free homotopy and usual homotopy of loops is that there is no mention of basepoints in the case of free homotopy.

Let X be a hyperbolic surface. We call a smooth map $\gamma : \mathbb{S}^1 \to X$ a *closed geodesic* if for every $t \in \mathbb{S}^1$ there exists an open set $U \subset \mathbb{S}^1$ with $t \in U$ so that

$$\mathrm{d}_X(\gamma(s), \gamma(s')) = \mathrm{d}_{\mathbb{S}^1}(s, s'),$$

where the metric $d_{\mathbb{S}^1} : \mathbb{S}^1 \times \mathbb{S}^1 \to [0, \infty)$ is the metric coming from the quotient $\mathbb{S}^1 = \mathbb{R}/(\ell(\gamma)\mathbb{Z})$ (so $\mathbb{S}^1$ has total length $\ell(\gamma)$). Just like with geodesics in $\mathbb{H}^2$, we will often identify a closed geodesic with its image.

Finally, we will need the following fact, which we shall not prove. A convex subset of $\mathbb{H}^2$ here is a subset $C \subset \mathbb{H}^2$ so that the geodesic segment between x and y lies in C for all $x, y \in C$.

Theorem 8.5. *(a) Let X be closed hyperbolic surface. Then there exists a covering map*

$$p : \mathbb{H}^2 \to X$$

that is a local isometry.

(b) Let X be a hyperbolic surface with boundary. Then there exists a closed convex subset $\widetilde{X} \subset \mathbb{H}^2$ and a covering map

$$p : \widetilde{X} \to X$$

that is a local isometry.

A proof of (a) for instance be found in [CE08, Theorem 1.37] and (b) is proved in [Bus10, Theorem 1.4.2]. Note that it follows from the fact that

$\mathbb{H}^2$ and convex subsets in $\mathbb{H}^2$ are simply connected that the covers above are unversal covers.

To see that every closed hyperbolic surface can be constructed by gluing pairs of pants together, we need the following proposition. Here, a *simple* closed curve on a hyperbolic surface X is a closed curve $\gamma : \mathbb{S}^1 \to X$ that is an embedding.

Proposition 8.6. *Let X be a closed hyperbolic surface and let $\gamma : \mathbb{S}^1 \to X$ be smooth map (a closed curve) that is not freely homotopic to a constant map. There exists a (up to reparameterization) unique closed geodesic $\overline{\gamma} : \mathbb{S}^1 \to X$ that is freely homotopic to γ. This geodesic is the curve of minimal length among all curves that are freely homotopic to γ. Moreover, if γ is simple then so is $\overline{\gamma}$.*

Proof sketch. We will prove everything, except the statement about simplicity. The proof will however assume some general covering theory, see [Hat02, Section 1.3] for details. Let

$$C := \left\{\gamma' : \mathbb{S}^1 \to X;\ \gamma' \text{ freely homotopic to } \gamma\right\}$$

and set

$$L = \inf\left\{\ell(\gamma');\ \gamma' \in C\right\}.$$

Now consider a sequence $\{\gamma_n\}_n$ so that $\ell(\gamma_n) \to L$. It follows from the Arzelà-Ascoli theorem ([Bus10, Theorem A.19]) that there exists a subsequence $\{\gamma_{n_k}\}_k$ and a simple closed curve $\overline{\gamma} : \mathbb{S}^1 \to X$ so that $\gamma_{n_k} \to \widetilde{\gamma}$ uniformly as $k \to \infty$. Because $\widetilde{\gamma}$ minimizeanys length, it needs to be a geodesic (up to reparameterization).

To show uniqueness, suppose there are two freely homotopic geodesics $\gamma_1, \gamma_2 : \mathbb{S}^1 \to X$. Consider the universal cover $p : \mathbb{H}^2 \to X$. Because γ_1 and γ_2 are freely homotopic, we can lift them to continuous maps $\widetilde{\gamma}_1, \widetilde{\gamma}_2 : \mathbb{R} \to \mathbb{H}^2$ that are homotopic. The fact that γ_1 and γ_2 are geodesics implies that $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ are as well.

By general covering theory, the subgroup of the deck group $\pi_1(X)$ that leaves $\widetilde{\gamma}_1$ invariant also leaves $\widetilde{\gamma}_2$ invariant (because they are homotopic). By a compactness argument, this implies that

$$\max_{t\in\mathbb{R}}\{\mathrm{d}(\widetilde{\gamma}_1(t), \widetilde{\gamma}_2(t)\} < \infty.$$

Now we note that when geodesics have at least one pair of distinct endpoints, the above does not hold. This implies that $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ have the same endpoints, which in turn implies they have coincide. $\square$

We note that the three boundary components of a pair of pants are simple closed geodesics.

Example 8.7. φ in Example 8.4 is determined up to 'twist'. That is, if we parameterize δ_1 by a simple closed geodesic $x : \mathbb{R}/(\ell(\delta_1)\mathbb{Z}) \to \delta_1$ and $\varphi' : \delta_1 \to \gamma_1$ is a different orientation reversing isometry, then there exists some $t_0 \in \mathbb{R}$ so that

$$\varphi'(x(t)) = \varphi(x(t_0 + t))$$

for all $t \in \mathbb{R}/(\ell(\delta_1)\mathbb{Z}) \to \delta_1$.

8.3 Exercises

Exercise 8.1. Let H be a right angled hexagon with three non consecutive sides of the same length $a > 0$.

(a) Show without computing their lengths that the lengths of the other three sides are also all the same.

(b) Compute the length of the other three sides.

Bibliography

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