Random Methods in Geometry

V5D4 - Selected Topics in Geometry - Summer semester 2017

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Introduction

Random methods provide a way to answer questions of the form: "what does a typical [insert your favorite object here] look like?" A classical example is the theory of random graphs.

In fact, Random graphs have not only been used to study the typical behavior of large graphs but also to provide existence proofs. The latter application is often called "the probabilistic method". The idea of this method is that it is sometimes easier to show that certain behavior occurs with a non-zero probability (in a suitable model) than to explicitly construct objects that exhibit this behavior.

The goal of this course will be to discuss some applications of these methods in geometry. Most of the course will focus on graphs and surfaces. In the last lecture we will discuss a selection of results on higher dimensional manifolds that are known.

We will treat the following topics:

- Some basic probability theory, in particular Poisson approximation (Lectures 1 and 2).
- Random graphs: counting the number of regular graphs and expansion (Lectures 3 - 6).
- Random hyperbolic surfaces: The genus distribution and pants decompositions (Lectures 7 - 10).
- The geometry of random surfaces and higher dimensional random hyperbolic manifolds (Lecture 11).

Lecture 1

Probability theory

In this chapter, we recall some basic probability theory that is needed later on. The overview below will be very incomplete, as we will only cover the parts of the theory that we need. For a comprehensive reference, we refer to [Ven13], most of the material covered below can be found in Chapters XV and XVIII of [Ven13].

1.1 Definitions

We start with the definition of a probabilty space. In this definition $\mathcal{P}(A)$ denotes the *power set* of a set A: the set of all subsets of A.

Definition 1.1. A *probability space* is a triple $(\Omega, \Sigma, \mathbb{P})$, where

- Ω is a set,
- $\Sigma \subset \mathcal{P}(\Omega)$ is a σ-algebra and
- $\mathbb{P} : \Sigma \to [0, 1]$ is a probability measure. That is, a measure such that $\mathbb{P}[\Omega] = 1$.

Given a probability space $(\Omega, \Sigma, \mathbb{P})$ and an element $A \in \Sigma$ such that $\mathbb{P}[A] \neq 0$, we define the *conditional probability of $B \in \Sigma$, conditioned on A* to be

$$\mathbb{P}[B|A] = \frac{\mathbb{P}[B \cap A]}{\mathbb{P}[A]}.$$

An important example to us will be the following:

Example 1.2. Let Ω be a finite set. We can turn Ω into a probability space by setting $\Sigma = \mathcal{P}(\Omega)$ and

$$\mathbb{P}[A] = \frac{|A|}{|\Omega|} \text{ for all } A \subseteq \Omega,$$

where $|A|$ denotes the cardinality of A. This probability measure is called the *uniform probability measure* on Ω. Note that

$$\mathbb{P}[B|A] = \frac{|B \cap A|}{|A|}$$

for any $A, B \subset \Omega$ so that $A \neq \emptyset$.

From hereon, we will fix the following convention regarding σ-algebras: If Ω is a finite set, then we set $\Sigma = \mathcal{P}(\Omega)$ and if Ω is a topological space then we set $\Sigma = \mathcal{B}(\Omega)$, the Borel algebra of Ω. This convention will allow us to forget about σ-algebras in most of what follows.

Definition 1.3. Given a probability space $(\Omega, \Sigma, \mathbb{P})$ and a measure space E, an E-valued *random variable* is a measurable function $X : \Omega \to E$. The *expected value* $\mathbb{E}[X]$ of an $\mathbb{R}$-valued random variable X is defined by

$$\mathbb{E}[X] = \int_\Omega X(\omega) d\mathbb{P}(\omega).$$

The *variance* of X is given by

$$\mathbb{V}\text{ar}[X] = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2.$$

Two random variables $X, Y : \Omega \to \mathbb{R}$ are called *independent* if

$$\mathbb{P}[X \leq x \text{ and } Y \leq y] = \mathbb{P}[X \leq x] \cdot \mathbb{P}[Y \leq y]$$

for all $x, y \in \mathbb{R}$.

Finally, let $k \in \mathbb{N}$, the quantity

$$\mathbb{E}\left[X^k\right]$$

is called the k^{th} *moment of* X.

Almost all random variables we will consider are real-valued. We remark that in general, the moments of a random variable are not necessarily finite.

Example 1.4. Let $A \subset \Omega$ be measurable. The function $\chi_A : \Omega \to \mathbb{R}$ defined by

$$\chi_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise.} \end{cases}$$

is a random variable that satisfies

$$\mathbb{E}\left[\chi_A\right] = \mathbb{P}\left[A\right].$$

Random variables of this form are called *Bernoulli* random variables.

Often it turns out to be easier to compute expected values than probabilities. In these cases, the following inequality relating the two is useful.

Lemma 1.5. Markov's inequality: *Let $X : \Omega \to \mathbb{R}$ be a random variable such that $\mathbb{E}\left[X\right] < \infty$ and*

$$X(\omega) \geq 0 \ \textit{ for all } \omega \in \Omega$$

Then for all $x \in (0, \infty)$ we have

$$\mathbb{P}\left[X \geq x\right] \leq \mathbb{E}\left[X\right]/x.$$

Proof. We have:

$$x \cdot \mathbb{P}\left[X \geq x\right] = x \cdot \mathbb{E}\left[\chi_{\{\omega \in \Omega;\, X(\omega) \geq x\}}\right] = \mathbb{E}\left[x \cdot \chi_{\{\omega \in \Omega;\, X(\omega) \geq x\}}\right] \leq \mathbb{E}\left[X\right],$$

where the last inequality follows from the fact that

$$x \cdot \chi_{\{\omega \in \Omega;\, X(\omega) \geq x\}}(\omega) \ \leq \ X(\omega)$$

for all $\omega \in \Omega$. □

1.2 The Chen-Stein method

In what follows we will give a self contained account of the Chen-Stein method. Some of the ingredients will however seem to come out of thin air. There is a good motivation for these ingredients, which we will skip in the interest of time.

The goal of this section will be to bound the distance between a random variable X and a Poisson distributed variable. Let us first recall the definition of a Poisson variable.

Definition 1.6. Let $\lambda \in (0, \infty)$. A random variable $X : \Omega \to \mathbb{N}$ is said to be *Poisson distributed with mean* λ if

$$\mathbb{P}[X = k] = \frac{\lambda^k e^{-\lambda}}{k!}$$

for all $k \in \mathbb{N}$.

We also need a notion of distance between random variables:

Definition 1.7. Let E be a measure space with σ-algebra $\mathcal{E}$. Furthermore, let $X : \Omega \to E$ and $Y : \Omega' \to E$ be random variables. The *total variational distance* between X and Y is defined as

$$\mathrm{d_{TV}}(X, Y) = \sup\{|\mathbb{P}[X \in A] - \mathbb{P}[Y \in A]|\,;\ A \in \mathcal{E}\}.$$

If $(X_n)_{n\in\mathbb{N}}$ is a sequence of E-valued random variables so that $\mathrm{d_{TV}}(X_n, X) \to 0$ as $n \to \infty$ then we say that the sequence $(X_n)_{n\in\mathbb{N}}$ *converges to* X *in total variational distance* and write

$$X_n \xrightarrow{\mathrm{TV}} X \ \text{ as } \ n \to \infty.$$

1.2.1 Stein's equation

It turns out that $X : \Omega \to \mathbb{N}$ is Poisson distributed if and only if

$$\mathbb{E}[\lambda g(X + 1) - X g(X)] = 0$$

for all bounded functions $g : \mathbb{N} \to \mathbb{R}$ (See Exercise 1.4 for the easier direction and [Ven13, Chapter XVIII] for the other direction). The basic idea of the

Chen-Stein method is that if $\mathbb{E}[\lambda g(X+1) - Xg(X)]$ is close to 0 for all bounded functions g, then X must be close to Poisson distributed.

Given $A \subset \mathbb{N}$, *Stein's equation* is the equation

$$\lambda g(k+1) - kg(k) = \chi_A(k) - \mathbb{E}[\chi_A(Z_\lambda)] \text{ for all } k \in \mathbb{N} \tag{1.1}$$

where $\chi_A : \mathbb{N} \to \{0,1\}$ is defined by

$$\chi_A(n) = \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{otherwise.} \end{cases}$$

We claim that this equation has a unique bounded solution that has value 0 at 0, which we shall denote by $g_A : \mathbb{N} \to \mathbb{R}$. Note that g_A also depends on λ. In order to keep the notation light we shall however suppress λ. The fact that a unique solution g_A exists is clear from the recursive nature of the equation, the proof that it is bounded we shall postpone to Proposition 1.9.

Our earlier remark about using $\mathbb{E}[\lambda g(X+1) - Xg(X)]$ as a measure of the distance to a Poisson variable is made precise by the following theorem.

Theorem 1.8. *Suppose $X : \Omega \to \mathbb{N}$ is a random variable and $Z_\lambda : \Omega \to \mathbb{N}$ is a Poisson distributed random variable with mean $\lambda > 0$. Then*

$$\mathrm{d_{TV}}(X, Z_\lambda) = \sup\{|\mathbb{E}[\lambda g_A(X+1) - Xg_A(X)]|;\ A \subset \mathbb{N}\}.$$

Proof. Given $A \subset \mathbb{N}$, (1.1) implies that

$$\mathbb{E}[\lambda g_A(X+1) - Xg_A(X)] = \mathbb{E}[\chi_A(X)] - \mathbb{E}[\chi_A(Z_\lambda)] = \mathbb{P}[X \in A] - \mathbb{P}[Z_\lambda \in A].$$

As such, filling in the definition of total variational distance leads to the theorem. □

1.2.2 Bounds on Stein's equation, part I

Theorem 1.8 gives us a way to bound the distance between any random variable X and a Poisson variable. Our next and final goal will be to express this bound in terms of more immediate information on X. That is, we will bound $|\mathbb{E}[\lambda g_A(X) - Xg_A(X)]|$ in terms of moment(-like expression)s of X.

To this end, we need a bound on solutions of (1.1). Given $A \subset \mathbb{N}$, let us write

$$||g_A|| = \sup_{k \in \mathbb{N}} \{|g_A(k)|\}.$$

In the next lecture, we will prove the following bound on $||g_A||$:

Proposition 1.9. *Let $A \subset \mathbb{N}$. Then*

$$||g_A|| \leq 1.$$

The main ingredient for our bound on $||g_A||$, is the following lemma:

Lemma 1.10. *Let $r, s \in \mathbb{N}$ so that $2r \leq s$. Then:*

$$\sum_{i=0}^{r} \binom{s}{i} \leq \frac{s-r+1}{s-2r+1} \cdot \binom{s}{r}.$$

Proof. Indeed, we have

$$\begin{aligned}
\binom{s}{r}^{-1} \sum_{i=0}^{r} \binom{s}{i} &= \sum_{i=0}^{r} \frac{(s-r)!r!}{(s-i)!i!} \\
&= \sum_{i=0}^{r} \frac{(s-r)!r!}{(s-r+i)!(r-i)!} \\
&= \sum_{i=0}^{r} \frac{r(r-1)\cdots(r-i+1)}{(s-r+i)\cdots(s-r+1)} \\
&\leq \sum_{i=0}^{r} \left(\frac{r}{s-r+1}\right)^{i}.
\end{aligned}$$

This is a geometric series, so we obtain that

$$\begin{aligned}
\binom{s}{r}^{-1} \sum_{i=0}^{r} \binom{s}{i} &\leq \frac{1 - \left(\frac{r}{s-r+1}\right)^{r+1}}{1 - \frac{r}{s-r+1}} \\
&\leq \frac{s-r+1}{s-2r+1}
\end{aligned}$$

for all r, s, where we have used the fact that $r \leq s/2$ for the last step. $\square$

1.3 Exercises

Exercise 1.1. Let $X : \Omega \to \mathbb{R}$ be a random variable so that

$$\mathbb{E}[X] < \infty \text{ and } \mathbb{E}\left[X^2\right] < \infty.$$

Chebyshev's inequality states that

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq x] \leq \frac{\mathbb{V}\text{ar}[X]}{x^2},$$

for all $x > 0$. Prove Chebyshev's inequality.

Exercise 1.2. Let $X : \Omega \to \mathbb{R}$ be a random variable so that

$$\mathbb{E}[X] < \infty \text{ and } \mathbb{E}\left[X^2\right] < \infty$$

and

$$X(\omega) \geq 0 \text{ for all } \omega \in \Omega$$

Prove that

$$\mathbb{P}[X > 0] \geq \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}.$$

This inequality is called the *second moment method*.

Exercise 1.3.

(a) Show that two Bernoulli variables $\chi_A, \chi_B : \Omega \to \mathbb{R}$ corresponding to measurable sets $A, B \subseteq \Omega$ are independent if and only if

$$\mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B].$$

(b) Give an example of a probability space Ω and three random variables $X, Y, Z : \Omega \to \mathbb{R}$ so that all pairs of random variables among $\{X, Y, Z\}$ are independent, but there exists $x, y, z \in \mathbb{R}$ so that

$$\mathbb{P}[X \leq x \text{ and } Y \leq y \text{ and } Z \leq z] \neq \mathbb{P}[X \leq x] \cdot \mathbb{P}[Y \leq y] \cdot \mathbb{P}[Z \leq z].$$

Exercise 1.4. Show that if a random variable $Z : \Omega \to \mathbb{N}$ is Poisson distributed with mean $\lambda \in (0, \infty)$ then:

$$\mathbb{E}[\lambda g(Z+1) - Zg(Z)] = 0$$

for all bounded functions $g : \mathbb{N} \to \mathbb{R}$.

Lecture 2

The Chen Stein method

2.1 Bounds on Stein's equation, part II

We now have the following bound on $||g_A||$:

Proposition 1.9. *Let $A \subset \mathbb{N}$. Then*

$$||g_A|| \leq 1.$$

Proof. To simplify matters, we define a new function $f : \mathbb{N} \to \mathbb{R}$ by

$$f(k) = \chi_A(k) - \mathbb{E}\left[\chi_A(Z_\lambda)\right]$$

for all $k \in \mathbb{N}$. Note that by definition

$$\mathbb{E}\left[f(Z_\lambda)\right] = 0.$$

Set $g_A(0) = 0$. From (1.1) we obtain that for all $k \in \mathbb{N}$:

$$g_A(k+1) = \frac{1}{\lambda} f(k) + \frac{k}{\lambda} g_A(k).$$

Hence

$$g_A(k+1) = \frac{1}{\lambda} \sum_{j=0}^{k} \frac{k(k-1)\cdots(k-j+1)}{\lambda^j} f(k-j) = \frac{k!}{\lambda^{k+1}} \sum_{i=0}^{k} \frac{\lambda^i}{i!} f(i).$$

Thus

$$g_A(k+1) = \frac{1}{\lambda \cdot \mathbb{P}\left[Z_\lambda = k\right]} \sum_{j=0}^{k} \mathbb{P}\left[Z_\lambda = j\right] f(j),$$

where $Z_\lambda : \Omega \to \mathbb{N}$ is a Poisson variable with mean λ. Filling in definition of f we get

$$\chi_{[0,k]}(Z_\lambda) f(Z_\lambda) = \chi_{A \cap [0,k]}(Z_\lambda) - \chi_{[0,k]}(Z_\lambda) \mathbb{P}\left[Z_\lambda \in A\right].$$

To shorten notation, let us write:

$$p_\lambda(B) = \mathbb{P}\left[Z_\lambda \in B\right]$$

for all $B \subset \mathbb{N}$ and $U_k = [0,k] \cap \mathbb{N}$. We get

$$\begin{aligned}
\mathbb{E}\left[\chi_{U_k}(Z_\lambda) f(Z_\lambda)\right] &= \mathbb{E}\left[\chi_{A \cap U_k}(Z_\lambda)\right] - \mathbb{E}\left[\chi_{U_k}(Z_\lambda)\right] p_\lambda(A) \\
&= p_\lambda(A \cap U_k) - p_\lambda(U_k) p_\lambda(A) \\
&= p_\lambda(A \cap U_k) \cdot p_\lambda(\mathbb{N} \setminus U_k) - p_\lambda(A \setminus U_k) \cdot p_\lambda(U_k).
\end{aligned}$$

So we obtain

$$g_A(k+1) = \frac{p_\lambda(A \cap U_k) \cdot p_\lambda(\mathbb{N} \setminus U_k) - p_\lambda(A \setminus U_k) \cdot p_\lambda(U_k)}{\lambda \cdot p_\lambda(k)}.$$

Hence

$$\begin{aligned}
|g_A(k+1)| &\leq \frac{\max\left\{p_\lambda(A \cap U_k) \cdot p_\lambda(\mathbb{N} \setminus U_k), p_\lambda(A \setminus U_k) \cdot p_\lambda(U_k)\right\}}{\lambda \cdot p_\lambda(k)} \\
&\leq \frac{p_\lambda(U_k) \cdot p_\lambda(\mathbb{N} \setminus U_k)}{\lambda \cdot p_\lambda(k)}.
\end{aligned}$$

Filling in the Poisson probabilities, we obtain:

$$\begin{aligned}
|g_A(k+1)| &\leq \frac{k! \cdot e^{-\lambda}}{\lambda^{k+1}} \cdot \sum_{i=0}^{k} \frac{\lambda^i}{i!} \sum_{j=k+1}^{\infty} \frac{\lambda^j}{j!} \\
&= k! \cdot e^{-\lambda} \cdot \sum_{i=0}^{k} \frac{\lambda^i}{i!} \sum_{j=0}^{\infty} \frac{\lambda^j}{(j+k+1)!}
\end{aligned}$$

Now we reorder the terms and get:

$$\begin{aligned} |g_A(k+1)| &\leq k! \cdot e^{-\lambda} \cdot \sum_{n=0}^{\infty} \lambda^n \sum_{i=0}^{\min\{n,k\}} \frac{1}{i!(n+k+1-i)!} \\ &= k! \cdot e^{-\lambda} \cdot \sum_{n=0}^{\infty} \frac{\lambda^n}{(n+k+1)!} \sum_{i=0}^{\min\{n,k\}} \binom{n+k+1}{i}. \end{aligned}$$

Note that $2 \cdot \min\{n,k\} < n+k+1$ for all $n,k \in \mathbb{N}$, so Lemma 1.10 applies. Hence we get:

$$|g_A(k+1)| \leq k! \cdot e^{-\lambda} \cdot \sum_{n=0}^{\infty} \frac{\lambda^n \cdot \binom{n+k+1}{\min\{n,k\}}}{(n+k+1)!} \frac{n+k+2-\min\{n,k\}}{n+k+2-2\cdot\min\{n,k\}}.$$

A straightforward computation shows that

$$\frac{\binom{n+k+1}{n}}{(n+k+1)!} = \frac{1}{n!(k+1)!} \quad \text{and} \quad \frac{\binom{n+k+1}{k}}{(n+k+1)!} = \frac{1}{(n+1)!k!}.$$

This implies that

$$\begin{aligned} |g_A(k+1)| &\leq e^{-\lambda} \cdot \left(\sum_{n=0}^{k} \frac{\lambda^n}{n!} \frac{k+2}{(k+1)(k-n+2)} \right. \\ &\quad \left. + \sum_{n=k+1}^{\infty} \frac{\lambda^n}{n!} \frac{n+2}{(n+1)(n-k+2)} \right) \\ &\leq e^{-\lambda} \cdot e^{\lambda} \\ &= 1. \end{aligned}$$

□

2.2 Approximation theorems

We are now ready to put all the above together into concrete approximation theorems. The type of random variables we will be considering later on are counting variables. In particular, they will be variables that are obtained as

the sum of (not necessarily mutually independent) Bernoulli variables. To this end, let $\mathcal{I}$ be a set and let $X_i : \Omega \to \mathbb{N}$ be a Bernoulli variable with

$$\mathbb{E}[X_i] = \mathbb{P}[X_i = 1] = p_i$$

for all $i \in \mathcal{I}$. We will be interested in approximating the random variable

$$W = \sum_{i \in \mathcal{I}} X_i$$

with a Poisson variable with mean $\mathbb{E}[W]$.

We will treat two types of approximation theorems for such variables, one type based on dependence neighborhoods and one on so called couplings.

2.2.1 Dependence neighborhoods

For all $i, j \in \mathcal{I}$, set

$$p_{ij} = \mathbb{E}[X_i X_j]$$

and, given $i \in \mathcal{I}$, define

$$\mathcal{D}_i = \{j \in \mathcal{I};\ j \neq i,\ X_i \text{ and } X_j \text{ not independent}\}.$$

Given this data, we define the following three quantities

$$B_1 = \sum_{i \in \mathcal{I}} p_i^2,\ \ B_2 = \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{D}_i} p_i p_j \ \text{ and } \ B_3 = \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{D}_i} p_{ij}.$$

The first approximation theorem we state is the following:

Theorem 2.1. *Let W be as above such that $\lambda = \mathbb{E}[W] \in (0, \infty)$. Furthermore, let $Z_\lambda : \Omega \to \mathbb{N}$ be Poisson distributed random variable with mean λ. Then*

$$\mathrm{d_{TV}}(W, Z_\lambda) \leq 2 \cdot (B_1 + B_2 + B_3).$$

Proof. Recall from Theorem 1.8 that we need to bound $\mathbb{E}[\lambda g_A(W+1) - W g_A(W)]$ for sets $A \subset \mathbb{N}$. Our first step will be to find a convenient decomposition of this quantity. First note that

$$\begin{aligned} \mathbb{E}[\lambda g_A(W+1) - W g_A(W)] &= \sum_{i \in \mathcal{I}} \mathbb{E}[\mathbb{E}[X_i]\, g_A(W+1) - X_i g_A(W)] \\ &= \sum_{i \in \mathcal{I}} \mathbb{E}[X_i]\, \mathbb{E}[g_A(W+1)] - \mathbb{E}[X_i g_A(W)] \end{aligned}$$

Given $i \in \mathcal{I}$, define two new random variables $S_i, T_i : \Omega \to \mathbb{N}$ by

$$S_i = \sum_{j \in \mathcal{I} \setminus (\mathcal{D}_i \cup \{i\})} X_j \text{ and } T_i = \sum_{j \in \mathcal{D}_i} X_j$$

and observe that

$$W = S_i + T_i + X_i$$

for all $i \in \mathcal{I}$. Notice that

$$\mathbb{E}\left[X_i g_A(W)\right] = \mathbb{E}\left[X_i g_A(S_i + T_i + X_i)\right] = \mathbb{E}\left[X_i g_A(S_i + T_i + 1)\right].$$

Moreover

$$\begin{aligned} \mathbb{E}\left[X_i g_A(W)\right] &= \mathbb{E}\left[X_i\right]\mathbb{E}\left[g_A(S_i+1)\right] + \mathbb{E}\left[(X_i - \mathbb{E}\left[X_i\right]) \cdot g_A(S_i+1)\right] \\ &\quad + \mathbb{E}\left[X_i(g_A(S_i+T_i+1) - g_A(S_i+1))\right]. \end{aligned}$$

Independence of X_i and S_i implies that

$$\mathbb{E}\left[(X_i - \mathbb{E}\left[X_i\right]) \cdot g_A(S_i+1)\right] = \mathbb{E}\left[X_i - \mathbb{E}\left[X_i\right]\right] \cdot \mathbb{E}\left[g_A(S_i+1)\right] = 0.$$

So we obtain

$$\begin{aligned} \mathbb{E}\left[\lambda g_A(W+1) - W g_A(W)\right] &= \sum_{i \in \mathcal{I}} \mathbb{E}\left[X_i\right]\mathbb{E}\left[g_A(W+1)\right] - \mathbb{E}\left[X_i\right]\mathbb{E}\left[g_A(S_i+1)\right] \\ &\quad - \mathbb{E}\left[X_i(g_A(S_i+T_i+1) - g_A(S_i+1))\right] \\ &= \sum_{i \in \mathcal{I}} \mathbb{E}\left[X_i\right]\mathbb{E}\left[g_A(T_i+S_i+X_i+1) - g_A(S_i+1)\right] \\ &\quad - \mathbb{E}\left[X_i(g_A(S_i+T_i+1) - g_A(S_i+1))\right]. \end{aligned}$$

Now we will apply the fact that $||g_A|| \leq 1$ (Proposition 1.9). Given any $j, k \in \mathbb{N}$, we have that

$$|g_A(k+j) - g(k)| \leq 2 \cdot \chi_{\mathbb{N}_{>0}}(j) \leq 2j.$$

As such

$$\left|\mathbb{E}\left[g_A(T_i+S_i+X_i+1) - g_A(S_i+1)\right]\right| \leq 2 \cdot \mathbb{E}\left[T_i + X_i\right]$$

and

$$\left|\mathbb{E}\left[X_i(g_A(S_i+T_i+1) - g_A(S_i+1))\right]\right| \leq 2 \cdot \mathbb{E}\left[X_i T_i\right].$$

Plugging this into the inequality above, we obtain

$$\begin{aligned}\mathbb{E}\left[\lambda g_A(W+1) - W g_A(W)\right] &\leq 2 \cdot \sum_{i \in \mathcal{I}} \mathbb{E}\left[X_i\right] \mathbb{E}\left[T_i + X_i\right] + \mathbb{E}\left[X_i T_i\right] \\ &= 2 \cdot (B_1 + B_2 + B_3).\end{aligned}$$

□

The set up for the multivariate version of Theorem 2.1 is as follows. First we generalize our set up. Again, we let $\mathcal{I}$ be a set and $X_i : \Omega \to \mathbb{N}$ a Bernoulli variable for all $i \in \mathcal{I}$. Now suppose that $d \in \mathbb{N}$ and

$$\mathcal{I} = \mathcal{I}_1 \sqcup \ldots \sqcup \mathcal{I}_d.$$

We will now be interested in approximating the random variable $W : \Omega \to \mathbb{N}^d$, coordinate-wise defined by

$$W_k = \sum_{i \in \mathcal{I}_k} X_i$$

with a vector of independent Poisson variables. Again we set

$$p_i = \mathbb{E}\left[X_i\right], \quad p_{ij} = \mathbb{E}\left[X_i X_j\right]$$

and, given $i \in \mathcal{I}$, define

$$\mathcal{D}_i = \left\{j \in \mathcal{I};\ j \neq i,\ X_i \text{ and } X_j \text{ not independent}\right\}.$$

Note that this set may intersect with multiple of the sets $\mathcal{I}_k$.

We will also use three similar quantities to those before:

$$B_{1,k} = \sum_{i \in \mathcal{I}_k} p_i^2, \quad B_{2,k} = \sum_{i \in \mathcal{I}_k} \sum_{j \in \mathcal{D}_i} p_i p_j \ \text{ and } \ B_{3,k} = \sum_{i \in \mathcal{I}_k} \sum_{j \in \mathcal{D}_i} p_{ij},$$

for $k = 1, \ldots, d$.

The approximation theorem now states:

Theorem 2.2. *Let W be as above such that $\lambda_k = \mathbb{E}\left[W_k\right] \in (0, \infty)$ for $k = 1, \ldots, d$. Furthermore, let $Z_k : \Omega \to \mathbb{N}$ be Poisson distributed random variable with mean λ_k for $k = 1, \ldots, d$. Then*

$$\mathrm{d_{TV}}\left(W, Z\right) \leq 2 \cdot \sum_{k=1}^{d} B_{1,k} + B_{2,k} + B_{3,k},$$

where $Z : \Omega \to \mathbb{N}^d$ is defined by $Z = (Z_1, \ldots, Z_d)$.

Proof. Our strategy will be to apply the proof of Theorem 2.1 recursively. First note that for $A \subset \mathbb{N}^d$

$$\begin{aligned}\mathbb{P}[W \in A] - \mathbb{P}[Z \in A] &= \mathbb{E}[\chi_A(W)] - \mathbb{E}[\chi_A(Z)] \\ &= \sum_{r=1}^{d} \mathbb{E}[\chi_A(Z_1, \ldots, Z_{r-1}, W_r, W_{r+1} \ldots, W_d)] \\ &\qquad -\mathbb{E}[\chi_A(Z_1, \ldots, Z_{r-1}, Z_r, W_{r+1}, \ldots, W_d)].\end{aligned}$$

Let us write

$$\begin{aligned}t_r &= \mathbb{E}[\chi_A(Z_1, \ldots, Z_{r-1}, W_r, W_{r+1} \ldots, W_d)] \\ &\quad -\mathbb{E}[\chi_A(Z_1, \ldots, Z_{r-1}, Z_r, W_{r+1}, \ldots, W_d)].\end{aligned}$$

Let $g_{A,r} : \mathbb{N}^d \to \mathbb{R}$ be the function satisfying $g_{A,r}(k) = 0$ when $k_r = 0$ and

$$\lambda_r g_{A,r}(k + e_r) - k_r g_{A,r}(k) = \chi_A(k) - \mathbb{E}[\chi_A(k_1, \ldots, k_{r-1}, Z_r, k_{r+1}, \ldots, k_d)] \tag{2.1}$$

for all $k \in \mathbb{N}^d$, where $e_r \in \mathbb{N}^d$ is defined by

$$(e_r)_j = \begin{cases} 1 & \text{if } j = r \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned}t_r &= \mathbb{E}[\lambda_r g_{A,r}(Z_1, \ldots, Z_{r-1}, W_r + 1, W_{r+1}, \ldots, W_d)] \\ &\quad -\mathbb{E}[W_r g_{A,r}(Z_1, \ldots, Z_{r-1}, W_r, W_{r+1}, \ldots, W_d)]\end{aligned}$$

Like in the proof of Theorem 2.1, we define random variables $S_{i,r}, T_{i,r} : \Omega \to \mathbb{N}$ by

$$S_{i,r} = \sum_{j \in \mathcal{I}_r \setminus (\mathcal{D}_i \cup \{i\})} X_j \text{ and } T_{i,r} = \sum_{j \in \mathcal{D}_i \cap \mathcal{I}_r} X_j$$

for all i, r. We again observe that

$$W_r = S_{i,r} + T_{i,r} + X_i$$

for all i, r. As such, with a similar computation to the one in the proof of Theorem 2.1, we obtain

$$\begin{aligned}t_r &= \sum_{i \in \mathcal{I}_r} \mathbb{E}[X_i]\, \mathbb{E}[g_{A,r}(Z_1, \ldots, Z_{r-1}, S_{i,r} + T_{i,r} + X_i + 1, S_{i,r+1} + T_{i,r+1}, \\ &\qquad \ldots, S_{i,d} + T_{i,d}) - g_{A,r}(Z_1, \ldots, Z_{r-1}, S_{i,r} + 1, \ldots, S_{i,d})] \\ &\quad -\mathbb{E}[X_i(g_{A,r}(Z_1, \ldots, Z_{r-1}, S_{i,r} + T_{i,r} + 1, S_{i,r+1} + T_{i,r+1}, \ldots, \\ &\qquad S_{i,d} + T_{i,d}) - g_{A,r}(Z_1, \ldots, Z_{r-1}, S_{i,r} + 1, S_{i,r+1}, \ldots, S_{i,d}))].\end{aligned}$$

Note that for all $k_1, \ldots, k_{r-1}, k_{r+1}, \ldots, k_d$:

$$\chi_A(k_1, \ldots, k_{r-1}, Z_r, k_{r+1}, \ldots, k_d) = \chi_{A'}(Z_r),$$

where $A' \subset \mathbb{N}$ is defined by

$$A' = \{k \in \mathbb{N};\ (k_1, \ldots, k_{r-1}, k, k_{r+1}, \ldots, k_d) \in A\}.$$

As such (2.1) is an instance of Stein's equation and Proposition 1.9 applies, from which we obtain

$$\sup_{k \in \mathbb{N}^d} \{|g_{A,r}(k)|\} \leq 1.$$

This means that for all $k, j \in \mathbb{N}^d$:

$$|g_{A,r}(k+j) - g_{A,r}(k)| \leq 2 \sum_{s=1}^{d} j_s.$$

Hence

$$|t_r| \leq 2 \sum_{i \in \mathcal{I}_r} \mathbb{E}[X_i] \mathbb{E}[X_i + T_{i,r} + \ldots T_{i,d}] + \mathbb{E}[X_i(T_{i,r} + \ldots + T_{i,d})].$$

Filling this in in our original bound, we obtain

$$\mathbb{P}[W \in A] - \mathbb{P}[Z \in A] \leq 2 \cdot \sum_{k=1}^{d} B_{1,k} + B_{2,k} + B_{3,k}.$$

□

2.2.2 Coupling

The approximation theorems from the previous section only work if we can find enough pairs of independent variables. It might occur that this is not the case, but that the variable in question is still approximately Poisson distributed, especially if the dependences are very weak. In the latter situation, the so called coupling method will do the trick.

First, we need to define what a positive coupling is. Suppose that for every $i \in \mathcal{I}$ we can define a partition

$$\mathcal{I} = \{i\} \sqcup \mathcal{J}_{i,1} \sqcup \mathcal{J}_{i,2}$$

and random variables $X'_{i,j} : \Omega \to \{0, 1\}$ for all $j \in \mathcal{I} \setminus \{i\}$ so that

1. $X'_{i,j}$ has the same distrbution as X_j when conditioned on $X_i = 1$. That is
$$\mathbb{P}[X'_{i,j} = 1] = \mathbb{P}[X_j = 1 |\ X_i = 1].$$
2. $X'_{i,j}(\omega) \geq X_j(\omega)$ for all $\omega \in \Omega$ and all $j \in \mathcal{J}_{i,1}$.

In this situation, the set of variables $\{X'_{i,j}\}_{i,j\in\mathcal{I}}$ is called a *positive coupling* for $\{X_i\}_{i\in\mathcal{I}}$.

Theorem 2.3. *Let $W : \Omega \to \mathbb{N}$ be as above such that $\lambda = \mathbb{E}[W] \in (0, \infty)$ and suppose that a positive coupling for $\{X_i\}_{i\in\mathcal{I}}$ exists. Furthermore, let $Z_\lambda : \Omega \to \mathbb{N}$ be Poisson distributed random variable with mean λ. Then*

$$\mathrm{d_{TV}}(W, Z_\lambda) \leq 2 \cdot \sum_{i\in\mathcal{I}} \left(p_i^2 + \sum_{j\in\mathcal{J}_{i,1}} p_{ij} - p_i p_j + \sum_{j\in\mathcal{J}_{i,2}} p_{ij} + p_i p_j \right).$$

Proof. Write

$$\begin{aligned}
\mathbb{E}[\lambda \cdot g_A(W+1) - X \cdot g_A(W)] &= \sum_{i\in\mathcal{I}} p_i \mathbb{E}[g_A(X+1)] - \mathbb{E}[X_i \cdot g_A(X)] \\
&= \sum_{i\in\mathcal{I}} p_i \Bigg(\mathbb{E}[g_A(X+1)] \\
&\qquad - \mathbb{E}\left[g_A\left(1 + \sum_{j\in\mathcal{I}\setminus\{i\}} X_j\right) \mid X_i = 1 \right] \Bigg),
\end{aligned}$$

Now we use the coupling assumption and obtain that

$$\begin{aligned}
\mathbb{E}[\lambda \cdot g_A(X+1) - X \cdot g_A(X)] &\leq \sum_{i\in\mathcal{I}} p_i \Bigg(\mathbb{E}[g_A(X+1)] \\
&\qquad - \mathbb{E}\left[g_A\left(1 + \sum_{j\in\mathcal{I}\setminus\{i\}} X'_{i,j}\right) \right] \Bigg) \\
&= \sum_{i\in\mathcal{I}} p_i \cdot \mathbb{E}\left[g_A(X+1) - g_A\left(1 + \sum_{j\in\mathcal{I}\setminus\{i\}} X'_{i,j}\right) \right],
\end{aligned}$$

Using the fact that $||g_A|| \leq 1$ (Proposition 1.9), we get

$$\begin{aligned}
|\mathbb{E}\left[\lambda \cdot g_A(X+1) - X \cdot g_A(X)\right]| &\leq 2 \cdot \sum_{i \in \mathcal{I}} p_i \cdot \mathbb{E}\left[\left|X - \sum_{j \in \mathcal{I} \setminus \{i\}} X'_{\alpha,\beta}\right|\right] \\
&= 2 \cdot \sum_{i \in \mathcal{I}} p_i \cdot \mathbb{E}\left[\left|X_i - \sum_{j \in \mathcal{I} \setminus \{i\}} (X'_{i,j} - X_j)\right|\right] \\
&\leq 2 \cdot \sum_{i \in \mathcal{I}} p_i \cdot \mathbb{E}\left[X_i\right] + p_i \cdot \mathbb{E}\left[\sum_{j \in \mathcal{J}_{i,1}} (X'_{i,j} - X_j)\right] \\
&\quad + \sum_{i \in \mathcal{I}} p_i \cdot \mathbb{E}\left[\sum_{j \in \mathcal{J}_{i,2}} (X'_{i,j} + X_i)\right]
\end{aligned}$$

where we used property (2) to obtain the last inequality. Now note that

$$p_i \cdot \mathbb{E}\left[X'_{i,j}\right] = \mathbb{E}\left[X_i X_j\right].$$

We obtain:

$$\mathrm{d_{TV}}\left(W, Z_\lambda\right) \leq 2 \cdot \sum_{i \in \mathcal{I}} \left(p_i^2 + \sum_{j \in \mathcal{J}_{i,1}} p_{ij} - p_i p_j + \sum_{j \in \mathcal{J}_{i,2}} p_{ij} + p_i p_j \right),$$

which is what we set out to prove. □

Note that none of the quantities above depend on the variables $X'_{i,j}$, it is only important that they exist. Furthermore, the only difference with Theorem 2.1 are the terms $p_{ij} - p_i p_j$. These terms measure the dependence of X_i and X_j. As such, we will want to choose $\mathcal{J}_{i,1}$ so that X_j is 'close to independent' of X_i for all $j \in \mathcal{J}_{i,1}$.

Finally, we will also need a multivariate version of the coupling method. Again, we let $\mathcal{I}$ be a set and $X_i : \Omega \to \mathbb{N}$ a Bernoulli variable for all $i \in \mathcal{I}$. Now suppose that $d \in \mathbb{N}$ and

$$\mathcal{I} = \mathcal{I}_1 \sqcup \ldots \sqcup \mathcal{I}_d$$

and again define the random variable $W : \Omega \to \mathbb{N}^d$ coordinate-wise by

$$W_k = \sum_{i \in \mathcal{I}_k} X_i.$$

Theorem 2.4. *Let $W : \Omega \to \mathbb{N}^k$ be as above such that $\lambda_k = \mathbb{E}[W_k] \in (0, \infty)$ for $k = 1, \dots, d$ and suppose that a positive coupling for $\{X_i\}_{i \in \mathcal{I}}$ exists. Furthermore, let $Z_k : \Omega \to \mathbb{N}$ be Poisson distributed random variable with mean λ_k for $k = 1, \dots, d$. Then*

$$\mathrm{d_{TV}}(W, Z) \leq 2 \cdot \sum_{i \in \mathcal{I}} \left(p_i^2 + \sum_{j \in \mathcal{J}_{i,1}} p_{ij} - p_i p_j + \sum_{j \in \mathcal{J}_{i,2}} p_{ij} + p_i p_j \right).$$

where $Z : \Omega \to \mathbb{N}^d$ is defined by $Z = (Z_1, \dots, Z_d)$.

The proof of the multivariate version of the statement above is very similar to the proof of Theorem 2.2 and we will skip it.

2.3 Exercises

Exercise 2.1.

(a) Let $X_1, X_2 : \Omega \to \mathbb{N}^r$ be random variables. Show that

$$\mathrm{d_{TV}}(X_1, X_2) = \frac{1}{2} \sum_{k \in \mathbb{N}^r} |\mathbb{P}[X_1 = k] - \mathbb{P}[X_2 = k]|.$$

(b) Let $r \in \mathbb{N}$ and $\lambda_1, \lambda_2 \in (0, \infty)^r$ and let $\mathbf{X}_1 = (X_{1,1}, \dots, X_{1,r}), \mathbf{X}_2 = (X_{2,1}, \dots, X_{2,r}) : \Omega \to \mathbb{N}^r$ be random variables so that $X_{i,j}$ are Poisson distributed with mean $\lambda_{i,j}$ and pairwise independent. Show that

$$\mathrm{d_{TV}}(\mathbf{X}_1, \mathbf{X}_2) = \mathcal{O}\left(\sum_{i=1}^{r} |\lambda_{1,i} - \lambda_{2,i}| \right)$$

as $\sum_{i=1}^{r} |\lambda_{1,i} - \lambda_{2,i}| \to 0$.

Exercise 2.2. Let $\lambda \in (0, \infty)$. For all $n \in \mathbb{N}$, let $\{X_{i,n}\}_{i=1}^{n}$ be independent Bernoulli variables so that

$$\mathbb{E}[X_{i,n}] = \lambda / n.$$

Furthermore, define

$$W_n = \sum_{i=1}^{n} X_{i,n}$$

and let Z_λ be a Poisson ditributed random variable with mean λ. Show that

$$W_n \xrightarrow{\text{TV}} Z_\lambda$$

as $n \to \infty$.

Exercise 2.3. Random geometric graphs: let $\mathbb{T}^2$ denote the 2-dimensional torus. That is

$$\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2,$$

where $\mathbb{Z}^2 \curvearrowright \mathbb{R}^2$ by translations. Figure 2.1 shows a cartoon of $\mathbb{T}^2$.

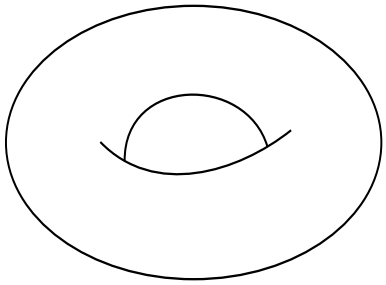

Figure 2.1: A torus.

The Lebesgue measure on $\mathbb{R}^2$ is invariant under the $\mathbb{Z}^2$ action and hence descends to a measure on $\mathbb{T}^2$. $[0,1]^2 \subset \mathbb{R}^2$ forms a fundamental domain for this action. As such the total area of $\mathbb{T}^2$ under this measure is 1. In other words, we obtain a probability space $(\mathbb{T}^2, \mathbb{P})$. $\mathbb{T}^2$ also comes with a distance: the Euclidean distance function $\mathrm{d}_{\mathbb{R}^2} : \mathbb{R}^2 \times \mathbb{R}^2 \to [0,\infty)$ is also invariant under the $\mathbb{Z}^2$ action and hence descends to a distance

$$\mathrm{d}_{\mathbb{T}^2} : \mathbb{T}^2 \times \mathbb{T}^2 \to [0,\infty).$$

Set $r_n = n^{-3/4}$. Given $x_1, \ldots, x_n \in \mathbb{T}^2$ we define a graph as follows. The points $x_1, \ldots, x_n$ will be the vertices of our graph. We connect x_i and x_j by an edge if and only if $\mathrm{d}_{\mathbb{T}^2}(x_i, x_j) \leq r_n$.

(a) Let $X_n : (\mathbb{T}^2)^n \to \mathbb{N}$ count the number of triangles (triples of vertices that are all connected by an edge) in the graph associated to the points $x_1, \ldots, x_n$. Show that

$$\mathbb{E}[X_n] \to \frac{1}{6} \int_{B_1(0)} \int_{B_1(0)} h(y_1, y_2) dy_1 dy_3$$

as $n \to \infty$, where $B_1(0) \subset \mathbb{R}^2$ denotes the unit ball around the origin in $\mathbb{R}^2$ and

$$h(y_1, y_2) = \begin{cases} 1 & \text{if } \mathrm{d}_{\mathbb{R}^2}(y_1, y_2) \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(b) Set

$$\lambda = \frac{1}{6} \int_{B_1(0)} \int_{B_1(0)} h(y_1, y_2) dy_1 dy_3$$

and let $Z_\lambda : \Omega \to \mathbb{N}$ be a Poisson random variable with mean λ. Show that

$$X_n \xrightarrow{\text{TV}} Z_\lambda$$

as $n \to \infty$.

Lecture 3

Random graphs I: Basics and cycle counts

The material in this lecture is mainly based on [Bol85], [Wor99].

3.1 Basic definitions

3.1.1 Graphs

Let us first fix our definition of what a graph is. There are multiple definitions available to capture the intuitive idea that a graph is a set of vertices and a set of edges between these vertices. We will want to allow multiple edges between pairs of vertices and loops, so we choose the following definition, in which we write $|X|$ for the cardinality of a set X.

Definition 3.1. A graph is a triple $G = (V, E, \mathcal{I})$ where V is a set, called the set of *vertices* of G, E is a set, called the set of *edges* of G and

$$\mathcal{I} \subset E \times V$$

is called the *incidence relation* of G and satisfies the condition that for all $e \in E$ we have

$$|\{v \in V;\ (e, v) \in \mathcal{I}\}| \in \{1, 2\}.$$

An edge that is incident to a single vertex is called a *loop*. A graph without loops in which every pair of vertices has at most one edge incident to it is called *simple*. If $v, w \in V$ and there exists and $e \in E$ so that both $(e, v) \in \mathcal{I}$ and $(e, w) \in \mathcal{I}$ we say that v and w are *adjacent* or that v and w *share an edge*.

The *degree* or *valence* of a vertex $v \in V$ is given by

$$\deg(v) = |\{e \in E;\ (e, v) \in \mathcal{I}\}| + \left| \left\{ e \in E;\ \begin{array}{c} (e, v) \in \mathcal{I} \text{ and} \\ (e, w) \notin \mathcal{I},\ \forall w \in V \text{ with } w \neq v \end{array} \right\} \right|.$$

An *isomorphism* of between graphs $G_1 = (V_1, E_1, \mathcal{I}_1)$ and $G_2 = (V_2, E_2, \mathcal{I}_1)$ is a pair of bijective maps $f_V : V_1 \to V_2,\ f_E : E_1 \to E_2$ such that

$$(e, v) \in \mathcal{I}_1 \ \Leftrightarrow\ (f_E(e), f_V(e)) \in \mathcal{I}_2.$$

An *automorphism* of a graph $G = (V, E, \mathcal{I})$ is an isomorphism between G and itself. The group formed by all automorphisms of G will be denoted by $\mathrm{Aut}(G)$.

A *walk* between vertices $v, w \in V$ is sequence of vertices $(v_1, v_2, \ldots, v_r)$ with $v_1 = v$, $v_r = w$ and so that for all $i = 1, \ldots r-1$ the vertices v_i and v_{i+1} are adjacent. A *cycle* in G is a walk between v and itself for some vertex $v \in V$.

G is called *connected* is there exists a walk between every pair of vertices $v, w \in V$.

Some remarks:

- The condition on $|\{v \in V;\ (e, v) \in \mathcal{I}\}|$ guarantees that every edge connects to either one or two vertices.
- In the definition above, a loop at a vertex (an edge that connects to only that vertex) adds 2 to the degree of this vertex.
- Given a graph G, we will often write $V(G)$ and $E(G)$ for the sets of its vertices and edges respectively.

The above serves as a formal definition of what a graph is. It is however not always the easiest way to describe graphs. Often we will just think in terms of pictures. Let us give an example of a graph.

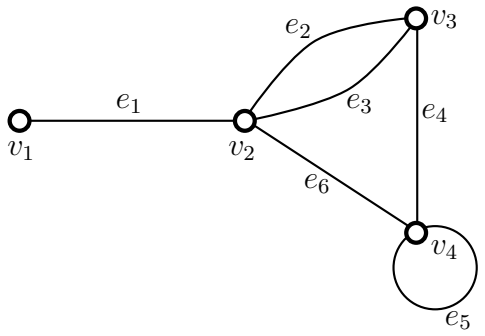

Figure 3.1: A graph.

The graph $G = (V, E, \mathcal{I})$ above is given by $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and

$$\begin{aligned}\mathcal{I} \;=\; & \{(e_1, v_1),\ (e_1, v_2),\ (e_2, v_2),\ (e_2, v_3),\ (e_3, v_2),\ (e_3, v_3),\ (e_4, v_3),\ (e_4, v_4),\\ & (e_5, v_4)\ (e_6, v_4),\ (e_6, v_2)\}\end{aligned}$$

3.1.2 Random graphs

There are multiple models of random graphs around. The most widely studied model is probably that of the Erdős-Rényi random graph: fix $p \in (0, 1)$, take n vertices and add each of the possible edges between these vertices to the graph with probability p and leave it out with probability $1 - p$.

We will however be interested in regular graphs:

Definition 3.2. Let $k \in \mathbb{N}$. A graph $G = (V, E, \mathcal{I})$ is called k-regular if

$$\deg(v) = k$$

for all $v \in V$.

Our goal now is to pick a graph at random among all k-regular graphs on a given set of vertices. Of course, the set of k-regular graphs on n vertices up to isomorphism is a finite set for all $k, n \in \mathbb{N}$. So, we could just pick one at random from this finite set. This model however turns out to be hard to control in general. Instead we will study the *configuration model* for random regular graphs.

First let us fix once and for all disjoint sets $W_1(n), \dots, W_n(n)$ with

$$|W_i(n)| = k$$

for $i = 1, \dots, n$, for every $n, k \in \mathbb{N}_{\geq 1}$ so that $n \cdot k$ is even. Furthermore, we will write

$$W(n) = \bigsqcup_{i=1}^{n} W_i(n).$$

We can now define configurations:

Definition 3.3. Let $n, k \in \mathbb{N}$ so that $n \cdot k$ is even. Then, a k-regular *configuration* on n vertices is a set of pairs

$$C = \{\{a_i, b_i\} \subset W(n)\}_{i=1}^{n \cdot k/2}$$

so that

$$\bigcup_{i=1}^{n \cdot k/2} \{a_i, b_i\} = W(n).$$

We will write $\mathcal{G}_{n,k}$ for the (finite) set of k-regular configurations on n vertices.

Note that the last condition guarantees that every element of $W(n)$ appears exactly once in a pair of the configuration C.

Definition 3.4. Let $n, k \in \mathbb{N}$ so that $n \cdot k$ is even. Furthermore, let $C = \{\{a_i, b_i\} \subset W(n)\}_{i=1}^{n \cdot k/2}$ be a k-regular configuration on n vertices.

The graph $G(C) = (V, E, \mathcal{I})$ associated with C is given by

$$V = \{v_1, \dots, v_n\},\ E = \{e_1, \dots, e_{n \cdot k/2}\}$$

and

$$(e_i, v_j) \in \mathcal{I} \ \Leftrightarrow\ \{a_i, b_i\} \cap W_j(n) \neq \emptyset.$$

In other words, our finite sets $W_i(n)$ represent the vertices of $G(C)$ and we connect two of them if and only if two of their elements appear as a pair in the configuration. As such, we will often think of the elements in C as labels on half-edges of $G(C)$. Figure 3.2 gives an example:

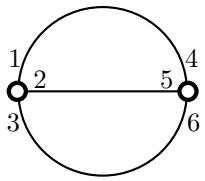

Figure 3.2: The graph $G(C)$ corresponding to the configuration $C = \{\{1,4\},\ \{2,5\},\ \{3,6\}\}$.

The number of configurations is easy to count. For $n \in \mathbb{N}$ even we write

$$n!! = (n-1)(n-3)\cdots 3 \cdot 1.$$

We have:

Lemma 3.5. *Let $n, k \in \mathbb{N}$ so that $n \cdot k$ is even. Then:*

$$|\mathcal{G}_{n,k}| = (n \cdot k)!!.$$

Proof. See Exercise 3.2. □

We can now define the configuration model for random regular graphs. Recall that, given a set X, $\mathcal{P}(X)$ denotes the power set of X.

Definition 3.6. *The configuration model.* Let $n, k \in \mathbb{N}$ so that $n \cdot k$ is even. We define a probability measure

$$\mathbb{P}_{n,k} : \mathcal{P}(\mathcal{G}_{n,k}) \to [0,1]$$

by

$$\mathbb{P}_{n,k} = \frac{|A|}{|\mathcal{G}_{n,k}|}, \text{ for all } A \subset \mathcal{G}_{n,k}.$$

Because configurations give rise to regular graphs, the definition above allows us to speak of random regular graphs. That is, if we say "the probability that a k-regular graph on n vertices has property P", we will mean the probability with respect to the probability measure $\mathbb{P}_{n,k}$.

Note however that while it is clear that every graph can be obtained from some configuration (we just label the vertices and edges), some graphs might be given a higher probability by $\mathbb{P}_{n,k}$ than others. Later on, we will see that while $\mathbb{P}_{n,k}$ indeed prefers certain graphs, the differences between the probabilities assigned are small enough so that $\mathbb{P}_{n,k}$ can still be used to make statements about graphs picked uniformly at random among isomorphism classes.

3.2 Counting regular graphs

Our main application of the configuration model is counting regular graphs. That is, we will first control the cycle counts and the number of automorphisms of a random regular graph and then use this to give asymptotic estimates on the number of regular graphs of a fixed degree on a large number of vertices.

3.2.1 Cycles

Let $r \in \mathbb{N}$. A *cycle* of length r (or r-cycle) in a graph G is sequence of vertices $(v_1, v_2, \dots, v_r)$ so that for all $i = 1, \dots r-1$ the vertices v_i and v_{i+1} share an edge and so do the vertices v_r and v_1. Cycles that are obtained from one another by cyclic permutation or 'reading backwards' will be considered the same. A cycle in which all the vertices are distinct is called a *circuit*.

Our first goal is to understand the number of cycles of a given length in a random regular graph. To this end, define random variables $X_{n,k,r} : \mathcal{G}_{n,k} \to \mathbb{N}$ defined by

$$X_{n,k,r}(C) = |\{r\text{-cycles in } G(C)\}|$$

We will use the Chen-Stein method to prove a Poisson limit theorem (due to Bollobás [Bol80]) for these random variables. Our proof is a simplified version of Johnson's proof in [Joh15].

Let us also define Poisson distributed random variables $X_{k,r} : \Omega \to \mathbb{N}$ with means

$$\lambda_{k,r} = \frac{(k-1)^r}{2r}$$

Finally, for a finite set $R \subset \mathbb{N}$, we define vectors of random variables

$$\mathbf{X}_{n,k,R} = (X_{n,k,r})_{r \in R} \text{ and } \mathbf{X}_{k,R} = (X_{k,r})_{r \in R}$$

Theorem 3.7. *Fix $k \in \mathbb{N}_{\geq 3}$. For any finite set $R \subset \mathbb{N}$ there exists a constant $C_R > 0$ so that*

$$\mathrm{d_{TV}}\left(\mathbf{X}_{n,k,R}, \mathbf{X}_{k,R}\right) \leq C_R/n$$

for all $n \in \mathbb{N}$.

Proof. We will apply the coupling version Chen-Stein method to prove this. Let us first analyse the possible labelings of an r-cycle in the graph corresponding to a configuration. Given an r-cycle in such a graph, we can traverse it and record the $2r$ labels that appear in it in order. If we also group every pair consecutive labels corresponding to the same edge, we obtain a list of the form

$$((a_1, b_1), \ldots, (a_r, b_r)) \in \left(W(n)^2\right)^r.$$

Note however that this does not define a map from cycles to lists of labels: to obtain the list, we need to know where to start traversing the cycle and in which direction to traverse it.

Let us write $A_{n,r}$ for the set of all such lists of labels that could possibly appear as an r-cycle in a configuration. In other words, $A_{n,r}$ is the set of lists $((a_1, b_1), \ldots, (a_r, b_r)) \in (W(n)^2)^r$ so that:

- The labels form a cycle: b_i and a_{i+1} lie in the same set $W_j(n)$ for all $1 \leq i \leq r-1$, as do b_r and a_1
- The pairs of labels that appear as edges are consitent: if (a_i, b_i) appears in a pair, then neither a_i nor b_i appears in a pair with another label (a pair is however allowed to appear multiple times).

Given $\alpha \in A_{n,r}$, we write

$$X_\alpha : \mathcal{G}_{n,k} \to \{0,1\},$$

where $X_\alpha(C)$ counts the number of appearances of α in C, which is either 0 or 1. So X_α is a Bernoulli variable for all $\alpha \in A_{n,r}$ and all $r \in W$.

However, $\sum_{\alpha \in A_{n,r}} X_\alpha$ is not equal to $X_{n,k,r}$. Indeed we over count by going througth all the $\alpha \in A_{n,r}$: in $A_{n,r}$ each cycle artificially has a starting point and direction of travel. This implies that every labeled r-cycle is counted $2r$ times in $A_{n,r}$.

In the end we will want to deal with the set of labeled cycles and not the set of labeled directed cycles with a starting point, we could try to find a convenient description of the set $A_{n,r}/\sim$, where $\sim$ is some equivalence relation that takes care of the symmetry. Another course of action, the one we will actually pursue, is to simply divide all the quantities we need to compute by $2r$.

Let us first compute the means. Like we said, we have:

$$\mathbb{E}\left[X_{n,k,r}\right] = \frac{1}{2r} \sum_{\alpha \in A_{n,r}} \mathbb{E}\left[X_\alpha\right] = \frac{1}{2r} \sum_{\alpha \in A_{n,r}} \mathbb{P}\left[C \text{ contains the pairs in } \alpha\right].$$

We note that the probability $\mathbb{P}\left[C \text{ contains the pairs in } \alpha\right]$ only depends on the number of distinct pairs in α. Indeed, if this number of pairs is e, then

$$\mathbb{P}\left[C \text{ contains the pairs in } \alpha\right] = \frac{1}{(n \cdot k - 1)(n \cdot k - 3) \cdots (n \cdot k - 2 \cdot e + 1)}.$$

As such, it makes sense to divide $\mathbb{E}\left[X_{n,k,r}\right]$ into terms: each term corresponding to an isomorphism type of cycles. We can then write

$$\mathbb{E}\left[X_{n,k,r}\right] = \frac{1}{2r} \sum_{\mathcal{C}} a_{n,k}(\mathcal{C}) \cdot p_{n,k}(\mathcal{C}),$$

where the sum runs over isomorphism types $\mathcal{C}$, $a_{n,k}(\mathcal{C})$ is the number of lists in $A_{n,r}$ that gives a cycle of the isomorphism type $\mathcal{C}$ and

$$p_{n,k}(\mathcal{C}) = \mathbb{P}\left[C \text{ contains the pairs in } \alpha_{\mathcal{C}}\right]$$

for any $\alpha_{\mathcal{C}} \in A_{n,r}$ that gives rise to a cycle of the isomorphism type $\mathcal{C}$.

It will turn out that $\mathbb{E}[X_{n,k,r}]$ is dominated by the term corresponding to circuits, so let us first compute that term. If $\alpha \in A_{n,r}$ corresponds to a circuit, then it contains exactly r distinct pairs, as such

$$p_{n,k}(r\text{-circuit}) = \frac{1}{(n \cdot k - 1)(n \cdot k - 3) \cdots (n \cdot k - 2 \cdot r + 1)}.$$

Furthermore, to count the number of lists in $A_{n,r}$ giving rise to r-circuits, we note that all we need to choose is which distinct r vertices we use and which of the labels of these vertices to connect to each other. This gives a total of

$$a_{n,k}(r\text{-circuit}) = n \cdot (n-1) \cdots (n-r+1) \cdot (k(k-1))^r$$

options.

For the other terms, we note that by the same reasoning as above

$$a_{n,k}(\mathcal{C}) \cdot p_{n,k}(\mathcal{C}) \leq \frac{k^{2r} n^v}{(n - 2 \cdot r + 1)^e},$$

where v is the number of vertices in $\mathcal{C}$ and e the number of edges. If $\mathcal{C}$ is not a circuit, it has more edges than vertices, which implies that

$$0 \leq \mathbb{E}[X_{n,k,r}] - \frac{1}{2r} \frac{n \cdot (n-1) \cdots (n-r+1) \cdot (k(k-1))^r}{(n \cdot k - 1)(n \cdot k - 3) \cdots (n \cdot k - 2 \cdot r + 1)} \leq \frac{C}{n},$$

where $C > 0$ is a constant that depends on r and k (it for instance contains the number of isomorphism classes $\mathcal{C}$ we need to sum over) but not on n. We have

$$\lambda_{k,r} \left(\frac{n \cdot k - r \cdot k}{n \cdot k - 1} \right)^r \leq \frac{1}{2r} \frac{n \cdot (n-1) \cdots (n-r+1) \cdot (k(k-1))^r}{(n \cdot k - 1)(n \cdot k - 3) \cdots (n \cdot k - 2 \cdot r + 1)} \leq \lambda_{k,r}$$

Because

$$\left(\frac{n \cdot k - r \cdot k}{n \cdot k - 1} \right)^r = 1 + \mathcal{O}\left(n^{-1}\right)$$

as $n \to \infty$, we obtain that

$$|\mathbb{E}[X_{n,k,r}] - \lambda_{k,r}| = \mathcal{O}\left(n^{-1}\right)$$

as $n \to \infty$.

In order to derive a bound on the total variational distance between $X = X_{n,k,R}$ and a vector of Poisson variables with the correct means, we will use Theorem 2.4.

Set $A = \sqcup_{r \in R} A_{n,r}$. Our first task is now to find a coupling for the variables $\{X_\alpha\}_{\alpha \in A}$. So we need partitions $A = A_{\alpha,1} \sqcup A_{\alpha,2} \sqcup \{\alpha\}$ and variables $X'_{\alpha,\beta}$.

We start with the variables. Given $C \in \mathcal{G}_{n,k}$ and $\alpha \in A_{n,r}$, we obtain a new configuration $C'_\alpha \in \mathcal{G}_{n,k}$ as follows:

1. All pairs of labels in C that contain no labels from α become pairs of labels in C'_α.
2. If (i,j) appears in α but $\{i,j\} \notin C$, then that means that there are two pairs $\{i,x\}, \{j,y\} \in C$ with $x \neq j$ and $y \neq i$. We replace these pairs in C by the pairs $\{i,j\}$ and $\{x,y\}$. We do this until all the pairs in α appear in the configuration obtained.

Now set $X'_{\alpha,\beta}(C) = X_\beta(C'_\alpha)$. The partition we choose is given by:

$$A_{\alpha,1} = \{\beta \in A;\ \beta \text{ shares no vertices with } \alpha\}.$$

We claim that these random variables satisfy the desired properties. Property (1) for couplings follows from the fact that, given α, the map $C \to C'_\alpha$ is constant to 1. This follows from symmetry: the actual labels involved play no role. As such

$$\begin{aligned} \mathbb{P}[X'_{\alpha,\beta}(C) = 1] &= \mathbb{P}[X_\beta(C'_\alpha) = 1] \\ &= \frac{1}{|\mathcal{G}_{n,k}|} \sum_{C' \in \mathcal{G}_{n,k} : X_\alpha(C) = 1} |\{C \in \mathcal{G}_{n,k};\ C'_\alpha = C'\}| \cdot X_\beta(C') \end{aligned}$$

Because the map $C \to C'_\alpha$ is constant to 1, we obtain

$$|\mathcal{G}_{n,k}| = |\{C' \in \mathcal{G}_{n,k};\ X_\alpha(C) = 1\}| \cdot |\{C \in \mathcal{G}_{n,k};\ C'_\alpha = C'\}|,$$

for any $C' \in \{C' \in \mathcal{G}_{n,k};\ X_\alpha(C) = 1\}$. Hence

$$\begin{aligned} \mathbb{P}[X'_{\alpha,\beta}(C) = 1] &= \frac{1}{|\{C' \in \mathcal{G}_{n,k};\ X_\alpha(C) = 1\}|} \sum_{C' \in \mathcal{G}_{n,k} : X_\alpha(C) = 1} X_\beta(C') \\ &= \mathbb{P}[X_\beta = 1 |\ X_\alpha = 1]. \end{aligned}$$

Property (2) for couplings follows directly from the definition of $A_{\alpha,1}$. Indeed if $\beta \in A_{\alpha,1}$ and $X_\beta(C) = 1$ then $X_\beta(C'_\alpha) = 1$, just because β has no labels in common with α, so C'_α still contains the pairs in β.

Since the coupling exists, we now only need to bound the quantities

$$p_\alpha = \mathbb{E}[X_\alpha] \text{ and } p_{\alpha\beta} = \mathbb{E}[X_\alpha X_\beta],\ \alpha, \beta \in A_{n,r}.$$

To bound the sums in Theorem 2.4 we use similar observations as in the computation of $\mathbb{E}[X_{n,k,r}]$. If α forms a cycle of e edges and $v \leq e$ vertices, then

$$p_\alpha = \mathcal{O}\left(n^{-e}\right)$$

However, the number of terms with the same isomorphism type as α is $\mathcal{O}(n^v)$. So the first sum is $\mathcal{O}(n^{v-2e}) = \mathcal{O}(n^{-1})$ (using that the number of isomorphism classes we are considering is finite and depends on R only). Similar arguments work for the sums corresponding to $A_{\alpha,2}$.

If α and β are vertex-disjoint, the probabilities $p_{\alpha\beta}$ and $p_\alpha p_\beta$ are readily computed. It follows that

$$p_{\alpha\beta} - p_\alpha p_\beta \leq \frac{C}{n} \cdot p_\alpha p_\beta.$$

This means that the sums corresponding to the sets $A_{\alpha,1}$ contribute at most $C \cdot \sum_{r\in R} \lambda_r'^2/n$. where $\lambda'_r = \mathbb{E}[X_{n,k,r}]$.

All in all, we obtain that

$$\mathrm{d_{TV}}\left(\mathbf{X}_{n,k,R}, \mathbf{X}'_{k,R}\right) \leq C/n,$$

where $\mathbf{X}'_{k,R}$ is a vector of independent Poisson variables with means λ'_r.

Using the triangle inequality, we see that

$$\mathrm{d_{TV}}\left(\mathbf{X}_{n,k,R}, \mathbf{X}'_{k,R}\right) \leq \mathrm{d_{TV}}\left(\mathbf{X}_{n,k,R}, \mathbf{X}'_{k,R}\right) + \mathrm{d_{TV}}\left(\mathbf{X}_{k,R}, \mathbf{X}'_{k,R}\right).$$

The above controls the first term, Exercise 2.1 controls the second. □

As an immediate consequence we obtain:

Corollary 3.8. *Fix $k \in \mathbb{N}_{\geq 3}$. For any finite set $R \subset \mathbb{N}$ we have*

$$\mathbf{X}_{n,k,R} \xrightarrow{\mathrm{TV}} \mathbf{X}_{k,R}$$

as $n \to \infty$.

3.3 Exercises

Exercise 3.1. Let $G = (V, E, \mathcal{I})$ be a graph. Show that

$$\sum_{v \in V} \deg(v) = 2\,|E|\,.$$

Exercise 3.2. Prove Lemma 3.5.

Exercise 3.3. Let $k \geq 3$. Show that

$$\lim_{n \to \infty} \mathbb{P}_{n,k}\left[\text{The graph is connected}\right] = 1.$$

Hint: try to estimate $\mathbb{E}_{n,k}[X]$, where $X : \mathcal{G}_{n,k} \to \mathbb{N}$ counts the number of connected compondents of at most $n/2$ vertices.

Lecture 4

Random graphs II: The number of regular graphs and the basics of expansion

4.1 Automorphisms

It turns out that a typical regular graph on a large number of vertices does not have any non-trivial symmetries. This is orginally due to Bollobás [Bol82] and independently McKay and Wormald [MW84] (see also [Wor86]).

Theorem 4.1. *Let $k \in \mathbb{N}_{\geq 3}$. We have*

$$\lim_{n\to\infty} \mathbb{E}_{n,k} \left[|\mathrm{Aut}(G)| \right] = 1.$$

We will skip the proof of this theorem.

4.2 The number of simple graphs

Given $n, k \in \mathbb{N}$ so that $n \cdot k$ is even, let $\mathcal{U}_{n,k}$ denote the set of ismorphism classes of simple k-regular graphs on n vertices. The following count is independently due to Bender and Canfield [BC78], Bollobás [Bol80] and Wormald [Wor78].

Theorem 4.2. *Let $k \in \mathbb{N}_{\geq 3}$. Then:*

$$|\mathcal{U}_{n,k}| \sim \frac{e^{-(k^2-1)/4}(n \cdot k)!!}{(k!)^n \cdot n!}$$

as $n \to \infty$.

Proof. Let $\mathcal{G}^*_{n,k}$ denote the subset of $\mathcal{G}_{n,k}$ consisting of configurations that give rise to a simple graph. We have an obvious map $\mathcal{G}^*_{n,k} \to \mathcal{U}_{n,k}$ that consists of forgetting the labels. This map is far from injective. However, it will turn out the cardinality of the fibers depends only on n, k and the number of automorphisms. As such, Theorem 4.1 tells us that up to a small error, we may assume that this cardinality is constant.

Let us work this idea out. The first thing we will do is add an intermediate step to the map $\mathcal{G}^*_{n,k} \to \mathcal{U}_{n,k}$. Let $\mathcal{V}_{n,k}$ denote the set of k-regular graphs with vertex set $\{1, \ldots, n\}$. We obtain maps

$$\mathcal{G}^*_{n,k} \to \mathcal{V}_{n,k} \to \mathcal{U}_{n,k}$$

by first forgetting the labels of the half-edges and then the labels on the vertices.

First note that the map $\mathcal{G}^*_{n,k} \to \mathcal{V}_{n,k}$ is constant to 1. Indeed, the number of pre-images of an element $G \in \mathcal{V}_{n,k}$ is equal to the number of ways to label the half edges at every vertex (note that this uses that G has no loops and no multiple edges). As such $\mathcal{G}^*_{n,k} \to \mathcal{V}_{n,k}$ is $(k!)^n$ to 1.

We have a natural action of $\mathfrak{S}_n \curvearrowright \mathcal{V}_{n,k}$, where $\mathfrak{S}_n$ denotes the symmetric group on n letters, by permuting the labels of the vertices. Furthermore

$$\mathcal{U}_{n,k} = \mathcal{V}_{n,k} / \mathfrak{S}_n \,.$$

By Burnside's lemma (see Exercise 4.1), we have

$$|\mathcal{U}_{n,k}| = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} |\{G \in \mathcal{V}_{n,k};\ \pi \cdot G = G\}|$$

Regrouping the terms sum, we get

$$\begin{aligned} |\mathcal{U}_{n,k}| &= \frac{1}{n!} \sum_{G \in \mathcal{V}_{n,k}} |\mathrm{Aut}(G)| \\ &= \frac{1}{n!} \sum_{a=1}^{n!} a \cdot |\{G \in \mathcal{V}_{n,k};\ |\mathrm{Aut}(G)| = a\}| \end{aligned}$$

Now we use that the map $\mathcal{G}^*_{n,k} \to \mathcal{V}_{n,k}$ is constant to 1 to obtain

$$|\mathcal{U}_{n,k}| = \frac{1}{(k!)^n \cdot n!} \sum_{a=1}^{n!} a \cdot \left|\left\{G \in \mathcal{G}^*_{n,k};\ |\mathrm{Aut}(G)| = a\right\}\right|$$

Hence

$$\frac{\left|\left\{G \in \mathcal{G}^*_{n,k};\ |\mathrm{Aut}(G)| = 1\right\}\right|}{(k!)^n \cdot n!} \leq |\mathcal{U}_{n,k}|$$

and

$$|\mathcal{U}_{n,k}| \leq \frac{\left|\mathcal{G}^*_{n,k}\right|}{(k!)^n \cdot n!} + \frac{1}{(k!)^n \cdot n!} \sum_{a=2}^{n!} a \cdot \left|\{G \in \mathcal{G}_{n,k};\ |\mathrm{Aut}(G)| = a\}\right|.$$

Let us now first work out the lower bound:

$$\frac{\left|\left\{G \in \mathcal{G}^*_{n,k};\ |\mathrm{Aut}(G)| = 1\right\}\right|}{(k!)^n \cdot n!} \geq \frac{\left|\mathcal{G}^*_{n,k}\right| - \left|\{G \in \mathcal{G}_{n,k};\ |\mathrm{Aut}(G)| > 1\}\right|}{(k!)^n \cdot n!}.$$

Note that a configuration $C \in \mathcal{G}_{n,k}$ gives rise to a simple graph if and only if $X_{n,k,1}(C) = X_{n,k,2}(C) = 0$. As such Corollary 3.8 implies that

$$\left|\mathcal{G}^*_{n,k}\right| = \mathbb{P}_{n,k}\left[X_{n,k,1}(C) = X_{n,k,2}(C) = 0\right] \cdot |\mathcal{G}_{n,k}| \sim e^{-\lambda_{k,1}-\lambda_{k,2}} \cdot |\mathcal{G}_{n,k}|$$

as $n \to \infty$. On the other hand, Theorem 4.1 implies that

$$\left|\{G \in \mathcal{G}_{n,k};\ |\mathrm{Aut}(G)| > 1\}\right| / |\mathcal{G}_{n,k}| \to 0$$

as $n \to \infty$.

For the upper bound we note that

$$\sum_{a=2}^{n!} a \cdot \frac{\left|\{G \in \mathcal{G}_{n,k};\ |\mathrm{Aut}(G)| = a\}\right|}{|\mathcal{G}_{n,k}|} = \mathbb{E}_{n,k}\left[|\mathrm{Aut}(G)|\right] - \mathbb{P}_{n,k}\left[|\mathrm{Aut}(G)| = 1\right].$$

Thus

$$\sum_{a=2}^{n!} a \cdot \frac{\left|\{G \in \mathcal{G}_{n,k};\ |\mathrm{Aut}(G)| = a\}\right|}{|\mathcal{G}_{n,k}|} \to 0$$

as $n \to \infty$ by Theorem 4.1.

Putting all of the above together, we see that

$$|\mathcal{U}_{n,k}| \sim e^{-\lambda_{k,1}-\lambda_{k,2}} \cdot \frac{|\mathcal{G}_{n,k}|}{(k!)^n \cdot n!} = \frac{e^{-(k-1)/2-(k-1)^2/4}(n \cdot k)!!}{(k!)^n \cdot n!} = \frac{e^{-(k^2-1)/4}(n \cdot k)!!}{(k!)^n \cdot n!}.$$

□

4.3 Expansion

This section is mainly based on [HLW06].

4.3.1 Definition

Loosely speaking, an expander graph is a sequence of graphs that is both sparse and well-connected. One of the earliest contexts in which they came up is in a problem from computer science: building a large network of computers in which it's not possible to disconnect a large piece of the network by cutting a small number of cables (well-connectedness) but not connecting too many computers to each other (sparseness). By now these sequences of graphs have many applications in pure mathematics as well.

There are two things to be made precise: sparseness and well-connectedness of a graph. A good candidate for the notion of sparseness is of course a uniform bound on the degree. We will set a stronger condition and assume k-regularity for some fixed k.

The idea of well-connectedness can be made precise with the Cheeger constant. In the following definition, given a graph G and a set of vertices $U \subset V(G)$, we will denote the set of edges that connect U to $V(G) \setminus U$ by ∂U.

Definition 4.3. Let G be a finite connected graph. The *Cheeger constant* or *Expansion ratio* of G is given by:

$$h(G) = \min\left\{ \frac{|\partial U|}{|U|};\ U \subset V(G),\ |U| \leq |V(G)|/2 \right\}.$$

An expander graph will be a sequence of graphs that is both sparse and well-connected:

Definition 4.4. Fix $k \in \mathbb{N}_{\geq 3}$. An *expander graph* is a sequence $(G_n)_{n\in\mathbb{N}}$ of connected k-regular graphs so that

$$|V(G_n)| \to \infty$$

as $n \to \infty$ and there exists a $\varepsilon > 0$ so that

$$h(G_n) > \varepsilon$$

for all $n \in \mathbb{N}$.

4.3.2 Eigenvalues

There exists an equivalent characterisation of expander graphs in terms of eigenvalues. We first need to define what an adjacency matrix is.

Definition 4.5. Given a graph G on the vertex set $\{1, \ldots, n\}$. The adjacency matrix $A(G) \in \mathrm{M}_n(\mathbb{R})$ is given by

$$A(G)_{ij} = m \text{ if and only if } i \text{ and } j \text{ share } m \text{ edges.}$$

Note that $A(G)$ is a self-adjoint matrix and as such has real eigenvalues, let us denote these by $\lambda_1(G) \geq \lambda_2(G) \geq \ldots \geq \lambda_n(G)$. Note that these eigenvalues do not depend on the labelling of the vertices. As such it makes sense to speak of the eigenvalues associated to a graph G.

Lemma 4.6. *Let G be a finite k-regular graph*

(a) $\lambda_1(G) = k$.

(b) G is connected if and only if $\lambda_1(G) > \lambda_2(G)$.

Proof. Exercise 4.2. □

This lemma implies that, given a connected regular graph, the first non-trivial eigenvalue is given by

$$\lambda(G) = \lambda_2(G).$$

4.4 Exercises

Exercise 4.1. (Burnside's lemma) Let G be a finite group and X a finite set so that $G \curvearrowright X$. Prove that

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |\{x \in X;\ g \cdot x = x\}|.$$

Exercise 4.2.

(a) Let G be a graph and $A(G)$ its adjacency matrix. Show that

$$(A^r)_{ij}$$

records the number of walks of length r between vertices i and j.

(b) Prove Lemma 4.6. Hint for part (b) of the lemma: consider the eigenvalues of the matrix

$$\mathrm{Id}_n - \frac{1}{k}A(G),$$

where Id_n denotes the $n \times n$ identity matrix. In particular: show that the eigenfunctions corresponding to the eigenvalue 0 of this matrix are constant on connected components.

Lecture 5

Random graphs III: Expansion

5.1 Expansion through eigenvalues

The fact that expansion can also be measured using eigenvalues is the content of the following theorem by Dodziuk [Dod84], Alon-Milman [AM85] and Alon [Alo86]:

Theorem 5.1. *Let G be a finite connected k-regular graph, then*

$$\frac{k-\lambda(G)}{2} \leq h(G) \leq \sqrt{2k(k-\lambda(G))}.$$

Proof. We follow the proof from [HLW06, Theorem 4.11] and start with the lower bound on $h(G)$. Let us (arbitrarily) label the vertices of our graph by $\{1,\dots,n\}$ and let $A = A(G)$ be the adjacency matrix of our graph G. Given vector $f \in \mathbb{R}^n \setminus \{0\}$, the *Rayleigh quotient* of f is given by

$$R_A(f) = \frac{\langle f, Af\rangle}{\langle f, f\rangle},$$

where $\langle \cdot,\cdot\rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the inner product.

Given $S \subset \{1,\dots,n\}$, we define $f_S \in \mathbb{R}^n$ by

$$(f_S)_i = \begin{cases} n-|S| & \text{if } i \in S \\ -|S| & \text{if } i \notin S. \end{cases}$$

Note that

$$\langle f_S, (1, \ldots, 1) \rangle = 0$$

for all $S \subsetneq \{1, \ldots, n\}$ so that $S \neq \emptyset$. Now write

$$f_S = \sum_{i=2} \langle f_S, g_i \rangle \cdot g_i,$$

where $\{g_i\}_{i=1}^n$ is an orthonormal basis of eigenvectors of A, so that g_i correpsonds to λ_i for $i = 1, \ldots, n$. Note that we may choose $g_1 = \frac{1}{\sqrt{n}} \cdot (1, \ldots, 1)$. This implies that

$$\langle f_S, A f_S \rangle = \sum_{i=2} \langle f_S, g_i \rangle^2 \cdot \lambda_i \leq \lambda(G) \cdot ||f_S||^2 .$$

So we obtain

$$R_A(f_S) \leq \lambda(G).$$

On the other hand, an easy computation gives that

$$||f_S||^2 = n \, |S| \, (n - |S|)$$

and

$$\langle f_S, A f_S \rangle = n \cdot k \cdot |S| \, (n - |S|) - n^2 \, |\partial S| .$$

Filling this in in the Rayleigh quotient for a set $S \subset \{1, \ldots, n\}$ so that $|S| \leq n/2$ and $h(G) = |\partial S| \, / \, |S|$, we obtain that

$$\lambda(G) \geq \frac{n \cdot k \cdot |S| \, (n - |S|) - n^2 \, |\partial S|}{n \, |S| \, (n - |S|)} = k - \frac{n \, |\partial S|}{|S| \, (n - |S|)} \geq k - 2 \cdot h(G),$$

which proves the lower bound.

For the upper bound on $h(G)$ we will use the Laplacian matrix $L \in \mathrm{M}_n(\mathbb{R})$ given by

$$L = k \cdot \mathrm{Id}_n - A,$$

where $\mathrm{Id}_n \in \mathrm{M}_n(\mathbb{R})$ denotes the n-dimensional identity matrix. Note that L has eigenvalues $k - \lambda_i(G)$, corresponding to the same eigenvectors $g_i \in \mathbb{R}^n$ for all $i = 1, \ldots, n$. We will again consider the associated Rayleigh quotients, given by

$$R_L(f) = \frac{\langle f, Lf \rangle}{\langle f, f \rangle},$$

for all $f \in \mathbb{R}^n \setminus \{0\}$.

In what follows, we assume that at most half of the entries of $g_2 \in \mathbb{R}^n$ are positive (we may assume this, because we can replaye g_2 by $-g_2$). Let Define $f \in \mathbb{R}^n$ by

$$f_i = \max\{(g_2)_i, 0\},$$

for $i = 1, \ldots, n$.

We now make two claims:

Claim 1. We have:

$$R_L(f) \leq k - \lambda(G).$$

Claim 2. We have:

$$\frac{h(G)^2}{2k} \leq R_L(f).$$

Note that if we prove these two claims, we prove the theorem.

Proof of Claim 1. Let us write $\mathrm{supp}(f) = V^+ \subset \{1, \ldots, n\}$. For $i \in V^+$ we have:

$$\begin{aligned}
(Lf)_i &= k \cdot f_i - \sum_{j=1}^{n} A_{ij} f_j \\
&= k \cdot (g_2)_i - \sum_{j \in V^+} A_{ij} (g_2)_j \\
&\leq k \cdot (g_2)_i - \sum_{j=1}^{n} A_{ij} (g_2)_j \\
&= (Lg_2)_j \\
&= (k - \lambda(G)) \cdot (g_2)_i.
\end{aligned}$$

As such

$$\langle f, Lf \rangle \;=\; \sum_{i=1}^{n} f_i (Lf)_i \;\leq\; (k - \lambda(G)) \sum_{i \in V^+} (g_2)_i^2,$$

where we used that $f_i = 0$ for $i \notin V^+$ in the second step. This means that

$$\langle f, Lf \rangle \leq (k - \lambda(G)) \cdot ||f||^2,$$

which proves our claim. □

Proof of Claim 2. To prove this claim, we note that

$$\langle f, Lf\rangle = \sum_{i=1}^{n}\left(k\cdot f_i^2 - \sum_{r=1}^{k}\sum_{\substack{j\in\{1,\ldots,n\},\ i \text{ and } j\\ \text{share exactly } r \text{ edges}}} r\cdot f_i f_j\right)$$

By reordering the terms above, we can see this sum as a sum over the edges $E(G)$ of G to obtain:

$$\langle f, Lf\rangle = \sum_{e\in E(G)} (f_{v_1(e)} - f_{v_2(e)})^2,$$

where $v_1(e)$ and $v_2(e)$ are the (not necessarily distinct) endpoints of e (in arbitrary order).

Now we assume that the vertices $\{1,\ldots,n\}$ are labelled so that $f_1 \geq f_2 \geq \ldots \geq f_n$. We have

$$\begin{aligned} h(G)\cdot ||f||^2 &= h(G)\sum_{i=1}^{n} f_i^2 \\ &= h(G)\cdot \sum_{i\in V^+} (f_i^2 - f_{i+1}^2)\cdot i \end{aligned}$$

The second equality follows from a telescoping argument and the fact that by assumption $f_{i+1} = 0$ for $i = |V^+|$. Set $[i] = \{1,\ldots i\}$. By definition of the Cheeger constant, we have that

$$h(G) \leq |\partial[i]|\,/i.$$

So we obtain:

$$\begin{aligned} h(G)\cdot ||f||^2 &\leq \sum_{i\in V^+} (f_i^2 - f_{i+1}^2)\cdot |\partial[i]| \\ &= \sum_{i=1}^{n-1} (f_i^2 - f_{i+1}^2)\cdot |\partial[i]| \\ &= \sum_{\substack{e\in E(G)\\ v_1(e)<v_2(e)}} \sum_{i=v_1(e)}^{v_2(e)-1} (f_i^2 - f_{i+1}^2). \end{aligned}$$

With another telescoping argument, we get:

$$\begin{aligned} h(G) \cdot ||f||^2 &\leq \sum_{\substack{e \in E(G) \\ v_1(e) < v_2(e)}} (f_{v_1(e)}^2 - f_{v_2(e)}^2) \\ &= \sum_{\substack{e \in E(G) \\ v_1(e) < v_2(e)}} (f_{v_1(e)} + f_{v_2(e)}) \cdot (f_{v_1(e)} - f_{v_2(e)}) \end{aligned}$$

Now we use the Cauchy-Schwarz inequality, which says that

$$\begin{aligned} &\sum_{\substack{e \in E(G) \\ v_1(e) < v_2(e)}} (f_{v_1(e)} + f_{v_2(e)}) \cdot (f_{v_1(e)} - f_{v_2(e)}) \\ \leq &\sqrt{\sum_{\substack{e \in E(G) \\ v_1(e) < v_2(e)}} (f_{v_1(e)} + f_{v_2(e)})^2} \cdot \sqrt{\sum_{\substack{e \in E(G) \\ v_1(e) < v_2(e)}} (f_{v_1(e)} - f_{v_2(e)})^2}. \end{aligned}$$

We have:

$$\sqrt{\sum_{\substack{e \in E(G) \\ v_1(e) < v_2(e)}} (f_{v_1(e)} + f_{v_2(e)})^2} \leq \sqrt{2 \sum_{\substack{e \in E(G) \\ v_1(e) < v_2(e)}} f_{v_1(e)}^2 + f_{v_2(e)}^2} = \sqrt{2k} \cdot ||f|| \,.$$

So, using our earlier observation on $\langle f, Lf \rangle$, we obtain:

$$h(G) \cdot ||f||^2 \leq \sqrt{2k} \cdot ||f|| \cdot \sqrt{\langle f, Lf \rangle},$$

which proves Claim 2. □

Putting Claims 1 and 2 together yields the theorem. □

We already alluded to the following immediate consequence of the theorem above:

Corollary 5.2. *Let $k \geq 3$. A sequence $(G_n)_n$ of k-regular graphs G_n on n vertices is an expander if and only if there exists an $\varepsilon > 0$ so that*

$$k - \lambda(G_n) > \varepsilon$$

for all $n \in \mathbb{N}$ (n even if k is odd).

$k - \lambda(G)$ is often called the *spectral gap* of G, note that it is also the smallest non-zero eigenvalue of the Laplacian

$$L(G) = k \cdot \mathrm{Id}_n - A(G)$$

of the graph G that we used in the proof of the theorem above.

5.2 Existence

We have not yet discussed whether or not expander graphs exist. The first proof of the existence of expanders actually was a random construction and is due to Pinsker [Pin73]. For instance due to work of Margulis [Mar73] and Lubotzky-Phillips-Sarnak [LPS88], there are also explicit examples of sequences of expander graphs.

We will give a probabilistic existence proof. In fact, it is known that random regular graphs are near optimal (their second eigenvalue $\lambda(G)$ is essentially as small as it could possibly be) expanders [Fri08] with probability tending to one. We will follow a shorter proof, due to Broder-Shamir [BS87], with a result that is less strong. Our exposition is based on that in [HLW06, Theorem 7.5].

5.2.1 A different model

We will consider a slightly different model than the configuration model that we have considered so far. This model is called the *permutation model* and works as follows. Given $k \in \mathbb{N}$ and elements $\pi_1, \ldots, \pi_k \in \mathfrak{S}_n$, where $\mathfrak{S}_n$ denotes the symmetric group on n letters, a $2k$-regular graph $G(\pi_1, \ldots, \pi_k) = (V, E, \mathcal{I})$ is obtained by setting

$$V = \{1, \ldots, n\},\ E = \left\{e_{i,\pi_j(i)};\ i = 1, \ldots, n, j = 1, \ldots, k\right\}$$

and

$$\mathcal{I} = \{(e_{i,j}, i),\ (e_{i,j}, j);\ i, j = 1, \ldots, n\}\,.$$

In other words, vertex i is connected to vertex $\pi_j(i)$ for all $j = 1, \ldots, k$. Figure 5.1 gives an example:

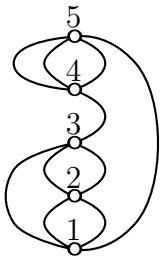

Figure 5.1: The graph corresponding to the permutations $\pi_1 = (1\ 2\ 3)(4\ 5)$ and $\pi_2 = (1\ 2\ 3\ 4\ 5)$.

As such, this model gives as a probability space

$$\Omega_{n,k}^{\text{perm}} = \mathfrak{S}_n^k$$

of $2k$-regular graphs with the usual uniform probability measure $\mathbb{P}_{n,k}^{\text{perm}}$. Furthermore, we have

$$\left|\Omega_{n,k}^{\text{perm}}\right| = (n!)^k.$$

It should be stressed that when we consider $\mathbb{P}_{n,k}^{\text{perm}}$ as a probability measure on the set of isomorphism classes of $2k$-regular graphs, we obtain a different measure than the measure $\mathbb{P}_{n,2k}$ coming from the configuration model.

It does turn out that $\mathbb{P}_{n,k}^{\text{perm}}$, $\mathbb{P}_{n,2k}$ and the uniform measure $\mathbb{P}_{n,2k}^{\text{unif}}$ on the set of isomorphism classes are all contiguous: if $(A_n)_n$ is a sequence of sets of isomorphism classes of $2k$-regular graphs on n vertices then

$$\mathbb{P}_{n,k}^{\text{perm}}[A_n] \to 0 \quad \Leftrightarrow \quad \mathbb{P}_{n,2k}^{\text{unif}}[A_n] \to 0 \quad \Leftrightarrow \quad \mathbb{P}_{n,2k}[A_n] \to 0$$

as $n \to \infty$. In particular, if we can prove that a random graph is an expander with probability tending to 1 in any of these models, we get the same statement for free for the other two models. In the proof of Theorem 4.2 we have essentially already proved the contiguity of $\mathbb{P}_{n,k}^{\text{unif}}$ and the restriction of $\mathbb{P}_{n,k}$ to simple graphs (see Exericise 5.2). We will not prove it for the permutation model in this course and will content ourselves with the statement that graphs coming from the permutation model are expanders. The interested reader is refered to [Wor99, Section 4] for details on contiguity.

5.3 Exercises

Exercise 5.1. Fix $k \geq 3$ and let $(G_n)_n$ be any sequence of k-regular graphs so that G_n has n vertices. Show that

$$\lambda(G_n) \geq \sqrt{k} \cdot (1 - o(1))$$

as $n \to \infty$.

Exercise 5.2. Recall that $\mathcal{U}_{n,k}$ denotes the set of isomorphism classes of simple k-regular on n vertices and that

$$\mathcal{G}^*_{n,k} = \{C \in \mathcal{G}_{n,k};\ G(C) \text{ is simple}\}\,.$$

Let $\mathbb{P}^{\text{unif}}_{n,k}$ denote the uniform probability measure on $\mathcal{U}_{n,k}$. Furthermore, let $\mathbb{P}^*_{n,k}$ denote the measure on $\mathcal{U}_{n,k}$ obtained from restricting $\mathbb{P}_{n,k}$ to simple graphs in $\mathcal{G}_{n,k}$ and then pushing it forward to $\mathcal{U}_{n,k}$. In other words, if $\pi : \mathcal{G}^*_{n,k} \to \mathcal{U}_{n,k}$ is the map that forgets all labels, then

$$\mathbb{P}^*_{n,k}[A] = \mathbb{P}_{n,k}[\pi^{-1}(A) |\ \mathcal{G}^*_{n,k}]$$

for all $A \subset \mathcal{U}_{n,k}$. Show that $\mathbb{P}^*_{n,k}$ and $\mathbb{P}^{\text{unif}}_{n,k}$ are contiguous.

Lecture 6

Random graphs IV: The existence of expanders

6.1 Expansion in the permutation model

We split the proof of the fact that random graphs are expanders over a couple of lemmas.

The first lemma will be about random walks on the symmetric group. Given a finite set $W = \{a_1, \ldots, a_k\}$, we will write

$$W^r = \left\{ a_{i_1}^{\varepsilon_1} \cdots a_{i_k}^{\varepsilon_k};\ i_j \in \{1, \ldots, k\},\ \varepsilon_i \in \{\pm 1\} \right\}.$$

We have

$$|W^r| = (2k)^r.$$

Let $\mathbb{P}_r$ denote the uniform probability measure on this finite set.

Furthermore, given $\pi_1, \ldots, \pi_k \in \mathfrak{S}_n$ and $w \in W^r$,

$$w(\pi_1, \ldots, \pi_k) \in \mathfrak{S}_n$$

will be the permutation obtained by replacing a_i^{ε} by π_i^{ε} for all $i = 1, \ldots, k$ and $\varepsilon \in \{\pm 1\}$. As such, we can speak of the set of fixed points of $w(\pi_1, \ldots, \pi_k)$, which we will denote by

$$\mathrm{Fix}(w(\pi_1, \ldots, \pi_k))$$

The relation between eigenvalues and random walks we will use is:

Lemma 6.1. *Let $\pi_1, \ldots, \pi_k \in \mathfrak{S}_n$ and set $G = G(\pi_1, \ldots, \pi_k)$. Furthermore, set $\rho(G) = \lambda(G)/2k$. Then*

$$\rho(G)^{2r} \leq \mathbb{E}_{2r}\left[|\mathrm{Fix}(w(\pi_1, \ldots, \pi_k))|\right] - 1$$

for all $r \in \mathbb{N}$.

Proof. Given a $2k$-regular graph G on n vertices, set $P(G) = A(G)/2k$. Note that

$$\mathrm{tr}\left(P(G)^{2r}\right) = \frac{1}{(2k)^{2r}}\mathrm{tr}\left(A(G)^{2r}\right) = \frac{1}{(2k)^{2r}}\sum_{i=1}^{n}\lambda_i(G)^{2r} = 1 + \frac{1}{(2k)^{2r}}\sum_{i=2}^{n}\lambda_i(G)^{2r}$$

for all $r \in \mathbb{N}$. Recall that all eigenvalues $\lambda_i(G)$ are real. Since even powers of real numbers are positive, we obtain that

$$\rho(G)^{2r} \leq \mathrm{tr}\left(P(G)^{2r}\right) - 1.$$

Recall from Exercise 4.2(a) that $\left(A(G)^{2r}\right)_{ij}$ counts the number of walks from vertex i to vertex j in $2r$ steps. This means that $\mathrm{tr}\left(A(G)^{2r}\right)$ counts the number of closed walks of $2r$ steps. Because $(2k)^{2r}$ counts all walks of $2r$ steps, $\mathrm{tr}\left(P(G)^{2r}\right)$ can be interpreted probabilistically. Indeed we have

$$\mathrm{tr}\left(P(G)^{2r}\right) = \sum_{i=1}^{n} p_{i,2r},$$

where $p_{i,2r}$ is the probability that a random walk of $2r$ steps on G, starting at vertex i ends at vertex i again.

Now we use that our graphs are built out of permutations. We can think of the edges in $G(\pi_1, \ldots, \pi_k)$ as being labeled by the permutations π_i and their inverses π_i^{-1}. As such, the walks of length $2r$ on $G(\pi_1, \ldots, \pi_k)$ correspond one to one to words in the $2k$ letters $\{\pi_1, \pi_1^{-1}, \ldots, \pi_k, \pi_k^{-1}\}$. Furthermore, if

$$w(\pi_1, \ldots, \pi_k) = \pi_{i_1}^{\varepsilon_1} \cdots \pi_{i_{2r}}^{\varepsilon_{2r}} \in \mathfrak{S}_n$$

is such a word of length $2r$ (so $\varepsilon_i \in \{\pm 1\}$), then the vertex we reach by starting at i and tracing w is given by $w(i)$. Because we are considering closed walks, we need to understand how often $w(i) = i$. In other words, we have

$$\mathrm{tr}\left(P(G)^{2r}\right) = \mathbb{E}_{2r}\left[|\mathrm{Fix}(w)|\right].$$

Putting this together with our upper estimate on $\rho(G)$ proves the lemma. $\square$

The lemma above translates the question of the expansion of a random graph into a question on random words of a given length in random permutations. As such, we will break the argument into two parts: first we consider the structure of a random word and then consider the number of fixed points of a word with this given structure.

We start with the properties of words. Let $w \in W_r$, we call $w = (w_1, \dots w_r)$ *reduced* if there is no $i \in \{1, \dots, r-1\}$ so that $w_i = w_{i+1}^{-1}$. Given a word that is not reduced, we can reduce it by successively removing all the pairs of consecutive letters that are each others inverses. The shorter word we obtain by doing this, will be called $\mathrm{red}(w)$.

We call a reduced word w *bad* if there exist words w_a and w_b so that $w = w_a w_b^j w_a^{-1}$ for some $j \geq 2$.

Lemma 6.2. *Let $k \geq 2$ and $r \in \mathbb{N}$. We have*

$$\mathbb{P}_{2r}\left[\mathrm{red}(w) \text{ is bad or empty}\right] \leq \frac{r+1}{(k/2)^r}.$$

Proof. To count the number of words with bad reductions, we first count words in the three letters $\{(,), *\}$. A word $w = w_1 \cdots w_{2r}$ in these letters is called *admissible* if the following conditions hold:

1. w contains as many open as closed brackets:
$$\left|\{i;\ w_i = \text{ (}\}\right| = \left|\{i;\ w_i = \text{) }\}\right|.$$
2. No bracket in w is closed before it is opened: for any $i = 1, \dots, 2r$:
$$\left|\{j \leq i;\ w_j = \text{ (}\}\right| \geq \left|\{j \leq i;\ w_j = \text{) }\}\right|.$$
3. Stars only appear when all brackets have been closed: if $w_i = *$ then
$$\left|\{j < i;\ w_j = \text{ (}\}\right| = \left|\{j < i;\ w_j = \text{) }\}\right|.$$

For example, the word $(())() * * * *(())$ is admissible, whereas the words $((*))$ and $)($ are not.

Now suppose $w \in W^{2r}$. We obtain an admissible word in $\{(,), *\}$ from a reduction of w to $\mathrm{red}(w)$ as follows: place a $*$ in place of every letter that

remains and place a pair of brackets (and) for every pair of inverses that cancel against each other at some point in the reduction. For example, if

$$w = a_1 \cdot a_2^{-1} \cdot a_2 \cdot a_1^{-1} \cdot a_3 \cdot a_1$$

then a corresponding word in $\{(,), *\}$ is

$$(()) * *.$$

Note however that the word in $\{(,), *\}$ depends on the reduction. For instance, from different reductions of $a_1 \cdot a_2^{-1} \cdot a_2 \cdot a_1^{-1} \cdot a_1 \cdot a_3$, we obtain $(()) * *$ or $*()()*$.

Nonetheless, we can obtain all words $w \in W^{2r}$ that reduce to a word of length $2l$ by filling in admissible words in $\{(,), *\}$ with $2l$ stars (we just obtain some words multiple times). Note that the case $l = 0$ corresponds to words with an empty reduction.

To count the number of words with a bad reduction, our strategy will be to count the number of admissible words in $\{(,), *\}$ with the right number of stars and then count the number of ways to fill them in.

Note that if a word in $\{(,), *\}$ is admissible and has $2l$ stars, then the position of the $r - l$ open brackets determine the word entirely. Indeed, once the open brackets have been filled in, every remaining place needs to be either a closed bracket or a star. To fill these letters in, we read the open places from right to left. If a star is allowed, we put a star in that place, if not, we put in a closed bracket. As such, the number of admissible words of length $2r$ with $2l$ stars is

$$\binom{2r}{r-l}$$

All that remains is to bound the number of choices for the bad word we put in place of the stars and the pairs of letters we put in place of the brackets.

Let us start with the former. We need to build a word of length $2l$ of the form $w_a \cdot w_b^j w_a^{-1}$. Note that once the lengths of w_a and w_b are given, j is determined. Furthermore, the length l_b of w_b satisfies

$$l_b \leq (2l - 2l_a)/2 = l - l_a$$

So we obtain that the number of bad words of length $2l$ is at most

$$\sum_{l_a=0}^{l}(2k)^{l-l_a}\cdot(2k)^{l_a}=(l+1)\cdot(2k)^l.$$

After this, we need to count how many ways there are to fill in the brackets. Once the letters corresponding to the open brackets have been chosen, the letters corresponding to the closed brackets are fixed. As such there are

$$(2k)^{r-l}$$

choices for this.

This means that the number of words with a bad or empty reduction can be bounded by

$$\sum_{l=0}^{r}\binom{2r}{r-l}(l+1)(2k)^r\leq(r+1)(2k)^r\sum_{l=0}^{r}\binom{2r}{r-l}\leq(r+1)2^{2r}\cdot(2k)^r.$$

Dividing by the total number of words of length r, we obtain the bound. □

Our next intermediate goal will be to bound the probability

$$\mathbb{P}_{n,k}^{\mathrm{perm}}[w(\pi_1,\ldots,\pi_k)(1)=1]$$

for any fixed word $w\in W^s$ for some $s\leq 2k$.

We will control this probability using the following heuristic that goes through the word w step by step (or letter by letter):

- Set $v_0=1$.
- Let w_i denote the i^{th} letter of w (read from the right). Suppose $w_i(\pi_1,\ldots,\pi_k)=\pi_j^{\varepsilon}$
  - If $\pi_j^{\varepsilon}(v_{i-1})$ has not yet been chosen, randomly pick $\pi_j^{\varepsilon}(v_{i-1})$ (among the vertices that have not yet been assigned to the range of π_j) and set
    $$v_i=\pi_j^{\varepsilon}(v_{i-1}).$$
    In this case the i^{th} step is called *free*.

- If $\pi_j^\varepsilon(v_{i-1})$ has already been chosen, set

$$v_i = \pi_j^\varepsilon(v_{i-1}).$$

In this case the i^{th} step is called *forced*.

Note that

$$\mathbb{P}[v_s = 1] = \mathbb{P}_{n,k}^{\text{perm}}[w(\pi_1, \ldots, \pi_k)(1) = 1].$$

We claim:

Lemma 6.3. *Let $0 < s \leq 2r$ and let $w \in W^s$ be a good reduced word. Then*

$$\mathbb{P}_{n,k}^{\text{perm}}[w(\pi_1, \ldots, \pi_k)(1) = 1] \leq \frac{1}{n-2r} + \frac{16 \cdot r^4}{(n-2r)^2}.$$

Proof. Let us call the i^{th} step in the process above a *coincidence* if it is free and moreover if v_i is a vertex that has already been seen before.

Note that the fact that w is reduced implies that if $w(\pi_1, \ldots, \pi_k)(1) = 1$ then at least one coincidence must occur. As such, we obtain the bound

$$\mathbb{P}_{n,k}^{\text{perm}}[w(1) = 1] \leq \mathbb{P}\begin{bmatrix}\text{Exactly one coincidence} \\ \text{occurs and } v_s = 1\end{bmatrix} + \mathbb{P}\begin{bmatrix}\text{At least two} \\ \text{coincidences occur}\end{bmatrix}.$$

Let us denote the event that a coincidence occurs at step i by C_i. Before the i^{th} step there are at most i vertices that have already been visited. Likewise, there are at least $n - i$ vertices that have not yet been assigned to the range of π_j. As such:

$$\mathbb{P}[C_i |\ v_0 = 1, \ldots, v_{i-1} = u_{i-1}] \leq \frac{i}{n-i} \leq \frac{2r}{n-2r}.$$

Note that this bound does not depend on the conditioning. Hence:

$$\begin{aligned}
\mathbb{P}\begin{bmatrix}\text{At least two} \\ \text{coincidences occur}\end{bmatrix} &\leq \sum_{1 \leq i < j \leq 2r} \mathbb{P}[C_i \text{ and } C_j] \\
&= \sum_{1 \leq i < j \leq 2r} \mathbb{P}[C_i]\, \mathbb{P}[C_j |\ C_i] \\
&\leq \sum_{1 \leq i < j \leq 2r} \frac{4 \cdot r^2}{(n-2r)^2} \\
&\leq \frac{16 \cdot r^4}{(n-2r)^2}. \qquad (6.1)
\end{aligned}$$

So, we need to show that

$$\mathbb{P}\begin{bmatrix}\text{Exactly one coincidence}\\ \text{occurs and } v_s = 1\end{bmatrix} \leq \frac{1}{n-2r}.$$

If $w(\pi_1, \ldots, \pi_k)(1) = 1$, then w gives rise to a closed cycle in the graph $G(\pi_1, \ldots, \pi_k)$. If only one coincidence occurs, then this cycle must look like the cycle in Figure 6.1: a (possibly empty) "tail" starting at vertex 1 with a circuit attached to it. The coincidence happens at the vertex where the tail is attached (v in the image), after which all the steps are forced.

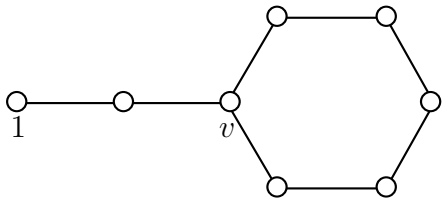

Figure 6.1: A cycle with a tail.

The word w could run through this cycle in multiple ways: it can run through the circuit multiple times. However, we necessarily have $w = w_a w_b w_a^{-1}$, where w_a corresponds to the tail and w_b is non-empty, since w is reduced. If the word were to run through the circuit multiple times, the word w_b would be a power. This can't happen since we are assuming w is good. Finally, we claim that the decomposition $w = w_a w_b w_a^{-1}$ is uniquely determined by w. Indeed, we claim w_b (the word corresponding to the cycle) is not of the form $tw't^{-1}$ for any $t \in W$. Indeed, if this were the case, then the step where the coincidence is supposed to occur would be forced. This observation uniquely determines w_b and hence also w_a.

Let t_1 be the step where the coincidence occurs. That is, t_1 is obtained by adding the number of letters of w_a and w_b together (again, this is fixed once w is fixed). Likewise, let t_0 be the number of letters of w_a. So

$$v = v_{t_0}.$$

We need that step t_1 is free and that the vertex v is chosen. We have

$$\mathbb{P}\begin{bmatrix}\text{Step } t_1 \text{ if free}\\ \text{and } v_{t_1} = v_{t_0}\end{bmatrix} \leq \frac{1}{n-t_1} \leq \frac{1}{n-2r}.$$

Note that the latter bound does not depend on when the coincidence occurs. So we obtain

$$\mathbb{P}\left[\begin{matrix}\text{Exactly one coincidence}\\ \text{occurs and } v_s = 1\end{matrix}\right] \leq \frac{1}{n-2r}. \tag{6.2}$$

Adding the bounds (6.1) and (6.2) together, we obtain the lemma. □

We are now ready to prove that random graphs are expander graphs:

Theorem 6.4. *[BS87] Fix $k \in \mathbb{N}$. We have*

$$\mathbb{E}_{n,k}^{\text{perm}}[\lambda(G)] \leq 2^{1/2} \cdot (2k)^{3/4} \cdot (1 + o(1))$$

as $n \to \infty$.

Proof. From Lemma 6.1 we obtain:

$$\mathbb{E}_{n,k}^{\text{perm}}[\rho(G)] \leq \mathbb{E}_{n,k}^{\text{perm}}[\rho(G)^{2r}]^{1/2r} \leq (\mathbb{E}_{n,k}^{\text{perm}}[\mathbb{E}_{2r}\left[|\text{Fix}(w(\pi_1, \ldots, \pi_k))|\right]] - 1)^{1/2r}$$

We have

$$\begin{aligned}
\mathbb{E}_{n,k}^{\text{perm}}[\mathbb{E}_{2r}\left[|\text{Fix}(w(\pi_1, \ldots, \pi_k))|\right]] &= \frac{\sum_{w \in W^{2r}} \sum_{\pi_1, \ldots, \pi_k \in \mathfrak{S}_n} |\text{Fix}(w(\pi_1, \ldots, \pi_k))|}{(2k)^{2r} \cdot (n!)^k} \\
&= \frac{1}{(2k)^{2r}} \sum_{w \in W^{2r}} \sum_{i=1}^{n} \mathbb{P}_{n,k}^{\text{perm}}[w(i) = i] \\
&= \frac{1}{(2k)^{2r}} \sum_{w \in W^{2r}} n \cdot \mathbb{P}_{n,k}^{\text{perm}}[w(1) = 1]
\end{aligned}$$

Using Lemma 6.3, we obtain

$$\begin{aligned}
\mathbb{E}_{n,k}^{\text{perm}}[\mathbb{E}_{2r}\left[|\text{Fix}(w(\pi_1, \ldots, \pi_k))|\right]] \leq\ & \frac{n}{n-2r} + \frac{16 \cdot r^4 \cdot n}{(n-2r)^2} \\
& + \mathbb{P}_{2r}[\text{red}(w) \text{ is bad or empty}].
\end{aligned}$$

Now we apply Lemma 6.2 and get

$$\mathbb{E}_{n,k}^{\text{perm}}[\mathbb{E}_{2r}\left[|\text{Fix}(w(\pi_1, \ldots, \pi_k))|\right]] \leq \frac{n}{n-2r} + \frac{16 \cdot r^4 \cdot n}{(n-2r)^2} + \frac{r+1}{(k/2)^r},$$

for all $n, r \in \mathbb{N}$. Now all we have to do is make a clever choice for r. $r(n) = 2\log_{k/2}(n)$ will do. We obtain

$$\mathbb{E}_{n,k}^{\text{perm}}[\mathbb{E}_{2r}[|\text{Fix}(w(\pi_1, \ldots, \pi_k))|]] - 1 = \frac{2r(n)}{n - 2r(n)} + O(\log(n)^4/n)$$

So, if we plug this into our earlier bound, we get that there exists a constant $C > 0$ so that

$$\begin{aligned}
\mathbb{E}_{n,k}^{\text{perm}}[\rho(G)] &\leq \left(\frac{C\log(n)^4}{n}\right)^{\frac{1}{4\log_{k/2}(n)}} \\
&= \exp\left(\frac{\log(n) + \log(C) + 4\log(\log(n))}{4\log(n)}\log(k/2)\right) \\
&= \left(\frac{2}{k}\right)^{\frac{1}{4} - \frac{\log(C) + 4\log(\log(n))}{4\log(n)}} \\
&= \left(\frac{2}{k}\right)^{\frac{1}{4}} \cdot (1 + o(1)),
\end{aligned}$$

as $n \to \infty$.

This implies that

$$\mathbb{E}_{n,k}^{\text{perm}}[\lambda(G)] = 2k \cdot \mathbb{E}_{n,k}^{\text{perm}}[\rho(G)] = 2^{1/2} \cdot (2k)^{3/4} \cdot (1 + o(1)),$$

as $n \to \infty$. □

Like we said before, this proves the existence of expander graphs:

Corollary 6.5. *Let $k \geq 3$. There exists a sequence $(G_n)_n$ of $2k$-regular graphs that forms an expander graph.*

Proof. For every $n \in \mathbb{N}$ there must be a set of permutations $\pi_1, \ldots, \pi_k \in \mathfrak{S}_n$ so that

$$\lambda(G(\pi_1, \ldots, \pi_k)) \leq \mathbb{E}_{n,k}^{\text{perm}}[\lambda(G)] < 2k,$$

where we used $k \geq 3$ for the strict inequality. If we choose one such graph for every n, we obtain an expander sequence. □

6.2 Exercises

Exercise 6.1. *The diameter of an expander.* Given a connected graph G, the diameter of G is given by:

$$\operatorname{diam}(G) = \sup\left\{\mathrm{d}(v,w);\ v,w \in V(G)\right\},$$

where $\mathrm{d}(v,w)$ denotes the distance between v and w: the number of edges in the shortest path between v and w. Moreover, given $v \in V(G)$ and $r \in \mathbb{N}$, let

$$B_r(v) = \{w \in V(G);\ \mathrm{d}(v,w) \leq r\}$$

denote the ball of radius r around v.

(a) Let $k \geq 3$ and let G be a connected k-regular graph on n vertices. Furthermore suppose G has at least one pair of vertices that do not share an edge. Show that

$$\operatorname{diam}(G) \geq \log_{k-1}(n) + \log(e/k).$$

Hint: Use the fact that $B_{\operatorname{diam}(G)}(v)$ covers the whole graph G for any $v \in V(G)$.

(b) For every $k \geq 3$ give an example of a sequence of k-regular graphs $(G_n)_n$ so that $|V(G_n)| \to \infty$ as $n \to \infty$ and $\operatorname{diam}(G_n)$ is linear in $V(G_n)$.

(c) Let $k \geq 3$ and let G be a connected k-regular graph on n vertices. Show that

$$|B_r(v)| \geq \min\left\{\frac{n}{2}, \left(1 + \frac{h(G)}{k}\right)^r\right\}.$$

(d) Let $k \geq 3$ and let G be a connected k-regular graph on n vertices. Set $\beta = 1 + h(G)/k$. Show that

$$\operatorname{diam}(G) \leq 2\log_\beta(n) + 3$$

and conclude that if $(G_n)_n$ is an expander graph, then there exist a constant $C > 0$ (independent of n) so that

$$\frac{1}{C}\log(|V(G_n)|) \leq \operatorname{diam}(G_n) \leq C\log(|V(G_n)|)$$

for all $n \in \mathbb{N}$.

Hint: Take two vertices $v, w \in G$ that realize the diameter and expand balls around them of a radius r so that $|B_r(v)| \geq n/2$ and $|B_r(w)| \geq n/2$, where r is the minimal radius with this property.

Exercise 6.2. A simple random walk on a graph G starting at $v \in V(G)$ is a sequence of random variables $X_r : \Omega \to V(G)$, $r \in \mathbb{N}$ with

$$X_0(\omega) = v$$

for all $\omega \in \Omega$ and

$$\mathbb{P}[X_{r+1} = v |\, X_r = w] = A(G)_{vw} / \deg(w),$$

where $A(G)$ is the adjacency matrix of G.

In what follows, fix $k \geq 3$ and let G be a connected k-regular graph on vertices $\{1, \ldots, n\}$.

(a) Set $P(G) = A(G)/k$. Show that

$$\mathbb{P}[X_r = i |\, X_0 = j] = \left(P(G)^r\right)_{ij}.$$

(b) Show that

$$\left| \left(P(G)^r\right)_{ij} - \frac{1}{n} \right| \leq \left(\frac{\lambda(G)}{k} \right)^r \cdot (n-1).$$

(c) Fix $j \in \{1, \ldots, n\}$ and let $\mathbb{P}^{\star r} : \mathcal{P}(V(G)) \to [0,1]$ be the probability measure on $V(G)$ given by:

$$\mathbb{P}^{\star r}[S] = \mathbb{P}[X_r \in S |\, X_0 = j]$$

for all $S \subset V(G)$. Furthermore, let $\mathbb{U} : \mathcal{P}(V(G)) \to [0,1]$ denote the uniform probability measure on $V(G)$. That is:

$$\mathbb{U}[S] = |S| / n$$

for all $S \subset V(G)$. Show that

$$\mathrm{d_{TV}}\left(\mathbb{P}^{\star r}, \mathbb{U}\right) \leq \frac{n \cdot (n-1)}{2} \left(\frac{\lambda(G)}{k} \right)^r.$$

Hint: Use Exercise 2.1(a).

Lecture 7

A crash course the geometry of hyperbolic surfaces

The goal of this lecture is to introduce hyperbolic surfaces. We will however not have time to go through all the details. For a more complete reference, we refer to [Bea95, Chapter 7].

7.1 The hyperbolic plane

Hyperbolic geometry originally developed in the early 19^{th} century to prove that the parallel postulate in Euclidean geometry is independent of the other postulates. From this perspective, the hyperbolic plane can be seen as a geometric object satisfying a collection of axioms very similar to Euclid's axioms for Euclidean geometry, but with the parallel postulate replaced by something else. From a more modern perspective, hyperbolic geometry is the study of manifolds that admit a Riemannian metric of constant curvature -1.

From the classical point of view, any concrete description of the hyperbolic plane is a *model* for two-dimensional hyperbolic geometry, in the same way that $\mathbb{R}^2$ is a model for Euclidean geometry.

Because this is a crash course, we will describe only one model for the hyperbolic plane: the upper half plane model. We note however that other

models (like for instance the Klein model, the Poincaré model and the hyperboloid model) do exist.

Given a smooth manifold M, let TM denote its tangent bundle. Recall that a Riemannian manifold (M, g) is a manifold M equipped with a smooth map

$$g : TM \times TM \to \mathbb{R},$$

called the *Riemannian metric*, so that the restriction $g_p : T_pM \times T_pM \to \mathbb{R}$ is a real inner product.

Definition 7.1. The hyperbolic plane $\mathbb{H}^2$ is the complex domain

$$\mathbb{H}^2 = \{z \in \mathbb{C};\ \Im(z) > 0\}$$

equipped with the Riemannian metric $g_{x+iy} : T_{x+iy}\mathbb{H}^2 \times T_{x+iy}\mathbb{H}^2 \to \mathbb{R}$ given by

$$g_{x+iy}(v, w) = \frac{1}{y^2}\big(dx(v) \cdot dx(w) + dy(v) \cdot dy(w)\big)$$

for all $x \in \mathbb{R}$ and $y \in (0, \infty)$

Because they are convenient, we will almost always work in local coordinates $x = \Re(z)$ and $y = \Im(z)$ for all $z \in \mathbb{H}^2$. We will denote the corresponding tangent vector fields by $\partial/\partial x$ and $\partial/\partial y$ respectively.

Let us first note that even thought distances in $\mathbb{H}^2$ behave very differently than in Euclidean geometry, the angles are the same. Indeed, locally the metric is just a scalar multiple of the usual inner product, so angles are no different.

Example 7.2. Let us compute the hyperbolic length of the straight line segment between $ai \in \mathbb{H}^2$ and $bi \in \mathbb{H}^2$ (denoted $[ai, bi]$) for $0 < a < b \in \mathbb{R}$. We may parameterize this segment by

$$\gamma : [0, 1] \to [ai, bi] \ \text{ given by } \ \gamma(t) = (1 - t) \cdot ai + t \cdot bi.$$

We have

$$\frac{d}{dt}\gamma(t) = -a\frac{\partial}{\partial y}_{\gamma(t)} + b\frac{\partial}{\partial y}_{\gamma(t)} = (b - a)\frac{\partial}{\partial y}_{\gamma(t)}.$$

So

$$g\left(\frac{d}{dt}\gamma(t), \frac{d}{dt}\gamma(t)\right) = \frac{(b - a)^2}{(a + t(b - a))^2}.$$

This means that the length of the line segment is given by

$$\begin{aligned}\ell([ai,bi]) &= \int_0^1 \sqrt{g\left(\frac{d}{dt}\gamma(t),\frac{d}{dt}\gamma(t)\right)}dt \\ &= \int_0^1 \frac{b-a}{a+t(b-a)}dt \\ &= [\log(a+t(b-a))]_0^1 \\ &= \log(b/a).\end{aligned}$$

Recall that given a connected Riemannian manifold (M,g), the *distance* between two points $p,q\in M$ is given by

$$\mathrm{d}(p,q)=\inf\left\{\ell(\gamma);\ \gamma:[0,1]\to M \text{ smooth},\ \gamma(0)=p \text{ and } \gamma(1)=q\right\}.$$

Example 7.3. We claim that for $ai,bi\in\mathbb{H}^2$ with $0<a<b\in\mathbb{R}$ we have

$$\mathrm{d}(ai,bi)=\log(b/a).$$

In Example 7.2 we have already shown that

$$\mathrm{d}(ai,bi)\leq\log(b/a),$$

so all we have to do is show the other inequality. Let $\gamma:[0,1]\to\mathbb{H}^2$ be any other smooth path with $\gamma(0)=ai$ and $\gamma(1)=bi$. Write

$$x(t)=\Re(\gamma(t)) \text{ and } y(t)=\Im(\gamma(t)),$$

so $\gamma(t)=x(t)+iy(t)$. We have

$$\begin{aligned}\ell(\gamma) &= \int_0^1 \sqrt{g\left(\frac{d}{dt}\gamma(t),\frac{d}{dt}(\gamma(t)\right)}dt \\ &= \int_0^1 \frac{1}{y(t)}\sqrt{\dot{x}(t)^2+\dot{y}(t)^2}dt,\end{aligned}$$

where $\dot{x}(t)=dx(t)/dt$ and $\dot{y(t)}=dy(t)/dt$. As such

$$\ell(\gamma)\geq\int_0^1\frac{\dot{y}(t)}{y(t)}dt=\log(b/a),$$

which proves our claim.

Let $\mathrm{Mat}(2,\mathbb{R})$ denote the set of 2×2 real matrices and define the group

$$\mathrm{PSL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2,\mathbb{R});\ ad-bc=1 \right\} / \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

The group $\mathrm{PSL}(2,\mathbb{R})$ acts on $\mathbb{H}^2$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az+b}{cz+d} \tag{7.1}$$

for all $z \in \mathbb{H}^2$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{R})$. Note that the expression above is well-defined, that is, it does not depend on the representative matrix we choose. In Exercise 7.1 we prove that this actually defines a $\mathrm{PSL}(2,\mathbb{R})$-action on $\mathbb{H}^2$ and that the action is by *isometries*. That is

$$\mathrm{d}(Az, Aw) = \mathrm{d}(z,w)$$

for all $z, w \in \mathbb{H}^2$ and $A \in \mathrm{PSL}(2,\mathbb{R})$. When acting on $\mathbb{H}^2$, the elements of $\mathrm{PSL}(2,\mathbb{R})$ are called Möbius transformations. We claim (but will not prove) that all orientation preserving isometries of $\mathbb{H}^2$ are Möbius transformations.

Proposition 7.4. *Let $A : \mathbb{H}^2 \to \mathbb{H}^2$ be a smooth map that preserves orientation so that*

$$\mathrm{d}(Az, Aw) = \mathrm{d}(z,w)$$

for all $z, w \in \mathbb{H}^2$, then A is a Möbius transformation.

A consequence of this is the following:

Proposition 7.5. *Let $z, w \in \mathbb{H}^2$. Then*

$$\mathrm{d}(z,w) = \cosh^{-1}\left(1 + \frac{|z-w|^2}{2\cdot \Im(z)\cdot \Im(w)}\right).$$

Proof. First of all, for z and w on the imaginary axis, this formula restricts to the formula from Example 7.3. As such, our strategy will be to prove that the expression on the right is invariant under Möbius transformations (as well as the expression on the left) and then to show that every pair of elements $z, w \in \mathbb{H}^2$ can be mapped to the imaginary axis by Möbius transformations.

The first fact comes down to checking that

$$\frac{|z-w|^2}{2\cdot\Im(z)\cdot\Im(w)} = \frac{|Az-Aw|^2}{2\cdot\Im(Az)\cdot\Im(Aw)}$$

for all $A \in \mathrm{PSL}(2,\mathbb{R})$ and $z, w \in \mathbb{H}^2$. This is a straightforward computation that we leave to the reader.

To show that we can move every pair of points to the imaginary axis with a Möbius transformation, we may assume that not both z and w are on the imaginary axis.

First suppose that z and w lie on a vertical line $\{x = b\}$. In this case the Möbius transformation $z \mapsto z - b$ maps both points to the imaginary axis.

Now suppose that z and w do not lie on a vertical line. Let C be the unique Euclidean circle through z and w that is perpendicular to the real line. Let α be one of the two points on the intersection $C \cap \mathbb{R}$.

$$z \mapsto \frac{-1}{z-\alpha}$$

is a Möbius transformation. We claim that it sends C to a straight line. One way to check this is by parameterization. Indeed, suppose C has center $\beta \in \mathbb{R}$ and suppose $\beta > \alpha$. We can then parameterize

$$C(t) = \beta + e^{2\pi i t}(\beta - \alpha),\ t \in \left(0, \frac{1}{2}\right)$$

It is a straightforward computation to check that

$$\Re\left(\frac{-1}{C(t)-\alpha}\right) = \frac{-1}{2(\beta-\alpha)}.$$

As such, our Möbius transformation sends z and w to two elements that lie on a vertical line and we are done. □

We note that Möbius transformations preserve the set of half circles orthogonal to $\mathbb{R}$ and vertical lines in $\mathbb{H}^2$ (see Exercise 7.2).

Recall that a *geodesic* $\gamma : \mathbb{R} \to \mathbb{H}^2$ is a smooth path so that

$$\mathrm{d}(\gamma(t), \gamma(s)) = |t - s|$$

for all $t, s \in \mathbb{R}$.

It follows from the proof and the two examples above that:

Proposition 7.6. *The image of a geodesic $\gamma : \mathbb{R} \to \mathbb{H}^2$ is a vertical line or a half circle orthogonal to $\mathbb{R}$. Moreover, every vertical line and half circle orthogonal to the real line can parameterized as a geodesic.*

We will often forget about the parametrization and call the image of a geodesic a geodesic as well. Note that it follows from the proposition above that given any two distinct points $z, w \in \mathbb{H}^2$ there exists a unique geodesic $\gamma \subset \mathbb{H}^2$ so that both $z \in \gamma$ and $w \in \gamma$. Furthermore, it also follows given a point $z \in \mathbb{H}^2$ and a geodesic γ that does not contain it, there is a unique perpendicular from z to γ (a geodesic γ' that intersects γ once perpendicularly and contains z)

The final fact we will need about the hyperbolic plane is:

Proposition 7.7. *Let $z \in \mathbb{H}$ and let $\gamma \subset \mathbb{H}^2$ be a geodesic so that $z \notin \gamma$ then*

$$\mathrm{d}(z, \gamma) := \inf \{\mathrm{d}(z, w);\ w \in \gamma\}$$

is realized by the intersection point of the perpendicular from z to γ.

Proof. This follows from Pythagoras' theorem for hyperbolic triangles. Indeed, given three points in $\mathbb{H}^2$ so that the three geodesics through them form a right angled hyperbolic triangle with sides of length a, b and c (where c is the side opposite the right angle), we have

$$\cosh(a)\cosh(b) = \cosh(c)$$

(see Exercise 7.3). This means in particular that $c > b$.

So, any other point on γ is further away from z than the point w realizing the perpendicular. Because that other point forms a right angled triangle with w and z. □

7.2 Surfaces

A *surface* is a smooth two-dimensional manifold. We call a surface *closed* if it is compact and has no boundary. A surface is said to be of *finite type* if it can be obtained from a closed surface by removing a finite number of points and (smooth) open disks. In what follows, we will always assume our surfaces to be orientable.

Example 7.8. To properly define a manifold, one needs to not only describe the set but also give smooth charts. In what follows we will content ourselves with the sets (Exercise 7.4 completes the picture).

(a) The 2-*sphere* is the surface

$$\mathbb{S}^2 = \left\{(x, y, z) \in \mathbb{R}^3;\ x^2 + y^2 + z^2 = 1\right\}.$$

(b) Let $\mathbb{S}^1$ denote the circle. The 2-*torus* is the surface

$$\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$$

(c) Given two (oriented) surfaces S_1, S_2, their *connected sum* $S_1 \# S_2$ is defined as follows. Take two closed sets $D_1 \subset S_1$ and $D_2 \subset S_2$ that are both diffeomorphic to closed disks, via diffeomorphisms

$$\varphi_i : \left\{(x, y) \in \mathbb{R}^2;\ x^2 + y^2 \leq 1\right\} \to D_i, \quad i = 1, 2,$$

so that φ_1 is orientation preserving and φ_2 is orientation reversing.

Then

$$S_1 \# S_2 = \left(S_1 \smallsetminus \mathring{D}_1 \sqcup S_2 \smallsetminus \mathring{D}_2\right) / \sim$$

where $\mathring{D}_i$ denotes the interior of D_i for $i = 1, 2$ and the equivalence relation $\sim$ is defined by

$$\varphi_1(x, y) \sim \varphi_2(x, y) \quad \text{for all } (x, y) \in \mathbb{R}^2 \text{ with } x^2 + y^2 = 1.$$

The figure below gives an example.

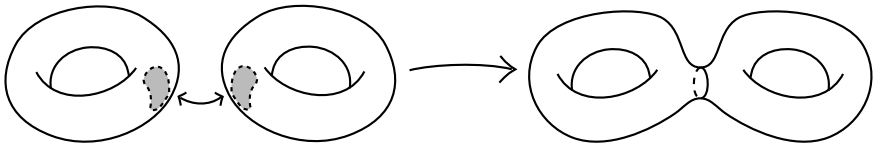

Figure 7.1: A connected sum of two tori.

Like our notation suggests, the manifold $S_1 \# S_2$ is independent (up to diffeomorphism) of the choices we make (the disks and diffeomorphisms φ_i). This is a non-trivial statement, the proof of which we will skip. Likewise, we will also not prove that the connected sum of surfaces is an associative operation and that $\mathbb{S}^2 \# S$ is diffeomorphic to S for all surfaces S.

A classical result from the 19^{th} century tells us that the three simple examples above are enough to understand all finite type surfaces up to diffeomorphism.

Theorem 7.9. Classification of closed surfaces *Every closed surface is diffeomorphic to the connected sum of a 2-sphere with a finite number of tori.*

Indeed, because the diffeormorphism type of a finite type surface does not depend on where we remove the points and open disks (another claim we will not prove), the theorem above tells us that a finite type surface is (up to diffeomorphism) determined by a triple of positive integers (g, b, n), where

- g is the number of tori in the connected sum and is called the *genus* of the surface.
- b is the number of disks removed and is called the number of *boundary components* of the surface.
- n is the number of points removed and is called the number of *punctures* of the surface.

we will denote the corresponding surface by $\Sigma_{g,b,n}$ and will write $\Sigma_g = \Sigma_{g,0,0}$.

7.3 Hyperbolic surfaces

For this section we will mainly follow [Bus10]. A hyperbolic surface will be a finite type surface equipped with a metric that locally makes it look like $\mathbb{H}^2$.

Because we will want to deal with surfaces with boundary, we need half spaces. Let $\gamma \subset \mathbb{H}^2$ be a geodesic. $\mathbb{H}^2 \setminus \gamma$ consists of two connected components C_1 and C_2. We will call $\mathcal{H}_i = C_i \cup \gamma$ a *closed half space* $(i = 1, 2)$. So for example

$$\left\{ z \in \mathbb{H}^2;\ \Re(z) \leq 0 \right\}$$

is a closed half space.

We formalize the notion of a hyperbolic surface as follows:

Definition 7.10. A finite type surface S with atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ is called a *hyperbolic surface* if $\varphi_\alpha(U_\alpha) \subset \mathbb{H}^2$ for all $\alpha \in A$ and

1. for each $p \in S$ there exists an $\alpha \in A$ so that $p \in U_\alpha$ and

   - If $p \in \partial S$ then
   $$\varphi_\alpha(U_\alpha) = V \cap \mathcal{H}$$
   for some open set $V \subset \mathbb{H}^2$ and some closed half space $\mathcal{H} \subset \mathbb{H}^2$.
   - If $p \in \mathring{S}$ then $\varphi_\alpha(U_\alpha) \subset \mathbb{H}^2$ is open.

2. For every $\alpha, \beta \in A$ and for each connected component C of $U_\alpha \cap U_\beta$ we can find a Möbius transformation $A : \mathbb{H}^2 \to \mathbb{H}^2$ so that
   $$\varphi_\alpha \circ \varphi_\beta^{-1}(z) = A(z)$$
   for all $z \in \varphi_\beta(C) \subset \mathbb{H}^2$.

Note that every hyperbolic comes with a metric: every chart is identified with an open set of $\mathbb{H}^2$ which gives us a metric. Because the chart transitions are restrictions of isometries of $\mathbb{H}^2$, this metric does not depend on the choice of chart and hence is well defined.

Definition 7.11. A hyperbolic surface S is called *complete* if the induced metric is complete.

7.4 Exercises

Exercise 7.1. (a) Show that the action of $\mathrm{PSL}(2, \mathbb{R})$ on $\mathbb{H}^2$ defined in (7.1) is indeed an action. That is, show that if $A, B \in \mathrm{PSL}(2, \mathbb{R})$ and $z \in \mathbb{H}^2$ then
$$Az \in \mathbb{H}^2 \text{ and } (A \cdot B)z = A(Bz).$$

(b) Recall that if M is a manifold and $f : M \to M$ a diffeomorphism, then we obtain a linear map
$$Df_p : T_pM \to T_{f(p)}M$$

called the differential of f. One way to describe this map is as follows. Given $v \in T_pM$, Take $\gamma : (-1, 1) \to M$ so that

$$\gamma(0) = p \text{ and } \frac{d}{dt}\gamma(0) = v$$

and define

$$(Df)v = \frac{d}{dt}(f \circ \gamma)(0).$$

Given $A \in \mathrm{PSL}(2, \mathbb{R})$, show that its derivative DA_z (as a map from $\mathbb{H}^2$ to itself) satisfies

$$g_{Az}(DA_z v, DA_z w) = g_z(v, w)$$

for all $z \in \mathbb{H}^2$ and $v, w \in T_z\mathbb{H}^2$.

(c) Given a smooth path $\gamma : [0, 1] \to \mathbb{H}^2$ and $A \in \mathrm{PSL}(2, \mathbb{R})$, we obtain a new smooth path $A \circ \gamma : [0, 1] \to \mathbb{H}^2$. Show that

$$\ell(\gamma) = \ell(A \circ \gamma).$$

Conclude that

$$\mathrm{d}(Az, Aw) = \mathrm{d}(z, w)$$

for all $z, w \in \mathbb{H}^2$ and $A \in \mathrm{PSL}(2, \mathbb{R})$.

Exercise 7.2. Let $C \subset \mathbb{H}^2$ be a half circle orthogonal to $\mathbb{R}$ or a vertical line and let $A : \mathbb{H}^2 \to \mathbb{H}^2$ be a Möbius transformation. Show that $A(C)$ is a vertical line or half circle orthogonal to $\mathbb{R}$.

Hint: consider what a Möbius transformation does to the endpoints (NB: ∞ is a possible endpoint) of half circles orthogonal to $\mathbb{R}$ and vertical lines

Exercise 7.3. *Pythagoras' theorem:* Suppose $x, y, z \in \mathbb{H}^2$ form a right angled triangle (that is, the geodesic between x and y intersects that between y and z perpendicularly) and let

$$a = \mathrm{d}(x, y),\ b = \mathrm{d}(y, z) \text{ and } c = \mathrm{d}(z, x).$$

Prove that

$$\cosh(a) \cdot \cosh(b) = \cosh(c).$$

Hint: just like in the proof of Proposition 7.5 you may assume that the geodesic between y and z is the imaginary axis.

Exercise 7.4. Define an atlas for $\mathbb{S}^2$ and $\mathbb{T}^2$.

Lecture 8

Pairs of pants and simple closed curves

8.1 Pairs of pants

Even though Definition 7.10 is a complete definition, it is not very descriptive. In what follows we will describe a concrete cutting and pasting construction for hyperbolic surfaces.

We start with right angled hexagons. Let $\gamma_1, \ldots, \gamma_6 \subset \mathbb{H}^2$ be consistently oriented geodesics so that

$$|\gamma_i \cap \gamma_j| = \begin{cases} 1 & \text{if } |i-j| = 1 \text{ or if } \{i,j\} = \{1,6\} \\ 0 & \text{otherwise.} \end{cases}$$

and the oriented angle at every intersection point is $\pi/2$. Now let $\mathcal{H}_1, \ldots, \mathcal{H}_6$, be half spaces defined by the geodesics $\gamma_1, \ldots, \gamma_6$ so that the intersection $\cap_{i=1}^6 \mathcal{H}_i$ is non-empty and compact. Then $\cap_{i=1}^6 \mathcal{H}_i \subset \mathbb{H}^2$ is called a right angled hexagon.

The picture to have in mind is:

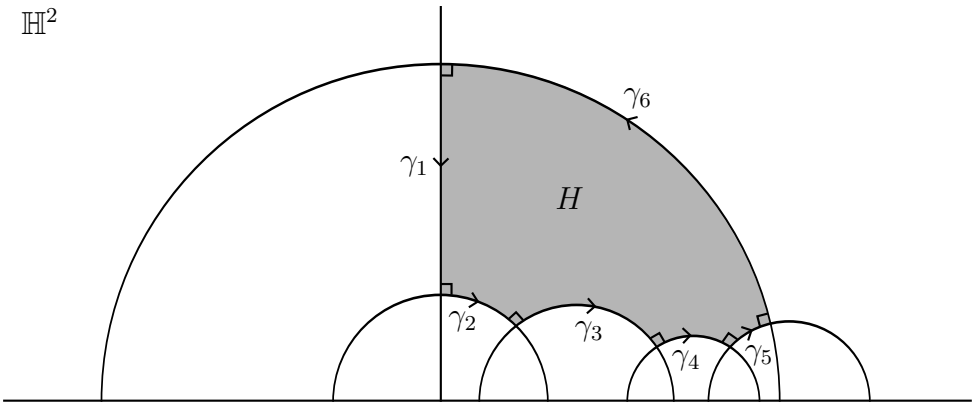

Figure 8.1: A right angled hexagon H.

It turns out that the lengths of three non-consecutive sides determine a right angled hexagon up to isometry.

Proposition 8.1. *Let $a, b, c \in (0, \infty)$. Then there exists a right angled hexagon $H \subset \mathbb{H}^2$ with three non-consecutive sides of length a, b and c respectively. Moreover, if H' is another right angled hexagon with this property, then there exists a Möbius transformation $A : \mathbb{H}^2 \to \mathbb{H}^2$ so that*

$$A(H) = H.$$

Proof. Let us start with the existence. Let γ_{im} denote the positive imaginary axis and set

$$B = \left\{ z \in \mathbb{H}^2;\ \mathrm{d}(z, \gamma_{im}) = c \right\}.$$

B is a one-dimensional submanifold of $\mathbb{H}^2$. Because the map $z \mapsto \lambda z$ is an isometry that preserves γ_{im} for every $\lambda > 0$, it must also preserve B. This means that B is a (straight Euclidean) line.

Now construct the following picture:

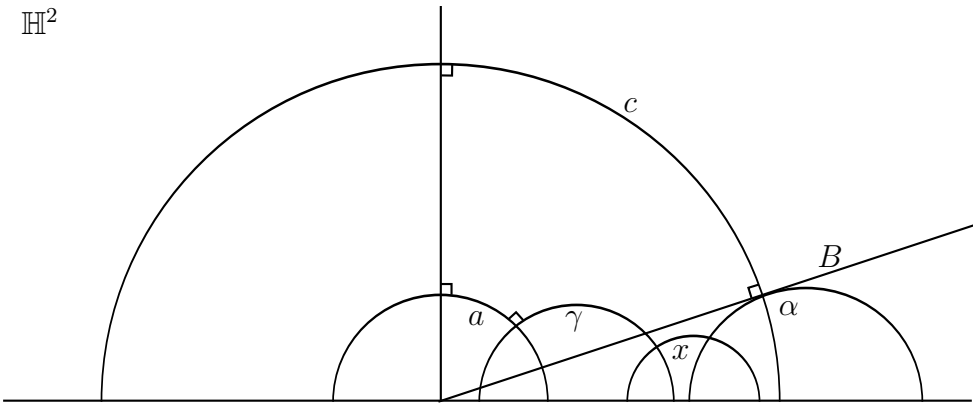

Figure 8.2: Constructing a right angled hexagon $H(a, b, c)$.

That is, we take the geodesic though the point $i \in \mathbb{H}^2$ perpendicular to γ_{im} and at distance a draw a perpendicular geodesic γ. furthermore, for any $p \in B$, we draw the geodesic α that realizes a right angle with the perpendicular from p to γ_{im}. Now let

$$x = \mathrm{d}(\alpha, \gamma) = \inf \{\mathrm{d}(z, w);\ z \in \gamma,\ w \in \alpha\}.$$

Because of Proposition 7.7, x is realized by the common perpendicular to α and γ. By moving p over B, we can realize any positive value for x and hence obtain our hexagon $H(a, b, c)$.

We also obtain uniqueness from the picture above. Indeed, given any right angled hexagon H' with three non-consequtive sides of length a, b and c, apply a Möbius transformation $A : \mathbb{H}^2 \to \mathbb{H}^2$ so that the geodesic segment of length a starts at i and is orthogonal to the imaginary axis. This implies that the geodesic after a gets mapped to the geodesic γ. Furthermore, one of the endpoints of the geodesic segment of length c needs to lie on the line B. We now know that the the geodesic α before that point needs to be tangent to B. Because α and β have a unique common perpendicular. The tangency point of α to B determines the picture entirely. Because the function that assigns the length x of the common perpendicular to the tangency point is injective, we obtain that there is a unique solution. □

One of our main building blocks for hyperbolic surfaces is the following:

Definition 8.2. Let $a, b, c \in (0, \infty)$. A *pair of pants* is a hyperbolic surface that is diffeomorphic to $\Sigma_{0,3,0}$ such that the boundary components have length a, b and c respectively.

Proposition 8.3. *Let $a, b, c \in (0, \infty)$ and let P and P' be pairs of pants with boundary curves of lengths a, b and c. Then there exists an isometry $\varphi : P \to P'$.*

Proof sketch. There exists a unique orthogonal geodesic (this essentially follows from Proposition 7.7, in Proposition 8.6 we will do a similar proof in full) between every pair of boundary components of P.

These three orthogonals decompose P into right-angled hexagons out of which three non-consecutive sides are determined. Proposition 8.1 now tells us that this determines the hexagons up to isometry and this implies that P is also determined up to isometry. □

Note that it also follows from the proof sketch above that the unique perpendiculars cut each boundary curve on P into two geodesic segments of equal length. Moreover, we obtain a standard parmeterization of the boundary pair of pants.

If P is a pair of pants and $\delta \subset \partial P$ is one of its boundary components, let us write $\ell(\delta)$ for the length of δ. Recall that an isometry between Riemannian manifolds M and N is a diffeomorphism $\varphi : M \to N$ so that

$$\mathrm{d}_M(x, y) = \mathrm{d}_N(\varphi(x), \varphi(y))$$

for all $x, y \in M$.

Example 8.4. Given two pairs of pants P_1 with boundary components δ_1, δ_2 and δ_3 and P_2 with boundary components γ_1, γ_2 and γ_3 so that

$$\ell(\delta_1) = \ell(\gamma_1),$$

we can choose an orientation reversing isometry $\varphi : \delta_1 \to \gamma_1$ and from that obtain a hyperbolic surface

$$S = P_1 \sqcup P_2 / \sim,$$

where $\varphi(x) \sim x$ for all $x \in \delta_1$. Note that S is diffeomorphic to $\Sigma_{0,4,0}$.

8.2 Simple closed curves

Given a manifold M, recall that two embeddings $\gamma_1, \gamma_2 : \mathbb{S}^1 \to M$ are called *freely homotopic*, if there exists a continuous map:

$$H : \mathbb{S}^1 \times [0,1] \to X$$

so that

$$H(t,0) = \gamma_1(t) \text{ and } H(t,1) = \gamma_2(t)$$

for all $t \in \mathbb{S}^1$. The difference between free homotopy and usual homotopy of loops is that there is no mention of basepoints in the case of free homotopy.

Let X be a hyperbolic surface. We call a smooth map $\gamma : \mathbb{S}^1 \to X$ a *closed geodesic* if for every $t \in \mathbb{S}^1$ there exists an open set $U \subset \mathbb{S}^1$ with $t \in U$ so that

$$\mathrm{d}_X(\gamma(s), \gamma(s')) = \mathrm{d}_{\mathbb{S}^1}(s, s'),$$

where the metric $d_{\mathbb{S}^1} : \mathbb{S}^1 \times \mathbb{S}^1 \to [0, \infty)$ is the metric coming from the quotient $\mathbb{S}^1 = \mathbb{R}/(\ell(\gamma)\mathbb{Z})$ (so $\mathbb{S}^1$ has total length $\ell(\gamma)$). Just like with geodesics in $\mathbb{H}^2$, we will often identify a closed geodesic with its image.

Finally, we will need the following fact, which we shall not prove. A convex subset of $\mathbb{H}^2$ here is a subset $C \subset \mathbb{H}^2$ so that the geodesic segment between x and y lies in C for all $x, y \in C$.

Theorem 8.5. *(a) Let X be closed hyperbolic surface. Then there exists a covering map*

$$p : \mathbb{H}^2 \to X$$

that is a local isometry.

(b) Let X be a hyperbolic surface with boundary. Then there exists a closed convex subset $\widetilde{X} \subset \mathbb{H}^2$ and a covering map

$$p : \widetilde{X} \to X$$

that is a local isometry.

A proof of (a) for instance be found in [CE08, Theorem 1.37] and (b) is proved in [Bus10, Theorem 1.4.2]. Note that it follows from the fact that

$\mathbb{H}^2$ and convex subsets in $\mathbb{H}^2$ are simply connected that the covers above are unversal covers.

To see that every closed hyperbolic surface can be constructed by gluing pairs of pants together, we need the following proposition. Here, a *simple* closed curve on a hyperbolic surface X is a closed curve $\gamma : \mathbb{S}^1 \to X$ that is an embedding.

Proposition 8.6. *Let X be a closed hyperbolic surface and let $\gamma : \mathbb{S}^1 \to X$ be smooth map (a closed curve) that is not freely homotopic to a constant map. There exists a (up to reparameterization) unique closed geodesic $\overline{\gamma} : \mathbb{S}^1 \to X$ that is freely homotopic to γ. This geodesic is the curve of minimal length among all curves that are freely homotopic to γ. Moreover, if γ is simple then so is $\overline{\gamma}$.*

Proof sketch. We will prove everything, except the statement about simplicity. The proof will however assume some general covering theory, see [Hat02, Section 1.3] for details. Let

$$C := \left\{\gamma' : \mathbb{S}^1 \to X;\ \gamma' \text{ freely homotopic to } \gamma\right\}$$

and set

$$L = \inf\left\{\ell(\gamma');\ \gamma' \in C\right\}.$$

Now consider a sequence $(\gamma_n)_n$ so that $\ell(\gamma_n) \to L$. It follows from the Arzelà-Ascoli theorem ([Bus10, Theorem A.19]) that there exists a subsequence $(\gamma_{n_k})_k$ and a simple closed curve $\overline{\gamma} : \mathbb{S}^1 \to X$ so that $\gamma_{n_k} \to \widetilde{\gamma}$ uniformly as $k \to \infty$. Because $\widetilde{\gamma}$ minimizes length, it needs to be a geodesic (up to reparameterization).

To show uniqueness, suppose there are two freely homotopic geodesics $\gamma_1, \gamma_2 : \mathbb{S}^1 \to X$. Consider the universal cover $p : \mathbb{H}^2 \to X$. Because γ_1 and γ_2 are freely homotopic, we can lift them to continuous maps $\widetilde{\gamma}_1, \widetilde{\gamma}_2 : \mathbb{R} \to \mathbb{H}^2$ that are homotopic. The fact that γ_1 and γ_2 are geodesics implies that $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ are as well.

By general covering theory, the subgroup of the deck group $\pi_1(X)$ that leaves $\widetilde{\gamma}_1$ invariant also leaves $\widetilde{\gamma}_2$ invariant (because they are homotopic). By a compactness argument, this implies that

$$\max_{t\in\mathbb{R}}\{\mathrm{d}(\widetilde{\gamma}_1(t), \widetilde{\gamma}_2(t)\} < \infty.$$

Now we note that when geodesics have at least one pair of distinct endpoints, the above does not hold. This implies that $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ have the same endpoints, which in turn implies they have coincide. $\square$

We note that the three boundary components of a pair of pants are simple closed geodesics.

Example 8.7. φ in Example 8.4 is determined up to 'twist'. That is, if we parameterize δ_1 by a simple closed geodesic $x : \mathbb{R}/(\ell(\delta_1)\mathbb{Z}) \to \delta_1$ and $\varphi' : \delta_1 \to \gamma_1$ is a different orientation reversing isometry, then there exists some $t_0 \in \mathbb{R}$ so that

$$\varphi'(x(t)) = \varphi(x(t_0 + t))$$

for all $t \in \mathbb{R}/(\ell(\delta_1)\mathbb{Z}) \to \delta_1$.

8.3 Exercises

Exercise 8.1. Let H be a right angled hexagon with three non consecutive sides of the same length $a > 0$.

(a) Show without computing their lengths that the lengths of the other three sides are also all the same.

(b) Compute the length of the other three sides.

Lecture 9

Models of random surfaces

9.1 Pants decompositions

A direct consequence of Proposition 8.6 is that every closed hyperbolic surface of genus $g \geq 2$ can be obtained from iterating Example 8.4. That is, every closed hyperbolic surface can be built by gluing together pairs of pants. Indeed, just take a system of homotopy classes of closed curves that cut the surface into pairs of pants. Proposition 8.6 tells us that each of these homotopy classes contain unique geodesics along which we can cut the surface into hyperbolic pairs of pants.

Let us formalize the notion of a pants decomposition:

Definition 9.1. Let X be a closed surface. A *pants decomposition* of X is a set of pairwise disjoint simple closed curves $\{\gamma_1, \ldots, \gamma_n\}$ so that

$$X \setminus (\bigcup_{i=1}^{n} \gamma_i)$$

is a disjoint union of pairs of pants.

An Euler characteristic argument shows that if X is a closed surface of genus g, then the number of curves in a pants decomposition is necessarily equal to $3g - 3$.

Figure 9.1 gives some examples:

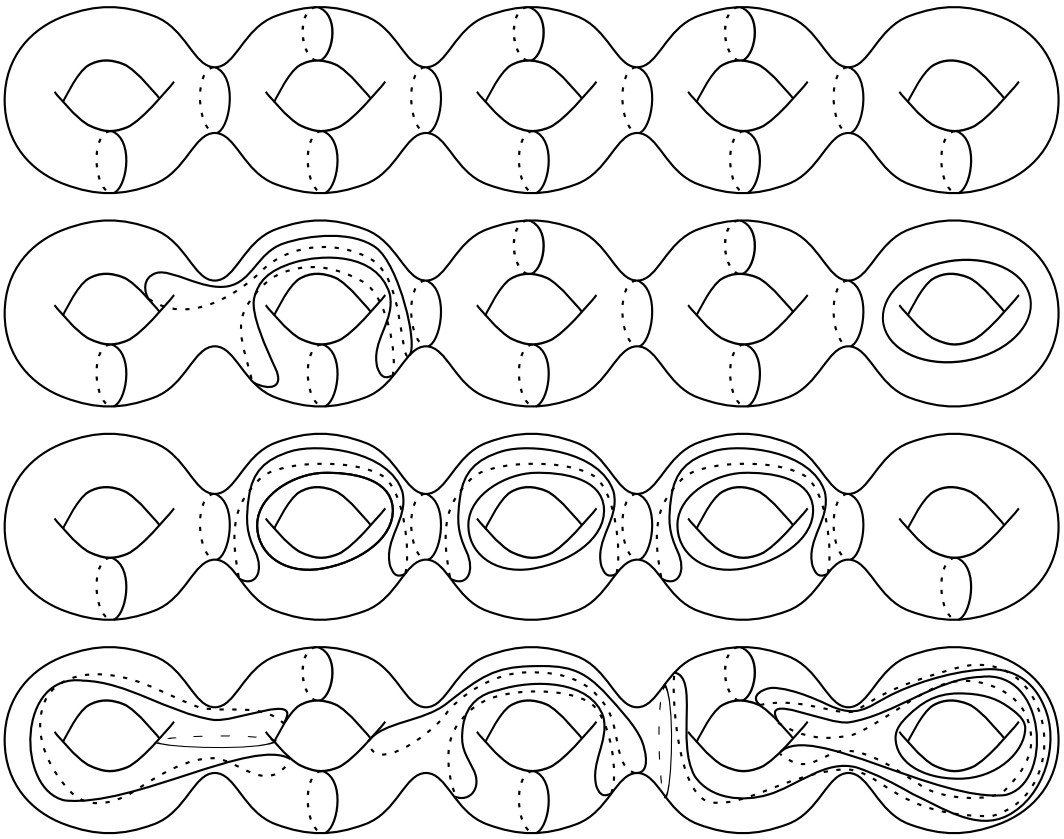

Figure 9.1: Four pants decompositions of a closed surface of genus 5.

To decide whether or not pants decompositions are the same up to diffeomorphism, the following graph is very useful:

Definition 9.2. Let X be a closed surface and let $\mathcal{P} = \{\gamma_1, \ldots, \gamma_n\}$ be a pants decomposition of X. The *dual graph* $G_\mathcal{P}$ to $\mathcal{P}$ is the graph obtained by setting

- $V(G_\mathcal{P})$ to be the set of connected components of $X \setminus \mathcal{P}$,
- $E(G_\mathcal{P}) = \mathcal{P}$ and

- γ_i is incident to a conneceted component $C \in V(G_{\mathcal{P}})$ if and only if it is a boundary component of C.

Note that $G_{\mathcal{P}}$ is a connected 3-regular graph with $3g-3$ edges and hence $2g-2$ vertices.

Proposition 9.3. *Let X be a closed surface and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be pants decompositions. There exists a diffeomorphism $\varphi : X \to X$ so that $\varphi(\mathcal{P}_1) = \varphi(\mathcal{P}_2)$ if and only if $G_{\mathcal{P}_1}$ and $G_{\mathcal{P}_2}$ are isomorphic graphs.*

Proof. Exercise 9.1. □

Note that this proposition implies that there are finitely many pants decompositions of a given surface up to diffeomorphism.

We already noted that every closed hyperbolic surface can be obtained by gluing pairs of pants together. In fact, only pairs of pants with boundary of a bounded length (in terms of the genus) are needed, this is a theorem by Bers. The best bound is due to Parlier [Par14], we will state a slightly weaker bound.

Theorem 9.4. *Every closed hyperbolic surface of genus g has a pants decomposition in which every curve has length at most*

$$20 \cdot g.$$

9.2 Teichmüller and moduli space

For applications later on, we will need a nice space to parameterize our hyperbolic surfaces. This role will be played by Teichmüller space. The definition we give is not the usual definition and in a course on Teichmüller theory would be a theorem (originally proven by Fenchel and Nielsen)

Definition/Theorem 9.5. Let $g \geq 2$. *Teichmüller space* is the manifold

$$\mathcal{T}_g = (0,\infty)^{3g-3} \times \mathbb{R}^{3g-3}.$$

The first $3g-3$ coordinates $(\ell_i)_{i=1}^{3g-3}$ are called the *length coordinates* and the last $3g-3$ coordinates $(\tau_i)_{i=1}^{3g-3}$ are called the *twist coordinates*.

Given a closed surface X with a pants decomposition $\mathcal{P} = \{\gamma_1, \ldots, \gamma_{3g-3}\}$, we define a hyperbolic surface for every point $(\ell, \tau) \in \mathcal{T}_g$ as follows.

First of all, assign the length ℓ_i to γ_i for $i = 1, \ldots, 3g-3$. Because of Proposition 8.3, this completely determines the geometry of the pair of pants $P_1, \ldots P_{2g-2}$ in the decomposition, we only need to decide how to glue them together (we need to pick diffeomorphisms φ_{i,τ_i} between the corresponding boundary components). Let P and P' be the (not necessarily distinct) pairs of pants that meet at γ_i and use the standard parameterization described earlier to parameterize the corresponding boundary components $\delta_{i,1} : \mathbb{S}^1 \to P$ and $\delta_{i,2} : \mathbb{S}^2 \to P'$. Now define

$$\varphi_i : \delta_{i,1} \to \delta_{i,2}$$

by

$$\varphi_i(\delta_{i,1}(t)) = \delta_{i,2}(\tau_i - t).$$

The picture to have in mind is the following.

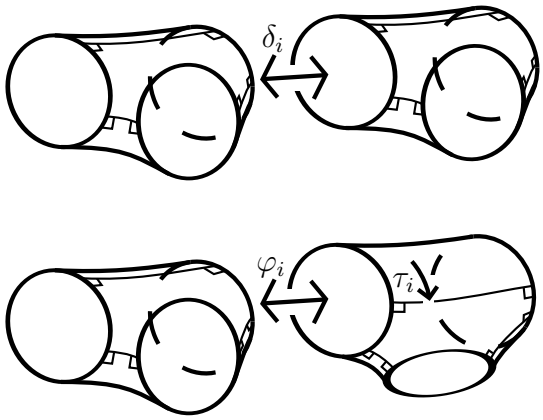

Figure 9.2: Twist.

The surface corresponding to $(\ell, \tau) \in \mathcal{T}_g$ is now given by

$$X(\ell, \tau) = \bigsqcup_{i=1}^{2g-2} P_i / \sim$$

where $\varphi_i(x) \sim x$ for all $x \in \delta_{i,1}$ and all $i = 1, \ldots, 3g-3$. Because this surface depends on the pants decomposition $\mathcal{P}$, we will sometimes denote it by $X_{\mathcal{P}}(\ell, \tau)$.

We note however that different points in $\mathcal{T}_g$ can give rise to isometric hyperbolic surfaces. For instance, there is an isometry

$$X(\ell_1, \ldots, \ell_{3g-3}, \tau_1, \ldots, \tau_{3g-3}) \to X(\ell_1, \ldots, \ell_{3g-3}, \tau_1 + \ell_1, \tau_2 \ldots, \tau_{3g-3}).$$

However, this is not the only way in which two points in Teichmüller space can give rise to isometric surfaces.

The quotient space $\mathcal{M}_g$ obtained by identifying points in Teichmüller space that define isometric surfaces is called the *moduli space* of closed hyperbolic surfaces of genus g. Note that from our definitions it is not clear that the induced topology on $\mathcal{M}_g$ is independent of the pants decomposition we use to define it. We will not prove the fact that it is indeed independent of the pants decomposition.

We also note that, even though we will not pursue this issue in this course, whereas the topology of Teichmüller space is very well understood, the toplogy of moduli space is a lot more complicated, so much so that many questions on it remain open.

In order to do probability theory later on, we need a volume form. To this end we will define the Weil-Petersson volume form. Again, we will not use the standard definition but rely on a theorem by Wolpert [Wol82] to define it.

Definition/Theorem 9.6. Let $A \subset \mathcal{T}_g$ be measurable. The *Weil-Petersson volume* of A is given by

$$\mathrm{vol}_{\mathrm{WP}}(A) = \int_A d\ell_1 \cdots d\ell_{3g-3} \cdot d\tau_1 \cdots d\tau_{3g-3}.$$

This measure descends to $\mathcal{M}_g$.

It is easy to see from the definition that the Weil-Petersson volume of $\mathcal{T}_g$ is infinite. From work by Wolpert [Wol82], it turns out that the Weil-Petersson volume of $\mathcal{M}_g$ is finite. The explicit bounds we need are due to Schumacher and Trappani [ST01], based on work of Penner [Pen92] and Grushevsky [Gru01].

Theorem 9.7. *There exist constants $a_1, a_2 > 0$ so that*

$$a_1^g \cdot g^{2g} \leq \mathrm{vol}_{\mathrm{WP}}(\mathcal{M}_g) \leq a_2^g \cdot g^{2g}.$$

The fact that $\mathrm{vol}_{\mathrm{WP}}(\mathcal{M}_g) < \infty$ leads to the following notion of random surfaces:

Definition 9.8. Let $g \in \mathbb{N}_{\geq 2}$ and let $\mathcal{B}(\mathcal{M}_g)$ be the Borel algebra of $\mathcal{M}_g$. We define the probability measure $\mathbb{P}_{\mathrm{WP}} : \mathcal{B}(\mathcal{M}_g) \to [0,1]$ by

$$\mathbb{P}_{\mathrm{WP}}[B] = \frac{\mathrm{vol}_{\mathrm{WP}}(B)}{\mathrm{vol}_{\mathrm{WP}}(\mathcal{M}_g)},$$

for all $B \in \mathcal{B}(\mathcal{M}_g)$.

9.3 Minimal total pants length

The goal of this section is to apply the probabilistic method to study the lengths of pants decompositions. Concretely, we will present a proof, due to Guth, Parlier and Young [GPY11], that there are surfaces that do not allow short pants decompositions.

Let us start with the definition of a random variable, which we will call *minimal total pants length*, $PL : \mathcal{M}_g \to \mathbb{R}$ by

$$\mathrm{PL}(X) = \min \left\{ \sum_{i=1}^{3g-3} \ell(c_i);\ \{c_i\}_{i=1}^{3g-3} \text{ forms a pants decomposition of } X \right\}.$$

As a direct corollary of Theorem 9.4, we obtain:

Corollary 9.9. *Let $X \in \mathcal{M}_g$, then*

$$\mathrm{PL}(X) \leq 60 \cdot g^2 - 60 \cdot g.$$

The main question in this section is how sharp this upper bound is. To this end, define $\mathrm{MPL} : \mathbb{N}_{\geq 2} \to \mathbb{R}$ by

$$\mathrm{MPL}(g) = \sup \{\mathrm{PL}(X);\ X \in \mathcal{M}_g\}.$$

We will prove the following theorem due to Guth, Parlier and Young [GPY11].

Theorem 9.10. *For all $\varepsilon > 0$ we have*

$$\lim_{g\to\infty} \mathbb{P}_{\mathrm{WP}}\left[X \in \mathcal{M}_g;\ \mathrm{PL}(X) \leq g^{7/6-\varepsilon}\right] = 0.$$

Proof. The upper bound follows directly from Corollary 9.9.

The main part of the proof consists of controlling the Weil-Petersson volume of sets of the form

$$\{X \in \mathcal{M}_g;\ \mathrm{PL}(X) \leq x\}.$$

Like we noted above, $\mathcal{M}_g$ is well-defined. The projection

$$\pi_{\mathcal{P}} : \mathcal{T}_g \to \mathcal{M}_g$$

does however depend on the pants decomposition $\mathcal{P}$. Note however that if for pants decompositions $\mathcal{P}$ and $\mathcal{P}'$ there exists a diffeomorphism $\varphi : \Sigma_g \to \Sigma_g$ such that

$$\varphi(\mathcal{P}) = \mathcal{P}'$$

then

$$\pi_{\mathcal{P}}(\ell, \tau) = \pi_{\mathcal{P}'}(\ell, \tau),$$

because $X_{\mathcal{P}}(\ell, \tau)$ and $X_{\mathcal{P}'}(\ell, \tau)$ are isometric. This means that if we let $\mathcal{I}_g$ denote the (finite) set of diffeomorphism types of pants decompositions of Σ_g, we have

$$\{X \in \mathcal{M}_g;\ \mathrm{PL}(X) \leq x\} \subset \bigcup_{\mathcal{P}\in\mathcal{I}_g} \pi_{\mathcal{P}}\left(\left\{(\ell,\tau) \in \mathcal{T}_g;\quad \begin{array}{c}\displaystyle\sum_{i=1}^{3g-3} \ell_i \leq x \\ \text{and } 0 \leq \tau_i \leq \ell_i\end{array}\right\}\right).$$

Write

$$A_{g,x} = \left\{(\ell,\tau) \in \mathcal{T}_g;\ \sum_{i=1}^{3g-3} \ell_i \leq x \text{ and } 0 \leq \tau_i \leq \ell_i\right\}.$$

Our observations above imply that

$$\mathrm{vol}_{\mathrm{WP}}(\{X \in \mathcal{M}_g;\ \mathrm{PL}(X) \leq x\}) \leq |\mathcal{I}_g| \cdot \mathrm{vol}_{\mathrm{WP}}\left(A_{g,x}\right).$$

The rest of the proof consists of two steps: bounding $|\mathcal{I}_g|$ and bounding the volume of the set $A_{g,x}$.

The bound on $|\mathcal{I}|$ we need is

$$|\mathcal{I}_g| \leq a^g g^g$$

for some $a > 0$ independent of g. Proving this is Exercise 9.4.

We have

$$\begin{aligned} \mathrm{vol}_{\mathrm{WP}}(A_{g,x}) &= \int_{\sum_i \ell_i \leq x} \prod_{i=1}^{3g-3} \left(\int_0^{\ell_i} d\tau_i \right) \, d\ell_1 \cdots d\ell_{3g-3} \\ &= \int_{\sum_i \ell_i \leq x} \prod_{i=1}^{3g-3} \ell_i \, d\ell_1 \cdots d\ell_{3g-3} \end{aligned}$$

By the arithmetic-geometric mean inequality, we have

$$\prod_{i=1}^{3g-3} \ell_i \leq \left(\frac{\sum_{i=1}^{3g-3} \ell_i}{3g-3} \right)^{3g-3} \leq b^g \frac{x^{3g}}{g^{3g}}$$

for some $b > 0$ independent of g. So we obtain

$$\mathrm{vol}_{\mathrm{WP}}(A_{g,x}) \leq b^g \frac{x^{3g}}{g^{3g}} \int_{\sum_{i=1} \ell_i \leq x} d\ell_1 \cdots d\ell_{3g-3}.$$

It can be proved by induction that

$$\int_{\sum_{i=1} \ell_i \leq x} d\ell_1 \cdots d\ell_{3g-3} = \frac{x^{3g-3}}{(3g-3)!} \leq c^g \frac{x^{3g}}{g^{3g}},$$

for some $c > 0$ independent of g.

Putting all our estimates together, we obtain that

$$\mathrm{vol}_{\mathrm{WP}}(A_{g,x}) \leq d^g \frac{x^{6g}}{g^{5g}}.$$

Using Theorem 9.6, we obtain that

$$\mathbb{P}_{\mathrm{WP}} \left[X \in \mathcal{M}_g ; \, \mathrm{PL}(X) \leq x \right] \leq r^g \frac{x^{6g}}{g^{7g}},$$

for some $r > 0$ independent of g. So, if $x \leq g^{7/6-\varepsilon}$ this probability tends to 0 as $g \to \infty$ and we are done. □

As a consequence we obtain:

Corollary 9.11. *For all $\varepsilon > 0$ there exists a $g_0 = g_0(\varepsilon) \in \mathbb{N}_{\geq 2}$ so that*

$$g^{7/6-\varepsilon} \leq \mathrm{MPL}(g) \leq 60 \cdot g^2 - 60 \cdot g$$

for all $g \geq g_0$.

9.4 Random triangulations

Another model for random surfaces is obtained from random triangulations. This is a model based on the configuration model of random 3-regular graphs on $2N$ vertices. Recall that the basis for the configuration model is a collection of disjoint sets $W_i(2N)$, $i = 1, \ldots, 2N$. For convenience, we will just set

$$W_1(2N) = \{1, 2, 3\},\ W_2(2N) = \{4, 5, 6\}, \ldots,$$
$$W_{2N}(2N) = \{6N - 2, 6N - 1, 6N\}.$$

We want to assign an oriented closed surface $S(C)$ without boundary to each 3-regular configuration C on $2N$ vertices. This goes as follows. Take $2N$ triangles (2-simplices) $\Delta_1, \ldots, \Delta_{2N}$, and label the sides of the first triangle with the labels 1, 2 and 3, those of the second 4, 5 and 6 and so forth (see the figure below).

Figure 9.3: $2N$ labeled triangles.

Each of these triangles naturally comes with an orientation (induced by the cyclic order of the labels on the sides). For each pair of labels $c = \{i, j\} \in C$ fix an orientation reversing simplicial map φ_c between the corresponding sides. We set

$$S(C) = \bigsqcup_{i=1}^{2N} \Delta_i / \sim$$

where the equivalence relation is given by the collection of maps $\{\varphi_c\}_{c\in C}$. From now on we will speak of configurations on $2N$ *triangles* instead of on $2N$ vertices.

Figure 9.4 gives some examples for $N = 1$.

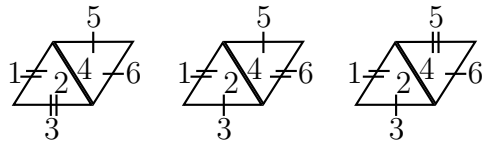

Figure 9.4: The surfaces corresponding to the configurations $\{\{1,3\},\{2,4\},\{5,6\}\}$, $\{\{1,6\},\{2,4\},\{3,5\}\}$ and $\{\{1,5\},\{2,4\},\{3,6\}\}$: a sphere, a torus and a sphere respectively.

Let us denote the set of all configurations on $2N$ triangles by Ω_N. We define a probability measure using the counting measure again:

Definition 9.12. Let $N \in \mathbb{N}$. We define the probability measure $\mathbb{P}_N : \mathcal{P}(\Omega_N) \to [0,1]$ by

$$\mathbb{P}_N[A] = \frac{|A|}{|\Omega_N|}$$

for all $A \subset \Omega_N$.

The main question we will work on in this course is the topology of these surfaces. This model can also be turned into a model for random hyperbolic surfaces, we will discuss some results on these random hyperbolic surfaces in Lecture 11.

The connectivity of a random triangulated surfaces is immediate from the connectivity of random trivalent graphs (due to Bollobás [Bol81] and Wormald [Wor81]) that we proved in Exercise 3.3:

Theorem 9.13. *We have*

$$\lim_{N\to\infty} \mathbb{P}_N[S \text{ is connected}] = 1.$$

Because of the classification of surfaces, this theorem implies that in order to understand the topology of these surfaces, the only thing that remains to be understood is the distribution of their genus, which is the content of the following lecture.

9.5 Exercises

Exercise 9.1. Prove Proposition 9.3.

Hint: for one of the directions, find a way to use Proposition 8.3 and the fact that a surjective distance preserving map between closed hyperbolic surfaces is automatically a diffeomorphism (this is a special case of what is called the Myers-Steenrod theorem).

Exercise 9.2. Which of the pants decompositions in Figure 9.1 are diffeomorphic?

Exercise 9.3. Show that the number of pants decompositions of a closed surface of genus g in which there are no two pairs of pants that share two boundary components and no curves incident to just one pair of pants is asymptotic to

$$\frac{e^{-2}(6g-6)!!}{6^{2g-2}\cdot(2g-2)!}$$

as $g \to \infty$.

Exercise 9.4. Let $\mathcal{I}_g$ denote the set of diffeomorphism classes of pants decompositions.

(a) Given a pants decomposition $\mathcal{P}$ of Σ_g, show that the number of automorphisms of the dual graph $G_{\mathcal{P}}$ can be bounded by

$$(2g-2)\cdot 6^{2g-2}.$$

Hint: suppose we know that an automorphism sends a vertex v in $G_{\mathcal{P}}$ to a vertex w, how many choices are left?

(b) Show that there exists a constant $a > 0$ so that

$$|\mathcal{I}_g| \leq a^g \cdot g^g$$

Exercise 9.5. Give an example of a sequence of configurations $(C_g)_{g=1}^{\infty}$ so that $S(C_g)$ is a connected surface of genus g for every g.

Lecture 10

Random triangulated surfaces

10.1 The Euler characteristic

In what follows we will estimate the genus of a random surface. We will follow Brooks and Makover's paper for this [BM04] ([DT06] give a similar proof).

We start with a definition. In this definition, a *triangulation* $\mathcal{T} = (V, E, F)$ of a closed surface S will be the data of a finite set of points $V = \{v_1, \ldots . v_k\} \in S$ (called *vertices*), a finite set of arcs $E = \{e_1, \ldots, e_l\}$ with endpoints in the vertices (called edges) so that the complement $S \setminus (\cup v_i \cup e_j)$ consists of a collection of disks $F = \{f_1, \ldots, f_m\}$ (called *faces*) that all connect to exactly 3 edges.

Note that a triangulation $\mathcal{T}$ here is a slightly more general notion than that of a simplicial complex (it's an example of what Hatcher calls a Δ-complex [Hat02, Page 102]). Figure 10.1 below gices an example of a triangulation of a torus that is not a simplicial complex.

We also note that the triangulated random surfaces we defined in the previous lecture (unsurprisingly) naturally come with a triangulation.

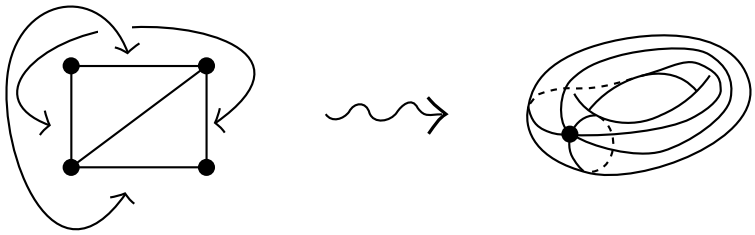

Figure 10.1: A torus with a triangulation

Definition 10.1. S be a surface with a triangulation $\mathcal{T} = (V, E, F)$. The *Euler characteristic* of S is given by

$$\chi(S) = |V| - |E| + |F|\,.$$

Because $\chi(S)$ can be defined entirely in terms of singular homology (see [Hat02, Theorem 2.4] for details), it is a homotopy invariant. In particular this implies it should only depend on the genus of our surface S. We have

Lemma 10.2. *Let S be a closed connected and oriented surface of genus g. We have*

$$\chi(S) = 2 - 2g.$$

Proof. See Exercise 10.1. □

Just like pants decompositions, triangulations also have dual graphs. The dual graph to a triangulation also comes with a cyclic order of the edges incident to every vertex. Such an order is called an *orientation*. The graph together with this order is sometimes called an oriented graph (*ribbon graph* and *fatgraph* are also terms that appear in the literature).

Definition 10.3. Let S be a closed surface and let $\mathcal{T} = \{V, E, F\}$ be a triangulation of X. The *dual graph* $G_\mathcal{T}$ to $\mathcal{T}$ is the graph obtained by setting

- $V(G_\mathcal{T}) = F$, the set of faces of $\mathcal{T}$,
- $E(G_\mathcal{T}) = E$ the set of edges of $\mathcal{T}$
- an edge in $E(G_\mathcal{T})$ is incident to a face if it is a boundary component of that face.

The orientation on $G_{\mathcal{T}}$ is induced by the orientation of S. That is, three edges incident to a face are said to be cyclicly oriented if their orientation agrees with that on the surface.

It is not hard to see that, just like the dual graph to a pants decomposition, the dual graph to a triangulation is 3-regular.

The dual graph to a triangulation can be embedded into the corresponding surface. Figure 10.2 gives an example:

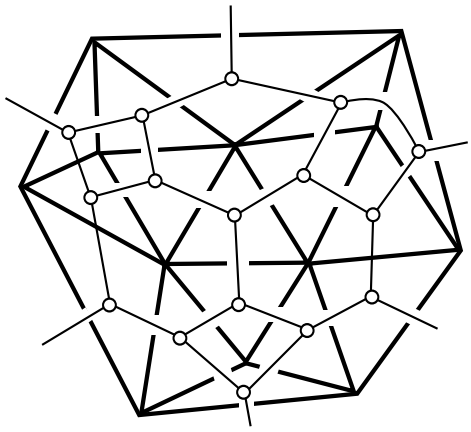

Figure 10.2: A part of a triangulation and its dual graph

Finally we note that if a 3-regular graph is oriented, we can make sense of left hand and right hand turns at a vertex. That is, if a path traverses a vertex in an oriented graph, then we say it turns left if it traverses the vertex in the direction opposite to the cyclic order and right if it traverses the vertex in the direction of the cyclic order. Figure 10.3 gives an example.

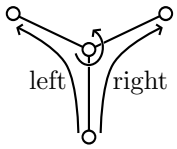

Figure 10.3: A left and a right turn at a vertex. The arrow at the middle vertex indicates the orientation at that vertex.

10.2 The genus of a random surface

Now we return to the random surfaces we associated to configurations. Because these random surfaces come with a triangulation, we can associate a dual graph $G(\omega)$ to each $\omega \in \Omega_N$. Of course, this is exactly the graph associated to ω in the configuration model for random 3-regular graphs. Moreoever the cyclic order at the vertices is by construction the cyclic order of the labels at those vertices.

Our first observation is the following:

Lemma 10.4. *Let $N \in \mathbb{N}$ and $\omega \in \Omega_N$ so that $S(\omega)$ is connected. Moreover, let $L(\omega)$ denote the number of cycles in $G(\omega)$ that consist of left hand turns exclusively. Then the genus $g(\omega)$ of $S(\omega)$ is given by*

$$g(\omega) = 1 + \frac{N}{2} - \frac{L(\omega)}{2}.$$

Proof. ω comes with a triangulation that has $2N$ faces and $3N$ edges (we start with $6N$ edges and pair them). The number of vertices $V(\omega)$ is not immediately clear. As such

$$2 - 2g(\omega) = \chi(S(\omega)) = V(\omega) + 2N - 3N = V(\omega) - N.$$

Hence

$$g(\omega) = 1 + \frac{N}{2} - \frac{V(\omega)}{2}.$$

The crucial observation is now that $V(\omega) = L(\omega)$. Indeed, the dual graph to the triangles around every vertex form a left hand turn cycle and conversely, gluing triangles along a left hand turn cycle leads to a vertex. □

We immediately conclude:

Lemma 10.5. *Let $N \in \mathbb{N}$ and $\omega \in \Omega_N$ so that $S(\omega)$ is connected. Then*

$$g(\omega) \leq \frac{N+1}{2}.$$

Proof. The number of left hand turn cycles (or equivalently the number of vertices) is at least 1. □

We actually claim that on average, the genus of a random surface is quite close to what it maximally can be. That is, it is equal to $N/2$ with a small error term. To make this precise, we have the following theorem due to Brooks and Makover [BM04]:

Theorem 10.6. *Let $N \in \mathbb{N}$. We have*

$$\mathbb{E}_N[L] \leq \frac{3}{2}\log(3N) + 3.$$

Proof. To prove this, we are going to slightly modify our probability space. Let Ω'_N be the set of ordered configurations. That is, Ω'_N contains the same configurations as Ω_N, but we make a distinction between the different order in which the pairs of labels appear in the configuration. As such

$$|\Omega'_N| = (3N)! \cdot |\Omega_N| .$$

The probability measure on Ω'_N is again just the uniform measure.

From the point of view of graphs or surfaces, the order in a configuration of course doesn't make a difference. As such, we might as well compute $\mathbb{E}_N[L]$ using Ω'_N.

The point of working with Ω'_N is that we can now speak of what the i^{th} pair of sides is that is glued together and what the graph looks like after the i^{th} step (for $i = 1, \ldots, 3N$). Figure 10.4 shows an example:

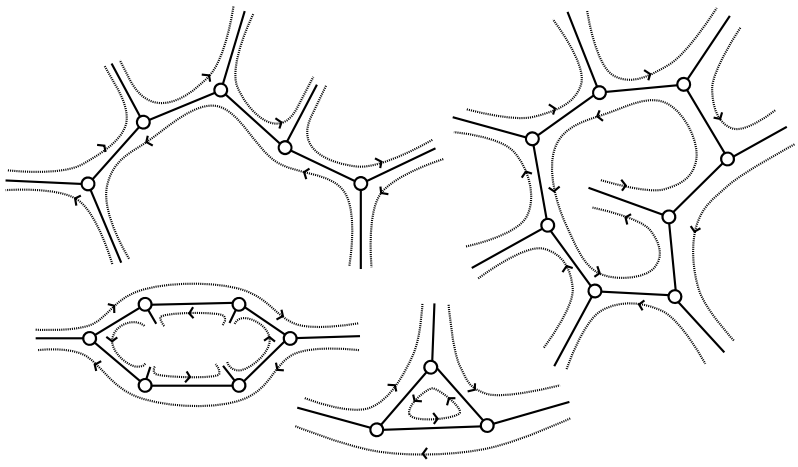

Figure 10.4: What the graph might look like after 21 steps. Half edges are added to each vertex in order to make the total degree 3.

Now let $L_i(\omega)$ be the number of left hand turn cycles created in the i^{th} step in $\omega \in \Omega'_N$. Clearly

$$L(\omega) = \sum_{i=1}^{3N} L_i(\omega)$$

and hence

$$\mathbb{E}_N[L] = \sum_{i=1}^{3N} \mathbb{E}[L_i(\omega)].$$

So, one strategy would be to try to control the distribution of $L_i(\omega)$. This turns out to be difficult as such, but a slight modification will work.

In step i we can create either 0, 1 or 2 extra left hand turn cycles. To see this, note that, when we draw all the left hand turn paths in the graph at before step i (the dotted paths in Figure 10.4), at every half-edge that is yet unpaired, one path starts and one path ends (note that these might be one and the same path, like in the component on the top right in Figure 10.4). As such, we can connect at most two pairs of paths into left hand turn cycles when pairing two edges. To see that two is possible, consider the component on the bottom left in Figure 10.4.

In the picture Figure 10.4 it is impossible to create two left hand turn cycles in the next step with most of the unpaired half edges (there is only one pair of half edges with which we can do this. Actually, with most of the unpaired half edges it is possible to create one left hand turn cycle in exactly two ways (the exceptions being one of unpaired half edges in the top right component and the two unpaired half edges in the bottom left component).

These special cases turn out to be the main issue in the proof. As such, let us give them a name. We will call an unpaired half edge e a *bottleneck* if the left hand turn path ending at it and the left hand turn path starting at it both connect it to the same half edge e' (we have already seen that the case $e = e'$ is possible).

Now, let $B_i(\omega)$ denote the number of bottle necks that are created at step i. If we want to create a left hand turn cycle with a bottle neck, we clearly need to pair it with another bottle neck. Hence

$$L(\omega) \leq \sum_{i=1}^{3N} L_i^*(\omega) + \frac{1}{2} \sum_{i=1}^{3N} B_i(\omega),$$

where $L_i^*(\omega)$ denotes the number of left hand turn cycles that are created without using bottle necks. The inequality here comes from the fact that at some point, a bottle neck may be destroyed again.

Now note that before the i^{th} step, there are $6N - 2i + 2$ half edges left to form the i^{th} pair with. We have the following cases

1. If the first edge that is chosen for the pair is not a bottleneck, then by the arguments above, we have that
$$\mathbb{E}[L_i^*] = \frac{2}{6N - 2i + 1}.$$
By a similar argument, there are at most two half edges with which we may create a bottleneck, hence
$$\mathbb{E}[B_i] \leq \frac{2}{6N - 2i + 1}.$$
2. If the first edge that is chosen is a bottleneck, then
$$\mathbb{E}[L_i^*] = \mathbb{E}[B_i] = 0.$$

So in either case, we have
$$\mathbb{E}[L_i^*] \leq \frac{2}{6N - 2i + 1} \text{ and } \mathbb{E}[B_i] \leq \frac{2}{6N - 2i + 1}.$$
Hence
$$L(\omega) \leq \frac{3}{2} \sum_{i=1}^{3N} \frac{2}{6N - 2i + 1} = \frac{3}{2} \sum_{i=1}^{3N} \frac{1}{(3N - i) + 1/2}.$$
Now we use that
$$\frac{1}{x + 1/2} \leq \log(x + 1) - \log(x)$$
for all $x \geq 1$ and we are done. □

We obtain:

Corollary 10.7. *Let $N \in \mathbb{N}$. We have*
$$\frac{N}{2} - \frac{3}{4}(\log(3N) + 2) \leq \mathbb{E}_N[g] \leq \frac{N}{2} + \frac{1}{2}.$$

Proof. The upper bound is direct from Lemma 10.5. The lower bound comes from putting together Lemma 10.4 and Theorem 10.6. □

10.3 Exercises

Exercise 10.1. In this exercise we prove Lemma 10.2 in two different ways.

1. (a) Describe a way to obtain a closed oriented surface of genus g from a polygon with $4g$ sides.

   (b) Use a triangulation of a $4g$-gon to prove Lemma 10.2.

2. (a) Show that if a closed oriented surface S is the connect sum of two closed oriented surfaces S_1 and S_2 then

$$\chi(S) = \chi(S_1) + \chi(S_2) - 2.$$

   (b) Compute the euler characteristic of the 2-sphere and the torus and use those to prove Lemma 10.2.

Exercise 10.2. Show that for a random surface in the configuration model we have

$$\mathbb{P}_N \left[g \leq \frac{N}{2} - x \right] \leq \frac{3\log(3N) + 6}{4x + 4}$$

Lecture 11

An outlook

In this last lecture we will discuss some more results on random hyperbolic geometry. Because this is intended as an outlook, we will only sketch the proofs of these results.

11.1 More on the genus of random triangulated surfaces

A lot more can be said about the asymptotic distribution of the genus of a random surface. In this section we discuss results due to Gamburd [Gam06] on this distribution.

To understand these results, we need an alternatrive description of the configuration model. This description uses permutations. It should however be stressed that the resulting model is *not* the permutation model we saw in Section 5.2.1.

The idea is as follows. First of all the orientation (the cyclic order of the labelled sides) of the triangles we start with can be captured in a permutation. This permutation consists of a product of three-cycles, one corresponding to each triangle. Figure 11.1 shows the idea:

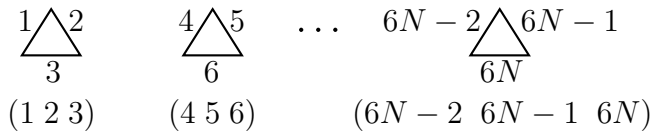

Figure 11.1: $2N$ labeled triangles.

This leads to a permutation

$$\sigma = (1\ 2\ 3)(4\ 5\ 6)\cdots(6N-2\ \ 6N-1\ \ 6N) \in \mathfrak{S}_{6N}\,.$$

Likewise, the configuration $C \in \Omega_N$ itself can also be recorded in a permutation. We simply write down a two-cycle $(a_i\ b_i)$ for each pair $\{a_i, b_i\} \in C$ and concatenate all these (disjoint) two-cycles. This leads to another permutation

$$\tau = (a_1\ b_1)(a_2\ b_2)\cdots(a_{3N}\ b_{3N}) \in \mathfrak{S}_{6N}$$

The cycle type of τ (the fact that τ has exactly $3N$ two-cycles and no cycles of any other length) determines a conjugacy class in the symmetric group $\mathfrak{S}_{6N}$. As such, Ω_N can be identified with a conjugacy class, which we shall denote by $K\left(2^{3N}\right) \subset \mathfrak{S}_{6N}$.

Let us denote the conjugacy class of σ by $K\left(3^{2N}\right) \subset \mathfrak{S}_{6N}$. In our model for random surfaces, σ is fixed. We could of course also randomly pick it in $K\left(3^{2N}\right)$. This would just come down to a random relabeling of the triangles and as such wouldn't change the probabilities of any graph theoretic or topological property. This leads to a probability space

$$\Omega''_N = K\left(3^{2N}\right) \times K\left(2^{3N}\right)$$

endowed with the uniform probability measure. Let us denote the surface corresponding to $(\sigma, \tau) \in \Omega''_N$ by $S(\sigma, \tau)$ and the corresponding triangulation by $\mathcal{T}(\sigma, \tau)$.

So far, this description using the symmetric group might sound a little artificial. The crux however is that the number of vertices of the triangulation $\mathcal{T}(\sigma, \tau)$ (the only part of the Euler characteristic that does not come directly from the set up) can be expressed in terms of a permutation. Indeed, we

claim that the number of vertices of $\mathcal{T}(\sigma, \tau)$ is exactly the number of cycles in a disjoint cycle decomposition of the permutation

$$\sigma\tau \in \mathfrak{S}_{6N} .$$

To see this, note that the permutation $\sigma\tau$ describes 'traversing the side of a triangle and then turning left'. Indeed, if σ is applied to a label l, then the label that comes out is exactly the label to the left of it on the same triangle. Likewise, if we apply τ, we obtain the label on the opposite side of the edge l represents. As such, the cycles in $\sigma\tau$ correspond one to one to left hand turn cycles in $G_{\mathcal{T}(\sigma,\tau)}$ and hence to vertices (like we've seen in the proof of Lemma 10.4).

Before we state Gamburd's result, let us note that when N is even, both σ and τ and hence also $\sigma\tau$ are even permutations. In other words, when N is even $\sigma\tau$ lies in the alternating group $\mathfrak{A}_{6N}$ on $6N$ letters. This means that we obtain a probability measure $\mathbb{P}_{\sigma\tau,N} : \mathcal{P}(\mathfrak{A}_{6N}) \to [0, 1]$ by

$$\mathbb{P}_{\sigma\tau,N}[A] = \frac{|\{(\sigma, \tau) \in \Omega''_N;\ \sigma\tau \in A\}|}{|\Omega''_N|}.$$

Let $\mathbb{U}_{\mathfrak{A}_{6N}}$ denote the uniform probability measure on the alternating group $\mathfrak{A}_{6N}$. In [Gam06], Gamburd proves:

Theorem 11.1. *We have*

$$\mathrm{d_{TV}}\left(\mathbb{P}_{\sigma\tau,N}, \mathbb{U}_{\mathfrak{A}_{6N}}\right) \to 0$$

as $N \to \infty$ over the even numbers.

The proof is based on a result called the Diaconis-Shahshahani upper bound lemma [DS81]. This result bounds the total variational distance between a probability measure on a finite group and the uniform measure on that group in terms of the Fourier transform of that probability measure (an object defined in terms of the irreducible representations of the given group). As such the proof of Gamburd's theorem goes through finding bounds on these Fourier transforms.

A consequence of the theorem above is that the number of vertices of a large random surface can be compared to the number of disjoint cycles in a permutation that is chosen uniformly at random in a large alternating group. This forms the basis for for example the following Corollary, also due to Gamburd:

Corollary 11.2. *Let $V_N : \Omega_N'' \to \mathbb{N}$ be the random variable that counts the number of vertices in $\mathcal{T}(\sigma, \tau)$. Moreover, let $W_N : \Omega \to \mathbb{R}$ denote a random variable that is normally distributed with mean $\log(N)$ and standard deviation $\sqrt{\log(N)}$. Then*

$$\mathrm{d_{TV}}\left(V_N, W_N\right) \to 0$$

as $N \to \infty$ over the even numbers.

11.2 The geometry of random triangulated surfaces

We have now seen that the configuration model gives rise to a model for random connected and oriented closed surfaces of large genus. In [BM04], Brooks and Makover used this a priori topological model to study random hyperbolic surfaces of large genus. This goes through hyperbolic triangles, which we will introduce first.

11.2.1 Hyperbolic triangles

Hyperbolic triangles can be defined in a similar way to how we defined right angled hexagons. Another option is to use convex hulls. Recall that a subset $C \subset \mathbb{H}^2$ is called convex if the geodesic segment between x and y lies in C for all $x, y \in C$. Given a subset $A \subset \mathbb{H}^2$, its *convex hull* is the set

$$\mathrm{conv}(A) := \bigcap_{\substack{C \subset \mathbb{H}^2 \\ C \text{ convex}}} C.$$

In Lecture 7 we have seen that every pair of distinct points $x, y \in \mathbb{H}^2 \cup \mathbb{R} \cup \{\infty\}$ define a unique geodesic γ_{xy}. Let us denote the segment on γ_{xy} between x and y with $[x, y]$ (note that this might be the whole geodesic γ). Now let $x, y, z \in \mathbb{H}^2 \cup \mathbb{R} \cup \{\infty\}$ be distinct points. We will call

$$T(x, y, z) = \mathrm{conv}\left(([x, y] \cup [y, z] \cup [z, x]) \cap \mathbb{H}^2\right)$$

a triangle. Figure 11.2 gives an example:

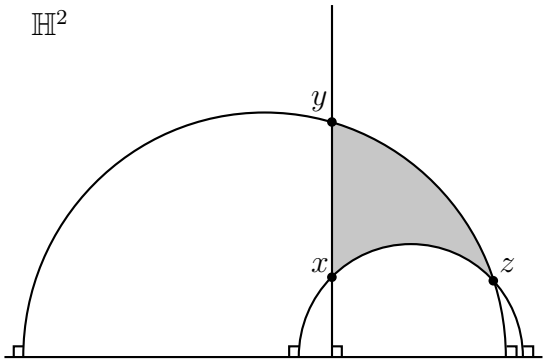

Figure 11.2: A hyperbolic triangle.

Note that (by definition) if one of the vertices x, y, z lies in $\mathbb{R} \cup \{\infty\}$, it is not part of the triangle. Such a vertex will be called an *ideal* vertex. If all three vertices are ideal, we will speak of an *ideal triangle*. Figure 11.3 contains multiple ideal triangles, all of which have ∞ as a vertex. Note that ideal triangles necessarily have angle 0 at each vertex,

We start with a uniqueness lemma:

Lemma 11.3. *Let T be a triangle with vertex angles $\alpha, \beta, \gamma \in [0, \pi/2]$.*

(a) Then the hyperbolic area of T is given by

$$\text{area}(T) = \pi - \alpha - \beta - \gamma.$$

(b) If T' is any other triangle with the same vertex angles, then there exists an isometry $A : \mathbb{H}^2 \to \mathbb{H}^2$ so that

$$A(T') = T.$$

The proof of (b) is similar to the proof we did for right angled hexagons, the proof of (a) can for instance be found in [Bea95, Chapter 7].

Corollary 11.4. *The area of a closed hyperbolic surface S of genus g is given by*

$$\text{area}(S) = 4\pi(g - 1).$$

Proof. Choose any triangulation $\mathcal{T} = (V, E, F)$ of the surface. By Lemma 10.2 we have

$$2 - 2g = |V| - |E| + |F| .$$

Because every face borders three edges and every edge has two sides, we get

$$|E| = \frac{3}{2} |F|$$

and hence

$$2 - 2g = |V| - \frac{1}{2} |F| .$$

On the other hand, if we denote the angles of a triangle $f \in F$ by $\alpha_1(f)$, $\alpha_2(f)$ and $\alpha_3(f)$ respectively, then Lemma 11.3 tells us that

$$\text{area}(S) = \sum_{f \in F} \pi - \alpha_1(f) - \alpha_2(f) - \alpha_3(f) = \pi |F| - \sum_{f \in F} \alpha_1(f) - \alpha_2(f) - \alpha_3(f).$$

We can replace the sum on the right hand side by a sum over the vertices. At each vertex the angles need to add up to 2π, so

$$\text{area}(S) = \pi |F| - 2\pi |V| = 2\pi \left(\frac{1}{2} |F| - |V| \right) = 4\pi(g - 1).$$

□

11.2.2 Random hyperbolic surfaces

The idea of the model now is as follows:

1. Glue ideal hyperbolic triangles according to the configuration. The result of this will be a hyperbolic surface with punctures (coming from the missing vertices of ideal triangles).
2. Compactify the surface, from which we obtain (generically) a closed hyperbolic surface.

Let us elaborate a little bit on how both steps work, starting with step 1. First of all, we will of course glue the ideal hyperbolic triangles together with isometries so that the hyperbolic metric on them descends. However,

because their sides have infinite length there is not just one isometry between a pair of sides. Figure 11.3 illustrates this issue with two gluings: $T(0,1,\infty)$ with $T(1,2,\infty)$ and $T(0,1,\infty)$ with $T(1,x,\infty)$.

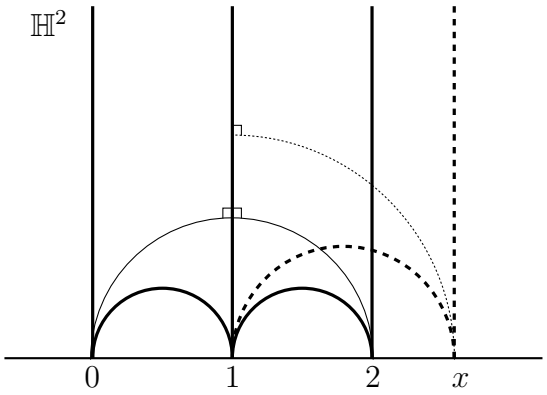

Figure 11.3: Shear.

Just like with pair of pants, we need one pair of points that are identified to determine the isometry between a pair of sides entirely. A natural candidate for this pair of points is constructed as follows. On both triangles involved in the gluing, take the vertex that is not part of the side in question and drop the unique perpendicular from the vertex to the side (in the figure above, these are the dotted lines). This defines two points on the side.

In a general gluing, the distance between the two special points will be called the *shear* of the gluing. We will always set the shear to 0 in our gluings. Together with this information, a configuration $C \in \Omega_N$ now specifies a hyperbolic surface $S_o(C)$ with punctures.

We will only sketch step 2 (for more details see [Bro99, BM04]), which if the genus $g(C)$ of $S_o(C)$ is at least 2 produces a closed hyperbolic surface $S_c(C)$. It relies on the fact that isometry classes of complete hyperbolic metrics on a finite type surface (of negative Euler characteristic) correspond one-to-one to complex structures up to biholomorphism (this is called the uniformization theorem).

The idea is that around every puncture we can find a region that is isometric/biholomorphic to

$$C_t = \left\{ z \in \mathbb{H}^2;\ \Im(z) > t \right\} / (z \mapsto z+1)$$

for some $t > 0$. This is not so hard to see from our construction. From this region we can find a biholomorphic map to the punctured unit disk

$$\{z \in \mathbb{C};\ 0 < |z| < 1\}.$$

Adding the point $z = 0$ in for every region gives us a closed surface with a complex structure on it. As such, the uniformization theorem gives us a hyperbolic metric when the genus is at least 2 (In Exercise 10.2 we have seen that the genus is almost never less than 2, so for all practical purposes, this is no restriction). We will denote the hyperbolic surface we obtain by $S_c(C)$.

The reason that this leads to an interesting model for random closed hyperbolic surfaces is the following theorem due to Belyĭ [Bel79]:

Theorem 11.5. *The inclusion*

$$\left(\bigcup_{N=1}^{\infty} \{S_c(C);\ C \in \Omega_N\} \bigcap \mathcal{M}_g\right) \subset \mathcal{M}_g$$

is dense for every $g \geq 2$.

Belyĭ's theorem is actually a theorem about when an algebraic curve over the complex numbers can be written as a curve over $\overline{\mathbb{Q}}$. The statement above relies on the identification of these curves with hyperbolic surfaces (see [JS96] for more information). It should also be noted that the analogous statement for the surfaces $S_o(C)$ is false: Lemma 10.4 shows that for every pair $(g, n) \in \mathbb{N}^2$ we only obtain finitely many surfaces of genus g with n punctures, whereas a similar construction to that in Section 9.2 shows that there are uncountably many hyperbolic surfaces of genus g and with n punctures.

11.2.3 Geometric properties

Let us now discuss some of the results that are known about the geometry of random triangulated surfaces. Of course we can ask questions about both the geometry of the closed and the punctured surfaces. However, in view of Theorem 11.5, the closed surfaces are a more interesting set of surfaces.

On the other hand, the punctured surface are the surfaces over which we have more geometric control. Indeed, by construction of these surfaces,

geometric questions about them can in general easily be translated into combinatorial questions about the triangulation. This is not the case for their closed counterparts. We obtain those by applying the uniformization theorem, which is an existence theorem that gives no explicit control over the hyperbolic metric that comes out.

As such, the first thing we need is a statement that compares the geometry of the punctured surface and its compactification. In [Bro99], Brooks proves such a statement. What he proves is that if the punctures on $S_o(C)$ are far enough apart and also do not contain any short curves in their neighborhood, then, outside the punctures and their images, the metric on $S_c(C)$ is close to that on $S_o(C)$. In [BM04], Brooks and Makover show that these conditions are satisfied with probability tending to 1 as $N \to \infty$. This allows one to prove statements about the geometry of the surfaces S_o and then transport them to statements about the geometry of the surfaces S_c.

Let us now define some random variables that we can investigate. First of all, note that we already know the distribution of the area of our surfaces. By Lemma 11.3 the area of S_o is just $2\pi \cdot N$ and Corollary 11.4 tells us that the distribution of $\text{area}(S_c)$ is determined by that of the genus.

Analogously to the case of graphs, we can ask about expansion like properties. That is, we can define a Cheeger constant. Given a connected hyperbolic surface S, define its *Cheeger constant* by:

$$h(S) = \inf\left\{ \frac{\ell(\partial A)}{\text{area}(A)};\ A \subset S \text{ a submanifold so that } \text{area}(A) \leq \text{area}(S)/2 \right\}.$$

Just like in the case of graphs we can also define a *Laplace* operator Δ (using the hyperbolic metric on the surface). If S is compact, then Δ has a discrete spectrum and it's lowest eigenvalue (corresponding to constant functions) is 0. Let us denote its first non-zero eigenvalue by $\lambda_1(S)$. Cheeger's inequality states that

$$\frac{h(S)^2}{4} \leq \lambda_1(S)$$

for all closed hyperbolic surfaces S. The proof, which uses a variational characterization of $\lambda_1(S)$, of this inequality is very similar to the proof in the case of graphs. Similarly to the case of graphs, there also exists an upper bound on $\lambda_1(S)$ in terms of $h(S)$, this is due to Buser [Bus82].

Other properties we can ask for are the lengths of curves. Let us define the *systole* of a closed hyperbolic surface to be the length of the shortest closed geodesic on the surface (see Figure 11.4 for an example).

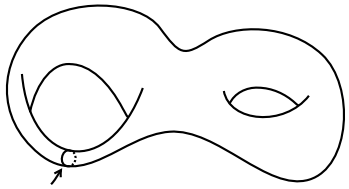

Figure 11.4: A curve that realizes the systole.

Finally, the diameter of a closed hyperbolic surface S is given by

$$\operatorname{diam}(S) = \max_{x,y \in S} \{\mathrm{d}(x, y)\}.$$

We could of course also define this quantity punctured hyperbolic surfaces. However, because it is always infinite, it wouldn't be a very interesting random variable to study.

In [BM04], Brooks and Makover prove the following theorem:

Theorem 11.6. *There exist positive constants C_1, C_2, C_3 and C_4 such that:*

(a) The first eigenvalue $\lambda_1(S_c(C))$ satisfies

$$\mathbb{P}_N[\lambda_1(S_c) \geq C_1] \to 1.$$

(b) The Cheeger constant $h(S_c(C))$ satisfies

$$\mathbb{P}_N[h(S_c) \geq C_2] \to 1.$$

(c) The shortest geodesic $\operatorname{sys}(S_c(C))$ satisfies

$$\mathbb{P}_N[\operatorname{sys}(S_c) \geq C_3] \to 1.$$

(d) The diameter $\operatorname{diam}(S_c(C))$ satisfies

$$\mathbb{P}_N[\operatorname{diam}(S_c) \leq C_4 \log(g)] \to 1.$$

First of all, (a) and (d) follow from (b). Indeed, (a) is just the application of Cheeger's inequality. (d) follows from (b) through an inequality that, both in statement and in proof, is very similar to the relation between the diameter and the Cheeger constant we proved in Exercise 6.1 (see for instance [Bro88, Bro86]).

The proof of point (b) relies on a comparison between the Cheeger constant of the graph and that of S_c due to Brooks [Bro88, Bro86]. Since we know that random 3-regular graphs are expanders (using that the permutation model is contiguous to the configuration model [Wor99, Section 4], this follows from Theorem 6.4), point (b) follows.

Point (c) is a consequence of the fact that the systole on the surfaces $S_o(C)$ is uniformly bounded from below. Using Brooks and Makover's comparison results of the metrics on S_o and S_c, the bound follows.

We also note that the diameter of a hyperbolic surface of genus g is at least logarithmic in g. This follows from the fact that a disk of radius $\mathrm{diam}(X)$ covers the whole surface X. Because the hyperbolic area of a disk is at most exponential in its radius and the area of a surface is linear in g (Corollary 11.4), we get a logarithmic bound. On the other hand, it is not hard to construct hyperbolic surface of a fixed genus with arbitrarily large diameter using a cleverly chosen pants decomposition with short pants curves. As such point (d) states that the diameter of a random surface is rather small.

Finally, we point out that Poisson aproximation results similar to Theorem 3.7, now about the number of closed geodesics up to a given length, are also available (see [Pet17, PT17] for details).

11.3 More on the geometry of Weil-Petersson random surfaces

In [Mir13], Mirzakhani proves several results about the geometry of large genus hyperbolic surfaces that are chosen at random using the Weil-Petersson volume form (like those in Section 9.2). The proof of these are in turn based on deep results on Weil-Petersson volumes that are also due to Mirzakhani [Mir07a, Mir07b].

Before we state some of the results, we need one more definition, namely that of a separating curve. A closed curve $\gamma \subset S$ on a surface S is called *separating* if $S \smallsetminus \gamma$ consists of multiple connected components. Figure 11.5 shows three examples of separating curves on a surface of genus 3.

The *separating systole* $\mathrm{sys}_{\mathrm{sep}}(S)$ of a closed hyperbolic surface S is the length of the shortest separating closed geodesic on that surface.

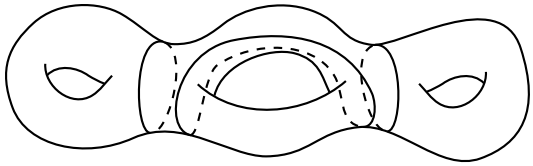

Figure 11.5: Three separating curves.

Mirzakhani [Mir13] proved the following:

Theorem 11.7. *(a) There exist genus-independent constants $c_1, c_2 > 0$ so that*

$$c_1 \cdot \varepsilon^2 \leq \mathbb{P}_{\mathrm{WP}}[X \in \mathcal{M}_g; \mathrm{sys}(X) < \varepsilon] \leq c_2 \cdot \varepsilon^2$$

for all $g \geq 2$.

(b) There exist genus-independent constants $c_3, c_4 > 0$ so that

$$c_3 \cdot \log(g) \leq \mathbb{E}_{\mathrm{WP}}[\mathrm{sys}_{\mathrm{sep}}(X); X \in \mathcal{M}_g] \leq c_4 \cdot \log(g)$$

for all $g \geq 2$.

(c) There exists a genus-independent constant $c_5 > 0$ so that

$$\mathbb{P}[X \in \mathcal{M}_g; h(X) \geq c_5] \to 1$$

as $g \to \infty$.

(d) There exists a genus-independent constant $c_6 > 0$ so that

$$\mathbb{E}_{\mathrm{WP}}[\mathrm{diam}(X); X \in \mathcal{M}_g] \leq c_6 \log(g)$$

for all $g \geq 2$.

It should be noted that these results are pretty similar to those from the combinatorial model. There currently is no a priori reason known for this fact.

Item (a) can also be interpreted as a geometric statement about $\mathcal{M}_g$. A classical theorem due to Mumford [Mum71] states that the set

$$\{X \in \mathcal{M}_g;\ \mathrm{sys}(X) \geq \varepsilon\} \subset \mathcal{M}_g$$

is compact for every $\varepsilon \geq 0$. This implies that Weil-Petersson concentrates on large compact subsets of $\mathcal{M}_g$. In fact, there exists a compactification of $\mathcal{M}_g$ called the Deligne-Mumford compactification, which coincides with the metric completion of $\mathcal{M}_g$ with respect to the Weil-Petersson metric. This completion can be thought of as attaching a boundary consisting of degenerate (with systole 0) surfaces to $\mathcal{M}_g$. In these terms item (a) gives us a decay rate of the Weil-Petersson volume as we move out to this boundary.

11.4 Random 3-manifolds

11.4.1 Generalizing the models for 2-manifolds

The first thing that one might want to try after having seen random surfaces is to generalize either of the two models to three-manifolds. We will start by explaining why this doesn't work.

The model based on Weil-Petersson volumes uses moduli spaces of hyperbolic metrics. This approach relies on two things. First of all it makes use of the classification of surfaces, one reason the model is interesting is that it says something about *all* hyperbolic surfaces. In dimension three there is no such thing as a classification of closed hyperbolic 3-manifolds (3-dimensional manifolds that locally look like hyperbolic 3-space). The second problem is Mostow's rigidity theorem [Mos68], which implies that any pair of hyperbolic metrics on a given closed 3-manifold is isometric. As such, the can never be an interesting deformation space of hyperbolic metrics (i.e. an analogue of $\mathcal{M}_g$), even if we were to content ourselves with random metrics on a given manifold.

The natural 3-dimensional analogue of Brooks and Makover's model is that of randomly gluing tetrahedra (3-simplices) along their faces. Since a

tetrahedron has 4 faces, we would use the configuration model for random 4-valent graphs for this. Once we have determined which face glues to which (the data of a configuration) we have (up to homeomorphism) three choices per glued pair of faces how they are to be glued if we want the resulting complex to be oriented. Let us denote the resulting probability space of complexes with n tetrahedra by Ω_n^3, equipped with a probability measure $\mathbb{P}_n$.

By doing this, we obtain a 3-dimensional Δ-complex (again, this is a very mild generalization of a simplicial complex) that automatically comes with a dual graph (see Figure 11.6 for how the dual graph looks in a single tetrahedron).

It is not immediately clear that the resulting complex is homeomorphic to a manifold. What we need to check for this is that every point in the complex has a neighborhood that is homeomorphic to $\mathbb{R}^3$. It is not hard to see that this is indeed the case for every point that is not a vertex in the complex. In [DT06], Dunfield and Thurston observed that at the vertices such a neighborhood can typically not be found:

Proposition 11.8. *We have*

$$\mathbb{P}_n[\textit{The resulting complex is a manifold}] \to 0$$

as $n \to \infty$.

Proof. Consider the following figure:

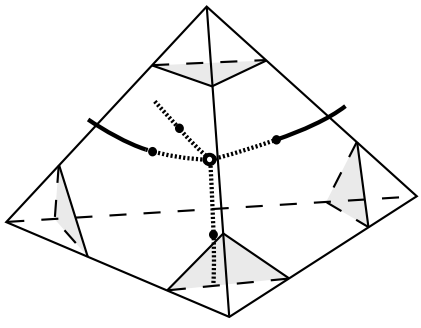

Figure 11.6: A tetrahedron with a bit of dual graph and triangles at each vertex.

It represents a tetrahedron in which we have drawn a triangle over every vertex. If a gluing of such tetrahedra is to be homeomorphic to a manifold, the triangles at all the vertices need to glue into spheres. These spheres naturally come with a triangulation.

Our first observation is that the average degree of a vertex in a triangulation of a sphere is at most 6. Indeed, given a triangulation $\mathcal{T} = (V, E, F)$ of a sphere, this average vertex degree is given by:

$$\frac{1}{|V|} \sum_{v \in V} \deg(v) = \frac{2\,|E|}{|V|}.$$

Moreoever, becase $\mathcal{T} = (V, E, F)$ is a triangulation, we have $|F| = \frac{2}{3}\,|E|$, hence

$$2 = |V| - |E| + |F| = |V| - \frac{1}{3}\,|E|\,.$$

So the average vertex degree is

$$\frac{2(3\,|V| - 2)}{|V|} \leq 6.$$

By Markov's inequality (Lemma 1.5), we get that at least $|V|/2$ vertices have degree less than 13. We also see from the fact that $|V| - \frac{1}{3}\,|E| = 2$ and the fact that the total number of edges in these spheres is equal to the total number of faces in our cell complex, which is $6n$. So we have at least $3n$ vertices with valence less than 13.

Now consider Figure 11.6 again. It shows that the vertices in our triangulated spheres correspond 2 to 1 to edges in the cell complex. As such, the cell complex contains at least $3n$ edges with at most 12 tetrahedra around them.

Every edge gives rise to a cycle in the dual graph. This means that the dual graph has at least $3n$ cycles of length at most 12. We know from Theorem 3.7 that the expected number of cycles of length at most 12 in a random 4-regular graph converges to a constant. Markov's inequality then implies that the probability that such a graph has at least $3n$ such cycles is $\mathcal{O}\left(n^{-1}\right)$ as $n \to \infty$. □

This proposition is bad news for the model based on triangulations. It shows that the set of manifolds is a subset that is asymptotically negligable

with respect to the measure $\mathbb{P}_n$. Since it is very hard in general to say something sensible about such sets, this renders the model unworkable. Of course one might try to modify the model so that a larger measure is assigned to the set of manifolds. So far no such model has been found. One obvious obstacle to this is that many of the well known models are contiguous [Wor99, Section 4].

11.4.2 Heegaard splittings

In [DT06], Dunfield and Thurston developed a model for random 3-manifolds based on so-called Heegaard splittings.

A *Handlebody* is a 3-manifold with boundary that can be obtained as the regular neighborhood of a graph in $\mathbb{R}^3$, like in Figure 11.7:

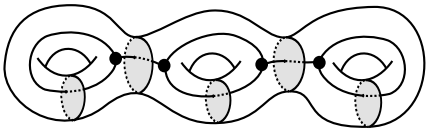

Figure 11.7: A handlebody.

Given a handlebody H, its boundary ∂H is necessarily homeomorphic to a closed surface. The *genus* of a handlebody will be the genus of that surface. Any pair of handlebodies of the same genus is homeomorphic. Given two handlebodies H_1 and H_2 of the same genus and an orientation reversing homeomorphism

$$\varphi : \partial H_1 \to \partial H_2$$

we can define a closed 3-manifold M_φ by

$$M_\varphi = (H_1 \sqcup H_2) / \sim$$

where $x \sim \varphi(x)$ for all $x \in \partial H_1$. Such a decomposition of a manifold into two handlebodies is called a *Heegaard splitting*. If we reverse the orientation on one of the handlebodies, we may also assume the homeomorphism to be orientation preserving, which is what we will do.

Initially, Heegaard splittings might seem like a very special way of building a closed 3-manifold. It however turns out that every closed 3-manifold can be obtained like this:

Lemma 11.9. *Every closed oriented* 3*-manifold admits a Heegaard splitting.*

Proof sketch. This relies on triangulations. A non-trivial fact due to Moise [Moi52] that every closed 3-manifold is homeomorphic to a simplicial complex. Thickening the 1-skeleton and the dual graph leads to two handlebodies that intersect in a closed surface and hence to a Heegaard splitting. □

Note however, that we only obtain all 3-manifolds if we allow for handlebodies of arbitrarily large genus.

It is not hard to see that if two homeomorphisms $\varphi_1, \varphi_2 : \partial H_1 \to \partial H_2$ are isotopic than M_{φ_1} and M_{φ_2} are homeomorphic. This allows us to consider the homeomorphism as an element of the so-called *mapping class group* of Σ_g:

$$\mathrm{MCG}(\Sigma_g) = \{\varphi : \Sigma_g \to \Sigma_g;\ \varphi \text{ an orientation preserving homeomorphism}\} / \sim$$

where the equivalence relation is given by isotopy. It turns out that $\mathrm{MCG}(\Sigma_g)$ is a finitely generated group. So given a finite generating set $S \subset \mathrm{MCG}(\Sigma_g)$, we can define a random walk $(\varphi_n)_n$ on $\mathrm{MCG}(\Sigma_g)$, where

$$\varphi_0 = e,$$

the unit in $\mathrm{MCG}(\Sigma_g)$ and φ_{n+1} is obtained from φ_n by randomly selecting an element in S and concatenating with that element.

A *random* 3*-manifold* is now just a manifold M_{φ_n} obtained from a random walk $(\varphi_n)_n$ on $\mathrm{MCG}(\Sigma_g)$.

Dunfield and Thurston introduced this model to study the so-called virtual Haken conjecture, which at that point had not yet been solved. We will give a different sample result. In this result, a 3-manifold is called *hyperbolic* if it admits a hyperbolic metric. If it does, then Mostow's rigidity theorem tells us that this metric is unique. Maher [Mah10] proved the following:

Theorem 11.10. *Let $(\varphi_n)_n$ be a random walk on $\mathrm{MCG}(\Sigma_g)$ for some $g \geq 2$ defined by some finite generating set S. Then*

$$\mathbb{P}[M_{\varphi_n} \text{ is hyperbolic}] \to 1$$

as $n \to \infty$.

Maher actually proves that a distance called the *Hempel distance* (which is related to the action of $\mathrm{MCG}(\Sigma_g)$ on sets of closed curves on Σ_g) grows linearly in the length of the walk. A theorem by Hempel [Hem01] in combination with Perelman's geometrization theorem then implies that the resulting manifold is hyperbolic.

11.5 Random higher-dimensional (hyperbolic) manifolds

There are also models around for higher (≥ 4) dimensional hyperbolic manifolds. First of all, in dimension n at least 4, the number of hyperbolic n-manifolds of volume $\leq v$ is finite for every $v \in \mathbb{R}$, this is a classical theorem due to Wang [Wan72]. The growth of this number and has been quantified by Raimbault [Rai13] and Gelander-Levit [GL14]. This leads to a natural probability space for higher dimensional manifolds.

Another model is that of Invariant Random Subgroups (IRSs), these are conjugation invariant probability measures on spaces of subgroups of a fixed Lie group, like for instance the group of orientation preserving isometries of hyperbolic n-space. When such a measure is supported on discrete torsion-free subgroups, it gives a model for random manifolds (see [ABB$^+$17] for more information).

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