Problem set 1: Reminder on Riemann surfaces

Exercise 1 (Group actions).

(a) Suppose $D \subset \mathbb{P}^1(\mathbb{C})$ is a domain and let $G < \mathrm{Aut}(\mathbb{P}^1(\mathbb{C})) = \mathrm{PGL}(2, \mathbb{C})$ be such that

- $g(D) = D$ for all $g \in G$
- if $g \in G \setminus \{e\}$ then the fixed points of g lie outside of D, i.e. the action on D is *free*,
- for each compact subset $K \subset D$, the set

$$\{g \in G;\ g(K) \cap K \neq \emptyset\}$$

is finite. That is, the action on D is *properly discontinuous*.

Show that the quotient space $G \backslash D$ can be equipped with the structure of a connected Riemann surface (using local inverses to the projection map $\pi : D \to G \backslash D$ as charts).

(b) Show that $G < \mathrm{PGL}(2, \mathbb{C})$ is discrete (with respect to the induced topology) if and only if the identity element $e \in G$ is isolated.

Exercise 2 (Hyperbolic geometry). Let $\mathbb{H}^2 = \{z \in \mathbb{C};\ \mathrm{Im}(z) > 0\}$.

(a) Show that

$$\{g \in \mathrm{PGL}(2, \mathbb{C});\ g(\mathbb{H}^2) = \mathbb{H}^2\} = \mathrm{PSL}(2, \mathbb{R}).$$

Remark: $\mathrm{PGL}(2, \mathbb{C}) = \mathrm{PSL}(2, \mathbb{C})$ so we can freely switch between the two. However $\mathrm{PSL}(2, \mathbb{R}) < \mathrm{PGL}(2, \mathbb{R})$ is a proper subgroup, the latter contains $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, whereas the former does not.

(b) We're going to prove the Iwasawa decomposition for $\mathrm{SL}(2, \mathbb{R})$. Namely, if we set

$$K = \mathrm{SO}(2, \mathbb{R}), \quad A = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}; \lambda > 0 \right\} \quad \text{and} \quad N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix};\ t \in \mathbb{R} \right\},$$

then $\mathrm{SL}(2, \mathbb{R}) = KAN$, i.e. for every matrix $g \in \mathrm{SL}(2, \mathbb{R})$, there exist a unique $k \in K$, $a \in A$ and $n \in N$ such that $g = kan$.

- Let (e_1, e_2) denote the standard basis of $\mathbb{R}^2$ and set $v_i = ge_i$ for $i = 1, 2$. Moreover, set

$$w_1 = v_1, \quad w_2 = v_2 - \frac{\langle w_1, v_2 \rangle}{\langle w_1, w_1 \rangle} \cdot w_1,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^2$. Show that there exist $k \in O(2)$ and $a = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ with $\lambda, \mu > 0$ such that

$$k^{-1} w_i = a e_i, \quad i = 1, 2.$$

- Show that there exists an $n \in N$ such that

$$g^{-1}w_i = n^{-1}e_i, \quad i = 1, 2$$

- Conclude that $g = kan$ and show that $k \in K$ and $a \in A$.
- Show that the decomposition $g = kan$ is unique.
- Bonus: Prove the Iwasawa decomposition of $\mathrm{SL}(n, \mathbb{R})$.

(c) Show that $\mathrm{PSL}(2, \mathbb{R})$ preserves the Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{|dz|^2}{\mathrm{Im}(z)} \quad \text{at } z = x + iy \in \mathbb{H}^2.$$

(d) Prove that the geodesics in $\mathbb{H}^2$ are vertical lines and half circles orthogonal to $\mathbb{R}$.

(e) Show that for $z, w \in \mathbb{H}^2$, their hyperbolic distance $\mathrm{d}(z, w)$ satisfies

$$\mathrm{dist}(z, w) = \cosh^{-1}\left(1 + \frac{|z - w|^2}{2\mathrm{Im}(z)\mathrm{Im}(w)}\right)$$

(f) Show that $g \in \mathrm{PSL}(2, \mathbb{R}) \setminus \{e\}$ acting on $\mathbb{H}^2 \cup \mathbb{R} \cup \{\infty\}$ has either one or two fixed points. Moreover, if g has two fixed points, they both lie on $\mathbb{R} \cup \{\infty\}$.

This leads to the following classification of elements $g \in \mathrm{PSL}(2, \mathbb{R}) \setminus \{e\}$

- g is called *elliptic* if it has a fixed point in $\mathbb{H}^2$
- g is called *parabolic* if it has a single fixed point in $\mathrm{R} \cup \{\infty\}$
- g is called *hyperbolic* (or *loxodromic*) if it has two fixed points in $\mathbb{R} \cup \{\infty\}$.

Show that g is elliptic, parabolic or hyperbolic if and only if it can be conjugated into $[K]$, $[N]$ or $[A]$ respectively. Finally, express the type of $g \in \mathrm{PSL}(2, \mathbb{R})$ in terms of $\mathrm{tr}(g)^2$.

(g) Show that the action of $G < \mathrm{PSL}(2, \mathbb{R})$ on $\mathbb{H}^2$ is properly discontinuous if and only if $G < \mathrm{PSL}(2, \mathbb{R})$ is discrete.

(h) Let

$$\begin{aligned}\Gamma(2) &:= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2, \mathbb{Z}); \begin{array}{ll} a \equiv d \equiv 1 & \mod 2 \\ b \equiv c \equiv 0 & \mod 2 \end{array} \right\} \\ &= \ker\left(\mathrm{PSL}(2, \mathbb{Z}) \overset{\text{reduction mod 2}}{\longrightarrow} \mathrm{PSL}(2, \mathbb{Z}/2\mathbb{Z})\right)\end{aligned}$$

Show that $\Gamma(2)\backslash\mathbb{H}^2$ is a Riemann surface.

Exercise 3 (Belyĭ maps). Suppose that X and Y are compact Riemann surfaces and $f : X \to Y$ is a non-constant holomorphic mapping. Recall that this implies that there exists a finite subset $C(f) \subset Y$ of *critical values* such that the cardinality $|f^{-1}(y)|$ is constant for $y \in Y \setminus C(f)$. The set $R(f) = Y \setminus C(f)$ is called the set of regular values of f. The cardinality of $|f^{-1}(y)|$ at a regular point $y \in R(f)$ is called the *degree* $\deg(f)$ of f. Note that $1 \leq |f^{-1}(y)| < \deg(f)$ for all $y \in C(f)$.

Belyĭ's theorem states that a Riemann surface X is biholomorphic to an algebraic curve defined over $\overline{\mathbb{Q}}$ if and only if there exists a holomorphic map $f : X \to \mathbb{P}^1(\mathbb{C})$ such that $C(f) \subset \{0, 1, \infty\}$. Such a map is called a Belyĭ map.

The goal of this exercise is to prove the direction that Belyĭ proved: if X is defined over $\overline{\mathbb{Q}}$, then it admits a Belyĭ map.

(a) Suppose that $X_1, \ldots, X_k$ are compact Riemann surfaces and suppose that $f_i : X_i \to X_{i+1}$ $(1 \leq i \leq k-1)$ is a sequence of non-constant holomorphic mappings. Set

$$f = f_{k-1} \circ \cdots \circ f_2 \circ f_1 : X_1 \to X_k.$$

Prove that $z \in C(f)$ if and only if there exists some $i \in \{1, \ldots, k-1\}$ such that the finite set $(f_{k-1} \circ \cdots \circ f_{i+1})^{-1}(z) \subset X_{i+1}$ contains a critical value for f_i.

(b) A *Belyĭ map* is a holomorphic map $f : X \to \mathbb{P}^1(\mathbb{C})$ that satisfies $C(f) \subset \{0, 1, \infty\}$. Show that the following maps are Belyĭ maps:

- The map $\beta_n : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ defined by $\beta_n(z) = z^n$.
- The map $\beta_{m,n} : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ defined by

$$\beta_{m,n}(z) = \frac{(m+n)^{m+n}}{m^m n^n} z^m (1-z)^n.$$

Where $m, n \in \mathbb{Z} \setminus \{0\}$ and $m + n \neq 0$.

- The map $\phi_d : \overline{X_d} \to \mathbb{P}^1(\mathbb{C})$ Defined as follows. Write $X_d = \{(x, y) \in \mathbb{C}^2;\ x^d + y^d = 1\}$, compactified by adding d points at ∞, corresponding to the d holomorphic branches of the d^{th} root. Define the projection map $\pi : X_d \to \mathbb{P}^1(\mathbb{C})$ by $\pi(x, y) = x$. Again, define $\beta_d : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ by $\beta_d(z) = z^d$. We set $\phi_d = \beta_d \circ \pi$.

(c) Now suppose

$$X = \overline{\{(x, y) \in \mathbb{C}^2;\ P(x, y) = 0\}}, \quad \text{with } P \in \overline{\mathbb{Q}}[x, y]$$

(and again the completion is performed by adding a suitable finite set of points at infinity)

- Show that there exists a branched covering $\pi : X \to \mathbb{P}^1(\mathbb{C})$ such that $C(\pi) \subset \mathbb{P}^1(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}} \cup \{\infty\}$.
- Since $C(\pi)$ is a finite set, we can assume (potentially by postcomposing with a Möbius transformation) that $\infty \notin C(\pi)$. Let f_1 be the minimal polynomial over $\mathbb{Q}$ of $C(\pi)$ – the monic polynomial of minimal degree that vanishes on all of $C(\pi)$. Construct the sequence of polynomials $f_2, f_3, \ldots$ by letting f_{i+1} be the minimal polynomial over $\mathbb{Q}$ on the set

$$C(f_i) \cap \mathbb{C} = \{f(z) \in \mathbb{C};\ f_i'(z) = 0\}$$

of finite critical values of f_i. Show that the degrees of these polynomials are strictly decreasing, that is: $\deg(f_{i+1}) < \deg(f_i)$ for $i \geq 1$.

- Since the degree is strictly decreasing, the sequence above terminates with a polynomial f_k of degree 1. Now set

$$f = f_k \circ \cdots \circ f_2 \circ f_1 \circ \pi : X \to \mathbb{P}^1(\mathbb{C}).$$

Show that $C(f) \cap \mathbb{C} \subset \mathbb{Q}$.

- Show that if $|C(f)| \geq 4$, we can postcompose f with a holomorphic map $g : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ such that $|C(g \circ f)| < |C(f)|$. Conclude that we have proved that X admits a Belyĭ map.

(d) Find Belyĭ maps on X_1, X_2, X_3 and X_4, given by:

$$X_1 = \{(x, y) \in \mathbb{C}^2;\ y^2 = x(x-1)(x-2/3)\}$$

$$X_2 = \{(x, y) \in \mathbb{C}^2;\ y^2 = (x-1)(x-\zeta_7)(x-\zeta_7^2)\} \quad \text{where } \zeta_7 = e^{2\pi i/7}$$
$$X_3 = \{(x, y) \in \mathbb{C}^2;\ y^2 = x(x-\zeta_7)(x-\zeta_7^2/\sqrt[7]{2})\} \quad \text{where } \zeta_7 = e^{2\pi i/7}$$
$$X_4 = \{(x, y) \in \mathbb{C}^2;\ y^2 = x(x+2)(x-\sqrt{31}+1)\} \cup \{\infty\}.$$

Remark: It's not so easy to find examples of curves defined of $\overline{\mathbb{Q}}$ but not $\mathbb{Q}$ for which the computation does not get out of hand. Don't hesitate to use your favorite computer algebra package to do some of the calculations.