Problem set 2: Teichmüller theory for tori

Exercise 1 (A fundamental domain and a generating set). Let

$$\mathcal{F} = \left\{ z \in \mathbb{H}^2;\ |z| \geq 1,\ -\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2} \right\}$$

(a) Prove that for all $\tau \in \mathbb{H}^2$ there exists an element $g \in \mathrm{PSL}(2,\mathbb{Z})$ so that $g\tau \in \mathcal{F}$. *Hint:* try maximizing $\mathrm{Im}(g\tau)$.

(b) Prove that

- if $\tau \in \mathring{\mathcal{F}}$ then
$$\Big(\mathrm{PSL}(2,\mathbb{Z}) \cdot \tau\Big) \cap \mathcal{F} = \{\tau\},$$
- if $\mathrm{Re}(\tau) = \frac{1}{2}$ then
$$\Big(\mathrm{PSL}(2,\mathbb{Z}) \cdot \tau\Big) \cap \mathcal{F} = \{g\tau, g\tau + 1\},$$
- if $\mathrm{Re}(\tau) = -\frac{1}{2}$ then
$$\Big(\mathrm{PSL}(2,\mathbb{Z}) \cdot \tau\Big) \cap \mathcal{F} = \{g\tau, g\tau - 1\}.$$
- and if $|\tau| = 1$ then
$$\Big(\mathrm{PSL}(2,\mathbb{Z}) \cdot \tau\Big) \cap \mathcal{F} = \{g\tau, -1/\tau\},$$

(c) Use the above to show that

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

generate $\mathrm{PSL}(2,\mathbb{Z})$. Hint: given $g \in \mathrm{PSL}(2,\mathbb{Z})$, bring $g \cdot 2i$ back to $\mathcal{F}$.

(d) Conclude that $\mathrm{SL}(2,\mathbb{Z})$ can be generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Exercise 2 (Moduli spaces of lattices in $\mathbb{R}^n$ and flat tori). A lattice $\Lambda < \mathbb{R}^n$ is a discrete subgroup of finite covolume. Alternatively, it's the $\mathbb{Z}$-linear span of a a basis of $\mathbb{R}^n$. The quotient $\mathbb{R}^n/\Lambda$ is an n-dimensional torus with a flat Riemannian metric (that descends from the Euclidean metric of $\mathbb{R}^n$). The goal of this exercise is to study moduli spaces of lattices and flat tori. Our end goal is to prove Mahler's compactness criterion. For more on these spaces and other spaces like them, we refer to Andrés Sambarino's courses *Géométrie des espaces globalement symétriques* and *Sous-groupes discrets des groupes de Lie.*

(a) Let $v_1, \dots, v_n \in \mathbb{R}^n$ form a basis and let A denote the matrix that has these vectors as columns. Show that $v_1, \dots, v_n$ generate a lattice of covolume 1 if and only if $\det(A) \in \{\pm 1\}$. Conclude that every lattice of covolume 1 in $\mathbb{R}^n$ is of the form

$$A \cdot \mathbb{Z}^n$$

for some $A \in \mathrm{SL}(n, \mathbb{R})$.

(b) Suppose that $v_1, \dots, v_n$ and $w_1, \dots, w_n$ are lattice bases for the same lattice $\Lambda < \mathbb{R}^n$ (i.e. $\mathrm{span}_{\mathbb{Z}}(v_1, \dots, v_n) = \mathrm{span}_{\mathbb{Z}}(w_1, \dots, w_n)$). Show that there exists a matrix A with integral entries such that $\det(A) \in \{\pm 1\}$ and

$$w_i = A \cdot v_i, \quad i = 1, \dots, n.$$

Conclude that the set of lattices of covolume 1 in $\mathbb{R}^n$ can be identified with

$$\mathrm{SL}(n, \mathbb{Z}) \backslash \mathrm{SL}(n, \mathbb{R})$$

(c) Show that the space of (isometry classes of) flat n-dimensional tori of volume 1 can be identified with

$$\mathrm{SL}(n, \mathbb{Z}) \backslash \mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n, \mathrm{R})$$

(d) (Blichfeld's theorem) Suppose that $\Lambda < \mathbb{R}^n$ is a lattice of covolume 1 and that $S \subset \mathbb{R}^n$ is measurable with $\mathrm{vol}(S) \geq 1$. Show that S contains two distinct points $v, w \in S$ with $v - w \in \Lambda$.

(e) (Minkowski's first theorem) Suppose that $\Lambda < \mathbb{R}^n$ is a lattice of covolume 1 and suppose that $C \subset \mathbb{R}^n$ is convex and centrally symmetric (i.e. $C = -C$). Suppose moreover that $\mathrm{vol}(C) > 2^n$. Show that C contains a non-zero lattice vector. *Hint:* Consider the set

$$\widehat{C} = \frac{1}{2} \cdot C = \left\{ \frac{1}{2} \cdot x;\ x \in C \right\}.$$

(f) (Minkowski's second theorem) Suppose that $\Lambda < \mathbb{R}^n$ is a lattice of covolume 1. Define the succesive minima of Λ by

$$m_k(\Lambda) = \min \left\{ r > 0;\ \dim\{\mathrm{span}_{\mathbb{R}} \left(\Lambda \cap \overline{\mathrm{B}}_r(0) \right) \geq k \right\},$$

where $\overline{\mathrm{B}}_r(0)$ denotes the closed ball of radius r around $0 \in \mathbb{R}^n$. Show that:

$$\prod_{k=1}^{n} m_k(\Lambda) \quad \leq \quad 2^n / \mathrm{vol}\left(\overline{\mathrm{B}}_1(0) \right).$$

(g) (Korkine–Zolotarev–Hermite reduction) Let $\Lambda < \mathbb{R}^n$ be a lattice. Our next goal is to find a "short" basis for Λ. The vectors realizing the successive minima (or rather some subset thereof) might seem like natural candidates. It however turns out that it's not always possible to extract a basis from this set of vectors. So we need something else.

- A lattice basis $(v_1, \ldots, v_n)$ is called *size reduced* if its Gram–Schmidt orthogonalization, defined recursively by

$$v_1^* = v_1 \quad \text{and} \quad v_j^* = v_j - \sum_{i=1}^{j-1} \frac{\langle v_j, v_i^* \rangle}{\|v_i^*\|^2} v_i^*, \text{ for } j \geq 2$$

satisfies

$$\left| \frac{\langle v_j, v_i^* \rangle}{\|v_i^*\|^2} \right| \leq \frac{1}{2} \quad \text{for all } 1 \leq i < j \leq n$$

Show that Λ admits a size reduced basis that has the same Gram–Schmidt orthogonalization.

- For $i = 1, \ldots n$, let $\pi_i : \mathbb{R}^n \to \mathbb{R}^n$ denote the orthogonal projection onto

$$\operatorname{span}_{\mathbb{R}}(v_1, \ldots, v_{i-1})^\perp = \operatorname{span}_{\mathbb{R}}(v_i^*, \ldots, v_n^*).$$

Observe that $\pi_i(\Lambda)$ is a lattice in $\pi_i(\mathbb{R}^n)$. A basis $(v_1, \ldots, v_n)$ is called *Korkine–Zolotarev–Hermite (KZH) reduced* if:

  * it's size reduced, and
  * for $i = 1, \ldots, n$, v_i^* is the shortest lattice vector in $\pi_i(\Lambda)$.

Show that every lattice admits a KZH reduced basis.

(h) Show that if $(v_1, \ldots, v_n)$ is a KZH reduced basis for the lattice Λ, then

$$\|v_i\|^2 \leq \frac{i+3}{4} \cdot m_i(\Lambda)^2$$

Hint: Show that

$$m_1(\pi_i(\Lambda)) \leq m_i(\Lambda).$$

(i) (Mahler's compactness criterion) Show that $C \subseteq \mathrm{SL}(n, \mathbb{Z}) \backslash \mathrm{SL}(n, \mathbb{R})$ is compact if and only if it's closed and

$$\inf\{\|v\|;\ v \in \Lambda \in C\} > 0.$$

(j) Show that $C \subset \mathrm{SL}(n, \mathbb{Z}) \backslash \mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n, \mathbb{R})$ is compact if and only if it's closed and

$$\inf\{\text{systole}(T);\ T \in C\} > 0,$$

here the systole of a flat torus T is the length of the shortest closed geodesic in T.

(k) Connect this up to the moduli space $\mathcal{M}_1$ of Riemann surfaces of genus 1.