Exercise 1 (A fundamental domain and a generating set). Let

$$\mathcal{F} = \left\{ z \in \mathbb{H}^2;\ |z| \geq 1,\ -\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2} \right\}$$

(a) Prove that for all $\tau \in \mathbb{H}^2$ there exists an element $g \in \mathrm{PSL}(2,\mathbb{Z})$ so that $g\tau \in \mathcal{F}$. *Hint:* try maximizing $\mathrm{Im}(g\tau)$.

Solution: Let $z \in \mathbb{H}^2$ and $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2, \mathrm{Z})$. A direct computation (that we have performed multiple times by now) shows that

$$\mathrm{Im}(gz) = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$

Now let $\tau \in \mathbb{H}^2$ and let $g \in \mathrm{PSL}(2,\mathbb{Z})$ be an element such that $\mathrm{Im}(g\tau)$ is maximal. Note that this is an honest maximum, since the number of integers so that $|cz+d| \leq K$ is finite for any $K > 0$.

Since

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{Z}),$$

we can always post-compose with T^k for some $k \in \mathbb{Z}$ to make sure that $-\frac{1}{2} \leq \mathrm{Re}(g\tau) \leq \frac{1}{2}$. Now suppose that $|g\tau| < 1$. Since

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{Z})$$

this leads to a contradiction, because $S(g\tau)$ would have a larger imaginary part (this follows directly from our formula for $\mathrm{Im}(S(gz))$). So, we conclude that we can indeed move every $\tau \in \mathbb{H}^2$ into $\mathcal{F}$ using $\mathrm{PSL}(2,\mathbb{Z})$.

(b) Prove that

- if $\tau \in \mathring{\mathcal{F}}$ then
$$\Big(\mathrm{PSL}(2,\mathbb{Z}) \cdot \tau\Big) \cap \mathcal{F} = \{\tau\},$$
- if $\mathrm{Re}(\tau) = \frac{1}{2}$ then
$$\Big(\mathrm{PSL}(2,\mathbb{Z}) \cdot \tau\Big) \cap \mathcal{F} = \{\tau, \tau+1\},$$
- if $\mathrm{Re}(\tau) = -\frac{1}{2}$ then
$$\Big(\mathrm{PSL}(2,\mathbb{Z}) \cdot \tau\Big) \cap \mathcal{F} = \{\tau, \tau-1\}.$$

– and if $|\tau| = 1$ then

$$\Big(\mathrm{PSL}(2,\mathbb{Z}) \cdot \tau\Big) \cap \mathcal{F} = \{\tau, -1/\tau\},$$

Solution: Suppose both z and gz lie in $\mathcal{F}$ for some $g \in \mathrm{PSL}(2,\mathbb{Z})$. Without loss of generality, we assume that $\mathrm{Im}(gz) \geq \mathrm{Im}(z)$. This implies that

$$|cz + d| \leq 1.$$

Using that $z \in \mathcal{F}$ this implies that $c \in \{-1, 0, 1\}$. If $c = 0$, then $d = \pm 1$ and we obtain

$$g = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

for some $k \in \mathbb{Z}$. Now using that $z \in \mathcal{F}$, we see that either $k \in \{\pm 1\}$ and $\mathrm{Re}(z) \in \{\pm\frac{1}{2}\}$ or $k = 0$.

If $c = \pm 1$, then $d = 0$ and $|z| = 1$ and hence

$$g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

which gives us the $|\tau| = 1$ case.

(c) Use the above to show that

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

generate $\mathrm{PSL}(2,\mathbb{Z})$. Hint: given $g \in \mathrm{PSL}(2,\mathbb{Z})$, bring $g \cdot 2i$ back to $\mathcal{F}$.

Solution: Observe that in the solution to (a), we used exactly the matrices T and S to bring a point any $\mathcal{F}$. Let h denote the word in $T^{\pm 1}$ and $S^{\pm 1}$ we generate to bring $g \cdot 2i$ to $\mathcal{F}$ and write

$$h \cdot g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Since $h \cdot g \cdot 2i \in \mathcal{F}$, we have

$$\frac{2}{4c^2 + d^2} = \mathrm{Im}(h \cdot g \cdot 2i) \geq \frac{\sqrt{3}}{2}$$

This means that $c = 0$ (since $2/4 < \sqrt{3}/2$). That in turn implies that $1 = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad$, so $a = d = \pm 1$. This means that

$$\mathrm{Re}(h \cdot g \cdot 2i) = \mathrm{Re}(2i + b) = b \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

and hence that $b = 0$. Which implies that $h \cdot g = 1$ in $\mathrm{PSL}(2,\mathbb{Z})$. So $g = h^{-1}$. h (and hence h^{-1}) being a word in $T^{\pm 1}$ and $S^{\pm 1}$, this writes g as a word in $T^{\pm 1}$ and $S^{\pm 1}$.

(d) Conclude that $\mathrm{SL}(2,\mathbb{Z})$ can be generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Solution: Because T and S generate $\mathrm{PSL}(2,\mathbb{Z})$, we can, for every $A \in \mathrm{SL}(2,\mathbb{Z})$, write at least one of $\{A, -A\}$ as a word in

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\pm 1} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\pm 1}.$$

Moreover

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

so we can in fact obtain A itself.

Exercise 2 (Moduli spaces of lattices in $\mathbb{R}^n$ and flat tori). A lattice $\Lambda < \mathbb{R}^n$ is a discrete subgroup of finite covolume. Alternatively, it's the $\mathbb{Z}$-linear span of a a basis of $\mathbb{R}^n$. The quotient $\mathbb{R}^n/\Lambda$ is an n-dimensional torus with a flat Riemannian metric (that descends from the Euclidean metric of $\mathbb{R}^n$). The goal of this exercise is to study moduli spaces of lattices and flat tori. Our end goal is to prove Mahler's compactness criterion. For more on these spaces and other spaces like them, we refer to Andrés Sambarino's courses *Géométrie des espaces globalement symétriques* and *Sous-groupes discrets des groupes de Lie.*

(a) Let $v_1, \ldots, v_n \in \mathbb{R}^n$ form a basis and let A denote the matrix that has these vectors as columns. Show that $v_1, \ldots, v_n$ generate a lattice of covolume 1 if and only if $\det(A) \in \{\pm 1\}$. Conclude that every lattice of covolume 1 in $\mathbb{R}^n$ is of the form

$$A \cdot \mathbb{Z}^n$$

for some $A \in \mathrm{SL}(n,\mathbb{R})$.

Solution: The set

$$\mathcal{F} = \left\{ \sum_i \lambda_i v_i;\ \lambda_1, \ldots \lambda_n \geq 0 \text{ and } \sum_i \lambda_i \leq 1 \right\} = A \cdot [0,1]^n$$

where A is the matrix whose columns are the vectors v_i, forms a fundamental domain for the action of Λ on $\mathbb{R}^n$. We have

$$1 = \mathrm{vol}(\mathcal{F}) = \int_{\mathcal{F}} dx_1 \cdots dx_n = |\det(A)| \cdot \int_0^1 \cdots \int_0^1 dy_1 \cdots dy_n = |\det(A)|$$

So $\det(A) \in \{\pm 1\}$. If $\det(A) = -1$, then we replace v_1 by $-v_1$ and obtain a matrix A' with $\det(A') = 1$. This doesn't change the lattice.

(b) Suppose that $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are lattice bases for the same lattice $\Lambda < \mathbb{R}^n$ (i.e. $\operatorname{span}_{\mathbb{Z}}(v_1, \ldots, v_n) = \operatorname{span}_{\mathbb{Z}}(w_1, \ldots, w_n)$). Show that there exists a matrix A with integral entries such that $\det(A) \in \{\pm 1\}$ and

$$w_i = A \cdot v_i, \quad i = 1, \ldots, n.$$

Conclude that the set of lattices of covolume 1 in $\mathbb{R}^n$ can be identified with

$$\mathrm{SL}(n, \mathbb{Z}) \backslash \mathrm{SL}(n, \mathbb{R})$$

Solution: Since the vectors $w_i \in \operatorname{span}_{\mathbb{Z}}(v_1, \ldots, v_n)$ there exists a matrix A with integer entries such that

$$w_i = Av_i, \quad i = 1, \ldots, n$$

Likewise, let B be the integral matrix such that

$$Bv_i = w_i, \quad i = 1, \ldots, n.$$

This implies that $B = A^{-1}$. Taking determinants, we obtain

$$1 = \det(BA) = \det(B)\det(A).$$

Since $\det(A), \det(B) \in \mathbb{Z}$, we obtain that $\det(A), \det(B) \in \{\pm 1\}$ as required.

We have seen that every lattice of covolume 1 is of the form $A \cdot \mathbb{Z}^n$ for some $A \in \mathrm{SL}(n, \mathbb{R})$. Now suppose $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ corresponding to matrices $A, B \in \mathrm{SL}(n, \mathbb{R})$ respectively generate the same lattice. Then $B = C \cdot A$ where C is some matrix in $\mathrm{SL}(n, \mathbb{Z})$ (the determinant needs to be 1 because of multiplicativity of determinants). So we conclude we can identify the set of lattices with $\mathrm{SL}(n, \mathbb{Z}) \backslash \mathrm{SL}(n, \mathbb{R})$

(c) Show that the space of (isometry classes of) flat n-dimensional tori of volume 1 can be identified with

$$\mathrm{SL}(n, \mathbb{Z}) \backslash \mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n, \mathrm{R})$$

Solution: If two lattices differ by a rotation, they yield the same flat torus.

Conversely, by the Killing–Hopf theorem, all flat n-dimensional tori are of the form

$$\mathbb{R}^n / \Lambda$$

for some lattice $\Lambda < \mathbb{R}^n$. Now suppose

$$\varphi : \mathbb{R}^n / \Lambda_1 \longrightarrow \mathbb{R}^n / \Lambda_2$$

is an isometry. We may lift φ to an isometry $\widetilde{\varphi} : \mathbb{R}^n \to \mathbb{R}^n$ that satisfies $\widetilde{\varphi}(\Lambda_1) = \Lambda_2$ and $\widetilde{\varphi}(0) = 0$. Since the only isometries of $\mathbb{R}^n$ that preserve the origin are linear isometries, we may identify $\widetilde{\varphi}$ with an element of $\mathrm{SO}(n, \mathbb{R})$.

(d) (Blichfeld's theorem) Suppose that $\Lambda < \mathbb{R}^n$ is a lattice of covolume 1 and that $S \subset \mathbb{R}^n$ is measurable with $\mathrm{vol}(S) > 1$. Show that S contains two distinct points $v, w \in S$ with $v - w \in \Lambda$.

<u>Solution:</u> Consider the projection of S to the quotient torus $\mathbb{R}^n/\Lambda$. Because the volume of S is more than that of $\mathbb{R}^n/\Lambda$, S contains two elements v and w that project to the same point in $\mathbb{R}^n/\Lambda$, i.e. $v - w \in \Lambda$.

(e) (Minkowski's first theorem) Suppose that $\Lambda < \mathbb{R}^n$ is a lattice of covolume 1 and suppose that $C \subset \mathbb{R}^n$ is convex and centrally symmetric (i.e. $C = -C$). Suppose moreover that $\mathrm{vol}(C) > 2^n$. Show that C contains a non-zero lattice vector. *Hint:* Consider the set

$$\widehat{C} = \frac{1}{2} \cdot C = \left\{ \frac{1}{2} \cdot x;\ x \in C \right\}.$$

<u>Solution:</u> We have

$$\mathrm{vol}\left(\widehat{C}\right) = \frac{1}{2^n}\mathrm{vol}(C) > 1,$$

so by (d), there exist $x, y \in \widehat{C}$ such that $x - y \in \Lambda \setminus \{0\}$. The vectors $2x, 2y \in C$. Moreover, because C is centrally symmetric, $-2y \in C$ as well. Finally, because C is convex:

$$\frac{1}{2} \cdot 2x + \frac{1}{2} \cdot -2y = x - y \in C.$$

(f) (Minkowski's second theorem) Suppose that $\Lambda < \mathbb{R}^n$ is a lattice of covolume 1. Define the succesive minima of Λ by

$$m_k(\Lambda) = \min\left\{ r > 0;\ \dim\{\mathrm{span}_{\mathbb{R}}\left(\Lambda \cap \overline{\mathrm{B}}_r(0)\right) \geq k \right\},$$

where $\overline{\mathrm{B}}_r(0)$ denotes the closed ball of radius r around $0 \in \mathbb{R}^n$. Show that:

$$\prod_{k=1}^{n} m_k(\Lambda) \quad \leq \quad 2^n / \mathrm{vol}\left(\overline{\mathrm{B}}_1(0)\right).$$

<u>Solution:</u> Let $v_1, \ldots v_n$ be a set of n vectors realizing the successive minima and let $\widetilde{v}_1, \ldots, \widetilde{v}_n$ denote their Gram-Schmidt orthonormalization.

Consider the ellipsoid

$$E = \left\{ \sum_{k=1}^{n} a_k \widetilde{v}_i \in \mathbb{R}^n;\ \sum_{k=1}^{n} \frac{a_k^2}{m_k(\Lambda)^2} < 1 \right\}.$$

Its volume is

$$\mathrm{vol}(E) = \mathrm{vol}\left(\overline{\mathrm{B}}_1(0)\right) \cdot \prod_{k=1}^{n} m_k(\Lambda).$$

So, if we show that it contains no non-zero lattice vectors, we obtain the inequality we're after from Minkowski's first theorem.

Given $w \in \Lambda \setminus \{0\}$, let $k = 1, \ldots, n$ be maximal such that $\|w\| \geq m_k(\Lambda)$. So, we can write

$$w = \sum_{i=1}^{k} a_i \widetilde{v}_i$$

with $a_i = \langle w, \widetilde{v}_i \rangle$ for $i = 1, \ldots, k$. We get

$$\sum_{i=1}^{n} \frac{a_i^2}{m_i(\Lambda)^2} = \sum_{i=1}^{k} \frac{a_i^2}{m_i(\Lambda)^2} \geq \frac{1}{m_k(\Lambda)^2} \sum_{i=1}^{k} a_i^2 = \frac{\|w\|^2}{m_k(\Lambda)^2} \geq 1.$$

So, indeed $w \notin E$ and hence Minkowski's first theorem applies.

(g) (Korkine–Zolotarev–Hermite reduction) Let $\Lambda < \mathbb{R}^n$ be a lattice. Our next goal is to find a "short" basis for Λ. The vectors realizing the successive minima (or rather some subset thereof) might seem like natural candidates. It however turns out that it's not always possible to extract a basis from this set of vectors. So we need something else.

- A lattice basis $(v_1, \ldots, v_n)$ is called *size reduced* if its Gram–Schmidt orthogonalization, defined recursively by

$$v_1^* = v_1 \quad \text{and} \quad v_j^* = v_j - \sum_{i=1}^{j-1} \frac{\langle v_j, v_i^* \rangle}{\|v_i^*\|^2} v_i^*, \text{ for } j \geq 2$$

satisfies

$$\left| \frac{\langle v_j, v_i^* \rangle}{\|v_i^*\|^2} \right| \leq \frac{1}{2} \quad \text{for all } 1 \leq i < j \leq n$$

Show that Λ admits a size reduced basis that has the same Gram–Schmidt orthogonalization.

Solution: Suppose $(v_1, \ldots, v_n)$ is any lattice basis for Λ. We will recursively change the basis so that it becomes size reduced. Suppose that

$$\left| \frac{\langle v_j, v_i^* \rangle}{\|v_i^*\|^2} \right| \leq \frac{1}{2} \quad \text{for all } 1 \leq i < j < k,$$

but $\left| \langle v_k, v_j^* \rangle \ / \ \|v_j^*\|^2 \right| > \frac{1}{2}$. We can change v_k into $\widetilde{v_k} = v_k + m v_j$ for some $m \in \mathbb{Z}$. Observe that the resulting n-tuple of vectors is still a lattice basis and moreover, $\widetilde{v_k}^* = v_k^*$. However, by orthogonality,

$$\widetilde{v_k}^* = \widetilde{v_k} - \sum_{i=1}^{k-1} \frac{\langle v_k + m v_j, v_i^* \rangle}{\|v_i^*\|^2} v_i^* = \widetilde{v_k} - \left(\frac{\langle v_k, v_i^* \rangle}{\|v_i^*\|^2} + m \right) \cdot v_j^* - \sum_{\substack{1 \leq i \leq k-1 \\ \neq j}} \frac{\langle v_k, v_i^* \rangle}{\|v_i^*\|^2} v_i^*.$$

So we're only influencing the coefficient $\langle v_k, v_j^* \rangle \ / \ \|v_j^*\|^2$, which we can but in the interval $[-1/2, 1/2]$ by choosing a suitable integer m.

- For $i = 1, \dots n$, let $\pi_i : \mathbb{R}^n \to \mathbb{R}^n$ denote the orthogonal projection onto

$$\mathrm{span}_{\mathbb{R}}(v_1, \dots, v_{i-1})^{\perp} = \mathrm{span}_{\mathbb{R}}(v_i^*, \dots, v_n^*).$$

Observe that $\pi_i(\Lambda)$ is a lattice in $\pi_i(\mathbb{R}^n)$. A basis $(v_1, \dots, v_n)$ is called *Korkine–Zolotarev–Hermite (KZH) reduced* if:

* it's size reduced, and
* for $i = 1, \dots, n$, v_i^* is the shortest lattice vector in $\pi_i(\Lambda)$.

Show that every lattice admits a KZH reduced basis.

Solution: We first recursively try to find candidates for the Gram–Schmidt orthogonalization of our basis. Observe that π_1, by definition, is the identity.

* Let v_1^* denote the shortest lattice vector of Λ, this defines π_2
* For $i \geq 2$, let v_i^* the shortest lattice vector of $\pi_i(\Lambda)$. This defines π_{i+1} (if $i < n$).

Now, for $i = 1, \dots, n$, we let $v_i \in \Lambda$ be some arbitrary vector such that $\pi_i(v_i) = v_i^*$. Observe that this implies that $(v_1^*, \dots, v_n^*)$ is the Gram–Schmidt orthogonalization of $(v_1, \dots, v_n)$.

We claim that $(v_1, \dots, v_n)$ forms a lattice basis for Λ. By construction, these vectors form a basis for $\mathbb{R}^n$. So, all we need to show is that the vectors in Λ have integral coefficients with respect to this basis. So, given $w \in \Lambda$, write

$$w = \sum_{i=1}^{n} \lambda_i v_i.$$

Suppose that $\lambda_i \notin \mathbb{Z}$ and i is maximal with respect to this (i.e. $\lambda_{i+1}, \dots, \lambda_n \in \mathbb{Z}$). Potentially exchanging w for $-w$, suppose that $\lambda_i > 0$. Because the vector

$$w' = w - \lfloor \lambda_i \rfloor \cdot v_i - \sum_{j=i+1}^{n} \lambda_j v_j$$

is a lattice vector as well, so we may assume that $0 < \lambda_i < 1$ and $\lambda_{i+1} = \ldots = \lambda_n = 0$. We have

$$\pi_i(w') = \lambda_i \pi_i(v_i) = \lambda_i v_i^*$$

which is strictly shorter than v_i^*, a contradiction. So indeed, $(v_1, \dots, v_n)$ forms a basis.

Finally, we can turn it into a size reduced basis, without changing the Gram–Schmidt orthogonalization, using the previous point.

(h) Show that if $(v_1, \dots, v_n)$ is a KZH reduced basis for the lattice Λ, then

$$\|v_i\|^2 \leq \frac{i+3}{4} \cdot m_i(\Lambda)^2$$

Hint: Show that

$$m_1(\pi_i(\Lambda)) \leq m_i(\Lambda)$$

Solution: Let us first show the inequality from the hint. Take n linearly independent vectors $w_1, \dots, w_n \in \Lambda$ with $\|w_i\| = m_i(\Lambda)$. Since π_i is a projection onto a space of dimension $n - i + 1$, there is at least one vector w_j among $w_1, \dots, w_i$ such that $\pi_i(w_i) \neq 0$. This means that

$$\|v_i^*\| = m_1(\pi_i(\Lambda)) \leq \|\pi_i(w_j)\| \leq m_j(\Lambda) \leq m_i(\Lambda),$$

thus proving the inequality from the hint.

Now we have

$$\begin{aligned}
\|v_i\|^2 &= \|v_i^* + \sum_{j=1}^{i-1} \frac{\langle v_i, v_j^* \rangle}{\|v_j^*\|^2} v_j^*\|^2 \\
&= \|v_i^*\|^2 + \sum_{j=1}^{i-1} \left| \frac{\langle v_i, v_j^* \rangle}{\|v_j^*\|^2} \right|^2 \|v_j^*\|^2 \\
&\leq \|v_i^*\|^2 + \frac{1}{4} \sum_{j=1}^{i-1} \|v_j^*\|^2 \\
&\leq \frac{i+3}{4} \cdot m_i(\Lambda)^2
\end{aligned}$$

as required.

(i) (Mahler's compactness criterion) Show that $C \subseteq \mathrm{SL}(n, \mathbb{Z}) \backslash \mathrm{SL}(n, \mathbb{R})$ is compact if and only if it's closed and

$$\inf\{\|v\|;\ v \in \Lambda \in C\} > 0.$$

Solution: First suppose that $C \subset \mathrm{SL}(n, \mathbb{Z}) \backslash \mathrm{SL}(n, \mathbb{R})$ is compact, which implies it's closed, so we only need to show that the shortest lattice vector is uniformly bounded from below on C. The key observation is that the function $m_1 : \mathrm{SL}(n, \mathbb{Z}) \backslash \mathrm{SL}(n, \mathbb{R}) \to (0, \infty)$ that associates the shortest lattice vector to the lattice $\Lambda \in \mathrm{SL}(n, \mathbb{Z}) \backslash \mathrm{SL}(n, \mathbb{R})$ is continuous. Indeed, at $\mathrm{SL}(n, \mathbb{Z}) \cdot A \in \mathrm{SL}(n, \mathbb{Z}) \backslash \mathrm{SL}(n, \mathbb{R})$, the shortest lattice vectors is realized by a finite number of lattice vectors ($\mathbb{Z}$-linear combinations of the columns of A) and all the other lattice vectors are stricly larger. So in some small open neighborhood of $\mathrm{SL}(n, \mathbb{Z}) \cdot A$, the shortest lattice vector is realized by one of these linear combinations. The length of a fixed linear combination of the columns of A is a continuous function. So locally, the shortest lattice vector is the minimum of a finite number of continuous functions, and hence continuous. This means that $m_1(C) \subset (0, \infty)$ is compact and thus that

$$\inf\{\|v\|;\ v \in \Lambda \in C\} > 0.$$

Conversely, if C is closed and $\inf\{\|v\|;\ v \in \Lambda \in C\} > 0$, then by (f) and (h), all lattices $\Lambda \in C$ admit a lattice basis whose vectors have norms uniformly bounded from below and above. In other words, they can all be represented by matrices $A \in \mathrm{SL}(n, \mathbb{R})$ the norm of whose columns is uniformly bounded from below and above, which implies that C is compact.

(j) Show that $C \subset \mathrm{SL}(n,\mathbb{Z})\backslash\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$ is compact if and only if it's closed and

$$\inf\{\text{systole}(T);\ T \in C\} > 0,$$

here the systole of a flat torus T is the length of the shortest closed geodesic in T.

Solution: First we observe that $\text{systole}(\mathbb{R}^n/\Lambda) = m_1(\Lambda)$. Moreover, since $\mathrm{SO}(n,\mathbb{R})$ is a compact group, $C \subset \mathrm{SL}(n,\mathbb{Z})\backslash\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$ is compact if and only if $\pi^{-1}(C) \subset \mathrm{SL}(n,\mathbb{Z})\backslash\mathrm{SL}(n,\mathbb{R})$ is compact, where $\pi : \mathrm{SL}(n,\mathbb{Z})\backslash\mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{Z})\backslash\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$ is the projection map. So the claim follows from Mahler's compactness criterion.

(k) Connect this up to the moduli space $\mathcal{M}_1$ of Riemann surfaces of genus 1.

Solution: We've seen that Riemann surfaces of genus 1 correspond one-to-one to flat 2-tori of area 1 up to isometry (we said metrics up to isometry and rescaling, but we can always pick a representative of area 1). So we should have

$$\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}^2 = \mathcal{M}_1 = \mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R}).$$

Indeed, we can identify $\mathbb{H}^2$ with $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$, because $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{H}^2$ by Möbius transformations (this is the same action as that of $\mathrm{PSL}(2,\mathbb{R})$, except that it's not faithful, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts trivially) and the stabilizer of a point is $\mathrm{SO}(2,\mathbb{R})$. For the same reasons, $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}^2 = \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2$. Mahler's compactness criterion implies that the systoles of a sequence of tori (normalized to have area 1) corresponding to a sequence of points in $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}^2$ tend to 0 if and only if the points go up into the cusp of $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}^2$.