Problem set 5: Hyperbolic surfaces

Exercise 1 (Unique geodesics). The goal of this exercise is to prove that free homotopy classes of essential curves on hyperbolic surfaces contain unique geodesics.

(a) Let us first state the Arzelà–Ascoli theorem. Recall that a map $f : X \to Z$ between metric spaces X and Z is called L-Lipschitz, for some $L > 0$ if

$$\mathrm{d}_Z(f(x), f(y)) \leq L \cdot \mathrm{d}_X(x, y)$$

for all $x, y \in X$. The Arzelà-Ascoli theorem now states:

Theorem (Arzelà-Ascoli) Let X be a metric space that has a countable dense subset and Z a compact metric space. Suppose $\gamma_n : X \to Z$ is an L-Lipschitz map for all $n \in \mathbb{N}$ and some fixed $L > 0$. Then there exists a subsequence $(\gamma_{n_k})_{k\in\mathbb{N}}$ that converges uniformly on compact sets in X to an L-Lipschitz map $\gamma : X \to Z$.

Use this theorem to show that every non-trivial free homotopy class on a closed hyperbolic surface X contains at least one closed geodesic that minimizes the length in the homotopy class.

(b) Show that this geodesic is unique. Hint: Suppose that there are two parallel geodesics and lift these to $\mathbb{H}^2$.

Exercise 2 (The band model).

(a) Show that

$$\mathbb{B} = \left\{ z \in \mathbb{C};\ |\mathrm{Im}(z)| < \frac{\pi}{2} \right\},$$

equipped with the metric

$$ds^2 = \frac{dx^2 + dy^2}{\cos^2(y)}$$

is isometric to $\mathbb{H}^2$. *Hint: this can be done without providing an isometry.*

(b) Show that $\mathbb{R} \subset \mathbb{B}$ is a geodesic.

Exercise 3 (Twist curves). Let $\mathcal{P}$ be a pants decomposition of a surface S. Show that there exists collection of disjoint simple closed curves Γ so that for each pair of pants P in $S \setminus \mathcal{P}$, $\Gamma \cap \mathcal{P}$ consists of three arcs, each connecting a different pair of boundary components of P.

Exercise 4 (Gauss–Bonnet for hyperbolic surfaces).

(a) Suppose T is a hyperbolic triangle with angles $\alpha, \beta, \gamma \geq 0$ at the vertices. Show that

$$\text{area}(T) = \pi - \alpha - \beta - \gamma$$

(b) Given a closed orientable hyperbolic surface X of genus g, equipped with a topological triangulation, we may straighten the edges to geodesic segments without moving the vertices. The result is still a triangulation (no intersections between edges will be created), this is a similar result to that of the first exrcise that we will assume. Prove that

$$\text{area}(X) = 4\pi \cdot (g - 1),$$

without using the Gauss–Bonnet formula (of which this is a special case).