Exercise 1 (Unique geodesics). The goal of this exercise is to prove that free homotopy classes of essential curves on hyperbolic surfaces contain unique geodesics.

(a) Let us first state the Arzelà–Ascoli theorem. Recall that a map $f : X \to Z$ between metric spaces X and Z is called L-Lipschitz, for some $L > 0$ if

$$\mathrm{d}_Z(f(x), f(y)) \leq L \cdot \mathrm{d}_X(x, y)$$

for all $x, y \in X$. The Arzelà-Ascoli theorem now states:

Theorem (Arzelà-Ascoli) Let X be a metric space that has a countable dense subset and Z a compact metric space. Suppose $\gamma_n : X \to Z$ is an L-Lipschitz map for all $n \in \mathbb{N}$ and some fixed $L > 0$. Then there exists a subsequence $(\gamma_{n_k})_{k\in\mathbb{N}}$ that converges uniformly on compact sets in X to an L-Lipschitz map $\gamma : X \to Z$.

Use this theorem to show that every non-trivial homotopy class on a closed hyperbolic surface X contains at least one closed geodesic.

Solution: Set

$$\mathcal{C} := \{\gamma' : \mathbb{S}^1 \to X;\ \gamma' \text{ freely homotopic to } \gamma\}$$

and set

$$L = \inf\{\ell(\gamma');\ \gamma' \in \mathcal{C}\}.$$

Now consider a sequence $(\gamma_n)_n$ so that $\ell(\gamma_n) \to L$. It follows from the Arzelà-Ascoli theorem that there exists a subsequence $(\gamma_{n_k})_k$ and a closed curve $\overline{\gamma} : \mathbb{S}^1 \to X$ such that $\gamma_{n_k} \to \widetilde{\gamma}$ uniformly as $k \to \infty$. Because $\overline{\gamma}$ minimizes length and cannot have length 0 (because then we would have contracted the curve), it needs to be a geodesic.

(b) Show that this geodesic is unique. Hint: Suppose that there are two parallel geodesics and lift these to $\mathbb{H}^2$.

Solution: Suppose there are two freely homotopic geodesics $\gamma_1, \gamma_2 : \mathbb{S}^1 \to X$. Consider the universal cover $\pi : C \to X$. Because γ_1 and γ_2 are freely homotopic, we can lift them to continuous maps $\widetilde{\gamma}_1, \widetilde{\gamma}_2 : \mathbb{R} \to \mathbb{H}^2$ that are homotopic. The fact that γ_1 and γ_2 are geodesics implies that $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ are as well.

Because γ_1 and γ_2 are homotopic, we can base them so that they induce the same element $g \in \Gamma$, where Γ is the deck group of the universal cover $\mathbb{H}^2 \to X$. The cyclic subgroup $G\langle g\rangle < \Gamma$ leaves both $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ invariant. Since $\widetilde{\gamma}_1/G$ and $\widetilde{\gamma}_2/G$ are compact, we obtain that

$$\sup_{t\in\mathbb{R}}\{\mathrm{d}(\widetilde{\gamma}_1(t), \widetilde{\gamma}_2(t)\} < \infty.$$

Now we note that when geodesics have at least one pair of distinct endpoints, the above does not hold. This implies that $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ have the same endpoints, which in turn implies they have coincide.

Exercise 2 (The band model).

(a) Show that
$$\mathbb{B} = \left\{ z \in \mathbb{C};\ |\mathrm{Im}(z)| < \frac{\pi}{2} \right\},$$
equipped with the metric
$$ds^2 = \frac{dx^2 + dy^2}{\cos^2(y)}$$
is isometric to $\mathbb{H}^2$. *Hint: this can be done without providing an isometry.*

<u>Solution:</u> Our goal is to apply the Killing–Hopf theorem, so we need to show that $\mathbb{B}$ is a complete Riemannian manifold of constant curvature -1. Recall that the Gaussian curvature of a metric
$$ds^2 = a(x,y)dx^2 + b(x,y)dy^2$$
is given by
$$K = -\frac{1}{2\sqrt{a(x,y)\cdot b(x,y)}}\left(\frac{\partial}{\partial y}\left(\frac{\partial a(x,y)/\partial y}{\sqrt{a(x,y)\cdot b(x,y)}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial b(x,y)/\partial x}{2\sqrt{a(x,y)\cdot b(x,y)}}\right)\right).$$
The derivatives with respect to x vanish and moreover,
$$\frac{\partial}{\partial y}\frac{1}{\cos^2(y)} = 2\frac{\sin(y)}{\cos^3(y)}.$$
We also have $\sqrt{a(x,y)\cdot b(x,y)} = \frac{1}{\cos^2(y)}$. So this gives:
$$K = -\cos^2(y)\frac{\partial}{\partial y}\tan(y) = -1.$$
Moreover, $\mathbb{B}$ for completeness, we will prove that bounded closed sets are compact (this is sometimes called the Heine–Borel property). To this end, let $\gamma : [0,1] \to \mathbb{B} : t \mapsto \gamma_1(t) + \gamma_2(t)\cdot i$ be a smooth curve with $\gamma_2(1) \in \{\pm\pi/2\}$, then
$$\ell(\gamma) = \int_0^1 \frac{1}{\cos(\gamma_2(t))}\sqrt{\left(\frac{\partial}{\partial t}\gamma_1(t)\right)^2 + \left(\frac{\partial}{\partial t}\gamma_2(t)\right)^2}\,dt$$
$$\geq \int_0^1 \frac{1}{\cos(\gamma_2(t))}\left|\frac{\partial \gamma_2(t)}{\partial t}\right| dt = \left|\int_0^{\pm\pi/2}\frac{1}{\cos(t)}\right| dt = \infty$$
In particular, the distance to $\partial\mathbb{B}$ is infinite and hence bounded closed sets are contained in closed Euclidean rectangles that fit in $\mathbb{B}$, and are thus compact.

(b) Show that $\mathbb{R} \subset \mathbb{B}$ is a geodesic.

Solution: Let $\gamma : [0,1] \to \mathbb{B} : t \mapsto \gamma_1(t) + \gamma_2(t) \cdot i$ be any curve with endpoints on $\mathbb{R}$. Then

$$\ell(\gamma) = \int_0^1 \frac{1}{\cos(\gamma_2(t))} \sqrt{\left(\frac{\partial}{\partial t}\gamma_1(t)\right)^2 + \left(\frac{\partial}{\partial t}\gamma_2(t)\right)^2}\, dt$$

$$\geq \int_0^1 \left|\frac{\partial}{\partial t}\gamma_1(t)\right| dt = \ell(\gamma^*)$$

where $\gamma^* : [0,1] \to \mathbb{B} : t \mapsto \gamma_1(t)$, which runs between the same endpoints.

Exercise 3 (Twist curves). Let $\mathcal{P}$ be a pants decomposition of a surface S. Show that there exists collection of disjoint simple closed curves Γ so that for each pair of pants P in $S \setminus \mathcal{P}$, $\Gamma \cap \mathcal{P}$ consists of three arcs, each connecting a different pair of boundary components of P.

Solution: One way to do this is by using hyperbolic geometry. Put a hyperbolic metric on the pairs of pants in the decomposition, say with all boundary components of length 1. Moreover, on each boundary component of such a pair of pants we have two special points that are the points at which the orthogeodesic segments connecting that component to the other two arrive. Use gluings that identify these points, so that all these orthodgeodesics glue into some collection of smooth closed geodesics. Because closed geodesics are homotopically non-trivial (if they were homotopically trivial, we could lift them to closed geodesics in $\mathbb{H}^2$, which don't exist), we get the curves we're after.

(a) Suppose T is a hyperbolic triangle with angles $\alpha, \beta, \gamma \geq 0$ at the vertices. Show that

$$\text{area}(T) = \pi - \alpha - \beta - \gamma$$

Solution: We first assume that $\gamma = 0$ and put the corresponding vertex at the point at ∞. We put the other vertices v and w, corresponding to the angles α and β respectively on the geodesic $|z| = 1$. A bit of Euclidean trigonometry shows that this means that the horizontal coordinates of v and w are $\cos(\pi - \alpha)$ and $\cos(\beta)$ respectively. This means that

$$\text{area}(T) = \int_{\cos(\pi-\alpha)}^{\cos(\beta)} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2} dy dx = \pi - \alpha - \beta.$$

Now any compact triangle T is the difference of two triangles with an ideal vertex. Indeed, again put v and w on the circle $|z| = 1$ and put the vertex u corresponding to the angle γ vertically above v. Then $T(u, v, w) = T(\infty, v, w) - T(\infty, u, w)$. So the formula follows.

(b) Given a closed orientable hyperbolic surface X of genus g, equipped with a topological triangulation, we may straighten the edges to geodesic segments without moving the vertices. The result is still a triangulation (no intersections between edges will be created), this is a similar result to that of the first exrcise that we will assume. Prove that

$$\text{area}(X) = 4\pi \cdot (g-1),$$

without using the Gauss–Bonnet formula (of which this is a special case).

Solution: The area of X can be computed as:

$$\text{area}(X) = \sum_{\substack{T \text{ triangle of the} \\ \text{triangulation}}} \text{area}(T) = \pi \cdot F - \sum_{\substack{T \text{ triangle of the} \\ \text{triangulation}}} \alpha_T + \beta_T + \gamma_T,$$

where F denotes the number of triangles and α_T, β_T and γ_T the angles at the vertices of T in some arbitrary order. Now we observe that

$$\sum_{\substack{T \text{ triangle of the} \\ \text{triangulation}}} \alpha_T + \beta_T + \gamma_T = 2\pi V,$$

where V denotes the number of vertices of the triangulation. So

$$\text{area}(X) = 2\pi(\frac{1}{2}F - V).$$

If we write E for the number of edges of the triangulation, then $E = 3F/2$. So $\frac{1}{2}F = E - F$. Thus

$$\text{area}(X) = 2\pi(-V + E - F) = -2\pi\chi(X) = 4\pi \cdot (g-1).$$