Problem set 6: The Teichmüller metric

Exercise 1 (Mapping class group invariance and completeness). Let X be a closed Riemann surface of genus at least 1.

(a) Show that the Teichmüller metric on $\mathcal{T}(X)$ is invariant under the action of the mapping class group.

(b) Show that the Teichmüller metric on $\mathcal{T}(X)$ is complete. Hint: Use the homeomorphism $\mathcal{E} : \mathrm{QD}_1(X) \to \mathcal{T}(X)$ defined in the course.

Exercise 2 (Teichmüller lines). Let X be a closed Riemann surface of genus at least 1. Recall that a geodesic segment σ in a metric space (M, d) is a segment such that for any $x, y, z \in \sigma$ such that y lies between x and z we have

$$\mathrm{d}(x,z) = \mathrm{d}(x,y) + \mathrm{d}(y,z).$$

(a) Show that every geodesic segment in $\mathcal{T}(X)$ is a subsegment of some Teichmüller line.

(b) Show that there is a unique geodesic in $\mathcal{T}(X)$ between any two points.

Exercise 3 (The torus). Recall that the Teichmüller space of the torus $\mathbb{R}^2/\mathbb{Z}^2$ can be identified with $\mathbb{H}^2 = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$, in which a point $[A]$ represents $[\mathbb{R}^2/(A \cdot \mathbb{Z}^2), f_A]$, where

$$f_A([(x,y)]) = [A \cdot (x,y)]$$

for all $[(x,y)] \in \mathbb{R}^2/\mathbb{Z}^2$. Prove that the Teichmüller metric d_{T} and the hyperbolic metric $\mathrm{d}_{\mathrm{hyp.}}$ on $\mathbb{H}^2$ satisfy

$$d_{\mathrm{T}} = \frac{1}{2} \cdot \mathrm{d}_{\mathrm{hyp.}}.$$