Exercise 1 (Mapping class group invariance and completeness). Let X be a closed Riemann surface of genus at least 1.

(a) Show that the Teichmüller metric on $\mathcal{T}(X)$ is invariant under the action of the mapping class group.

Solution: Let $[X_1, f_1], [X_2, f_2] \in \mathcal{T}(X)$ and $[\varphi] \in \mathrm{MCG}(X)$. The Teichmüller distance $\mathrm{d_T}([X_1, f_1], [X_2, f_2])$ is realized by the unique Teichmüller map in the homotopy class of $f_2 \circ f_1^{-1} : X_1 \to X_2$. .

Now consider the points $[X_1, f_1 \circ \varphi^{-1}], [X_2, f_2 \circ \varphi^{-1}] \in \mathcal{T}(X)$. Since

$$\left(f_2 \circ \varphi^{-1}\right) \circ \left(f_1 \circ \varphi^{-1}\right)^{-1} = f_2 \circ f_1^{-1},$$

we find the same Teichmüller map (and hence the same Teichmüller distance) when we compute the distance between $[X_1, f_1 \circ \varphi^{-1}]$ and $[X_2, f_2 \circ \varphi^{-1}]$.

(b) Show that the Teichmüller metric on $\mathcal{T}(X)$ is complete. Hint: Use the homeomorphism $\mathcal{E} : \mathrm{QD}_1(X) \to \mathcal{T}(X)$ defined in the course.

Solution: Our goal is to show that closed bounded sets are compact (the Heine–Borel property). Write $B_{\mathrm{T}}(X, R)$ for the Teichmüller ball of radius R around X and let $C \subset B_{\mathrm{T}}(Y, R)$ be some closed set. By the fact that the map $\mathcal{E}$ is a homeomorphism, C is homeomorphic to a closed set

$$\mathcal{E}^{-1}(C) \subset \mathcal{E}^{-1}(B_{\mathrm{T}}(X, R)) \quad \subset \mathrm{QD}_1(X)$$

But

$$\begin{aligned}\mathcal{E}^{-1}(B_{\mathrm{T}}(X, R)) &= \left\{ q \in \mathrm{QD}_1(X);\ \frac{1 + \|q\|}{1 - \|q\|} < \exp(2R) \right\} \\ &= \left\{ q \in \mathrm{QD}_1(X);\ \|q\| < \frac{e^{2R} + 1}{e^{2R} - 1} \right\},\end{aligned}$$

where $||q|| = \int_X |q|$. This being a norm on $\mathrm{QD}_1(X)$ as a $6g - 6$ dimensional real vector space implies that $\mathcal{E}^{-1}(C)$ and hence C is compact.

Exercise 2 (Teichmüller lines). Let X be a closed Riemann surface of genus at least 1. Recall that a geodesic segment σ in a metric space (M, d) is a segment such that for any $x, y, z \in \sigma$ such that y lies between x and z we have

$$\mathrm{d}(x, z) = \mathrm{d}(x, y) + \mathrm{d}(y, z).$$

(a) Show that every geodesic segment in $\mathcal{T}(X)$ is a subsegment of some Teichmüller line.

Solution: Suppose $[X_1, f_1], [X_2, f_2], [X_3, f_3] \in \mathcal{T}(X)$ are such that

$$\mathrm{d_T}([X_1, f_1], [X_3, f_3]) = \mathrm{d_T}([X_1, f_1], [X_2, f_2]) + \mathrm{d_T}([X_2, f_2], [X_3, f_3])$$

and write $h_{ij} : X_i \to X_j$ for the Teichmüller map in the homotopy class of $f_j \circ f_i^{-1}$ and K_{ij} for its quasiconformal dilatation, for $i, j \in \{1, 2, 3\}$.

From the relation between the distances, we obtain

$$\log(K_{13}) = \log(K_{12}) + \log(K_{23})$$

and hence

$$K_{13} = K_{12} \cdot K_{23}.$$

On the other hand, by the estimate on the dilatation of compositions we proved in problem set 4, $h_{23} \circ h_{12}$ has quasiconformal dilatation at most $K_{12} \cdot K_{23}$. Because h_{13} is the Teichmüller map in the homotopy class of $h_{23} \circ h_{12}$, $h_{23} \circ h_{12}$ also has quasiconformal dillatation at least K_{13} with equality if and only if $h_{23} \circ h_{12} = h_{13}$. In conclusion

$$h_{13} = h_{23} \circ h_{12}.$$

Moreover, again looking back at the proof of the submultiplicativity of quasiconformal dilatations from problem set 4, we see that the equality case implies that the horizontal foliation for the initial quadratic differential corresponding to h_{23} needs to coincide with the horizontal foliation of the terminal quadratic differential of h_{12}. This means that the three points must lie on a Teichmüller line.

(b) Show that there is a unique geodesic in $\mathcal{T}(X)$ between any two points.

Solution: It follows from the uniqueness statement in Teichmüller's theorem that there can be at most one Teichmüller line between any pair of points. This porves the claim combined with the previous item.

Exercise 3 (The torus). Recall that the Teichmüller space of the torus $\mathbb{R}^2/\mathbb{Z}^2$ can be identified with $\mathbb{H}^2 = \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2, \mathbb{R})$, in which a point $[A]$ represents $[\mathbb{R}^2/(A \cdot \mathbb{Z}^2), f_A]$, where

$$f_A([(x, y)]) = [A \cdot (x, y)]$$

for all $[(x, y)] \in \mathbb{R}^2/\mathbb{Z}^2$. Prove that the Teichmüller metric $\mathrm{d_T}$ and the hyperbolic metric $\mathrm{d_{hyp.}}$ on $\mathbb{H}^2$ satisfy

$$d_{\mathrm{T}} = \frac{1}{2} \cdot \mathrm{d_{hyp.}}.$$

Solution: Given $[A], [B] \in \mathrm{SL}(2, \mathbb{R})$, the hyperbolic distance is computed as

$$\delta = \mathrm{d_{hyp.}}([A], [B]) = \min \left\{ 2 \cosh^{-1} \left(\frac{|\mathrm{tr}(C)|}{2} \right) ;\ C \in [A \cdot B^{-1}] \right\}.$$

Indeed, the identification $\mathbb{H}^2 = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ is through the orbit map that assigns to $[A] \in \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ the element $A \cdot i$ (which is well-defined because $\mathrm{SO}(2,\mathbb{R})$ is exactly the stabilizer of i). So the distance between $[A]$ and $[B]$ is the distance between $A \cdot i$ and $B \cdot i$, which is exactly the minimal translation length among all elements that maps $A \cdot i$ to $B \cdot i$ and that is computed by the expression on the right.

The matrix C that realizes this minimal translation length is some hyperbolic matrix of the form

$$C = T \cdot \begin{pmatrix} e^{\delta/2} & 0 \\ 0 & e^{-\delta/2} \end{pmatrix} \cdot T^{-1}$$

for some $\mathrm{SL}(2,\mathbb{R})$-matrix T. So C is a Teichmüller map with $K = e^{\delta}$. This means that (by Teichmüller's uniqueness theorem)

$$\mathrm{d_T}([A],[B]) = \frac{1}{2}\log(e^{\delta}) = \frac{\delta}{2}.$$