Teichmüller Theory

Lecture notes for the course
V5D3 - Advanced Topics in Geometry

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Version: July 4, 2019

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LECTURE 1

Basics on surfaces

The Teichmüller space of a surface S is the deformation space of complex structures on S and can also be seen as a space of hyperbolic metrics on S. The aim of this course will be to study the geometry and topology of this space and its quotient: the moduli space of hyperbolic metrics on S.

Before we get to any of this, we need to talk about surfaces themselves. So, today we will discuss some of the basics on surfaces.

1.1. Preliminaries on surface topology

1.1.1. Examples and classification. A *surface* is a smooth two-dimensional manifold. We call a surface *closed* if it is compact and has no boundary. A surface is said to be of *finite type* if it can be obtained from a closed surface by removing a finite number of points and (smooth) open disks with disjoint closures. In what follows, we will always assume our surfaces to be orientable.

Example 1.1. To properly define a manifold, one needs to not only describe the set but also give smooth charts. In what follows we will content ourselves with the sets (Exercise 1.1 completes the picture).

(a) The 2-*sphere* is the surface

$$\mathbb{S}^2 = \left\{ (x,y,z) \in \mathbb{R}^3; \quad x^2+y^2+z^2 = 1 \ \right\}.$$

(b) Let $\mathbb{S}^1$ denote the circle. The 2-*torus* is the surface

$$\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$$

(c) Given two (oriented) surfaces S_1, S_2, their *connected sum* $S_1 \# S_2$ is defined as follows. Take two closed sets $D_1 \subset S_1$ and $D_2 \subset S_2$ that are both diffeomorphic to closed disks, via diffeomorphisms

$$\varphi_i : \left\{ (x,y) \in \mathbb{R}^2; \quad x^2+y^2 \leq 1 \ \right\} \to D_i, \quad i=1,2,$$

so that φ_1 is orientation preserving and φ_2 is orientation reversing.

Then

$$S_1 \# S_2 = \left(S_1 \setminus \mathring{D}_1 \sqcup S_2 \setminus \mathring{D}_2 \right) / \sim$$

where $\mathring{D}_i$ denotes the interior of D_i for $i=1,2$ and the equivalence relation $\sim$ is defined by

$$\varphi_1(x,y) \sim \varphi_2(x,y) \quad \text{for all } (x,y) \in \mathbb{R}^2 \text{ with } x^2+y^2=1.$$

The figure below gives an example.

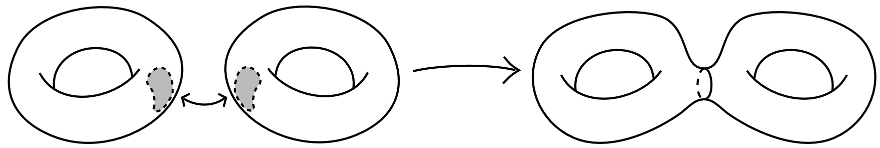

FIGURE 1. A connected sum of two tori.

> Like our notation suggests, the manifold $S_1 \# S_2$ is independent (up to diffeomorphism) of the choices we make (the disks and diffeomorphisms φ_i). This is a non-trivial statement, the proof of which we will skip. Likewise, we will also not prove that the connected sum of surfaces is an associative operation and that $\mathbb{S}^2 \# S$ is diffeomorphic to S for all surfaces S.

A classical result from the 19^{th} century tells us that the three simple examples above are enough to understand all finite type surfaces up to diffeomorphism.

Theorem 1.2 (Classification of closed surfaces)**.** *Every closed orientable surface is diffeomorphic to the connected sum of a* 2*-sphere with a finite number of tori.*

Indeed, because the diffeormorphism type of a finite type surface does not depend on where we remove the points and open disks (another claim we will not prove), the theorem above tells us that an orientable finite type surface is (up to diffeomorphism) determined by a triple of positive integers (g, b, n), where

- g is the number of tori in the connected sum and is called the *genus* of the surface.
- b is the number of disks removed and is called the number of *boundary components* of the surface.
- n is the number of points removed and is called the number of *punctures* of the surface.

Definition 1.3. The triple (g, b, n) defined above will be called the *signature* of the surface. We will denote the corresponding surface by $\Sigma_{g,b,n}$ and will write $\Sigma_g = \Sigma_{g,0,0}$.

1.1.2. Euler characteristic. The Euler characteristic is a useful topological invariant of a surface. There are multiple ways to define it. We will use triangulations. A *triangulation* $\mathcal{T} = (V, E, F)$ of a closed surface S will be the data of a finite set of points $V = \{v_1, \ldots . v_k\} \in S$ (called *vertices*), a finite set of arcs $E = \{e_1, \ldots, e_l\}$ with endpoints in the vertices (called edges) so that the complement $S \smallsetminus (\cup v_i \cup e_j)$ consists of a collection of disks $F = \{f_1, \ldots, f_m\}$ (called *faces*) that all connect to exactly 3 edges.

Note that a triangulation $\mathcal{T}$ here is a slightly more general notion than that of a simplicial complex (it's an example of what Hatcher calls a Δ-complex [**Hat02**, Page 102]). Figure 2 below gives an example of a triangulation of a torus that is not a simplicial complex.

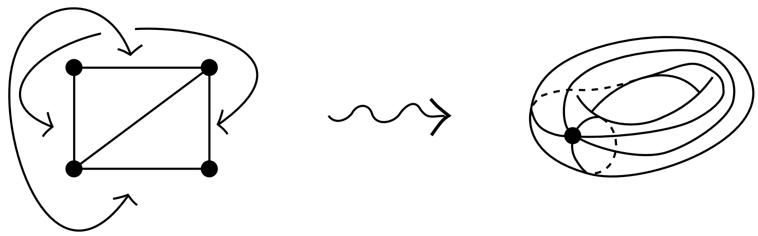

FIGURE 2. A torus with a triangulation

Definition 1.4. S be a closed surface with a triangulation $\mathcal{T} = (V, E, F)$. The *Euler characteristic* of S is given by

$$\chi(S) = |V| - |E| + |F|.$$

Because $\chi(S)$ can be defined entirely in terms of singular homology (see [**Hat02**, Theorem 2.4] for details), it is a homotopy invariant. In particular this implies it should only depend on the genus of our surface S. Indeed, we have

Lemma 1.5. *Let S be a closed connected and oriented surface of genus g. We have*

$$\chi(S) = 2 - 2g.$$

PROOF. See Exercise 1.2. □

For surfaces that are not closed, we can define

$$\chi(\Sigma_{g,b,n}) = 2 - 2g - b - n.$$

This can be computed with a triangulation as well. For surfaces with only boundary components, the usual definition still works. For surfaces with punctures there no longer is a finite triangulation, so the definition above no longer makes sense. There are multiple ways out. The most natural is to use the homological definition, which gives the formula above. Another option is to allow some vertices to be missing, that is, to allow edges to run between vertices and punctures. Both give the formula above.

1.2. Riemann surfaces

For the basics on Riemann surfaces, we refer to [**Bea84, FK92**] and for a reference on complex functions of a single variable, we refer to [**SS03**].

1.2.1. Definition and first examples. A Riemann surface is a one-dimensional complex manifold. That is,

Definition 1.6. A *Riemann surface* X is a connected Hausdorff topological space X, equipped with an open cover $\{U_\alpha\}_{\alpha \in A}$ of open sets and maps $\varphi_\alpha : U_\alpha \to \mathbb{C}$ so that

(1) $\varphi_\alpha(U_\alpha)$ is open and φ_α is a homeomorphism onto its image.
(2) For all $\alpha, \beta \in A$ so that $U_\alpha \cap U_\beta \neq \emptyset$ the map

$$\varphi_\alpha \circ (\varphi_\beta)^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

is holomorphic.

The pairs $(U_\alpha, \varphi_\alpha)$ are usually called *charts* and the collection $((U_\alpha, \varphi_\alpha))_{\alpha\in A}$ is usually called an *atlas*.

Note that we do not a priori assume a Riemann surface X to be a second countable space. It is however a theorem by Radó that every Riemann surface is second countable (for a proof, see [**Hub06**, Section 1.3]). Moreover every Riemann surface is automatically orientable (see for instance [**GH94**, Page 18]).

Example 1.7. (a) The simplest example is of course $X = \mathbb{C}$ equipped with one chart: the identity map.

(b) We set $X = \mathbb{C}\cup\{\infty\} = \widehat{\mathbb{C}}$ and give it the topology of the one point compactification of $\mathbb{C}$, which is homeomorphic to the sphere $\mathbb{S}^2$. The charts are

$$U_0 = \mathbb{C},\ \varphi_0(z) = z$$

and

$$U_\infty = X \setminus \{0\},\ \varphi_\infty(z) = 1/z.$$

So $U_0 \cap U_\infty = \mathbb{C}\setminus\{0\}$ and

$$\varphi_0 \circ (\varphi_\infty)^{-1}(z) = 1/z \quad \text{for all } z \in \mathbb{C}\setminus\{0\}$$

which is indeed holomorphic on $\mathbb{C}\setminus\{0\}$. $\widehat{\mathbb{C}}$ is usually called the *Riemann sphere*.

(c) Recall that a *domain* $D \subset \widehat{\mathbb{C}}$ is any connected and open set in $\widehat{\mathbb{C}}$. Any such domain inherits the structure of a Riemann surface from $\widehat{\mathbb{C}}$.

1.2.2. Quotients. To get a larger set of examples, we will consider quotients. First of all, we need the notion of a holomorphic map:

Definition 1.8. Let X and Y be Riemann surfaces, equipped with atlasses $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ and $\{(V_\beta, \psi_\beta)\}_{\beta\in B}$ respectively. A function $f : X \to Y$ is called *holomorphic* if

$$\psi_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap f^{-1}(V_\beta)) \to \psi_\beta(f(U_\alpha) \cap V_\beta)$$

is holomorphic for all $\alpha \in A, \beta \in B$ so that $f(U_\alpha) \cap V_\beta \neq \emptyset$. A bijective holomorphism is called a *biholomorphism* or *conformal*.

Note that when $\varphi : X \to Y$ is biholomorphic, then so is $\varphi^{-1} : Y \to X$.

The group

$$\mathrm{PSL}(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix};\quad a,b,c,d \in \mathbb{C}, \quad ad-bc = 1 \right\} \Big/ \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

acts on $\widehat{\mathbb{C}}$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{az+b}{cz+d}, \tag{1}$$

where some care needs to be taken with the point ∞ (see Exercise 1.3). It turns out that $\mathrm{PSL}(2,\mathbb{C})$ is the full group of biholomorphisms of $\widehat{\mathbb{C}}$, but we will skip over this for now. Moreover, each such map that is not the identity has two fixed points (that might coincide) (see Exercise 1.3). These maps are called Möbius transformations.

Theorem 1.9. *Let $D \subset \widehat{\mathbb{C}}$ be a domain and let $G < \mathrm{PSL}(2,\mathbb{C})$ so that*

(1) $g(D) = D$ *for all* $g \in G$
(2) If $g \in G \setminus \{e\}$ *then the fixed points of* g *lie ourside of* D.
(3) For each compact subset $K \subset D$, *the set*

$$\{g \in G; \quad g(K) \cap K \neq \emptyset \ \}$$

is finite.

Then the quotient space

$$D/G$$

has the structure of a Riemann surface.

A group that satisfies the second condition is said to act *freely* on D and a group that satisfies the thirs condition is said to act *properly discontinuously* on D. We will postpone the proof of the theorem to the next lecture.

1.2.3. Tori. The theorem from the previous section gives us a lot of new examples. The first is that of tori. Consider the elements

$$g_1 := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, g_\tau := \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \in \mathrm{PSL}(2, \mathbb{C}),$$

for some $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$, acting on the domain $\mathbb{C} \subset \widehat{\mathbb{C}}$ by

$$g_1(z) = z + 1 \quad \text{and} \quad g_\tau(z) = z + \tau$$

for all $z \in \mathbb{C}$.

We define the group

$$\Lambda_\tau = \langle g_1, g_\tau \rangle < \mathrm{PSL}(2, \mathbb{C}).$$

A direct computation shows that

$$\begin{bmatrix} 1 & p + q\tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & r + s\tau \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & p + q + (r + s)\tau \\ 0 & 1 \end{bmatrix},$$

for all $p, q, r, s \in \mathbb{Z}$, from which it follows that

$$\Lambda_\tau = \left\{ \begin{bmatrix} 1 & n + m\tau \\ 0 & 1 \end{bmatrix}; \quad m, n \in \mathbb{Z} \ \right\} \simeq \mathbb{Z}^2 .$$

Let us consider the conditions from Theorem 1.9. (1) is trivially satisfied: Λ_τ preserves $\mathbb{C}$. Any non-trivial element in Λ_τ is of the form

$$\begin{bmatrix} 1 & n + m\tau \\ 0 & 1 \end{bmatrix}$$

and hence only has the point $\infty \in \widehat{\mathbb{C}}$ as a fixed point (see Exercise 1.3), which gives us condition (2). To check condition (3), suppose $K \subset \mathbb{C}$ is compact. Write $d_K = \sup\{|z - w|; \quad z, w \in K \ \} < \infty$. Given $g \in \Lambda_\tau$, write

$$T_g = \inf\{|gz - z|; \quad z \in \mathbb{C} \ \}$$

for the *translation length* of g. Note that $T_g = |gz - z|$ for all $z \in \mathbb{C}$. This is quite special. We have

$$\{g \in \Lambda_\tau; \quad g(K) \cap K \neq \emptyset \ \} \subset \{g \in \Lambda_\tau; \quad T_g \leq 2d_K \ \}$$

and the latter is finite. So $\mathbb{C}/\Lambda_z$ is indeed a Riemann surface.

We claim that this is a torus. One way to see this is to note that the quotient map $\pi : \mathbb{C} \to \mathbb{C}/\Lambda_\tau$ restricted to the convex hull

$$\mathcal{F} = \mathrm{conv}(\{0, 1, \tau, 1+\tau) \\ := \left\{ \lambda_1 + \lambda_2\tau + \lambda_3(1+\tau);\quad \lambda_1, \lambda_1, \lambda_3 \in [0,1], \lambda_1 + \lambda_2 + \lambda_3 \leq 1 \right\}$$

is surjective. Figure 3 shows a picture of $\mathcal{F}$. On $\mathring{\mathcal{F}}$, π is also injective. So to understand what the quotient looks like, we only need to understand what happens to the sides of $\mathcal{F}$. Since the quotient map identifies the left hand side of $\mathcal{F}$ with the right hand side and the top with the bottom, the quotient is a torus.

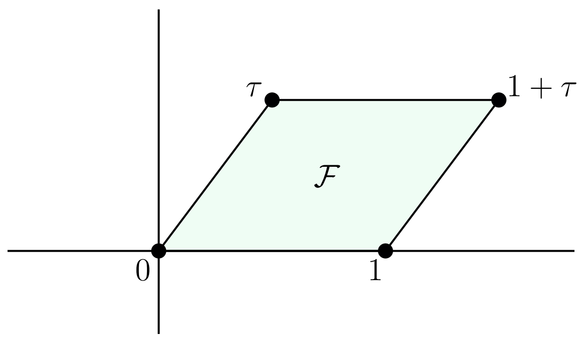

FIGURE 3. A fundamental domain for the action $\Lambda_\tau \curvearrowright \mathbb{C}$.

We can also prove that $\mathbb{C}/\Lambda_\tau$ is a torus by using the fact that for all $z \in \mathbb{C}$ there exist unique $x, y \in \mathbb{R}$ so that

$$z = x + y\tau.$$

The map $\mathbb{C}/\Lambda_\tau \to \mathbb{S}^1 \times \mathbb{S}^1$ given by

$$[x + y\tau] \mapsto (e^{2\pi i x}, e^{2\pi i y})$$

is a homeomorphism.

Note that we have not yet proven whether all these tori are distinct as Riemann surfaces. But it will turn out later that many of them are.

1.3. Hyperbolic surfaces

Set $\mathbb{H}^2 = \left\{ z \in \mathbb{C};\quad \mathrm{Im}(z) > 0 \right\}$, the upper half plane. It turns out that the group of biholomorphisms of $\mathbb{H}^2$ is $\mathrm{PSL}(2,\mathbb{R})$. We will see a lot more about this later during the course, but for now we will just note that there are many subgroups of $\mathrm{PSL}(2,\mathbb{R})$ that satisfy the conditions of Theorem 1.9.

It also turns out that $\mathrm{PSL}(2,\mathbb{R})$ is exactly the group of orientation preserving isometries of the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

This is a complete metric of constant curvature -1. So, this means that all these Riemann surfaces naturally come equipped with a complete metric of constant curvature -1.

1.4. Exercises

Exercise 1.1. Define an atlas for $\mathbb{S}^2$ and $\mathbb{T}^2$.

Exercise 1.2. In this exercise we prove Lemma 1.5 in two different ways.

1. (a) Describe a way to obtain a closed oriented surface of genus g from a polygon with $4g$ sides.
   (b) Use a triangulation of a $4g$-gon to prove Lemma 1.5.
2. (a) Show that if a closed oriented surface S is the connect sum of two closed oriented surfaces S_1 and S_2 then
   $$\chi(S) = \chi(S_1) + \chi(S_2) - 2.$$
   (b) Compute the euler characteristic of the 2-sphere and the torus and use those to prove Lemma 1.5.

Exercise 1.3. (a) Given $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2, \mathbb{C})$,

- what is $g(\infty)$?
- what is $g^{-1}(\infty)$?
- what are the fixed points of g?

(b) Show that (1) defines an action

LECTURE 2

Quotients, uniformization, tori and spheres

2.1. Quotients revisited

We begin by tying up a loose end from the previous lecture:

Theorem 1.9. *Let $D \subset \widehat{\mathbb{C}}$ be a domain and let $G < \mathrm{PSL}(2, \mathbb{C})$ so that*

(1) $g(D) = D$ for all $g \in G$
(2) If $g \in G \setminus \{e\}$ then the fixed points of g lie ourside of D.
(3) For each compact subset $K \subset D$, the set

$$\left\{ g \in G; \quad g(K) \cap K \neq \emptyset \ \right\}$$

is finite.

Then the quotient space

$$D/G$$

has the structure of a Riemann surface.

Proof. Let $\pi : D \to D/G$ denote the quotient map. First of all, since D is connected and π is continuous, D/G is connected.

In order to show that D/G is Hausdorff, we consider two distinct points

$$\pi(z_1) \neq \pi(z_2) \in D/G$$

where z_1 and z_2 are two pre-images in D. Define

$$A_n = \left\{ w \in D; \quad |w - z_1| < r/n \ \right\} \quad \text{and} \quad B_n = \left\{ w \in D; \quad |w - z_2| < r/n \ \right\},$$

where $r > 0$ is small enough so that

$$K = \overline{A_1} \cup \overline{B_1} \subset D.$$

Now, suppose that for all $n \geq 1$ we have

$$\pi(A_n) \cap \pi(B_n) \neq \emptyset$$

This means that we can find some sequence $a_n \in A_n$ and $g_n \in G$ so that

$$g_n(a_n) \in B_n$$

for all $n \in \mathbb{N}$. This means that

$$\emptyset \neq g_n(A_n) \cap B_n \subset g_n(K) \cap K$$

for all $n \in \mathbb{N}$ and hence by the third assumption, the set $\{g_n\}_{n \in \mathbb{N}}$ is finite. This means that there is a subsequence so that $g_n = g$ for some fixed $g \in G$ and all n large enough

$$z_2 = \lim_{n \to \infty} g_n(a_n) = \lim_{n \to \infty} g(a_n) = g(z_1),$$

which contradicts $\pi(z_1) \neq \pi(z_2)$ and hence proves that D/G is Hausdorff.

All that remains is to find an atlas. To this end, select a precompact open disk $K_z \subset D$ around each $z \in D$. By assumptions (2) and (3) we can choose K_z small enough so that no non-trivial translate $g(K_z)$ intersects it. This implies that the map

$$\pi|_{K_z} : K_z \to D/G$$

is a homeomorphism onto its image. So we set

$$U_z = \pi(K_z) \quad \text{and} \quad \varphi_z = (\pi|_{K_z})^{-1} : U_z \to D.$$

This means that the transition maps are of the form

$$\varphi_z \circ \varphi_w^{-1} = (\pi|_{K_z})^{-1} \circ (\pi|_{K_w}).$$

Given any element ζ in the domain of this map, we have

$$\varphi_z \circ \varphi_w^{-1}(\zeta) = g(\zeta) =: \xi$$

for some $g \in G$ and $\xi \in D$. Near ζ we have $\pi = \pi|_{K_w}$ while near ξ we have $\pi = \pi|_{K_z}$. Since $\pi = \pi \circ g$ for all $g \in G$, we obtain

$$\pi|_{K_z} = \pi|_{K_w} \circ g$$

and hence

$$\varphi_z \circ \varphi_w^{-1} = g$$

near ζ, which is holomorphic. □

2.2. The uniformization theorem and automorphism groups

The Riemann mapping theorem tells us that any pair of simply connected domains in $\mathbb{C}$ that are both not all of $\mathbb{C}$ are biholomorphic. In the early 20^{th} century this was generalized by Koebe and Poincaré to a classification of *all* simply connected Riemann surfaces:

Theorem 2.1 (Uniformization theorem)**.** *Let X be a simply connected Riemann surface. Then X is biholomorphic to exactly one of*

$$\widehat{\mathbb{C}}, \quad \mathbb{C} \quad \textit{or} \quad \mathbb{H}^2.$$

Proof. See for instance [**FK92**, Chapter IV]. □

For later use, we define:

Definition 2.2. Let X be a Riemann surface, its automorphism group is given by

$$\mathrm{Aut}(X) := \left\{ \varphi : X \to X; \quad \varphi \text{ is a biholomorphism } \right\}.$$

We record the following fact:

Proposition 2.3.
- $\mathrm{Aut}(\widehat{\mathbb{C}}) = \mathrm{PSL}(2,\mathbb{C})$ *acting by Möbius transformations,*
- $\mathrm{Aut}(\mathbb{C}) = \left\{ \varphi : z \mapsto az + b; \quad a \in \mathbb{C}\setminus\{0\}, b \in \mathbb{C} \right\}$,
- $\mathrm{Aut}(\mathbb{H}^2) = \mathrm{PSL}(2,\mathbb{R})$ *acting by Möbius transformations.*

Proof. See for instance [**Bea84**, Chapter 5] or [**IT92**, Section 2.3]. □

Note that in all three cases, we have

$$\mathrm{Aut}(X) = \left\{ g \in \mathrm{Aut}(\widehat{\mathbb{C}});\quad g(X) = X \right\},$$

that is, all the automorphisms of $\mathbb{C}$ and $\mathbb{H}^2$ extend to $\widehat{\mathbb{C}}$. However, not all automorphisms of $\mathbb{H}^2$ extend to $\mathbb{C}$.

The uniformization theorem implies that every Riemann surface is of the form of the surfaces in Theorem 1.9.

Corollary 2.4. *Let X be a Riemann surface. Then there exists a group $G < \mathrm{Aut}(D)$, where D is exactly one of $\mathbb{C}$, $\widehat{\mathbb{C}}$ or $\mathbb{H}^2$ so that*

- *G acts freely and properly discontinuously on D and*
- *$X = D/G$ as a Riemann surface.*

PROOF. Let $\widetilde{X}$ denote the universal cover of X and $\pi_1(X)$ its fundamental group. The fact that X is a Riemann surface, implies that $\widetilde{X}$ can be given the structure of a Riemann surface too, so that $\pi_1(X)$ acts freely and properly discontinuously on $\widetilde{X}$ by biholomorphisms (see for instance [**IT92**, Lemma 2.6]) and so that

$$\widetilde{X}/\pi_1(X) = X.$$

Since $\widetilde{X}$ is simply connected, it must be biholomorphic to exactly one of $\mathbb{C}$, $\widehat{\mathbb{C}}$ or $\mathbb{H}^2$. □

2.3. Quotients of the three simply connected Riemann surfaces

Now that we know that we can obtain all Riemann surfaces as quotients of one of three simply connected Riemann surfaces, we should start looking for interesting quotients.

2.3.1. Quotients of the Rieman sphere. It turns out that for the Riemann sphere there are none:

Proposition 2.5. *Let X be a Riemann surface. The universal cover of X is biholomorphic to $\widehat{\mathbb{C}}$ if and only if X is biholomorphic to $\widehat{\mathbb{C}}$.*

PROOF. The "if" part is clear. For the "only if" part, note that every element in $\mathrm{PSL}(2,\mathbb{C})$ has at least one fixed point on $\widehat{\mathbb{C}}$ (see Exercise 1.3(a)). Since, by assumption

$$X = \widehat{\mathbb{C}}/G,$$

where G acts properly discontinuously and freely, we must have $G = \{e\}$. □

2.3.2. Quotients of the plane. In Section 1.2.3, we have already seen that in the case of the complex plane, the list of quotients is a lot more interesting: there are tori. This however turns out to be almost everything:

Proposition 2.6. *Let X be a Riemann surface. The universal cover of X is biholomorphic to $\mathbb{C}$ if and only if X is biholomorphic to either $\mathbb{C}$, $\mathbb{C}\setminus\{0\}$ or*

$$\mathbb{C} \,/ \left\langle \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} \right\rangle$$

for some $\lambda, \mu \in \mathbb{C}\setminus\{0\}$ that are linearly independent over $\mathbb{R}$.

PROOF. First suppose $X = \mathbb{C}/G$. Since G acts properly discontinuously, G is one of the following three forms:

(1) $G = \{e\}$
(2) $G = \langle \varphi_b \rangle$, where $\varphi_b(z) = z + b$ for some $b \in \mathbb{C} \setminus \{0\}$
(3) $G = \langle \varphi_{b_1}, \varphi_{b_2} \rangle$ where $b_1, b_2 \in \mathbb{C}$ are independent over $\mathbb{R}$.

(see Exercise 2.1). We have already seen that the third case gives rise to tori. In the second case, the surface is biholomorphic to $\mathbb{C} \setminus \{0\}$ (see Exercise 2.2).

Now let us prove the converse. For $X = \mathbb{C}$ the statement is clear. Likewise, for $X = \mathbb{C} \setminus \{0\}$, we have just seen that the composition

$$\mathbb{C} \to \mathbb{C}/(z \sim z+1) \simeq \mathbb{C} \setminus \{0\}$$

is the universal covering map. So the only question is whether every Riemann surface structure on the torus comes from the complex plane. We have seen above that the universal cover cannot be the Riemann sphere, which means that (using the uniformization theorem) all we need to prove is that it cannot be the upper half plane either.

The fundamental group of the torus is isomorphic to $\mathbb{Z}^2$, so what we need to prove is that there is no subgroup of $\mathrm{Aut}(\mathbb{H}^2) = \mathrm{PSL}(2, \mathbb{R})$ that acts properly discontinuously and freely on $\mathbb{H}^2$. For this we use Lemma 2.7, the proof of which we leave to the reader. □

Lemma 2.7. *Suppose $G < \mathrm{PSL}(2, \mathbb{R})$ acts properly discontinuously and freely on $\mathbb{H}^2$ and suppose furthermore that G is abelian. Then $G \simeq \mathbb{Z}$.*

2.3.3. Quotients of the upper half plane, part I. It will turn out that the richest family of Riemann surfaces is that of quotients of $\mathbb{H}^2$. Indeed, looking at the clasification of closed orientable surfaces, we note that we have so far only seen the sphere and the torus. It turns out that all the other closed orientable surfaces also admit the structure of a Riemann surface. In fact, they all admit lots of different such structures. The two propositions above imply that they must all arise as quotients of $\mathbb{H}^2$.

We will not yet discuss how to construct all these surfaces but instead discuss an example (partially taken from [**GGD12**, Example 1.7]). Fix some distinct complex numbers $a_1, \ldots, a_{2g+1}$ and consider the following subset of $\mathbb{C}^2$:

$$\mathring{X} = \left\{ (z, w) \in \mathbb{C}^2; \quad w^2 = (z - a_1)(z - a_2) \cdots (z - a_{2g+1}) \right\}.$$

Let X denote the one point compactification of $\mathring{X}$ obtained by adjoining the point (∞, ∞).

As opposed to charts, we will describe inverse charts, or *parametrizations* around every $p \in \mathring{X}$:

- Suppose $p = (z_0, w_0) \in \mathring{X}$ is so that $z_0 \neq a_i$ for all $i = 1, \ldots, 2g+1$. Set

$$\varepsilon := \min_{i=1,\ldots,2g+1} \{ |z_0 - a_i| / 2 \}$$

Then define the map $\varphi^{-1} : \left\{ \zeta \in \mathbb{C}; \quad |\zeta| < \varepsilon \right\} \to \mathring{X}$ by

$$\varphi^{-1}(\zeta) = \left(\zeta + z_0, \quad \sqrt{(\zeta + z_0 - a_1) \cdots (\zeta + z_0 - a_2)} \right),$$

where the branch of the square root is chosen so that $\varphi^{-1}(0) = (z_0, w_0)$, gives a parametrization.

- For $p = (a_j, 0)$, we set

$$\varepsilon := \min_{i \neq j} \{ \sqrt{|z - a_i| / 2} \}$$

Then define the map $\varphi^{-1} : \{ \zeta \in \mathbb{C}; \quad |\zeta| < \varepsilon \ \} \to \mathring{X}$ by

$$\varphi^{-1}(\zeta) = \left(\zeta^2 + a_j, \quad \zeta \sqrt{\prod_{i \neq j} (\zeta^2 + a_j - a_i)} \right).$$

The reason that we need to take different charts around these points is that

$$\sqrt{z - a_j}$$

is not a well defined holomorphic function near $z = a_j$.

Also note that the choice of the branch of the root does not matter. By changing the branch we would obtain a new parametrization $\widetilde{\varphi}^{-1}$ that satisfies $\widetilde{\varphi}^{-1}(\zeta) = \widetilde{\varphi}^{-1}(-\zeta)$.

We leave the fact that this defines a Riemann surface structure as an exercise (Exercise 2.3).

It's not hard to see that $\mathring{X}$ is not bounded as a subset of $\mathbb{C}^2$. This means in particular that it's not compact. We can however compactify it in a similar fashion to how we compactified $\mathbb{C}$ in order to obtain the Riemann sphere. That is, we add a point (∞, ∞) and around this point define a parametrization:

$$\varphi_\infty^{-1}(\zeta) = \begin{cases} \left(\zeta^{-2}, \quad \zeta^{-(2g+1)} \sqrt{(1 - a_1 \zeta^2) \cdots (1 - a_{2g+1} \zeta^2)} \right) & \text{if } \zeta \neq 0 \\ (\infty, \infty) & \text{if } \zeta = 0, \end{cases}$$

for all $\zeta \in \{ |\zeta| < \varepsilon \}$ and some appropriate $\varepsilon > 0$.

2.4. Exercises

Exercise 2.1. Suppose $G < \mathrm{Aut}(\mathbb{C})$ acts properly discontinuously on $\mathbb{C}$.

(a) Show that G cannot contain a Möbius transformation of the form

$$\varphi : z \mapsto az + b$$

with $a \neq 1$. *Hint: Show that* $\varphi^n(z) = a^n z + \frac{1-a^n}{1-a} b$. *There are two cases to consider:* $|a| \neq 1$ *and* $|a| = 1$.

(b) Show that G is either generated by a single Möbius transformation or by two Möbius transformations that are linearly independent over $\mathbb{R}$.

Exercise 2.2. Let $b \in \mathbb{C} \setminus \{0\}$ and consider the Möbius transformation $\varphi_b : \mathbb{C} \to \mathbb{C}$ given by

$$\varphi_b(z) = z + b$$

for all $z \in \mathbb{C}$. Show that the map

$$[z] \in \mathbb{C} / \langle \varphi_b \rangle \mapsto e^{2\pi i z/b} \in \mathbb{C} \setminus \{0\}$$

is a biholomorphism.

Exercise 2.3. (a) Check that the parametrizations in Section 2.3.3 give rise to holomorphic charts.

LECTURE 3

More on quotients and conformal structures

3.1. Quotients of the upper half plane, part II

We continue the discussion from Section 2.3.3. The reason that the resulting surface X is compact is that we can write it as the union of the sets

$$\left\{(z,w) \in \mathring{X}; \quad |z| \le 1/\varepsilon^2 \right\} \cup \left(\left\{(z,w) \in \mathring{X}; \quad |z| \ge 1/\varepsilon^2 \right\} \cup \{(\infty,\infty)\}\right),$$

for some small $\varepsilon > 0$. The first set is compact beacause it's a bounded subset of $\mathbb{C}^2$. The second set is compact because it's $\varphi_\infty^{-1}(\{|\zeta| \le \varepsilon\})$.

To see that X is connected, we could proceed using charts as well. We would have to find a collection of charts that are all connected, overlap and cover X. However, it's easier to use complex analysis. Suppose $z_0 \neq a_i$ for all $i = 1, \ldots, a_{2g+1}$. In that case, we can define a path

$$z \mapsto \left(z, \quad \sqrt{\prod_{i=1}^{2g+1} (z - a_i)}\right)$$

between z_0 and a_i using analytic continuation.

To figure out the genus of X, note that there is a map $\pi : X \to \widehat{\mathbb{C}}$ given by

$$\pi(z,w) = z \quad \text{for all } (z,w) \in X.$$

This map is two-to-one almost everywhere. Only the points $z = a_i$, $i = 1, \ldots, 2g+1$ and the point $z = \infty$ have only one pre-image.

Now triangulate $\widehat{\mathbb{C}}$ so that the vertices of the triangulation coincide with the points $a_1, \ldots, a_{2g+1}, \infty$. If we lift the triangulation to X using π, we can compute the Euler characteristic of X. Every face and every edge in the triangulation of $\widehat{\mathbb{C}}$ has two pre-images, whereas each vertex has only one. This means that:

$$\chi(X) = 2\chi(\widehat{\mathbb{C}}) - (2g+2) = 2 - 2g.$$

Because X is an orientable closed surface, we see that it must have genus g (Lemma 1.5). In particular, if $g \ge 2$, these surfaces are quotients of $\mathbb{H}^2$. Note that this also implies that for $g \ge 1$, the Riemann surface $\mathring{X}$ is also a quotient of $\mathbb{H}^2$.

To get a picture of what X looks like, draw a closed arc α_1 between a_1 and a_2 on $\widehat{\mathbb{C}}$, an arc α_2 between a_3 and a_4 that does not intersect the first arc and so on, and so forth. The last arc α_{g+1} goes between a_{2g+1} and ∞. Figure 1 shows a picture of what these arcs might look like.

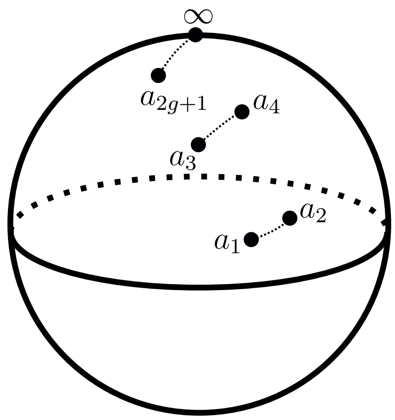

FIGURE 1. $\widehat{\mathbb{C}}$ with some intervals removed.

Let

$$D = \widehat{\mathbb{C}} \setminus \left(\bigcup_{i=1}^{g+1} \alpha_i \right).$$

The map

$$\pi|_{\pi^{-1}(D)} : \pi^{-1}(D) \to D$$

is now a two-to-one map. Moreover on the arcs, it's two-to-one on the interior and one-to-one on the boundary. Because it's also smooth, this means that the pre-image of the arcs is a circle. So, X may be obtained (topologically) by cutting $\widehat{\mathbb{C}}$ open along the arcs, taking two copies of that, and gluing these along their boundary. Figure 2 depicts this process.

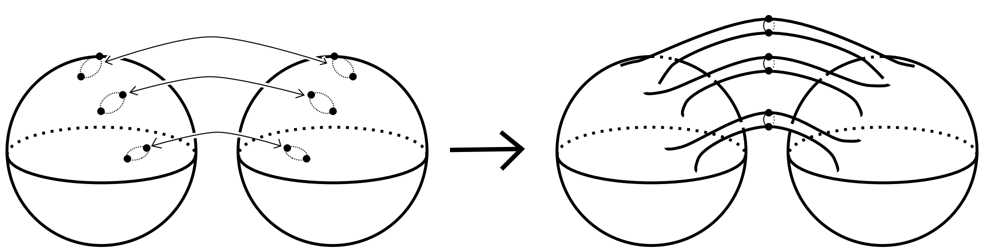

FIGURE 2. Gluing X out of two Riemann spheres.

Finally, we note that our Riemann surfaces come with an involution $\imath : X \to X$, given by

$$\imath(w) = \begin{cases} -w & \text{if } w \neq \infty \\ \infty & \text{if } w = \infty. \end{cases}$$

This map is called the *hyperelliptic involution* and the surfaces we described are hence called *hyperelliptic surfaces*. Note that $\pi : X \to \widehat{\mathbb{C}}$ is the quotient map $X \to X/\imath$.

3.2. Riemannian metrics and Riemann surfaces

We already noted that every Riemann surface comes with a natural Riemannian metric. Indeed the Riemann sphere has the usual round metric of constant curvature $+1$. Likewise, $\mathbb{C}$ has a flat metric, its usual Euclidean metric $\mathrm{Aut}(\mathbb{C})$ does not act by isometries. However, in the proof of Proposition 2.6, we saw that all the quotients are obtained by quotienting by a group that does act by Euclidean isometries. This means that the Euclidean metric descends. Finally, we will see later that $\mathrm{Aut}(\mathbb{H}^2)$ also acts by isometries of the hyperbolic metric defined in Section 1.3. So every quotient of $\mathbb{H}^2$ comes with a natural metric of constant curvature -1.

It turns out that we can also go the other way around. That is: Riemann surface structures on a given surface are in one-to-one correpsondence with complete metrics of constant curvature.

One way to see this uses the Killing-Hopf theorem. In the special case of surfaces, this states that every oriented surface equipped with a Riemannian metric of constant curvature $+1$, 0 or -1 can be obtained as the quotient by a group of orientation preserving isometries acting properly discontinuously and freely on $\mathbb{S}^2$ equipped with the round metric, $\mathbb{R}^2$ equipped with the Euclidean metric or $\mathbb{H}^2$ equipped with the hyperbolic metric respectively (see [**CE08**, Theorem 1.37] for a proof). For a Riemannian manifold M, let us write

$$\mathrm{Isom}^+(M) = \left\{\varphi : M \to M;\quad \varphi \text{ is an orientation preserving isometry } \right\}.$$

So, we need the fact that

(1) $\mathrm{Isom}^+(\mathbb{S}^2) = \mathrm{SO}(2,\mathbb{R})$ and this has no non-trivial subgroups that act properly discontinuously on $\mathbb{S}^2$.
(2) $\mathrm{Isom}^+(\mathbb{R}^2) = \mathrm{SO}(2,\mathbb{R}) \ltimes \mathbb{R}^2$, where $\mathbb{R}^2$ acts by translations. The only subgroups of this group that act properly discontinuously and freely are the fundamental groups of tori (see Exercise 2.1).
(3) $\mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}(2,\mathbb{R})$.

Assuming these facts for now, we get our one-to-one correspondence:

Proposition 3.1. *Given an orientable surface Σ of finite type with $\partial\Sigma = \emptyset$, the identification described above gives a one-to-one correspondence of sets*

$$\left\{ \begin{array}{c} \textit{Riemann surface} \\ \textit{structures on } \Sigma \end{array} \right\} \Big/ \textit{biholomorphism} \leftrightarrow \left\{ \begin{array}{c} \textit{Complete Riemannian} \\ \textit{metrics of constant} \\ \textit{curvature } \{-1,0,+1\} \\ \textit{on } \Sigma \end{array} \right\} \Big/ \textit{isometry.}$$

Whether the curvature is 0, $+1$ or -1 is determined by the topology of Σ. This for instance follows from the discussion above. It can also be seen from the Gauss-Bonnet theorem. Recall that in the case of a closed Riemannian surface X, this states that

$$\int_X K \, dA = 2\pi \, \chi(\Sigma),$$

where K is the Gaussian curvature on X and dA the area measure. For constant curvature κ, this means that

$$\kappa \cdot \text{area}(X) = 2\pi\ \chi(X)$$

So $\chi(X) = 0$ if and only of $\kappa = 0$ and otherwise $\chi(X)$ needs to have the same sign as κ. This last equality generalizes to finite type surfaces and we obtain:

Lemma 3.2. *Let X be a hyperbolic surface homeomorphic to $\Sigma_{g,b,n}$ then*

$$\text{area}(X) = 2\pi(2g + n + b - 2).$$

3.3. Conformal structures

There is another type of structures on a surface that is in one-to-one correspondence with Riemann surface structures, namely conformal structures.

We say that two Riemannian metrics ds_1^2 and ds_2^2 on a surface X are *conformally equivalent* is there exists a positive function $\rho : X \to \mathbb{R}_+$ so that

$$ds_1^2 = \rho \cdot ds_2^2.$$

So a conformal equivalence class of Riemannian metrics can be seen as a notion of angles on the surface.

We have already seen that a Riemann surface structure induces a Riemannian metric on the surface, so it certainly also induces a conformal class of metrics.

So, we need to explain how to go back. We will also only sketch this. First of all, suppose we are given a surface X with charts $(U_j, (u_j, v_j))_j$ equipped with a Riemannian metric that in all local coordinates (u_j, v_j) is of the form

$$ds^2 = \rho(u_j, v_j) \cdot (du_j^2 + dv_j^2),$$

where $\rho : X \to \mathbb{R}_+$ is some smooth function. Consider the complex-valued coordinate

$$w_j = u_j + i\ v_j.$$

We claim that this is holomorphic. Indeed, applying a coordinate change on $U_j \cap U_k$, we have

$$ds^2 = \rho(u_k, v_k) \cdot \left[\left(\left(\frac{\partial u_j}{\partial u_k}\right)^2 + \left(\frac{\partial v_j}{\partial u_k}\right)^2 \right) du_k^2 + \left(\left(\frac{\partial u_j}{\partial v_k}\right)^2 + \left(\frac{\partial v_j}{\partial v_k}\right)^2 \right) dv_k^2 \right.$$
$$\left. + 2 \left(\frac{\partial u_j}{\partial u_k}\frac{\partial v_j}{\partial v_k} + \frac{\partial u_j}{\partial v_k}\frac{\partial v_j}{\partial u_k} \right) du_k dv_k \right].$$

Our assumption implies that

$$\left(\frac{\partial u_j}{\partial u_k}\right)^2 + \left(\frac{\partial v_j}{\partial u_k}\right)^2 = \left(\frac{\partial u_j}{\partial v_k}\right)^2 + \left(\frac{\partial v_j}{\partial v_k}\right)^2$$

and

$$\frac{\partial u_j}{\partial u_k}\frac{\partial v_j}{\partial v_k} = \frac{\partial u_j}{\partial v_k}\frac{\partial v_j}{\partial u_k} = 0.$$

Some elementary, but tedious, manipulations show that these are equivalent to the Cauchy-Riemann equations for the chart transition $w_k \circ w_j^{-1}$, which means that these coordinates are indeed holomorphic. The coordinates (U_j, w_j) are usually called *isothermal coordinates*.

Also note that we have not used the factor ρ, so any metric that is conformal to our metric will give us the same structure.

This means that what we need to show is that for each Riemannian metric, we can find a set of coordinates so that our metric takes this form. So, suppose our metric is given by

$$ds^2 = A\,dx^2 + 2B\,dx\,dy + V\,dy^2$$

in some local coordinates (x, y).

Writing $z = x + iy$, we get that

$$ds^2 = \lambda\,|dz + \mu d\overline{z}|^2 := \lambda(dz + \mu d\overline{z})(d\overline{z} + \overline{\mu} dz),$$

where

$$\lambda = \frac{1}{4}\left(A + C + 2\sqrt{AC - B^2}\right) \quad \text{and} \quad \mu = \frac{A - C + 2i\,B}{A + C + 2\sqrt{AC - B^2}}.$$

We are looking for a coordinate $w = u + iv$ so that

$$ds^2 = \rho(du^2 + dv^2) = \rho\,|dw|^2 = \rho \cdot \left|\frac{\partial w}{\partial z}\right|^2 \cdot \left|dz + \frac{\partial w / \partial \overline{z}}{\partial w / \partial z} d\overline{z}\right|^2.$$

This means that isothermal coordinates exist if there is a solution to the partial differential equation

$$\frac{\partial w}{\partial \overline{z}} = \mu \cdot \frac{\partial w}{\partial z}.$$

It turns out this solution does indeed exist on a surface, which means that we obtain a Riemann surface structure. Moreover, it turns out this map is one-to-one. In particular, holmorphic maps are conformal. So we obtain

Proposition 3.3. *Given an orientable surface Σ of finite type with $\partial\Sigma = \emptyset$, the identification described above gives a one-to-one correspondence of sets*

$$\left\{ \begin{array}{c} \textit{Riemann surface} \\ \textit{structures on } \Sigma \end{array} \right\} \Big/ \textit{biholom.} \leftrightarrow \left\{ \begin{array}{c} \textit{Conformal classes} \\ \textit{of Riemannian} \\ \textit{metrics on } \Sigma \end{array} \right\} \Big/ \textit{diffeomorphism.}$$

3.4. Exercises

Exercise 3.1. Check that the map $\pi : X \to \widehat{\mathbb{C}}$ defined in Section 3.1 is holomorphic.

LECTURE 4

Hyperbolic surfaces

In order to describe the quotients of $\mathbb{H}^2$, we're going to use hyperbolic geometry. For this section we will mainly follow [**Bus10**] and [**Bea95**].

4.1. The hyperbolic plane

Hyperbolic geometry originally developed in the early 19^{th} century to prove that the parallel postulate in Euclidean geometry is independent of the other postulates. From this perspective, the hyperbolic plane can be seen as a geometric object satisfying a collection of axioms very similar to Euclid's axioms for Euclidean geometry, but with the parallel postulate replaced by something else. From a more modern perspective, hyperbolic geometry is the study of manifolds that admit a Riemannian metric of constant curvature -1.

4.1.1. The upper half plane model. From the classical point of view, any concrete description of the hyperbolic plane is a *model* for two-dimensional hyperbolic geometry, in the same way that $\mathbb{R}^2$ is a model for Euclidean geometry.

We start with the upper half plane model.

Definition 4.1. The hyperbolic plane $\mathbb{H}^2$ is the complex domain

$$\mathbb{H}^2 = \left\{ z \in \mathbb{C};\quad \mathrm{Im}(z) > 0 \ \right\}$$

equipped with the Riemannian metric $ds^2_{x+iy} : T_{x+iy}\,\mathbb{H}^2 \times T_{x+iy}\,\mathbb{H}^2 \to \mathbb{R}$ given by

$$ds^2_{x+iy}(v,w) = \frac{1}{y^2}\left(dx^2 + dy^2\right)$$

for all $x \in \mathbb{R}$ and $y \in (0,\infty)$

Because they are convenient, we will almost always work in local coordinates $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$ for all $z \in \mathbb{H}^2$. We will denote the corresponding tangent vector fields by $\partial/\partial x$ and $\partial/\partial y$ respectively.

Let us first note that even thought distances in $\mathbb{H}^2$ behave very differently than in Euclidean geometry, the angles are the same. Indeed, this follows from the fact that the hyperbolic metric on $\mathbb{H}^2$ is conformal to the Euclidean metric.

Example 4.2. Let us compute the hyperbolic length of the straight line segment between $ai \in \mathbb{H}^2$ and $bi \in \mathbb{H}^2$ (denoted $[ai, bi]$) for $0 < a < b \in \mathbb{R}$. We may parameterize this segment by

$$\gamma : [0,1] \to [ai, bi] \ \text{ given by } \ \gamma(t) = (1-t)\cdot ai + t \cdot bi.$$

We have

$$\frac{d}{dt}\gamma(t) = -a\frac{\partial}{\partial y}_{\gamma(t)} + b\frac{\partial}{\partial y}_{\gamma(t)} = (b-a)\frac{\partial}{\partial y}_{\gamma(t)}.$$

So, denoting the Riemannian metric by $g(\cdot,\cdot) : T_z\,\mathbb{H}^2 \times T_z\,\mathbb{H}^2 \to \mathbb{R}$, we have

$$g\left(\frac{d}{dt}\gamma(t), \frac{d}{dt}\gamma(t)\right) = \frac{(b-a)^2}{(a+t(b-a))^2}.$$

This means that the length of the line segment is given by

$$\begin{aligned} \ell([ai,bi]) &= \int_0^1 \sqrt{g\left(\frac{d}{dt}\gamma(t), \frac{d}{dt}\gamma(t)\right)}dt \\ &= \int_0^1 \frac{b-a}{a+t(b-a)}dt \\ &= [\log(a+t(b-a))]_0^1 \\ &= \log(b/a). \end{aligned}$$

Recall that given a connected Riemannian manifold (M,g), the *distance* between two points $p,q \in M$ is given by

$$\mathrm{d}(p,q) = \inf\left\{\ell(\gamma);\quad \gamma : [0,1] \to M \text{ smooth, } \gamma(0) = p \text{ and } \gamma(1) = q\ \right\}.$$

Example 4.3. We claim that for $ai, bi \in \mathbb{H}^2$ with $0 < a < b \in \mathbb{R}$ we have

$$\mathrm{d}(ai,bi) = \log(b/a).$$

In Example 4.2 we have already shown that

$$\mathrm{d}(ai,bi) \leq \log(b/a),$$

so all we have to do is show the other inequality. Let $\gamma : [0,1] \to \mathbb{H}^2$ be any other smooth path with $\gamma(0) = ai$ and $\gamma(1) = bi$. Write

$$x(t) = \mathrm{Re}(\gamma(t)) \text{ and } y(t) = \mathrm{Im}(\gamma(t)),$$

so $\gamma(t) = x(t) + iy(t)$. We have

$$\begin{aligned} \ell(\gamma) &= \int_0^1 \sqrt{g\left(\frac{d}{dt}\gamma(t), \frac{d}{dt}(\gamma(t)\right)}dt \\ &= \int_0^1 \frac{1}{y(t)}\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}dt, \end{aligned}$$

where $\dot{x}(t) = dx(t)/dt$ and $\dot{y(t)} = dy(t)/dt$. As such

$$\ell(\gamma) \geq \int_0^1 \frac{\dot{y}(t)}{y(t)}dt = \log(b/a),$$

which proves our claim.

Recall that $\mathrm{PSL}(2,\mathbb{R})$ acts on $\mathbb{H}^2$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az+b}{cz+d} \tag{2}$$

for all $z \in \mathbb{H}^2$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{R})$. In Exercise 4.1 we prove that this is an action by *isometries*. That is

$$\mathrm{d}(g\,z, g\,w) = \mathrm{d}(z,w)$$

for all $z, w \in \mathbb{H}^2$ and $g \in \mathrm{PSL}(2,\mathbb{R})$. Recall moreover that

$$\mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}(2,\mathbb{R}).$$

That is, there are no other orientation preserving isometries. We note that the assumption that the map preserves orientation is important here. The map $x+iy \mapsto -x+iy$ is an example of an isometry that does not preserve orientation and does not lie in $\mathrm{PSL}(2,\mathbb{R})$.

As a consequence we obtain:

Proposition 4.4. *Let $z, w \in \mathbb{H}^2$. Then*

$$\mathrm{d}(z,w) = \cosh^{-1}\left(1 + \frac{|z-w|^2}{2\cdot \mathrm{Im}(z)\cdot \mathrm{Im}(w)}\right).$$

Proof. First of all, for z and w on the imaginary axis, this formula restricts to the formula from Example 4.3. As such, our strategy will be to prove that the expression on the right is invariant under Möbius transformations (as well as the expression on the left) and then to show that every pair of elements $z, w \in \mathbb{H}^2$ can be mapped to the imaginary axis by Möbius transformations.

The first fact comes down to checking that

$$\frac{|z-w|^2}{2\cdot \mathrm{Im}(z)\cdot \mathrm{Im}(w)} = \frac{|Az-Aw|^2}{2\cdot \mathrm{Im}(Az)\cdot \mathrm{Im}(Aw)}$$

for all $A \in \mathrm{PSL}(2,\mathbb{R})$ and $z, w \in \mathbb{H}^2$. This is a straightforward computation that we leave to the reader.

To show that we can move every pair of points to the imaginary axis with a Möbius transformation, we may assume that not both z and w are on the imaginary axis.

First suppose that z and w lie on a vertical line $\{x = b\}$. In this case the Möbius transformation $z \mapsto z - b$ maps both points to the imaginary axis.

Now suppose that z and w do not lie on a vertical line. Let C be the unique Euclidean circle through z and w that is perpendicular to the real line. Let α be one of the two points on the intersection $C \cap \mathbb{R}$.

$$z \mapsto \frac{-1}{z-\alpha}$$

is a Möbius transformation. We claim that it sends C to a straight line. One way to check this is by parameterization. Indeed, suppose C has center $\beta \in \mathbb{R}$ and suppose

$\beta > \alpha$. We can then parameterize

$$C(t) = \beta + e^{2\pi i t}(\beta - \alpha),\ t \in \left(0, \frac{1}{2}\right)$$

It is a straightforward computation to check that

$$\mathrm{Re}\left(\frac{-1}{C(t) - \alpha}\right) = \frac{-1}{2(\beta - \alpha)}.$$

As such, our Möbius transformation sends z and w to two elements that lie on a vertical line and we are done. □

We note that Möbius transformations preserve the set of half circles orthogonal to $\mathbb{R}$ and vertical lines in $\mathbb{H}^2$ (see Exercise 4.2).

Recall that a *geodesic* $\gamma : \mathbb{R} \to \mathbb{H}^2$ is a smooth path so that for all $t_0 \in \mathbb{R}$, there exists a $\delta > 0$ such that

$$\mathrm{d}(\gamma(t), \gamma(s)) = |t - s| \quad \text{for all} \quad t, s \in (t_0 - \delta, t_0 + \delta).$$

That is, geodesics are locally length minimizing curves [1]. A *geodesic segment* is a smooth path $\gamma : (a, b) \to \mathbb{H}^2$ with the same property. We will often drop the distinction between geodesics and geodesic segments.

It follows from the proof and the two examples above that:

Proposition 4.5. *The image of a geodesic $\gamma : \mathbb{R} \to \mathbb{H}^2$ is a vertical line or a half circle orthogonal to $\mathbb{R}$. Moreover, every vertical line and half circle orthogonal to the real line can parameterized as a geodesic.*

We will often forget about the parametrization and call the image of a geodesic a geodesic as well. Note that it follows from the proposition above that given any two distinct points $z, w \in \mathbb{H}^2$ there exists a unique geodesic $\gamma \subset \mathbb{H}^2$ so that both $z \in \gamma$ and $w \in \gamma$. Furthermore, it also follows given a point $z \in \mathbb{H}^2$ and a geodesic γ that does not contain it, there is a unique perpendicular from z to γ (a geodesic γ' that intersects γ once perpendicularly and contains z)

The final fact we will need about the hyperbolic plane is:

Proposition 4.6. *Let $z \in \mathbb{H}$ and let $\gamma \subset \mathbb{H}^2$ be a geodesic so that $z \notin \gamma$. Then*

$$\mathrm{d}(z, \gamma) := \inf\left\{\mathrm{d}(z, w); \quad w \in \gamma\ \right\}$$

is realized by the intersection point of the perpendicular from z to γ.

Likewise, any two geodesics that don't intersect and are not asymptotic to the same point in $\mathbb{R} \cup \{\infty\}$ have a unique common perpendicular. Moreover, this perpendicular minimizes the distance between them.

Proof. The first claim follows from Pythagoras' theorem for hyperbolic triangles. Indeed, given three points in $\mathbb{H}^2$ so that the three geodesics through them form a right angled hyperbolic triangle with sides of length a, b and c (where c is the side opposite the right angle), we have

$$\cosh(a)\cosh(b) = \cosh(c)$$

[1] It turns out that geodesics in $\mathbb{H}^2$ are also globally length minimizing.

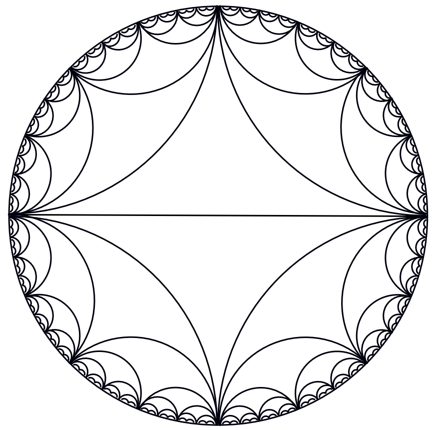

FIGURE 1. The Farey tesselation.

(see Exercise 4.3). This means in particular that $c > b$.

So, any other point on γ is further away from z than the point w realizing the perpendicular. Because that other point forms a right angled triangle with w and z.

The second claim follows from the first. □

4.1.2. The disk model.

Set

$$\Delta = \left\{ z \in \mathbb{C}; \quad |z| < 1 \right\}.$$

The map $f : \mathbb{H}^2 \to \Delta$ given by

$$f(z) = \frac{z - i}{z + i}$$

is a biholormorphism. We can also use it to push forward the hyperbolic metric to Δ. A direct computation tells us that the metric we obtain is given by

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}.$$

Since f is conformal, the angles in the disk model are still the same as Euclidean angles.

We have:

Proposition 4.7. *The hyperbolic geodesics in* Δ *are*

- *straight diagonals through the origin* $0 \in \Delta$
- $C \cap \Delta$ *where* $C \subset \mathbb{C}$ *is a circle that intersects* $\partial\Delta$ *orthogonally.*

PROOF. This follows from Proposition 4.5 and the fact that by definition f maps geodesics to geodesics (see Exercise 4.4). □

The following figure shows a collection of geodesics in Δ, known as the Farey tesselation.

4.2. Hyperbolic surfaces

A hyperbolic surface will be a finite type surface equipped with a metric that locally makes it look like $\mathbb{H}^2$.

Because we will want to deal with surfaces with boundary later on, we need half spaces. Let $\gamma \subset \mathbb{H}^2$ be a geodesic. $\mathbb{H}^2 \setminus \gamma$ consists of two connected components C_1 and C_2. We will call $\mathcal{H}_i = C_i \cup \gamma$ a *closed half space* $(i = 1, 2)$. So for example

$$\left\{ z \in \mathbb{H}^2; \quad \mathrm{Re}(z) \leq 0 \right\}$$

is a closed half space.

We formalize the notion of a hyperbolic surface as follows:

Definition 4.8. A finite type surface S with atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ is called a *hyperbolic surface* if $\varphi_\alpha(U_\alpha) \subset \mathbb{H}^2$ for all $\alpha \in A$ and

1. for each $p \in S$ there exists an $\alpha \in A$ so that $p \in U_\alpha$ and
   - If $p \in \partial S$ then
   $$\varphi_\alpha(U_\alpha) = V \cap \mathcal{H}$$
   for some open set $V \subset \mathbb{H}^2$ and some closed half space $\mathcal{H} \subset \mathbb{H}^2$.
   - If $p \in \mathring{S}$ then $\varphi_\alpha(U_\alpha) \subset \mathbb{H}^2$ is open.
2. For every $\alpha, \beta \in A$ and for each connected component C of $U_\alpha \cap U_\beta$ we can find a Möbius transformation $A : \mathbb{H}^2 \to \mathbb{H}^2$ so that
   $$\varphi_\alpha \circ \varphi_\beta^{-1}(z) = A(z)$$
   for all $z \in \varphi_\beta(C) \subset \mathbb{H}^2$.

Note that every hyperbolic surface comes with a metric: every chart is identified with an open set of $\mathbb{H}^2$ which gives us a metric. Because the chart transitions are restrictions of isometries of $\mathbb{H}^2$, this metric does not depend on the choice of chart and hence is well defined.

Definition 4.9. A hyperbolic surface S is called *complete* if the induced metric is complete.

4.3. Right angled hexagons

Even though Definition 4.8 is a complete definition, it is not very descriptive. In what follows we will describe a concrete cutting and pasting construction for hyperbolic surfaces.

We start with right angled hexagons. A right angled hexagon $H \subset \mathbb{H}^2$ is a compact simply connected closed subset whose boundary consists of 6 geodesic segments, that meet each other orthogonally.

The picture to have in mind is:

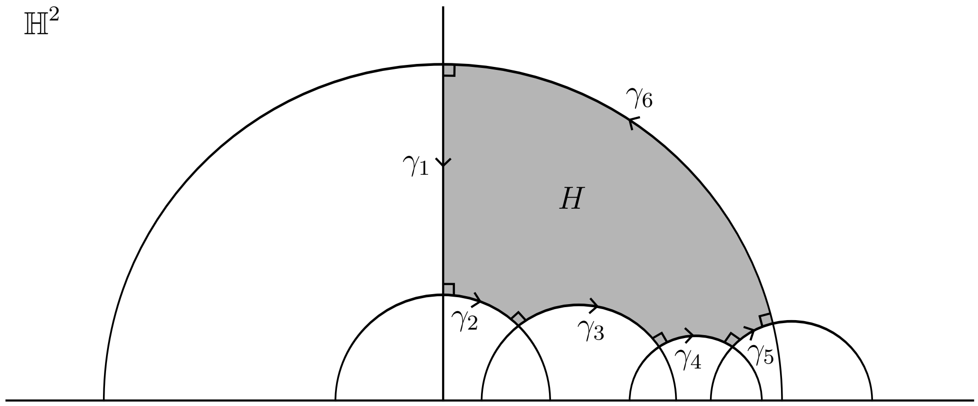

FIGURE 2. A right angled hexagon H.

It turns out that the lengths of three non-consecutive sides determine a right angled hexagon up to isometry.

Proposition 4.10. *Let $a, b, c \in (0, \infty)$. Then there exists a right angled hexagon $H \subset \mathbb{H}^2$ with three non-consecutive sides of length a, b and c respectively. Moreover, if H' is another right angled hexagon with this property, then there exists a Möbius transformation $A : \mathbb{H}^2 \to \mathbb{H}^2$ so that*

$$A(H) = H'.$$

PROOF. Let us start with the existence. Let γ_{im} denote the positive imaginary axis and set

$$B = \left\{ z \in \mathbb{H}^2; \quad \mathrm{d}(z, \gamma_{im}) = c \right\}.$$

B is a one-dimensional submanifold of $\mathbb{H}^2$. Because the map $z \mapsto \lambda z$ is an isometry that preserves γ_{im} for every $\lambda > 0$, it must also preserve B. This means that B is a (straight Euclidean) line.

Now construct the following picture:

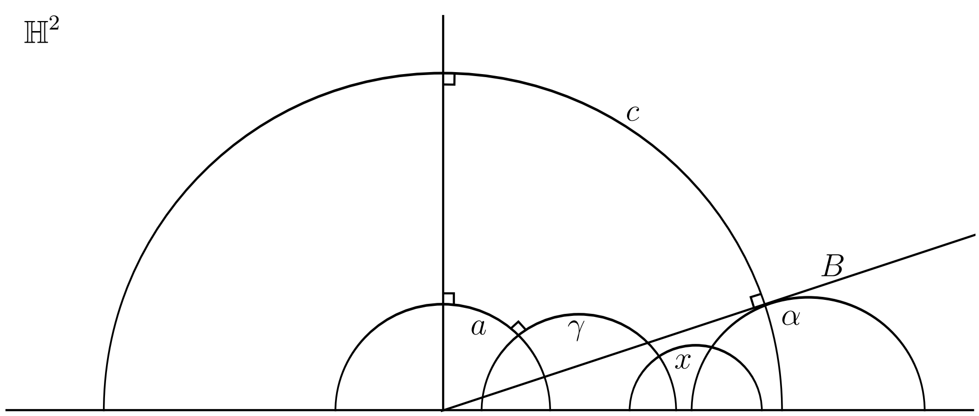

FIGURE 3. Constructing a right angled hexagon $H(a, b, c)$.

That is, we take the geodesic though the point $i \in \mathbb{H}^2$ perpendicular to γ_{im} and at distance a draw a perpendicular geodesic γ. furthermore, for any $p \in B$, we draw the geodesic α that realizes a right angle with the perpendicular from p to γ_{im}. Now let

$$x = \mathrm{d}(\alpha, \gamma) = \inf \left\{ \mathrm{d}(z, w); \quad z \in \gamma,\ w \in \alpha \ \right\}.$$

Because of Proposition 4.6, x is realized by the common perpendicular to α and γ. By moving p over B, we can realize any positive value for x and hence obtain our hexagon $H(a, b, c)$.

We also obtain uniqueness from the picture above. Indeed, given any right angled hexagon H' with three non-consequtive sides of length a, b and c, apply a Möbius transformation $A : \mathbb{H}^2 \to \mathbb{H}^2$ so that the geodesic segment of length a starts at i and is orthogonal to the imaginary axis. This implies that the geodesic after a gets mapped to the geodesic γ. Furthermore, one of the endpoints of the geodesic segment of length c needs to lie on the line B. We now know that the the geodesic α before that point needs to be tangent to B. Because α and β have a unique common perpendicular. The tangency point of α to B determines the picture entirely. Because the function that assigns the length x of the common perpendicular to the tangency point is injective, we obtain that there is a unique solution. □

4.4. Exercises

Exercise 4.1. (a) Given $A \in \mathrm{PSL}(2,\mathbb{R})$, show that its derivative D_zA satisfies

$$g_{Az}(D_zAv, D_zAw) = g_z(v,w)$$

for all $z \in \mathbb{H}^2$ and $v, w \in T_z\mathbb{H}^2$.

(b) Given a smooth path $\gamma : [0,1] \to \mathbb{H}^2$ and $A \in \mathrm{PSL}(2,\mathbb{R})$, we obtain a new smooth path $A \circ \gamma : [0,1] \to \mathbb{H}^2$. Show that

$$\ell(\gamma) = \ell(A \circ \gamma).$$

Conclude that

$$\mathrm{d}(Az, Aw) = \mathrm{d}(z,w)$$

for all $z, w \in \mathbb{H}^2$ and $A \in \mathrm{PSL}(2,\mathbb{R})$.

Exercise 4.2. Let $C \subset \mathbb{H}^2$ be a half circle orthogonal to $\mathbb{R}$ or a vertical line and let $A : \mathbb{H}^2 \to \mathbb{H}^2$ be a Möbius transformation. Show that $A(C)$ is a vertical line or half circle orthogonal to $\mathbb{R}$.

Hint: consider what a Möbius transformation does to the endpoints (NB: ∞ is a possible endpoint) of half circles orthogonal to $\mathbb{R}$ and vertical lines

Exercise 4.3. *Pythagoras' theorem:* Suppose $x, y, z \in \mathbb{H}^2$ form a right angled triangle (that is, the geodesic between x and y intersects that between y and z perpendicularly) and let

$$a = \mathrm{d}(x,y),\ b = \mathrm{d}(y,z) \text{ and } c = \mathrm{d}(z,x).$$

Prove that

$$\cosh(a) \cdot \cosh(b) = \cosh(c).$$

Hint: just like in the proof of Proposition 4.4 you may assume that the geodesic between y and z is the imaginary axis.

Exercise 4.4. Complete the proof of Proposition 4.7.

Exercise 4.5. Let H be a right angled hexagon with three non consecutive sides of the same length $a > 0$.

(a) Show without computing their lengths that the lengths of the other three sides are also all the same.
(b) Compute the length of the other three sides.

LECTURE 5

Pants decompositions, part I

For this lecture, we mainly follow [**Bus10**].

5.1. Pairs of pants and gluing

One of our main building blocks for hyperbolic surfaces is the following:

Definition 5.1. Let $a, b, c \in (0, \infty)$. A *pair of pants* is a hyperbolic surface that is diffeomorphic to $\Sigma_{0,3,0}$ such that the boundary components have length a, b and c respectively.

Proposition 5.2. *Let $a, b, c \in (0, \infty)$ and let P and P' be pairs of pants with boundary curves of lengths a, b and c. Then there exists an isometry $\varphi : P \to P'$.*

Proof sketch. There exists a unique orthogonal geodesic (this essentially follows from Proposition 4.6, in Proposition 6.1 we will do a similar proof in full) between every pair of boundary components of P.

These three orthogonals decompose P into right-angled hexagons out of which three non-consecutive sides are determined. Proposition 4.10 now tells us that this determines the hexagons up to isometry and this implies that P is also determined up to isometry. □

Note that it also follows from the proof sketch above that the unique perpendiculars cut each boundary curve on P into two geodesic segments of equal length.

In order to deal with non-compact surfaces, we will need non-compact polygons. To this end, we note that, looking at Proposition 4.5, complete geodesics in $\mathbb{H}^2$ are parametrized by their *endpoints*: pairs of distinct point in

$$\partial \mathbb{H}^2 := \mathbb{R} \cup \{\infty\}$$

(or $\mathbb{S}^1$ if we use the disk model).

A (not necessarily compact) polygon now is a closed connected simply connected subset $P \subset \mathbb{H}^2$, whose boundary consists of geodesic segments.

If two consecutive segments "meet" at a point in $\partial \mathbb{H}^2$, this point will be called an *ideal* vertex of the boundary. Note that the angle at an ideal vertex is always 0. A polygon all of whose vertices are ideal is called an *ideal polygon*.

We can also make sense of a pair of pants where some of the boundary components have "length" 0. In this case, we obtain a complete hyperbolic structure on a surface with boundary and punctures so that

$$\#\text{punctures} + \#\text{boundary components} = 3.$$

Such pairs of pants can be obtained by gluing either

- two pentagons with one ideal vertex each and right angles at the other vertices,
- two quadrilaterals with two ideal vertices each right angles at the other vertices or
- two ideal triangles.

Along the sides of infinite length there however is a gluing condition. We will come back to this later (see Proposition 6.3).

Moreover, we obtain a similar uniqueness statement to the proposition above. As always in the non-compact case, the adjective complete does need to be added.

If P is a pair of pants and $\delta \subset \partial P$ is one of its boundary components, let us write $\ell(\delta)$ for the length of δ. Recall that an isometry between Riemannian manifolds M and N is a diffeomorphism $\varphi : M \to N$ so that

$$\mathrm{d}_M(x,y) = \mathrm{d}_N(\varphi(x), \varphi(y))$$

for all $x, y \in M$.

Example 5.3. Given two pairs of pants P_1 with boundary components δ_1, δ_2 and δ_3 and P_2 with boundary components γ_1, γ_2 and γ_3 so that

$$\ell(\delta_1) = \ell(\gamma_1),$$

we can choose an orientation reversing isometry $\varphi : \delta_1 \to \gamma_1$ and from that obtain a hyperbolic surface

$$S = P_1 \sqcup P_2 / \sim,$$

where $\varphi(x) \sim x$ for all $x \in \delta_1$. One way to see that this surface comes with a well defined hyperbolic structure, is that locally it's obtained by gluing two half spaces in $\mathbb{H}^2$ together along their defining geodesics. Note that S is diffeomorphic to $\Sigma_{0,4,0}$.

Repeating the construction above, we can build hyperbolic surfaces of any genus and any number of boundary components.

In what follows we will prove that every hyperbolic surface can be obtained from this construction.

5.2. The universal cover of a hyperbolic surface with boundary

It will be useful to have a description of the Riemannian universal cover of a surface with boundary. To this end, we first prove:

Proposition 5.4. *Let X be a hyperbolic surface with non-empty boundary that consists of closed geodesics. Then there exists a complete hyperbolic surface X^* without boundary in which X can be isometrically embedded so that X is a deformation retract of X^*.*

PROOF. For each $\ell \in (0, \infty)$, we define a hyperbolic surface

$$F_\ell = [0, \infty) \times \mathbb{R} / \{t \sim t + 1\},$$

equipped with the metric

$$ds^2 = d\rho^2 + \ell^2 \cosh^2(\rho) \cdot dt^2$$

for all $(\rho, t) \in F_\ell$. We will call such a surface a *funnel*.

One can check that this is a metric of constant curvature -1, in which the boundary is totally geodesic.

We can glue funnels of the right length along the boundary components, in a similar way to Example 5.3. Figure 1 shows an example.

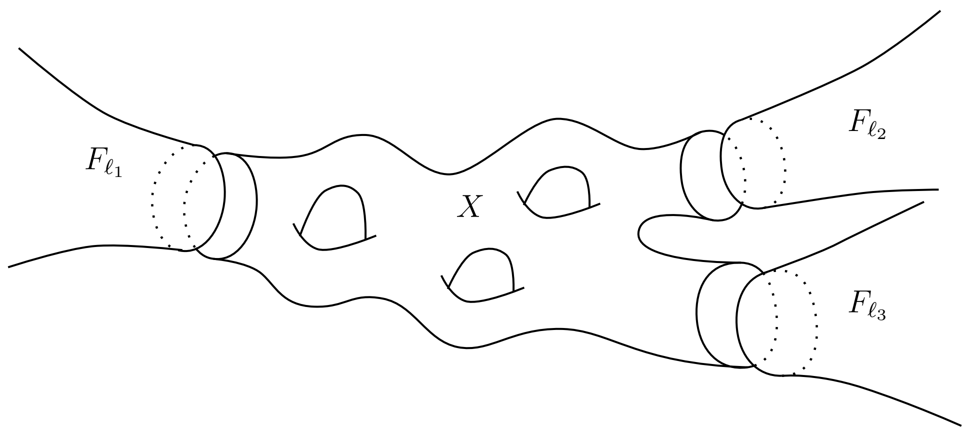

FIGURE 1. Attaching funnels

Since both F_ℓ and X are complete, the resulting surface X^* is complete.

Moreover, since F_ℓ retracts onto its boundary component, X is a deformation retract of X^*. □

See [**Bus10**, Theorem 1.4.1] for a version of the above to surfaces with more general types of boundary components.

Recall that a subset $C \subset M$ of a Riemannian manifold M is called *convex* if for all $p, q \in C$ there exists a length minimizing geodesic $\gamma : [0, \mathrm{d}(p,q)] \to M$ such that

$$\gamma(0) = p, \quad \gamma(\mathrm{d}(p,q)) = q \quad \text{and} \quad \gamma(t) \in C \; \forall \, t \in [0, \mathrm{d}(p,q)].$$

As a result of this construction we obtain:

Proposition 5.5. *Let X be a complete hyperbolic surface with non-empty boundary that consists of closed geodesics. Then the universal Riemannian cover of $\widetilde{X}$ of X is isometric to a convex subset of $\mathbb{H}^2$ whose boundary consists of complete geodesics.*

PROOF. The Killing-Hopf theorem tells us that the universal cover of X^* is the hyperbolic plane $\mathbb{H}^2$. Here X^* is the surface given by Proposition 5.4.

Let us denote the covering map by $\pi : \mathbb{H}^2 \to X^*$. Now let C be a connected component of $\pi^{-1}(X)$. The boundary of C consists of the lifts of ∂X and hence of a countable collection of disjoint complete geodesics in $\mathbb{H}^2$. As such, it's a countable intersection of half spaces (which are convex) and hence convex. □

5.3. Simple closed geodesics, part I

Definition 5.6. Let M be a smooth manifold.

- A *simple closed curve* is an continuous embedding
$$\gamma : \mathbb{S}^1 \to M.$$
- A *closed curve* is an continuous immersion
$$\gamma : \mathbb{S}^1 \to M$$
- Two closed curves $\gamma_1, \gamma_2 : \mathbb{S}^1 \to M$ are called *freely homotopic*, if there exists a continuous map:
$$H : \mathbb{S}^1 \times [0,1] \to X.$$
so that
$$H(t,0) = \gamma_1(t) \text{ and } H(t,1) = \gamma_2(t)$$
for all $t \in \mathbb{S}^1$.

So, the difference between free homotopy and usual homotopy of loops is that there is no mention of basepoints in the case of free homotopy. It turns out that two simple closed curves on a a surface are freely homotopic if and only if they are freely isotopic (see [**FM12**, Proposition 1.10] for a proof). Moreover, there is a one-to-one correspondence

$$\left\{ \begin{array}{c} \text{Conjugacy classes of} \\ \text{non-trivial elements in } \pi_1(X) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{free homotopy classes of} \\ \text{non-trivial closed curves on } X \end{array} \right\}.$$

Closed curves that are also geodesics will be called *closed geodesics*. These turn out to come from what are called hyperbolic elements in $\pi_1(X)$, we will describe how this works in what follows.

Definition 5.7. Let $g \in \mathrm{PSL}(2,\mathbb{R})$.

(1) If $\mathrm{tr}\,(g)^2 < 4$ then g is called *elliptic*.
(2) If $\mathrm{tr}\,(g)^2 = 4$ then g is called *parabolic*.
(3) If $\mathrm{tr}\,(g)^2 > 4$ then g is called *hyperbolic*.

Note that, since trace is conjugacy invariant, conjugate elements in $\mathrm{PSL}(2,\mathbb{R})$ are of the same type.

The classification can equivalently be described as:

Lemma 5.8. *Let $g \in \mathrm{PSL}(2,\mathbb{R})$. Then*

(1) g is elliptic if and only if g has a single fixed point inside $\mathbb{H}^2$.
(2) g is parabolic if and only if g has a single fixed point on $\mathbb{R}\cup\{\infty\}$.
(3) g is hyperbolic if and only if g has two distinct fixed points on $\mathbb{R}\cup\{\infty\}$.

Proof. Exercise 5.1(a). □

Given a hyperbolic isometry, we can define its translation distance as follows:

Definition 5.9. Let $g \in \mathrm{PSL}(2,\mathbb{R})$ be hyperbolic. Then its *translation distance* is given by
$$T_g := \inf\left\{ z \in \mathbb{H}^2;\quad \mathrm{d}(z, gz) \;\right\}.$$
Moreover, its *axis* is defined as
$$\alpha_g := \left\{ z \in \mathbb{H}^2;\quad \mathrm{d}(z, gz) = T_g \;\right\}.$$

We have:

Lemma 5.10. *Let* $g \in \mathrm{PSL}(2,\mathbb{R})$ *be hyperbolic with fixed points* $x_1, x_2 \in \partial\,\mathbb{H}^2$. *Then its axis* α_g *is the unique geodesic between* x_1 *and* x_2 *and its translation length is given by*

$$T_g = 2\cosh^{-1}\left(\frac{|\mathrm{tr}\,(g)|}{2}\right).$$

PROOF. Exercise 5.1(b). □

5.4. Exercises

Exercise 5.1. (a) Prove Lemma 5.8.
(b) Prove Lemma 5.10. *Hint: translate everything to the imaginary axis.*

Exercise 5.2. Suppose Γ acts properly discontinuously and freely on a closed convex subset $C \subset \mathbb{H}^2$. Suppose $g \in \Gamma$ is a hyperbolic element. Show that the axis α_g of g is contained in C.

LECTURE 6

Pants decompositions, part II

We will mainly follow [**Bus10**] for this lecture.

6.1. Simple closed geodesics part II

Now we can state the corrspondence that we are after. We will call a curve puncture parallel if it can be homotoped into a puncture.

Proposition 6.1. *Let X be a complete hyperbolic surface with totally geodesic boundary. Then there are one-to-one correspondences*

$$\left\{ \begin{array}{c} \textit{Non-trivial free homotopy classes of} \\ \textit{non puncture-parallel closed curves on } X \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{Conjugacy classes of} \\ \textit{hyperbolic elements in } \Gamma \end{array} \right\}.$$

and

$$\left\{ \begin{array}{c} \textit{Conjugacy classes of} \\ \textit{hyperbolic elements in } \Gamma \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{Oriented, unparametrized} \\ \textit{closed geodesics on } \mathbb{H}^2/\Gamma \end{array} \right\}.$$

Moreover, the unique geodesic minimizes length over the conjugacy class.

If the free homotopy class contains a simple closed curve, then the corresponding geodesic is also simple.

More generally, if γ and γ' are non-homotopic non-puncture parallel and non-trivial closed curves, then

- *The number of self-intersections of the unique geodesic $\overline{\gamma}$ homotopic to γ is mimimal among all closed curves homotopic to γ and*
- *$\#\overline{\gamma}\cap\overline{\gamma'}$ is minimal among all pairs of curves homotopic to γ and γ' respectively.*

Before we prove this, we recall the Arzelà-Ascoli theorem. First recall that a map $f : X \to Z$ between metric spaces X and Z is called L-Lipschitz, for some $L > 0$ if

$$\mathrm{d}_Z(f(x), f(y)) \leq L \cdot \mathrm{d}_X(x, y)$$

for all $x, y \in X$. The Arzelà-Ascoli theorem now states:

Theorem 6.2 (Arzelà-Ascoli)**.** *Let X be a metric space that has a countable dense subset and Z a compact metric space. Suppose $\gamma_n : X \to Z$ is an L-Lipschitz map for all $n \in \mathbb{N}$ and some fixed $L > 0$. Then there exists a subsequence $(\gamma_{n_k})_{k\in\mathbb{N}}$ that converges uniformly on compact sets in X to an L-Lipschitz map $\gamma : X \to Z$.*

See [**Bus10**, Theorem A.19] for a proof.

Proof of Proposition 6.1. In order to make our lives a little easier, we will assum X to be compact. The argument for the general case is similar. We will hence not worry about the assumption that the curve is non puncture parallel.

First of allm consider a conjugacy class $K \subset \Gamma$ of hyperbolic elements. Let us pick an element $g \in K$, with axis $\alpha_g \subset C$ (note that the axis necessarily lies in C, Exercise 5.2). The projection map $\pi : C \to X$ sends α_g to a closed geodesic of length T_g. Moreover, since

$$\pi\Big(\alpha_{hgh^{-1}}\Big) = \pi\Big(h\alpha_g\Big) = \pi\Big(\alpha_g\Big),$$

the resulting geodesic does not depend on the choice of g.

So, we need to go back. Let

$$\mathcal{C} := \left\{\gamma' : \mathbb{S}^1 \to X; \quad \gamma' \text{ freely homotopic to } \gamma\ \right\}$$

and set

$$L = \inf\left\{\ell(\gamma'); \quad \gamma' \in \mathcal{C}\ \right\}.$$

Now consider a sequence $(\gamma_n)_n$ so that $\ell(\gamma_n) \to L$. It follows from the Arzelà-Ascoli theorem ([**Bus10**, Theorem A.19]) that there exists a subsequence $(\gamma_{n_k})_k$ and a closed curve $\overline{\gamma} : \mathbb{S}^1 \to X$ so that $\gamma_{n_k} \to \widetilde{\gamma}$ uniformly as $k \to \infty$. Because $\overline{\gamma}$ minimizes length, it needs to be a geodesic (up to reparameterization).

To show uniqueness, suppose there are two freely homotopic geodesics $\gamma_1, \gamma_2 : \mathbb{S}^1 \to X$. Consider the universal cover $\pi : C \to X$. Because γ_1 and γ_2 are freely homotopic, we can lift them to continuous maps $\widetilde{\gamma}_1, \widetilde{\gamma}_2 : \mathbb{R} \to \mathbb{H}^2$ that are homotopic. The fact that γ_1 and γ_2 are geodesics implies that $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ are as well.

The cyclic subgroup G of the deck group Γ that leaves $\widetilde{\gamma}_1$ invariant also leaves $\widetilde{\gamma}_2$ invariant (because they are homotopic). Since $\widetilde{\gamma}_1/G$ and $\widetilde{\gamma}_2/G$ are compact, we obtain that

$$\sup_{t\in\mathbb{R}}\{\mathrm{d}(\widetilde{\gamma}_1(t), \widetilde{\gamma}_2(t)\} < \infty.$$

Now we note that when geodesics have at least one pair of distinct endpoints, the above does not hold. This implies that $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ have the same endpoints, which in turn implies they have coincide.

To get the conjugacy class, we note that the cyclic subgroup $G < \Gamma$ must be generated by a hyperbolic element g. Had we chosen another lift, we would have obtained a conjugate of g.

For the statement about simplicity, we first need to show that $\overline{\gamma}$ is primitive, that is, it is not a non-trivial power in Γ. To this end, suppose g^k is an element in the conjugacy class corresponding to γ so that g is primitive. Consider the cover

$$A := \mathbb{H}^2 / \langle g \rangle \to X^*,$$

where X^* is the surface provided by Proposition 5.4 and $g^k \in \Gamma$ is an element in the conjugacy class corresponding to γ. Since A is an orientable surface with fundamental group isomorphic to $\mathbb{Z}$, it must be an annulus. All simple closed curves in an annulus induce eaither g or g^{-1} in its fundamental group (Exercise 6.1).

Now that we know that $\overline{\gamma}$ is primitive, we need to show that it doesn't have any self intersecions. If $\overline{\gamma}$ were to have self intersections, this would mean that each lift $\widetilde{\overline{\gamma}}$ of $\overline{\gamma}$ is intersected by another lift $g\widetilde{\overline{\gamma}}$ of $\overline{\gamma}$. Consider the lifts $\widetilde{\gamma}$ and $g\widetilde{\gamma}$ of γ with the same enpoints as $\widetilde{\widetilde{\gamma}}$ and $g\widetilde{\widetilde{\gamma}}$ respectively. Since $\widetilde{\widetilde{\gamma}}$ and $g\widetilde{\widetilde{\gamma}}$ intersect, these entpoints alternate, which means that $\widetilde{\gamma}$ and $g\widetilde{\gamma}$ intersect as well (see Figure 1). This however implies that γ has a self-intersection, which isn't the case.

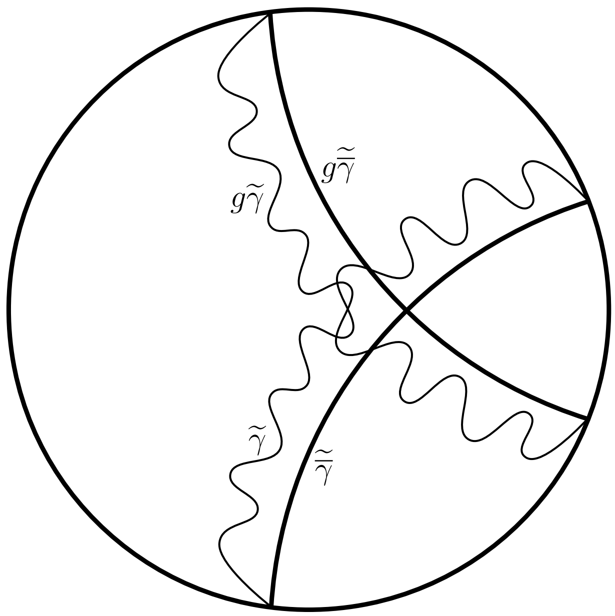

FIGURE 1. Intersecting curves in Δ.

We leave the proof of the rest to the reader. □

Before we get to pants decompositions, we record what happens to curves that are parallel to a puncture.

Proposition 6.3. *Let X be a complete hyperbolic surface. So that $X = C/\Gamma$ where C is a convex subset of $\mathbb{H}^2$, bounded by complete geodesics and $\Gamma < \mathrm{PSL}(2, \mathbb{R})$ acts prorperly discontinuously and freely on C. Then there are a one-to-one correspondences*

$$\left\{ \begin{array}{c} \textit{Conjugacy classes of} \\ \textit{parabolic elements in } \Gamma \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{Oriented, unparametrized} \\ \textit{puncture-parallel closed curves } \mathbb{H}^2 / \Gamma \end{array} \right\}.$$

This proposition gives us the gluing condition we spoke about in Section 5.1: the gluing needs to be so that the resulting puncture parallel curves give rise to parabolic elements, this turns out to uniquely determine the gluing.

6.2. Every hyperbolic surface admits a pants decomposition

As an immediate consequence to Proposition 6.1 we get that hyperbolic surfaces admit pants decompositions.

Definition 6.4. Let X be a complete, orientable hyperbolic surface of finite area. A pants decomposition of X is a collection of pairwise disjoint simple closed geodesics

$\mathcal{P} = \{\alpha_1, \ldots, \alpha_k\}$ in X so that each connected component of

$$X \setminus \left(\bigcup_{i=1}^{k} \alpha_i \right)$$

consists of hyperbolic pairs of pants whose boundary components have been removed.

We have the following:

Lemma 6.5. *Let $\mathcal{P}$ be a pants decomposition of a hyperbolic surface X that is homeomorphic to $\Sigma_{g,b,n}$ then*

- *$\mathcal{P}$ contains $3g+n+b-3$ closed geodesics and*
- *$X \setminus \mathcal{P}$ consists of $2g+n+b-2$ pairs of pants.*

PROOF. Exercise 6.2. □

Proposition 6.6. *Let X be a complete, orientable hyperbolic surface of finite area and totally geodesic boundary. Then X admits a pants decomposition.*

PROOF. Take any collection of simple closed curves on $\Sigma_{g,b,n}$ that decompose it into pairs of pants. Proposition 6.1 tells us that these curves can be realized by unique geodesics. □

Note that we actually get countably many such pants decompositions: given a pants decomposition we can apply a diffeomorphism not isotopic to the identity (of which there are many, we will see more about this later) to obtain a new topological pants decomposition, that is realized by different geodesics.

Finally, we remark, that lengths alone are not enough to determine the hyperbolic metric:

Example 6.7. φ in Example 5.3 is determined up to 'twist'. That is, if we parameterize δ_1 by a simple closed geodesic $x : \mathbb{R}/(\ell(\delta_1)\,\mathbb{Z}) \to \delta_1$ and $\varphi' : \delta_1 \to \gamma_1$ is a different orientation reversing isometry, then there exists some $t_0 \in \mathbb{R}$ so that

$$\varphi'(x(t)) = \varphi(x(t_0 + t))$$

for all $t \in \mathbb{R}/(\ell(\delta_1)\,\mathbb{Z}) \to \delta_1$.

Summarizing the above, we get the following parametrization of all hyperbolic surfaces:

Theorem 6.8. *Let (g, b, n) be so that*

$$\chi(\Sigma_{g,b,n}) < 0.$$

If we fix a pants decomposition $\mathcal{P}$ of $\Sigma_{g,b,n}$ and vary the lengths $\ell_i \in (0, \infty)$ and twist $\tau_i \in [0, \ell_i]$, we obtain all complete hyperbolic surfaces homeomorphic to $\Sigma_{g,b,n}$.

Note however that there is no guarantee that we don't obtain the same surface multiple times (and in fact we do).

6.3. Exercises

Exercise 6.1. Let A be an annulus and $\gamma : \mathbb{S}^1 \to A$ a simple closed curve. Make an identification $\pi_1(A) \simeq \mathbb{Z} \simeq \langle g \rangle$. Show that the conjugacy class in $\pi_1(A)$ corresponding to γ is either $\{g\}$ or $\{g^{-1}\}$. *Hint: assume it's not and use the intermediate value theorem.*

Exercise 6.2. Prove Lemma 6.5.

LECTURE 7

Spaces of tori

The goal of the rest of this course is to understand the deformation spaces associated to Riemann surfaces: Teichmüller and moduli spaces. We will mainly follow [**IT92**].

In general, the Teichmüller space associated to a surface will be a space of *marked* Riemann surface structures on that surface and the corresponding moduli space will be a space of isomorphism classes of Riemann surface structures. As such, the moduli space associated to a surface will be a quotient of the corresponding Teichmüller space.

First of all, note that the uniformization theorem tells us that there is only one Riemann surface structure on the sphere. This means that the corresponding moduli space will be a point. It turns out that the same holds for its Teichmüller space. This means that the lowest genus closed surface for which we can expect an intersting deformation space is the torus.

7.1. Riemann surface structures on the torus

So, let us parametrize Riemann surface structures on the torus. Recall from Proposition 2.6 that every Riemann surface structure on the torus is of the form

$$\mathbb{C} \,/ \left\langle \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} \right\rangle$$

for some $\lambda, \mu \in \mathbb{C} \setminus \{0\}$ that are linearly independent over $\mathbb{R}$.

First of all note that every such torus is biholomorphic to a torus of the form

$$R_\tau := \mathbb{C} \,/ \Lambda_\tau,$$

for some $\tau \in \mathbb{H}^2$, where

$$\Lambda_\tau = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \right\rangle$$

(see Exercise 7.1).

However, there are still distinct $\tau, \tau' \in \mathbb{H}^2$ that lead to holomorphic tori. We have:

Proposition 7.1. *Let $\tau, \tau' \in \mathbb{H}^2$. The two tori R_τ and $R_{\tau'}$ are biholomorphic if and only if*

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

for some $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$.

PROOF. First assume R_τ and $R_{\tau'}$ are biholomorphic and let $f : R_{\tau'} \to R_\tau$ be a biholomorphism. Lift f to a biholomorphism $\widetilde{f} : \mathbb{C} \to \mathbb{C}$. This means that

$$\widetilde{f}(z) = \alpha z + \beta$$

for some $\alpha, \beta \in \mathbb{C}$. By postcomposing with a biholomorphism of $\mathbb{C}$, we may assume that $\widetilde{f}(0) = 0$.

Because $\widetilde{f}$ is a lift, we know that both $\widetilde{f}(1)$ and $\widetilde{f}(\tau')$ are equivalent to 0 under Λ_τ. So

$$\widetilde{f}(\tau') = \alpha\tau' = a\tau + b$$

$$\widetilde{f}(1) = \alpha = c\tau + d$$

for some $a, b, c, d \in \mathbb{Z}$. So

$$\tau' = \frac{a\tau + b}{c\tau + d}.$$

So we only need to show that $ad - bc = 1$. If we apply the same argument to $\widetilde{f}^{-1}$ we obtain

$$\tau = \frac{a'\tau' + b'}{c'\tau' + d'},$$

for some $a', b', c', d' \in \mathbb{Z}$. Working out the relations $\widetilde{f}^{-1} \circ \widetilde{f}(1) = 1$ and $\widetilde{f}^{-1} \circ \widetilde{f}(\tau') = \tau'$, we obtain $ad - bc = \pm 1$. Since

$$\mathrm{Im}(\tau') = \frac{ad - bc}{|c\tau + d|^2} > 0,$$

we obtain $ad - bc = 1$.

Conversely, if

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

Then

$$f([z]) = [(c\tau + d)z]$$

gives a biholomorphic map $f : R_{\tau'} \to R_\tau$. ☐

7.2. The Teichmüller and moduli spaces of tori

Looking at Proposition 7.1, we see that we can parametrize all complex structures on the torus with the set

$$\mathcal{M}_1 = \mathbb{H}^2 \,/\, \mathrm{PSL}(2, \mathbb{Z}).$$

Moreover this set is the quotient of the hyperbolic place by a group of isometries that acts properly discontinuously on it. However, the group doesn't quite act freely, so it's not directly a hyperbolic surface.

So, let us investigate the structure of this quotient. One way of doing this is to find a fundamental domain for the action of $\mathrm{PSL}(2, \mathbb{Z})$ on $\mathbb{H}^2$. Set

$$\mathcal{F} = \left\{ z \in \mathbb{H}^2; \quad |z| \geq 1 \text{ and } -\tfrac{1}{2} \leq \mathrm{Re}(z) \leq \tfrac{1}{2} \;\right\}.$$

Figure 1 shows a picture of $\mathcal{F}$.

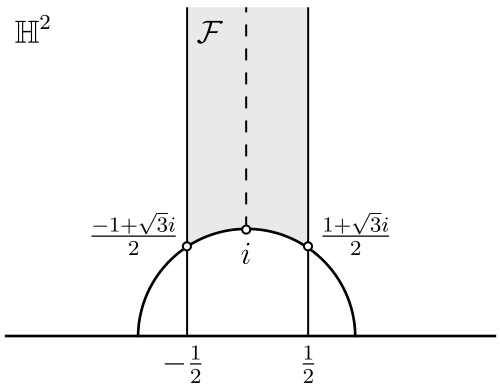

FIGURE 1. A fundamental domain for the action of $\mathrm{PSL}(2,\mathbb{Z})$ on $\mathbb{H}^2$.

We claim

Proposition 7.2. *For all $\tau \in \mathbb{H}^2$ there exists an element $g \in \mathrm{PSL}(2,\mathbb{Z})$ so that $g\tau \in \mathcal{F}$. Moreover,*

- *if $\tau \in \mathring{\mathcal{F}}$ then*
$$\Big(\mathrm{PSL}(2,\mathbb{Z}) \cdot \tau\Big) \cap \mathcal{F} = \{\tau\},$$
- *if $\mathrm{Re}(\tau) = \frac{1}{2}$ then*
$$\Big(\mathrm{PSL}(2,\mathbb{Z}) \cdot \tau\Big) \cap \mathcal{F} = \{g\tau, g\tau + 1\},$$
- *if $\mathrm{Re}(\tau) = -\frac{1}{2}$ then*
$$\Big(\mathrm{PSL}(2,\mathbb{Z}) \cdot \tau\Big) \cap \mathcal{F} = \{g\tau, g\tau - 1\}.$$
- *and if $|\tau| = 1$ then*
$$\Big(\mathrm{PSL}(2,\mathbb{Z}) \cdot \tau\Big) \cap \mathcal{F} = \{g\tau, -1/\tau\},$$

PROOF. Let $z \in \mathbb{H}^2$ and $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{Z})$. A direct computation shows that
$$\mathrm{Im}(gz) = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$
Now let $\tau \in \mathbb{H}^2$ and let $g \in \mathrm{PSL}(2,\mathbb{Z})$ be an element so that $\mathrm{Im}(g\tau)$ is maximal. Note that this is an honest maximum, since the number of integers so that $|cz + d| \leq K$ is finite for any $K > 0$. Since $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{Z})$, we can always post-compose with T^k for some $k \in \mathbb{Z}$ to make sure that $-\frac{1}{2} \leq \mathrm{Re}(g\tau) \leq \frac{1}{2}$. Now suppose that $|g\tau| < 1$. Since $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{Z})$ this leads to a contradiction, because $S(g\tau)$ would have a larger imaginary part (this follows from a direct computation).

Now suppose both z and gz lie in $\mathcal{F}$ for some $g \in \mathrm{PSL}(2,\mathbb{R})$. Without loss of generality, we assume that $\mathrm{Im}(gz) \geq \mathrm{Im}(z)$. This implies that

$$|cz + d| \leq 1.$$

Using that $z \in \mathcal{F}$ this implies that $c \in \{-1, 0, 1\}$. If $c = 0$, then $d = \pm 1$ and we obtain

$$g = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

for some $k \in \mathbb{Z}$. Now using that $z \in \mathcal{F}$, we see that either $k \in \{\pm 1\}$ and $\mathrm{Re}(z) \in \{\pm \frac{1}{2}\}$ or $k = 0$.

If $c = \pm 1$, then $d = 0$ and hence

$$g = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

which gives us the $|g\tau| = 1$ case. □

Since T maps the line $\mathrm{Re}(z) = -1/2$ to the line $\mathrm{Re}(z) = 1/2$ and S fixes i and swaps $(-1+\sqrt{3}i)/2$ and $(1+\sqrt{3}i)/2$ (which are in turn the fixed points of ST). These turn out to be the only side identifications and thus the quotient looks like Figure 2:

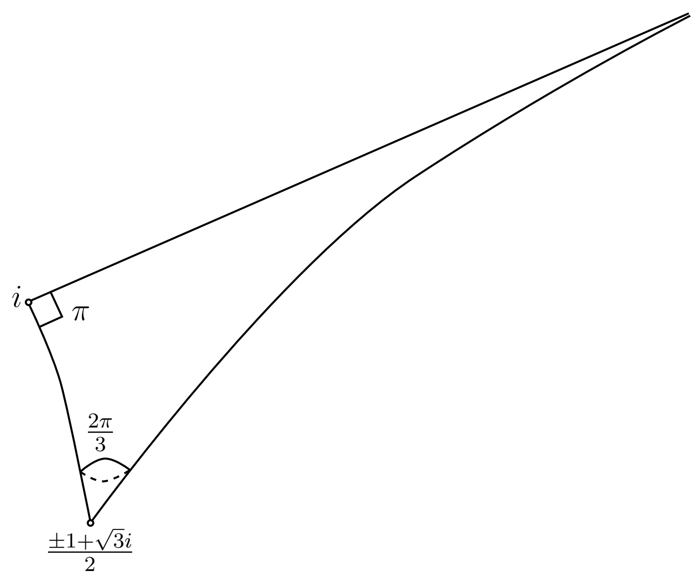

FIGURE 2. A cartoon of $\mathcal{M}_1$.

So $\mathcal{M}_1$ is a space that has the structure of a hyperbolic surface near almost every point. The only problematic points are the images of i and $(\pm 1 + \sqrt{3}i)/2$, where the $\mathcal{M}_1$ looks like a cone. The technical term for such a space is a hyperbolic *orbifold*.

$\mathcal{M}_1$ is called the *moduli space* of tori. $\mathcal{T}_1 = \mathbb{H}^2$ is called the *Teichmüller space* of tori.

Our next intermediate goal is to generalize this to all surfaces. Of course, we could use pants decompositions to parametrize Riemann surface structures on surfaces of higher genus. However, there is no a priori reason that the resulting deformation space should have anything to do with the Teichmüller space of the torus we just introduced.

To this end, we start by introducing different perspectives on $\mathcal{T}_1$, that generalize more naturally to higher genus surfaces.

7.3. $\mathcal{T}_1$ as a space of marked structures

Our objective in this section is to understand what the information is that is parametrized by $\mathcal{T}_1$.

7.3.1. Markings as a choice of generators for $\pi_1(R)$, part I. So, suppose $\tau \in \mathbb{H}^2$ and $\tau' = g\tau$ for some $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{Z})$. Let $f : R_{\tau'} \to R_\tau$ denote the biholomorphism from the proof of Proposition 7.1. We saw that we can find a lift $\widetilde{f} : \mathbb{C} \to \mathbb{C}$ so that $\widetilde{f}(z) = (c\tau + d)z$. In particular, using the relation between τ and τ', we see that

$$\widetilde{f}(\{1,\ \tau'\}) = \{c\tau + d,\ a\tau + b\}.$$

So, the biholomorphism corresponds to a base change (i.e. the change of a choice of generators) for Λ_τ.

Let us formalize this idea of a base change. First we take a base point $p_0 = [0] \in R_\tau$ for the fundamental group $\pi_1(R_\tau, p_0)$. The segments between 0 and 1 and between 0 and τ project to simple closed curves on R_τ and determine generators

$$[A_\tau],\ [B_\tau] \in \pi_1(R_\tau, p_0).$$

This now also gives us a natural choice of isomorphism $\Lambda_\tau \simeq \pi_1(R_\tau, p_0)$, mapping

$$1 \mapsto [A_\tau] \quad \text{and} \quad \tau \mapsto [B_\tau].$$

Likewise, for $R_{\tau'}$ we also obtain a natural system of generators $[A_{\tau'}], [B'_\tau] \in \pi_1(R_{\tau'}, p_0)$. Moreover, if $f_* : \pi_1(R_{\tau'}, p_0) \to \pi_1(R_\tau, p_0)$ denotes the map f induces on the fundamental group, then

$$f_*([A_{\tau'}]) \neq [A_\tau] \quad \text{and} \quad f_*([B_{\tau'}]) \neq [B_\tau].$$

Let us package these choices of generators:

Definition 7.3. Let R be a Riemann surface homeomorphic to $\mathbb{T}^2$.

(1) A *marking* on R is a generating set $\Sigma_p \subset \pi_1(R, p)$ consisting of two elements.
(2) Two markings Σ_p and $\Sigma'_{p'}$ are called *equivalent* if there exists a continuous curve α from p to p' so that the corresponding isomorphism $T_\alpha : \pi(R, p) \to \pi_1(R, p')$ satisfies
$$T_\alpha(\Sigma_p) = \Sigma'_{p'}.$$

Two pairs (R, Σ) and (R', Σ') of marked Riemann surfaces homeomorphic to $\mathbb{T}^2$ are called *equivalent* if there exists a biholomorphic mapping $h : R \to R'$ so that

$$h_*(\Sigma) \simeq \Sigma'.$$

Note that above we have *not* proved that $(R_\tau, \{[A_\tau], [B_\tau]\})$ and $(R_{\tau'}, \{[A_{\tau'}], [B_{\tau'}]\})$ are equivalent as marked Riemann surfaces, because our map f_* did not send the generators to each other, and in fact, they are not equivalent:

Theorem 7.4. *Let $\tau, \tau' \in \mathcal{T}_1$. Then the marked Riemann surfaces*

$$(R_\tau, \{[A_\tau], [B_\tau]\}) \quad \textit{and} \quad (R_{\tau'}, \{[A_{\tau'}], [B_{\tau'}]\})$$

are equivalent if and only if $\tau' = \tau$. Moreover, we have an identification

$$\mathcal{T}_1 = \left\{ (R, \Sigma_p); \quad \begin{matrix} R \textit{ a Riemann surface homemorphic to } \mathbb{T}^2 \\ p \in R,\ \Sigma_p \textit{ a marking on } R \end{matrix} \right\} \Big/ \sim .$$

7.4. Exercises

Exercise 7.1. Prove that every complex 1-dimensional torus is biholomorphic to a torus of the form

$$R_\tau := \mathbb{C}\,/\Lambda_\tau,$$

for some $\tau \in \mathbb{H}^2$.

LECTURE 8

Teichmüller space

We will mainly follow [**IT92**] for this lecture.

8.1. Markings as a choice of generators for $\pi_1(R)$, part II

We begin by proving the following theorem from the previous lecture:

Theorem 7.4. *Let $\tau, \tau' \in \mathcal{T}_1$. Then the marked Riemann surfaces*

$$(R_\tau, \{[A_\tau], [B_\tau]\}) \quad \textit{and} \quad (R_{\tau'}, \{[A_{\tau'}], [B_{\tau'}]\})$$

are equivalent if and only if $\tau' = \tau$. Moreover, we have an identification

$$\mathcal{T}_1 = \left\{ (R, \Sigma_p); \quad \begin{matrix} R \text{ a Riemann surface homemorphic to } \mathbb{T}^2 \\ p \in R,\ \Sigma_p \text{ a marking on } R \end{matrix} \right\} \Big/ \sim .$$

Proof. We begin by proving part of the second claim: every marked complex torus is equivalent to a marked torus of the form $(R_\tau, \{[A_\tau], [B_\tau]\})$. So, suppose (R, Σ) is a marked torus. We know that R is biholomorphic to R_τ for some $\tau \in \mathcal{T}_1$. Moreover, since $\Sigma = \{[A], [B]\}$ is a minimal generating set for Λ_τ, we can find a lattice isomorphism $\varphi : \Lambda_\tau \to \Lambda_\tau$ so that

$$\varphi([A]) = 1.$$

Moreover $R_{\varphi([B])}$ is biholomorphic to R_τ. So (R, Σ) is equivalent to

$$(R_{\varphi([B])}, \{A_{\varphi([B])}, B_{\varphi([B])}\}).$$

So, to prove the theorem, we need to show that $(R_\tau, \{[A_\tau], [B_\tau]\})$ and $(R_{\tau'}, \{[A_{\tau'}], [B_{\tau'}]\})$ are equivalent if and only if $\tau = \tau'$. Of course, if $\tau = \tau'$ then the two corresponding marked surfaces are equivalent, so we need to show the other direction.

So let $h : R_{\tau'} \to R_\tau$ be a biholomorphism that induces the equivalence. We may assume that $h([0]) = [0]$ and take a lift $\widetilde{h} : \mathbb{C} \to \mathbb{C}$ so that

$$\widetilde{h}(0) = 0.$$

This means that $\widetilde{h}(z) = \alpha z$ for some $\alpha \in \mathbb{C} \setminus \{0\}$. Hence $1 = \widetilde{h}(1) = \alpha$, which implies that $\tau = \widetilde{h}(\tau') = \tau'$. □

Note that so far, our alternate description of Teichmüller space only recovers the set $\mathcal{T}_1$ and not yet it topology. Of course we can use the bijection to define a topology. However, there is also an intrinsic defintion. We will discuss how to do this later.

8.2. Markings by diffeomorphisms

First, we give a third interpreation of $\mathcal{T}_1$. This goes through another (equivalent) way of marking Riemann surfaces.

To this end, fix once and for all a surface S diffeomorphic to $\mathbb{T}^2$. We define:

Definition 8.1. Let R and R' be Riemann surfaces and let

$$f : S \to R \quad \text{and} \quad f' : S \to R'$$

be orientation preserving diffeomorphisms. We say that the pairs (R, f) and (R', f') are *equivalent* if there exists a biholomorphism $h : R \to R'$ so that

$$(f')^{-1} \circ h \circ f : S \to S$$

is homotopic to the identity.

Note that if we pick a generating set $\{[A], [B]\}$ for the fundamental group $\pi_1(S, p)$ then every pair (R, f) as above defines a point

$$(R, \{f_*([A]), f_*([B])\}) \in \mathcal{T}_1 .$$

It turns out that this gives another description of the Teichmüller space of tori:

Theorem 8.2. *Fix S and $[A], [B] \in \pi_1(S, p)$ as above. Then the map*

$$\left\{ (R, f); \quad \begin{array}{c} R \text{ a Riemann surface, } f : S \to R \\ \text{an orientation preserving diffeomorphism} \end{array} \right\} \Big/ \sim \quad \to \quad \mathcal{T}_1$$

given by

$$(R, f) \mapsto (R, \{f_*([A]).f_*([B])\}),$$

is a well-defined bijection.

Proof. We start with well-definedness. Meaning, suppose (R, f) and (R', f') are equivalent. By definition, this means that there exists a biholomorphic map $h : R \to R'$ so that

$$h \circ f : S \to R' \quad \text{and} \quad f' : S \to R'$$

are homotopic. Now if α is a continuous arc between $f'(p)$ and $h(f(p))$, we see that T_α induces an equivalence between the markings

$$\{f'_*([A]),\ f'_*([B])\} \quad \text{and} \quad \{(h \circ f)_*([A]),\ (h \circ f)_*([B])\},$$

making $(R, \{f_*([A]),\ f_*[B]\})$ and $(R', \{f'_*([A]),\ f'_*([B])\})$ equivalent. This means that they correspond to the same point by the previous theorem. So, the map is well defined.

Moreover, the map is surjective. For any $\tau \in \mathcal{T}_1$ we can find an orientation preserving diffeomorphism $f : S \to R_\tau$. Indeed, we know that S is diffeomorphic to R_{τ_0} for some $\tau_0 \in \mathcal{T}_1$ so that $\{[A], [B]\} = \{[A_{\tau_0}], [B_{\tau_0}]\}$. One checks that the map $f_\tau : \mathbb{C} \to \mathbb{C}$ given by

$$f_\tau(z) = \frac{(\tau - \overline{\tau}_0)z + (\tau - \tau_0)\overline{z}}{\tau_0 - \overline{\tau}_0}$$

descends to an orientation preserving diffeomorphism that also induces the marking $\{[A_\tau], [B_\tau]\}$.

For the injectivity, suppose that

$$\Big[(R, \{f_*([A]), f_*([B])\})\Big] = \Big[(R', \{f'_*([A]), f'_*([B])\})\Big].$$

Take $\tau_0 \in \mathcal{T}_1$ so that

$$\Big[(S, \{[A], [B]\})\Big] = \Big[(R_{\tau_0}, \{([A_{\tau_0}]), [B_{\tau_0}]\})\Big].$$

Moreover, let $h : R \to R'$ be a holomorphism so that $h_*\{f_*([A]), f_*([B])\} = \{f'_*([A]), f'_*([B])\}$. We choose lattices $\Lambda, \Lambda' \subset \mathbb{C}$, generated by 1 adn a and 1 and a' respectively so that

$$R = \mathbb{C}\,/\Lambda \quad \text{and} \quad R' = \mathbb{C}\,/\Lambda',$$

and the generators induce the bases $\{f_*([A]), f_*([B])\}$ and $\{f'_*([A]), f'_*([B])\}$ respectively.

Now, let $\widetilde{f}, \widetilde{f}', \widetilde{h} : \mathbb{C} \to \mathbb{C}$ be lifts. We may assume that

$$\widetilde{f}(0) = \widetilde{f}'(0) = \widetilde{h}(0) = 0, \quad \widetilde{f}(1) = \widetilde{f}'(1) = \widetilde{h}(1) = 1,$$

and

$$\widetilde{f}(\tau_0) = a, \quad \widetilde{f}'(\tau_0) = a' \quad \text{and } \widetilde{h}(a) = a'$$

So we obtain a homotopy $F_t : \mathbb{C} \to \mathbb{C}$ defined by

$$F_t(z) = (1-t)\,\widetilde{h} \circ \widetilde{f}(z) + t\,\widetilde{f}'(z)$$

between $\widetilde{f}$ and $\widetilde{f}'$ that descends to a homotopy between $f : S \to R'$ and $f' : S \to R'$. □

8.3. The Teichmüller space of Riemann surfaces

The two description of the Teichmüller space of the torus above can be generalized to different Riemann surfaces. We will take the second one as a definition, as this is the most common definition in the literature. Moreover, it naturally leads to another key object in Teichmüller theory: the mapping class group.

Definition 8.3. Let S be a surface of finite type. Then the *Teichmüller space* of S is defined as

$$\mathcal{T}(S) = \left\{ (X, f);\ \begin{array}{c} X \text{ a Riemann surface }, f : S \to X \\ \text{an orientation preserving diffeomorphism} \end{array} \right\} \Big/ \sim,$$

where

$$(X, f) \sim (Y, g)$$

if and only if there exists a biholomorphism $h : X \to Y$ so that the map

$$g^{-1} \circ h \circ f : S \to S$$

is homotopic to the identity.

Remark 8.4. Note that by phrasing the definition in terms of Riemann surfaces, we cannot yet make sense of a Teichmüller space of surfaces with boundary. However, in the world of hyperbolic surfaces, such Teichmüller spaces *do* make sense.

We will often write

$$\mathcal{T}(\Sigma_{g,n}) = \mathcal{T}_{g,n} \quad \text{and} \quad \mathcal{T}(\Sigma_g) = \mathcal{T}_g\,.$$

In order to get to the analogous definition to the space of marked tori, we need to single out particularly nice generating sets for the fundamental group, just like we did for tori. We will stick to closed surfaces. Recall that the fundamental group of a surface of genus g satisfies:

$$\pi_1(\Sigma_g, p) = \left\langle a_1, \ldots, a_g, b_1, \ldots, b_g \;\middle|\; \prod_{i=1}^{g} [a_i, b_i] = e \right\rangle.$$

In what follows, a generating set $A_1, \ldots, A_g, B_1, \ldots, B_g$ of $\pi_1(\Sigma_g, p)$ that satisfies

$$\prod_{i=1}^{g} [A_i, B_i] = e,$$

will be called a *canonical* generating set. Note that this includes the torus case.

Definition 8.5. Let R be a closed Riemann surface.

(1) A *marking* on R is a canonical generating set $\Sigma_p \subset \pi_1(R, p)$.
(2) Two markings Σ_p and $\Sigma'_{p'}$ are called *equivalent* if there exists a continuous curve α from p to p' so that the corresponding isomorphism $T_\alpha : \pi(R, p) \to \pi_1(R, p')$ satisfies

$$T_\alpha(\Sigma_p) = \Sigma'_{p'}.$$

Two pairs (R, Σ) and (R', Σ') of marked closed Riemann surfaces are called *equivalent* if there exists a biholomorphic mapping $h : R \to R'$ so that

$$h_*(\Sigma) \simeq \Sigma'.$$

Just like in the case of the torus, the space of marked Riemann surfaces turns out to be the same as Teichmüller space:

Theorem 8.6. *Let S be a closed surface and Σ a marking on S. Then the map*

$$\mathcal{T}(S) \to \left\{ (R, \Sigma_p);\; \begin{array}{c} R \text{ a closed Riemann surface diffeomorphic to } S \\ p \in R,\; \Sigma_p \text{ a marking on } R \end{array} \right\} \Big/ \sim\,.$$

given by

$$[(R, f)] \mapsto [(R, f_*(\Sigma)]$$

is a bijection.

Proof sketch. Write $\Sigma = \{[A_1], \ldots, [A_g], [B_1], \ldots, [B_g]\}$, where A_i, B_i are simple closed curves based at a point $p_0 \in S$. Let us start with the injectivity. So, suppose

$$[(R, f_*(\Sigma)] = [(R', f'_*(\Sigma)].$$

This means that we can find a biholomorphic map $h : R \to R'$ and a self-diffeomorphism $g_0 : R' \to R'$ so that

$$g_1 = g_0 \circ h \circ f$$

corresponds with f' on the curves $A_1, \dots, A_g, B_1, \dots, B_g$. The domain obtained by deleting these curves from S is a disk. This implies that f and g_1 are homotopic, which in turn means that

$$[(R, f)] = [(R', f')] \in \mathcal{T}(S).$$

For surjectivity, suppose we are given a marked Riemann surface (R, Σ_p). So we need to find an orientation preserving homeomorphism $f : S \to R$ so that $f_*(\Sigma) = \Sigma_p$. So, let us take simple closed smooth curves $A'_1, \dots, A'_g, B'_1, \dots, B'_g$ so that $\Sigma_p = \{[A'_1], \dots, [A'_g], [B'_1], \dots, [B'_g]\}$. Moreover, we will set

$$C = \bigcup_{j=1}^{g} (A_j \cup B_j), \quad C' = \bigcup_{j=1}^{g} (A'_j \cup B'_j), \quad S_0 = S \setminus C, \quad \text{and} \quad R_0 = R \setminus C'.$$

R_0 and S_0 are diffeomorphic to polygons with $4g$ sides. So we can find a diffeomorphism by extending a diffeomorphism R_0, S_0. For more details, see [**IT92**, Theorem 1.4]. □

8.4. Exericises

Exercise 8.1. Flesh out the argument for injectivity in the proof of Theorem 8.6.

LECTURE 9

The mapping class group and moduli space

9.1. The mapping class group

Just like in the case of the torus, we have a natural group action on the Teichmüller space of a surface, by a group called the mapping class group:

Definition 9.1. Let S_0 be a compact surface of finite type and $\Sigma \subset S_0$ a finite set. Set $S = S_0 \setminus \Sigma$.

The *mapping class group* of S is given by

$$\mathrm{MCG}(S) = \mathrm{Diff}^+(S, \partial S, \Sigma)/\mathrm{Diff}_0^+(S, \partial S, \Sigma)$$

where

$$\mathrm{Diff}^+(S, \partial S, \Sigma) = \left\{ f : S \to S; \begin{array}{c} f \text{ an orientation preserving diffeomorphism} \\ \text{that preserves the boundary components of} \\ S \text{ setwise, and the puncture pointwise} \end{array} \right\}$$

and

$$\mathrm{Diff}_0^+(S, \partial S, \Sigma) = \left\{ f \in \mathrm{Diff}^+(S, \partial S, \Sigma); \quad f \text{ homotopic to the identity } \right\}.$$

The group operation is induced by composition of functions (see Exercise 9.1).

We will discuss some properties of this group. Since this is not a course on mapping class groups, we will not discuss the proofs of these properties. For more information, we refer to [**FM12**]. Some authors let go of the condition that $\mathrm{MCG}(S)$ fixes the punctures. The group we defined above is then often called the *pure mapping class group*.

9.1.1. Dehn twists. Let us start by describing some non-trivial elements. First, consider an annulus

$$A := [0,1] \times \mathbb{R} / \mathbb{Z}\,.$$

Define a map $T : A \to A$ by

$$T(t, [\theta]) = T(t, [\theta + t])$$

for all $t \in [0,1]$, $\theta \in \mathbb{R}$. This map is called a Dehn twist. Note that this map fixes ∂A pointwise. Figure 1 shows that this map does to a segment connecting the two boundary components of the annulus.

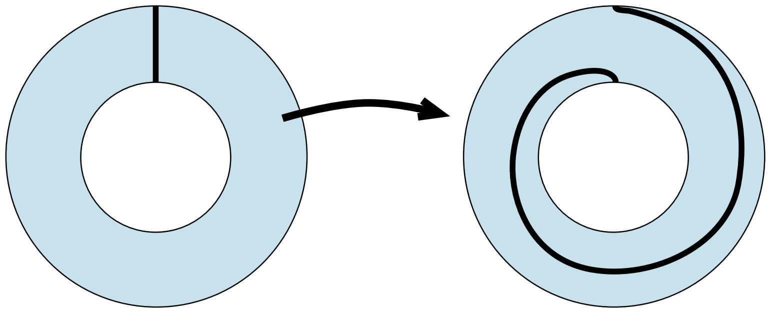

FIGURE 1. A Dehn twist on an annulus.

Now let α be an essential simple closed curve on S. Let N be closed regular neighborhood of α. Identifying N with A, we can define a map $T_\alpha : S \to S$ by

$$T_\alpha(p) = \begin{cases} T(p) & \text{if } p \in N \\ p & \text{if } p \in S \setminus N \end{cases}$$

Because $T|_{\partial A}$ is the identity map, this is a continuous map. To obtain an element in $\mathrm{MCG}(S)$, we need to start with a smooth map. There are multiple ways out at the moment. We could smoothen T. Or we could use surface topology to argue that T_α is homotopic to a smooth map. Since for the mapping class group, we only car about diffeomorphisms up to homotopy, the element we get in $\mathrm{MCG}(S)$ will not depend on how we do this.

Figure 2 shows an example of a Dehn twist.

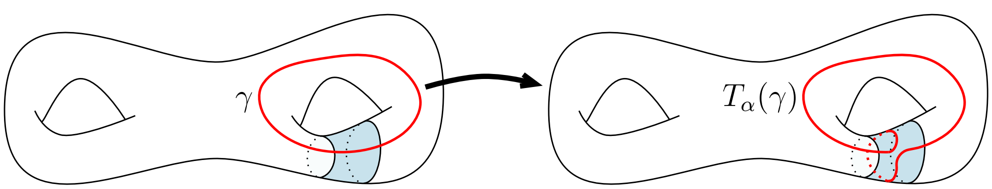

FIGURE 2. A Dehn twist on a surface of genus two.

We see that T_α maps a curve γ on the surface intersecting the defining curve α (of which we have only drawn the regular neighborhood) transversely to a curve that is not homotopic to γ. In particular, T_α is not homotopic to the identity and hence defines a non-trivial element in $\mathrm{MCG}(S)$.

9.1.2. Dehn-Lickorish. It actually turns out that the mapping class group is finitely generated. In the following theorem, a non-separating curve will be a curve α so that $S \setminus \alpha$ is connected. Figure 3 shows an example.

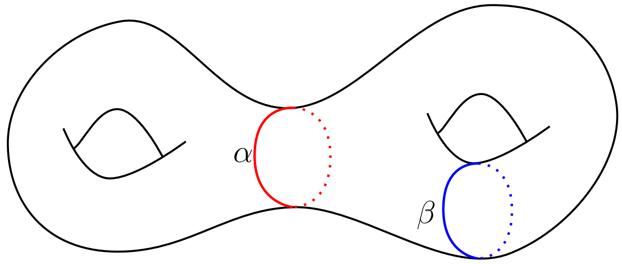

FIGURE 3. A separating curve (α) and a non-separating curve (β).

Theorem 9.2 (Dehn - Lickorish theorem)**.** *Let S be a surface of finite type, the mapping class group* $\mathrm{MCG}(S)$ *is generated by finitely many Dehn twists about nonseparating simple closed curves.*

For a proof see [**FM12**, Chapter 4].

9.2. Moduli space

Looking at Definition 8.3, we see there is a natural group action of the mapping class group of a surface on the corresponding Teichmüller space.

$$[g] \cdot [(R, f)] = [(R, f \circ g^{-1})].$$

The quotient is what will be called moduli space.

Definition 9.3. Let S be a surface of finite type . The *moduli space* of S is the space

$$\mathcal{M}(S) = \mathcal{T}(S) / \mathrm{MCG}(S).$$

We will write

$$\mathcal{M}(\Sigma_{g,n}) = \mathcal{M}_{g,n} \quad \text{and} \quad \mathcal{M}(\Sigma_g) = \mathcal{M}_g\,.$$

Remark 9.4. Note that by using the convention that the mapping class group fixes boundary components and punctures, we leave these "marked", i.e. if two surfaces are isometric, but any isometry between them permutes the punctures, these surfaces represent different points in moduli space.

9.3. The mapping class group of the torus

Recall that the moduli space of the torus also appeared as a quotient. Namely, we had

$$\mathcal{M}_1 = \mathbb{H}^2 / \mathrm{PSL}(2, \mathbb{Z}).$$

This makes one wonder whether the mapping class group of the torus is maybe $\mathrm{PSL}(2, \mathbb{Z})$. This turns out to be almost correct.

First we need to introduce the algebraic intersection number between oriented curves. Let α and β be two oriented closed curves on an oriented surface S that intersect each other transversely at every intersection point. Then the *algebraic intersection number* between α and β is given by

$$i(\alpha, \beta) = \sum_{p \in \alpha \cap \beta} \mathrm{sgn}(\omega(v_p(\alpha), v_p(\beta))),$$

where $\mathrm{sgn}: \mathbb{R} \to \{\pm 1\}$ denotes the sign function, ω is any volume form that induces the orientation and $v_p(\alpha)$ and $v_p(\beta)$ denote the unit tangent vectors to α and β respectively at p. Note that

$$i(\beta, \alpha) = -i(\alpha, \beta).$$

Figure 4 shows an exemple of a positive contribution to the intersection number.

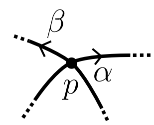

FIGURE 4. A positive contribution to $i(\alpha, \beta)$ if the orientation points out of the page.

We note that this form descends to homology. That is, it induces a form

$$i : H_1(S, \mathbb{Z}) \times H_1(S, \mathbb{Z}) \to \mathbb{Z}$$

called the *intersection form*, with the properties:

(1) i is bilinear.
(2) i is *alternating*, i.e.
$$i(a, b) = -i(b, a)$$
for all $a, b \in H_1(S, \mathbb{Z})$.
(3) i is *non-degenerate*, i.e. if $a \in H_1(S, \mathbb{Z})$ is such that
$$i(a, b) = 0 \quad \text{for all} b \in H_1(S, \mathbb{Z})$$
then $a = 0$.

(see [**FK92**, Section III.1] for more details). Such a form is called a *symplectic form.*

Recall that every diffeomorphism $f : S \to S$ induces an automorphism $f_* : H_1(S, \mathbb{Z}) \to H_1(S, \mathbb{Z})$. First of all note that the image preserves the intersection form. Moreover, isotopic maps give rise to the same automorphism. So this gives us a representation

$$\mathrm{MCG}(S) \to \mathrm{Aut}(H_1(S, \mathbb{Z}), i)$$

called the *homology representation* of the mapping class group. Recall that if S is a closed orientable surface of genus g, then $H_1(S, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$. Choosing an identification, the homology representation becomes a map

$$\mathrm{MCG}(S) \to \mathrm{Sp}(2g, \mathbb{Z}) = \left\{ A \in \mathrm{Mat}_{2g}(\mathbb{Z}); \quad i(Av, Aw) = i(v, w), \ \forall v, w \in \mathbb{Z}^{2g} \ \right\}.$$

Finally, there is an isomorphism

$$\mathrm{Sp}(2, \mathbb{Z}) \simeq \mathrm{SL}(2, \mathbb{Z})$$

(see Exercise 9.2).

We will show that for the torus, the homology representation is an isomorphism.

Theorem 9.5. *We have*

$$\mathrm{MCG}(\mathbb{T}^2) \simeq \mathrm{SL}(2,\mathbb{Z}).$$

The action of $\mathrm{MCG}(\mathbb{T}^2)$ *on* $\mathcal{T}_1$ *is that given by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau - b}{-c\tau + d}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ *and* $\tau \in \mathcal{T}_1$.

Proof. Let us first prove that the homology representation is surjective. We will identify

$$\mathbb{T}^2 = \mathbb{R}^2 \,/\, \mathbb{Z}^2$$

Every element $A \in \mathrm{SL}(2,\mathbb{Z})$ induces a linear map $\phi_A : \mathbb{R}^2 \to \mathbb{R}^2$. Moreover, since $\mathrm{SL}(2,\mathbb{Z})$ preserves $\mathbb{Z}^2 \subset \mathbb{R}^2$, the action on $\mathbb{R}^2$ descends to an action by diffeomorphisms

$$[\phi_A] : \mathbb{T}^2 \to \mathbb{T}^2.$$

Moreover, since the action of A on lattice vectors corresponds to the action on closed curves. $[\phi_A]_* = A$.

For injectivity, we note that for $\mathbb{T}^2$, we have

$$\pi_1(\mathbb{T}^2, [0]) \simeq H_1(\mathbb{T}^2, \mathbb{Z}).$$

So suppose, $\phi : \mathbb{T}^2 \to \mathbb{T}^2$ is a diffeomorphism so that $\phi_* = \mathrm{Id}$. This means that if we take a lift $\widetilde{\phi} : \mathbb{R}^2 \to \mathbb{R}^2$ we have

$$\widetilde{\phi}(x + (m,n)) = \widetilde{\phi}(x) + \phi_*(m,n) = \widetilde{\phi}(x) + (m,n),$$

for all $(m,n) \in \mathbb{Z}$. This means that

$$F_t(x) = tx + (1-t)\widetilde{\phi}(x), \quad t \in [0,1], x \in \mathbb{R}^2$$

gives a $\mathbb{Z}^2$-equivariant homotopy between $\widetilde{\phi}$ and the identity. In other words, ϕ is isotopic to the identity and in particular represents the trivial element in $\mathrm{MCG}(\mathbb{T}^2)$.

In order to prove the final statement, use a generating set

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

of $\mathrm{SL}(2,\mathbb{Z})$. Proving the claim for these two elements, proves it for all (using that the formula defines an action, the proof of which we leave to the reader). Let us write α and β for the curves corresponding to the vectors $(0,1), (1,0) \in \mathbb{Z}^2$ respectively. Above we essentially showed that T acts as T_α (see also Exercise 9.3). Now since

$$T_\alpha(X, f) = (X, f \circ T_\alpha^{-1}),$$

we see that $T_\alpha(\tau) = \tau - 1$.

Likewise, S acts by $T_\alpha \circ T_\beta \circ T_\alpha^{-1}$, which means that it sends the lattice with basis $(1,\tau)$ to a lattice with basis $(-\tau, 1)$. Now we need to rotate and scale the lattice back into position. And obtain $(1, -1/\tau)$. □

Remark 9.6. Note that the theorem above implies that the mapping class group action is not faithful. The kernel of the action is the center of $\mathrm{SL}(2, \mathbb{Z})$, i.e. the subgroup

$$\left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} < \mathrm{SL}(2, \mathbb{Z}).$$

On the other hand, we do have

$$\mathbb{H}^2 \,/\, \mathrm{PSL}(2, \mathbb{Z}) = \mathbb{H}^2 \,/\, \mathrm{SL}(2, \mathbb{Z}).$$

This means that the mapping class group action is indeed a generalization of the situation for the torus case.

9.4. Exericises

Exercise 9.1. Check that the group operation of the mapping class group of a surface is well defined.

Exercise 9.2. Show that

$$\mathrm{Sp}(2,\mathbb{Z}) \simeq \mathrm{SL}(2,\mathbb{Z}).$$

Exercise 9.3. How can

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

be realized by Dehn twists?

LECTURE 10

A topology on Teichmüller space

Our next goal is to put a topology on Teichmüller space. Of course, the goal is that marked complex structures that are more similar should be closer to each other. So that raises the question how one measures the how similar two marked complex structures are. This is where Beltrami differentials come in. We will mainly follow [**IT92, Hub06**]

10.1. Beltrami coefficients

In what follows, we will equip our base surface S with a complex structure as well. So, suppose

$$[R, f] \in \mathcal{T}(S).$$

Let (U, z) denote a local coordinate on S and (V, w) one on R so that $f(U) \subset V$. Define a map $F = w \circ f \circ z^{-1} : z(U) \to \mathbb{C}$ and consider the function

$$\mu = \left(\frac{\partial F}{\partial \overline{z}}\right) \Big/ \left(\frac{\partial F}{\partial z}\right)$$

on $z(U)$. This is a smooth function that is independent of the choice of coordinate (V, w) (Exercise 10.1). One computes that the Jacobian of F is given by

$$\left|\frac{\partial F}{\partial z}\right|^2 - \left|\frac{\partial F}{\partial \overline{z}}\right|^2.$$

Since F is orientation preserving, this is positive and hence

$$|\mu| < 1.$$

Moreover, the Cauchy-Riemann equations imply that F is holomorphic on $z(U)$ if and only if $\mu = 0$. In this sense, μ measures how far away f is from conformal in the chart (U, z). We will make this more geometric later. μ will be called the *Beltrami coefficient* of f with respect to (U, z).

Let's have a look at how the Beltrami coefficient depends on the coordinate. So, consider coordinate patches (U_j, z_j) and (U_k, z_k) on S so that $U_j \cap U_k \neq \emptyset$ and coordinate patches (V_j, w_j) and (V_k, w_k) on R so that

$$f(U_j) \subset V_j \quad \text{and} \quad f(U_k) \subset V_k.$$

Let μ_j and μ_k denote the corresponding Beltrami coefficients. Working out the chain rule, we get

$$\mu_j = (\mu_k \circ z_{kj}) \cdot \overline{\left(\frac{\partial z_{kj}}{\partial z_j}\right)} \Big/ \left(\frac{\partial z_{kj}}{\partial z_j}\right) \quad \text{on} \quad z_j(U_j \cap U_k),$$

where $z_{kj} = z_k \circ z_j^{-1}$. In particular, the absolute value $|\mu|$ is independent of the choice of coordinate patch. We will write $|\mu_f(z)|$ for this value at the point $z \in R$.

10.2. Quasiconformal mappings

Now let us consider what it is that $|\mu_f(z)|$ measures. Let $f : D \to D$ denote an orientation preserving diffeomorphism of some domain $D \subset \mathbb{C}$ containing 0. And write

$$Df(0) \cdot z = \frac{\partial f}{\partial z}(0) \cdot z + \frac{\partial f}{\partial \overline{z}}(0) \cdot \overline{z}$$

denote the first order Taylor expansion of f at 0. Writing $Df(0) \cdot z = a \cdot z + b \cdot \overline{z}$ (so $a = \frac{\partial f}{\partial z}(0)$ and $b = \frac{\partial f}{\partial \overline{z}}(0)$), it turns out that the determinant and operator norm of $Df(0)$ are given by

$$\det(Df(0)) = |a|^2 - |b|^2 \quad \text{and} \quad ||Df(0)|| = |a| + |b|.$$

The former is a direct computation and the latter we will see in a moment.

Let us consider the inverse image of a unit circle under this map. I.e. the solutions of the equation

$$|Df(0) \cdot z| = 1.$$

Let us write

$$z = r\ e^{i\theta},\ a = |a|\ e^{i\alpha},\ b = |b|\ e^{i\beta}.$$

The equation then becomes

$$\left|(|a| + |b|) \cos\left(\theta + \frac{\alpha - \beta}{2}\right) + i(|a| - |b|) \sin\left(\theta + \frac{\alpha - \beta}{2}\right)\right| = \frac{1}{r}.$$

This is the equation of an ellipse with

- minor axis at polar angle $\frac{\beta - \alpha}{2}$ of half length $\frac{1}{|a|+|b|}$
- major axis at polar angle $\frac{\beta - \alpha + \pi}{2}$ of half length $\frac{1}{|a|-|b|}$.

Note that the latter is positive since the Jacobian

$$\det(Df(0)) = |a|^2 - |b|^2 > 0.$$

Moreover, the former proves out claim about the operator norm.

Figure 1 shows a picture of the situation.

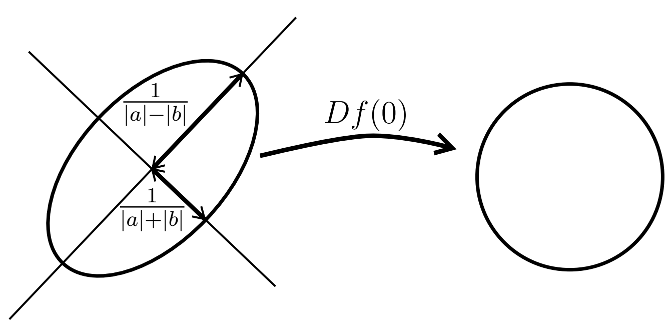

FIGURE 1. An ellipse that gets mapped to a circle.

The ratio of the axes of the ellipse is

$$\frac{|a|+|b|}{|a|-|b|} = \frac{1+|\mu_f(0)|}{1-|\mu_f(0)|}.$$

In particular, if $\mu_f(0)$ is close to 1, then the preimage is very far away from a circle and if $\mu_f(0) = 0$ (i.e. if f is biholomorphic) then the preimage is a circle. So this means that

- Biholomorphic maps locally send circles to circles (i.e. they are conformal)
- The $\mu_f(0)$ measures how far away the preimage of a circle (an ellipse) is from a circle.

For this reason, the Beltrami coefficient $\mu_f(z)$ is sometimes also called the *complex dillatation* of f at z. Since the ratio between the major and minor axis of the ellipse is a measure of how far f is from being conformal, this leads to the following definition:

Definition 10.1. Let R and R' be Riemann surfaces and let $f : R \to R'$ be an orientation preserving diffeomorphism. We say $f : R \to R'$ is a K-*quasiconformal mapping* if

$$K \geq K_f := \sup_{z\in R} \left\{ \frac{1+|\mu_f(z)|}{1-|\mu_f(z)|} \right\}.$$

K_f is called the quasiconformal dillatation of f.

Note that

$$K_f \geq 1$$

for all orientation preserving diffeomorphisms $f : R \to R'$. Moreover, the fact that holomorphic maps are exactly the maps with $\mu_f = 0$ implies that 1-quasiconformal maps are conformal.

Remark 10.2. The notion of a quasiconformal map can be generalized to maps of much lower regularity (see eg. [**IT92**, Chapter 4]).

The Beltrami coefficient behaves as follows with respect to precomposition with a diffeomorphism:

Proposition 10.3. *Let R, S and T be Riemann surfaces and let*

$$R \xrightarrow{f} S \xrightarrow{g} T$$

be orientation preserving diffeomorphisms. Then

- *the following relation holds*

$$\mu_g \circ f = \left(\frac{\partial f}{\partial z} \Big/ \overline{\frac{\partial f}{\partial z}} \right) \cdot \frac{\mu_{g\circ f} - \mu_f}{1 - \overline{\mu}_f \cdot \mu_{g\circ f}}.$$

- *In particular, if S_1 and S_2 are Riemann surfaces and*

$$f_1 : R \to S_1 \quad \text{and} \quad f_2 : R \to S_2$$

are orientation preserving diffeomorphisms. Then the map $f_2 \circ f_1^{-1} : S_1 \to S_2$ is biholomorphic if and only if

$$\mu_{f_1} = \mu_{f_2}.$$

Proof. This follows from the chain rule (Exercise 10.2). □

10.3. Topologizing Teichmüller space

Now that we have equipped our base surface S with the structure of a Riemann surface. We can define the set of functions

$$B(S)_1 := \left\{ \mu_f : S \to \mathbb{C};\ \begin{array}{c} f : S \to R \text{ an orientation} \\ \text{preserving diffeomorphism,} \\ R \text{ a Riemann surface} \end{array} \right\}.$$

This can be given a topology using the L^∞-norm. I.e.

$$||\mu_f||_\infty = \sup_{z \in S}\{|\mu_f(z)|\}.$$

Again writing $S = S_0 \setminus \Sigma$ where S_0 is a closed and orientable surface and Σ a finite set, we obtain an action of $\mathrm{Diff}^+(S, \Sigma)$ on $B(S)_1$ by pullback, i.e.

$$\varphi \cdot \mu_f = \mu_{f \circ \varphi^{-1}}$$

for all $\varphi \in \mathrm{Diff}^+(S, \Sigma)$ and $\mu_f \in B(S)_1$.

We now have the following:

Theorem 10.4. *Let S, R and R' be Riemann surfaces and*

$$f : S \to R \quad \textit{and} \quad f' : S \to R'$$

be orientation preserving diffeomorphisms. Then there exists a biholomorphic mapping

$$h : R \to R'$$

if and only if

$$\mu_f = \mu_{f' \circ \varphi^{-1}}$$

for some $\varphi \in \mathrm{Diff}^+(S, \Sigma)$. Moreover, the map

$$(f')^{-1} \circ h \circ f : S \to S$$

is homotopic to the identity if and only if $\varphi \in \mathrm{Diff}_0^+(S, \Sigma)$.

Proof. Suppose that there exists a biholormorphic map $h : R \to R'$. Then we set

$$\varphi = (f')^{-1} \circ h \circ f : S \to S.$$

Then

$$\mu_{f'} = \mu_{h \circ f \circ \varphi^{-1}} = \mu_{f \circ \varphi^{-1}},$$

where we have used Proposition 10.3 for the second equality.

Conversely, suppose there exists some $\varphi \in \mathrm{Diff}_0^+(S, \Sigma)$ so that

$$\mu_f = \mu_{f' \circ \varphi^{-1}}$$

Proposition 10.3 then shows that $f' \circ \varphi \circ f^{-1} : R \to R'$ is biholomorphic.

The second claim follows from the form of φ. □

A direct consequence is the following:

Corollary 10.5. *The map from the set of marked Riemann surfaces defined by*

$$(R, f) \mapsto \mu_f$$

induces a bijections

$$\mathcal{T}(S) \to B(S)_1 / \mathrm{Diff}^+_0(S, \Sigma)$$

and

$$\mathcal{M}(S) \to B(S)_1 / \mathrm{Diff}^+(S, \Sigma).$$

In particular, since $B(S)_1$ is a topological space, these bijections equip $\mathcal{T}(S)$ and $\mathcal{M}(S)$ with a topology. We will see later on that the choice of Riemann surface structure on S does not influence the topology on Teichmüller space.

10.4. Grötzsch's theorem

In order to have a non-trivial example of the quasi-conformal comparisson between two Riemann surfaces and also for applications later on, we will prove Grötzsch's theorem.

In this section A_m will denote the annulus

$$A_m := \left\{ z \in \mathbb{C}; \quad 0 < \mathrm{Im}(z) < m \ \right\} / \, \mathbb{Z}$$

for all $m > 0$. Here the $\mathbb{Z}$-action is given by $k \cdot z = z + k$ for all $k \in \mathbb{Z}$, $z \in \mathbb{C}$.

Theorem 10.6 (Grötzsch's theorem)**.** *Let $f : A_m \to A_{m'}$ be a K-quasiconformal map. Then*

$$\frac{1}{K} \leq \frac{m}{m'} \leq K.$$

Moreover, equality is realized if and only if f can be lifted to a map

$$\widetilde{f} : \left\{ z \in \mathbb{C}; \quad 0 < \mathrm{Im}(z) < m \ \right\} \to \left\{ z \in \mathbb{C}; \quad 0 < \mathrm{Im}(z) < m \ \right\}$$

given by

$$\widetilde{f}(x + iy) = b + x + i \frac{m'}{m} y$$

for some $b \in \mathbb{R}$.

We will prove this theorem in the next lecture.

10.5. Exericises

Exercise 10.1. Suppose

$$[R, f] \in \mathcal{T}(S).$$

Let (U, z) denote a local coordinate on S and (V, w) one on R so that $f(U) \subset V$. Define a map $F = w \circ f \circ z^{-1} : z(U) \to \mathbb{C}$ and consider the function

$$\mu = \left(\frac{\partial F}{\partial \overline{z}}\right) \Big/ \left(\frac{\partial F}{\partial z}\right)$$

on $z(U)$. Show that this is a smooth function that is independent of the choice of coordinate (V, w).

Exercise 10.2. Prove Proposition 10.3.

LECTURE 11

Grötzsch's theorem and quasiconformal maps

We will mainly follow [**Hub06**] for today's material.

11.1. Grötzsch's theorem

Recall that A_m denotes the annulus

$$A_m := \{ z \in \mathbb{C}; \quad 0 < \mathrm{Im}(z) < m \ \} \,/\, \mathbb{Z}$$

for all $m > 0$, where the $\mathbb{Z}$-action is given by $k \cdot z = z + k$ for all $k \in \mathbb{Z}$, $z \in \mathbb{C}$.

Theorem 10.6 (Grötzsch's theorem)**.** *Let $f : A_m \to A_{m'}$ be a K-quasiconformal map. Then*

$$\frac{1}{K} \leq \frac{m}{m'} \leq K.$$

Moreover, equality is realized if and only if f can be lifted to a map

$$\widetilde{f} : \{ z \in \mathbb{C}; \quad 0 < \mathrm{Im}(z) < m \ \} \to \{ z \in \mathbb{C}; \quad 0 < \mathrm{Im}(z) < m \ \}$$

given by

$$\widetilde{f}(x + iy) = b + x + i\frac{m'}{m}y$$

for some $b \in \mathbb{R}$.

Before we prove the theorem, we need two lemmas. In the first lemma, we assume the annuli $A_{m'}$ and A_m are equipped with the Euclidean metric that descends from $\mathbb{C}$. Moreover, for the rest of this section $||Df(z)||$ will denote the L^2-norm of $Df(z)$. I.e. if $f(x + iy) = u(x, y) + iv(x, y)$ then

$$||Df||^2 = \left| \frac{\partial u(x,y)}{\partial x} \right|^2 + \left| \frac{\partial u(x,y)}{\partial y} \right|^2 + \left| \frac{\partial v(x,y)}{\partial x} \right|^2 + \left| \frac{\partial v(x,y)}{\partial y} \right|^2 .$$

Lemma 11.1. *Let $f : A_m \to A_{m'}$ be a K-quasiconformal map. Then*

$$\frac{1}{\mathrm{area}(A_m)} \int_{A_m} ||Df(x,y)|| \, dxdy \ \geq \ \max\left\{ 1, \frac{m'}{m} \right\}.$$

Proof. The basic observation is that since the shortest non-trivial curve in A_m has length 1, f sends circles in $A_{m'}$ to circles of length at least 1. Likewise, vertical segments in A_m go to segments of length at least m'. This means that

$$\frac{1}{\mathrm{area}(A_m)} \int_{A_m} ||Df(x,y)|| \, dxdy \geq \frac{1}{m} \int_0^m \int_0^1 \left| \frac{\partial f}{\partial x} \right| dxdy \geq \frac{1}{m} \int_0^m dy = 1.$$

Likewise

$$\frac{1}{\text{area}(A_m)} \int_{A_m} ||Df(x,y)||\, dxdy \geq \frac{1}{m} \int_0^1 \int_0^m \left| \frac{\partial f}{\partial y} \right| dydx \geq \frac{1}{m} \int_0^1 m' dx = \frac{m'}{m}.$$

□

Moreover, we have:

Lemma 11.2. *Let R and R' be Riemann surfaces and let $f : R \to R'$ be a K quasi-conformal map. Then*

$$\det(Df)(z) \geq \frac{1}{K} ||Df(z)||^2$$

for all $z \in R$.

Proof. This is a matter of writing out the Jacobian and $||Df(z)||^2$. Indeed

$$\det(Df(z)) = \left| \frac{\partial f}{\partial z} \right|^2 - \left| \frac{\partial f}{\partial \overline{z}} \right|^2 \quad \text{and} \quad ||Df(z)||^2 = \left(\left| \frac{\partial f}{\partial z} \right| + \left| \frac{\partial f}{\partial \overline{z}} \right| \right)^2.$$

So

$$\frac{\det(Df(z))}{||Df(z)||^2} = \frac{1 - |\mu(z)|}{1 + |\mu(z)|} \geq \frac{1}{K}.$$

□

We can now prove Grötzsch's theorem.

Proof of Theorem 10.6. We have

$$m' = \text{area}(A'_m) = \int_{A_m} \det(Df)(x,y) dxdy.$$

Using the lemma above, we get

$$m' \geq \frac{1}{K} \int_{A_m} ||Df(x,y)||^2 \, dxdy = \frac{1}{mK} \int_{A_m} ||Df(x,y)||^2 \, dxdy \int_{A_m} 1^2 dxdy \geq \frac{1}{mK} \left(\int_{A_m} ||Df(x,y)|| \, dxdy \right)^2.$$

The last step is the Cauchy-Schwarz inequality. Now using Lemma 11.1, we get

$$m' \geq \frac{m}{K} \max \left\{ 1, \frac{m'}{m} \right\}^2,$$

which is equivalent to the theorem.

Moreover, to reach equality, we need both the Jacobian $\det(Df(z))$ and $||Df(z)||$ to be constant functions, which means that $\widetilde{f}$ needs to be an affine map. Moreover, since horizontal circles need to be mapped to horizontal circles and vertical segments to vertical segments, f needs to be of the form as claimed. □

11.2. Inverses and composition of quasi-conformal maps

It is not clear from Defintion 10.1 that being quasiconformal is closed under taking inverses and composition. It turns out it is and we will prove that in this section.

Instead of proving this directly, we will first give an equivalent definition of quasi-conformal mappings. In this definition, a *quadrilateral* (Q, p_1, p_2, p_3, p_4) in a Riemann surface R will be closed subset $Q \subset R$ so that

- Q is homeomorphic to a closed disk
- the points $p_1, p_2, p_3, p_4 \in \partial Q$ are called the vertices of Q and their cyclic order coincides with that induced by the orientation.

Figure 1 shows an example.

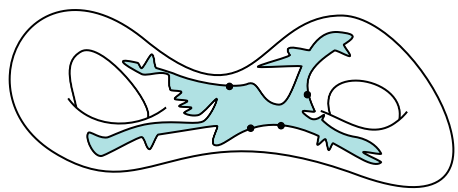

FIGURE 1. A quadrilateral.

First we note:

Lemma 11.3. *For every quadrilateral Q we can find a homeomorphism*

$$h : Q \to \left\{ z \in \mathbb{C};\quad 0 \leq \mathrm{Re}(z) \leq a,\ 0 \leq \mathrm{Im}(z) \leq b \ \right\}$$

for some $a, b > 0$ so that h is conformal on the interior $\mathring{Q}$ of Q and

$$h(p_1) = 0, \quad h(p_2) = a, \quad h(p_3) = a + ib \quad \textit{and} \quad h(p_4) = ib.$$

Moreover, a/b is independent of h.

PROOF. By the Riemann mapping theorem, we can find a conformal map

$$\mathring{Q} \to \mathbb{H}^2,$$

which by the Carathéodory theorem extends to a homeomorphism

$$Q \to \mathbb{H}^2 \cup \partial\, \mathbb{H}^2 .$$

By post-composing with a Möbius transformation, we obtain a map $h_1 : Q \to \mathbb{H}^2 \cup \partial\, \mathbb{H}^2$ so that also

$$h_1(p_1) = -1, \quad h_1(p_2) = 1 \quad \text{and} \quad h_1(p_3) = -h_1(p_4) > 1.$$

If we set $k = 1/h_1(p_3)$ and define $h_2 : \mathbb{H}^2 \to \mathbb{C}$ by

$$h_2(z) = \int_0^z \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}$$

for all $z \in \mathbb{H}^2 \cup \partial\, \mathbb{H}^2$, then h_2 turns out to be a conformal map between $\mathbb{H}^2 \cup \partial\, \mathbb{H}^2$ and some rectangle

$$\left\{ z \in \mathbb{C};\quad -K \leq \mathrm{Re}(z) \leq K,\ 0 \leq \mathrm{Im}(z) \leq K' \ \right\}.$$

For instance, to see that the image of $\mathbb{H}^2 \cup \partial\,\mathbb{H}^2$ is indeed given by this rectangle, we note that if we let ζ run over the real line the argument of $\sqrt{(1-\zeta)^2(1-k^2\zeta^2)}$ jumps by $\pi/2$ each time ζ crosses a root. As such, $\partial\,\mathbb{H}^2$ gets sent to a rectangular curve. So $\mathbb{H}^2$ itself gets mapped either to the interior or the exterior of this curve. Using that i gets mapped to a point with positive imaginary part, we see that it's the interior. For more information on this map and its cousins, see [**Neh75**, Section V.6].

The map we are after is now

$$h(z) = h_2 \circ h_1(z) + K$$

for all $z \in Q$.

To see that a/b is well defined, we suppose that $h' : Q \to \mathbb{C}$ is another mapping satisfying the conditions of the lemma. This means that

$$h' \circ h^{-1} : [0,a] + i[0,b] \to [0,a'] + [0,b']$$

is a biholormophism. Using the Schwartz reflection principle (see eg. [**SS03**, Theorem 5.6]), we can extend this map to an automorphism of $\mathbb{C}$. This means that there exist $\lambda \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$ so that

$$h' \circ h^{-1}(z) = \lambda z + \mu$$

for all $z \in [0,a] + i[0,b]$. The fact that

$$h'(p_1) = h(p_1) = 0, \quad h(p_2) > 0 \quad \text{and} \quad h'(p_2) > 0$$

implies that $\lambda \in (0,\infty)$ and $\mu = 0$. This implies that

$$a'/b' = a/b.$$

□

We will call the ratio a/b the *modulus* of Q and denote it by

$$M(Q).$$

We now claim the following

Theorem 11.4. *Let $D \subset \mathbb{C}$ be a domain. An orientation preserving embedding $f : D \to \mathbb{C}$ is K-quasiconformal if and only if*

$$\frac{1}{K} \cdot M(Q) \leq M(f(Q)) \leq K \cdot M(Q)$$

for all quadrilaterals $Q \subset D$.

Before we get to the proof of this theorem, we note the following immediate consequence:

Corollary 11.5. *Let R, S and T be Riemann surfaces and suppose*

$$R \xrightarrow{f_1} S \xrightarrow{f_2} T$$

are quasi-conformal with quasiconformal dilatations K_1 and K_2 then

- *The map*
$$f_2 \circ f_1 : R \to T$$
is quasiconformal with dilatation
$$K \leq K_1 \cdot K_2.$$
- *The map*
$$f_1^{-1} : S \to R$$
is K_1-quasiconformal.

Proof. Exercise 11.1. □

Proof of Theorem 11.4. The proof that if f is K-quasiconformal then the relation between the moduli holds is essentially the same as that of Grötzsch's theorem, so we leave it to the reader.

For the other direction, we expand f near 0 as
$$f(z) = f(0) + \frac{\partial f}{\partial z}(0) \cdot z + \frac{\partial f}{\partial \overline{z}}(0) \cdot \overline{z} + o(|z|).$$
Consider the rectangle
$$R_\varepsilon = [0, \varepsilon] + i[0, \varepsilon].$$
Up to a small error $f(R_\varepsilon)$ is the rectangle
$$\left[a, a + \frac{1}{2}\left(\frac{\partial f}{\partial z}(0) + \frac{\partial f}{\partial \overline{z}}(0)\right)\varepsilon\right] + i\left[b, b + \frac{1}{2}\left(\frac{\partial f}{\partial z}(0) - \frac{\partial f}{\partial \overline{z}}(0)\right)\varepsilon\right],$$
where $f(0) = a + ib$. So, from out assumption we obtain
$$K = KM(R_\varepsilon) \geq M(f(R_\varepsilon)) \geq \frac{\frac{\partial f}{\partial z}(0) + \frac{\partial f}{\partial \overline{z}}(0)}{\frac{\partial f}{\partial z}(0) - \frac{\partial f}{\partial \overline{z}}(0)} + o(1)$$
so if we let ε tend to 0, we get that
$$K \geq \frac{\frac{\partial f}{\partial z}(0) + \frac{\partial f}{\partial \overline{z}}(0)}{\frac{\partial f}{\partial z}(0) - \frac{\partial f}{\partial \overline{z}}(0)}$$
and hence, since the point 0 did not play a role in the above, that f is K-quasiconformal.
□

11.3. Exercises

Exercise 11.1. Prove Corollary 11.5.

LECTURE 12

Hyperbolic annuli and Fenchel-Nielsen coordinates

12.1. The Teichmüller metric

The topology on Teichmüller space is actually induced by a metric, called the Teichmüller metric. We won't dive into the (very interesting) theory of the Teichmüller metric too much in this course. However, it will be useful to have it around, so we will use this section to define it.

Definition 12.1. Let S be a surface of finite type without boundary. Then we define the *Teichmüller metric* on $\mathcal{T}(S)$ by

$$\mathrm{d_T}([R,f],[R',f']) = \inf_g \log(K_g)$$

where the infimum is taken over all quasi-conformal maps $g : R \to R'$ so that $g \circ f$ is isotopic to f' relative to the punctures.

Note that for non-compact surfaces, the existence of a single map g with the required properties is not clear. We will pretend this issue does not exist for now.

The fact that

$$\mathrm{d_T}([R,f],[R',f']) = 0$$

if $[R,f] = [R',f']$ follows from the fact that 1-quasiconformal maps are conformal. The fact that this is so only if $[R,f] = [R',f']$ can be proved with an apporximation argument. Corollary 11.5 implies the triangle inequality so the Teichmüller metric is indeed a metric. Finally, this metric induces the same topology on Teichmüller space (Exercise 12.1).

12.2. Hyperbolic annuli

Some of the annuli we saw in Section 11.1 can also be equipped with a hyperbolic metric. In order to do this, note that if $g \in \mathrm{PSL}(2,\mathbb{R})$ is a hyperbolic or parabolic isometry then the group $\langle g \rangle \simeq \mathbb{Z}$ acts on $\mathbb{H}^2$ properly disconinuously and freely. This means that

$$N_g = \mathbb{H}^2 / \langle \gamma \rangle$$

is an orientable hyperbolic surface with fundamental group $\mathbb{Z}$ and hence an annulus. First we note that the geometry of the annulus only depends on the translation length of g.

Lemma 12.2. *Let $g, h \in \mathrm{PSL}(2,\mathbb{R})$ be either both hyperbolic or both parabolic elements so that their translation lengths satisfy $T_g = T_h$. Then the annuli N_g and N_h are isometric. Moreover, every complete hyperbolic annulus is isometric to N_g for some parabolic or hyperbolic $g \in \mathrm{PSL}(2,\mathbb{R})$.*

PROOF. Exercise 12.2. □

Note that this includes the case where $T_g = T_h = 0$.

The question now becomes whether it is biholomorphic to A_m for some m and if so, to which. In order to solve this question, we introduce a new model for the hyperbolic plane the *band model*. Set

$$\mathbb{B} = \left\{ z \in \mathbb{C}; \quad |\mathrm{Im}(z)| < \tfrac{\pi}{2} \ \right\},$$

equipped with the metric

$$ds^2 = \frac{dx^2 + dy^2}{\cos^2(y)}.$$

This is another model for the hyperbolic plane, moreover the real line is a geodesic in $\mathbb{B}$ (see Exercise 12.3). Maps of the from $\varphi_b : \mathbb{B} \to \mathbb{B}$ defined by

$$z \mapsto z + b$$

for some $b > 0$ are isometries for this metric. Moreover $\langle \varphi_b \rangle \simeq \mathbb{Z}$ acts on $\mathbb{B}$ properly discontinuously, which means that

$$M_b = \mathbb{B} / \langle \varphi_b \rangle$$

is a hyperbolic annulus. Moreover, the translation length of φ_b is b, so using Lemma 12.2, we see that

$$M_b \simeq N_g$$

as hyperbolic surfaces, where $g \in \mathrm{PSL}(2, \mathbb{R})$ is any hyperbolic element with translation length b.

We now claim that:

Lemma 12.3. *Let $m > 0$. The annuli A_m and $M_{\pi/m}$ are biholomorphic.*

PROOF. Since the map $z \mapsto z - i\ m/2$ is a biholormophism of $\mathbb{C}$ that commutes with the $\mathbb{Z}$-action. A_m is biholomorphic to

$$\left\{ z \in \mathbb{C}; \quad |\mathrm{Im}(z)| < \tfrac{m}{2} \ \right\} / \, \mathbb{Z} \, .$$

The map $\left\{ z \in \mathbb{C}; \quad |\mathrm{Im}(z)| < \tfrac{m}{2} \ \right\} \to \mathbb{B}$ given by $z \mapsto \frac{\pi}{m} z$ is a $\mathbb{Z}$-equivariant biholomorphism and hence descends to a biholomorphism

$$\left\{ z \in \mathbb{C}; \quad |\mathrm{Im}(z)| < \tfrac{m}{2} \ \right\} / \, \mathbb{Z} \simeq M_{\pi/m}.$$

□

For the parabolic case we have:

Lemma 12.4. *let $g \in \mathrm{PSL}(2, \mathbb{R})$ be parabolic. The annuli N_g and $\mathbb{D} \setminus \{0\}$ are biholomorphic.*

PROOF. Using Lemma 12.2, we may assume that

$$g = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The map $\mathbb{H}^2 \to \mathbb{D}$ given by

$$z \mapsto e^{-2\pi i z}$$

induces the biholomorphism. □

12.3. Fenchel-Nielsen coordinates, part I

In what follows we will introduce a set of coordinates on Teichmüller spaces of hyperbolic surfaces. So in this section, we will assume that our base surface S admits a complete hyperbolic metric. Moreover, we will fix a (topological) pants decomposition $\mathcal{P}$ on S.

12.3.1. Lengths. Given any essential closed curve γ on S, we obtain a function

$$\ell_\gamma : \mathcal{T}(S) \to \mathbb{R}_+$$

called a *length function*, defined as follows. Each $[R, f] \in \mathcal{T}(S)$ can be seen as a marked hyperbolic surface. So, Proposition 6.1 implies that the homotopy class of $f(\gamma)$ on R contains a unique geodesic. $\ell_\gamma([R, f])$ is the length of this geodesic.

Hence, given S and $\mathcal{P}$ as above, we obtain a map

$$\ell_{\mathcal{P}} : \mathcal{T}(S) \to \mathbb{R}_+^{3g-3+n}$$

defined by

$$\ell_{\mathcal{P}}([R, f]) = \Big(\ell_\gamma([R, f])\Big)_{\gamma\in\mathcal{P}}.$$

We have:

Lemma 12.5. *Let S and γ be as above. The function*

$$\log \circ \ell_\gamma : \mathcal{T}(S) \to \mathbb{R}$$

is 1-Lipschitz with respect to the Teichmüller metric, i.e.

$$|\log(\ell_\gamma([R, f])) - \log(\ell_\gamma([R', f']))| \leq \mathrm{d_T}([R, f], [R', f'])$$

for all $[R, f], [R', f'] \in \mathcal{T}(S)$.

Proof. Fix a basepoint $p \in S$ so that we can identify γ with an element of $\pi_1(S, p)$, that we will also denote by γ. The infinite cyclic subgroup of $\pi_1(S, p)$ generated by γ induces a $\mathbb{Z}$-cover

$$S_\gamma \to S.$$

We will write A and A' for the corresponding covering spaces of R and R'. Just like in the proof of Proposition 6.1, these are annuli and by Lemma 12.3, they are biholomorphic to $A_{\pi/\ell_\gamma([R,f])}$ and $A_{\pi/\ell_\gamma([R',f'])}$ respectively. K-quasiconformal maps between R and R' lift to K-quasiconformal maps of A and A'. So we have

$$\begin{aligned} \mathrm{d_T}([R, f], [R', f']) &= \inf_{\substack{g \text{ homotopic} \\ \text{to } 'f\circ f^{-1}}} \log(K_g) \\ &= \inf_{\substack{g \text{ homotopic} \\ \text{to } f'\circ f^{-1}}} \log(K_{\widetilde{g}:A\to A'}) \\ &\geq \left|\log\left(\frac{\ell_\gamma([R, f])}{\ell_\gamma([R', f'])}\right)\right|, \end{aligned}$$

where the last line follows from Grötzsch's theorem (Theorem 10.6). □

12.3.2. Twists. So, given S and $\mathcal{P}$ as above, we have a continuous map

$$\ell_{\mathcal{P}} : \mathcal{T}(S) \to \mathbb{R}_+^{3g-3+n}.$$

It's however not quite injective. The problem is that we can still rotate the hyperbolic metric along the curves in the pants decomposition. Twist coordinates will remedy this.

First we pick a collection of disjoint simple closed curves Γ so that for each pair of pants P in $S \setminus \mathcal{P}$, $\Gamma \cap \mathcal{P}$ consists of three arcs, each connecting a different pair of boundary components of P. Figure 1 shows an example.

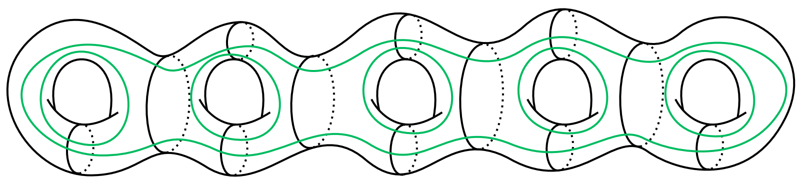

FIGURE 1. A pants decomposition $\mathcal{P}$ with a set of curves Γ.

Regardless of our choice of pants decomposition $\mathcal{P}$, such a system of curves Γ always exists (Exercise 12.4).

Now let $\gamma \in \mathcal{P}$ be a pants curve. Then γ bounds either one P or two pairs of pants P_1 and P_2 in the decomposition. Let us assume the latter for simplicity, the other case is analogous. The left hand side of Figure 2 shows an example of such a curve γ.

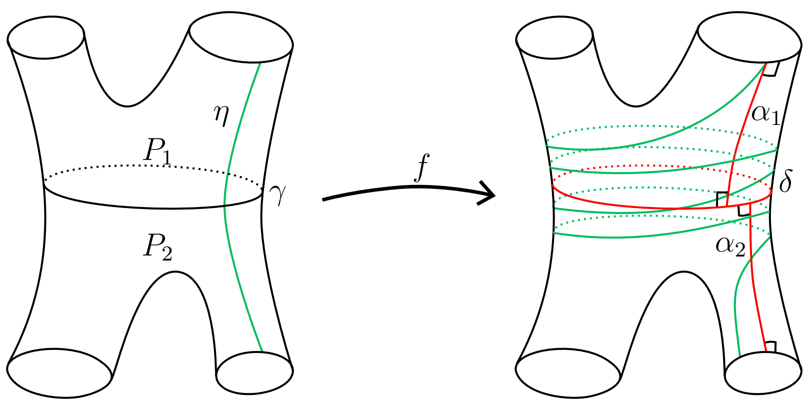

FIGURE 2. The image of an arc under a diffeomorphism.

If $f : S \to R$ is an orientation preserving diffeomorphism, then it maps $\mathcal{P}$ to some pants decomposition of R. Moreover, if η is one of the (two) components of $(P_1 \cup P_2) \cap \Gamma$ that intersects γ, then $f(\eta)$ is some arc between boundary components of $f(P_1)$ and $f(P_2)$ (like on the right hand side of Figure 2). Now

- δ will be the unique simple closed geodesic in the free homotopy class of $f(\gamma)$ on R.
- α_1 and α_2 the two unique perpendiculars between the boundary components between which $f(\eta)$ runs and δ (see Figure 2).

Then relative to the boundary of $f(P_1 \cup P_2)$, the arc $f(\eta)$ is freely homotopic to $\alpha_2 \cdot \delta^k \cdot \alpha_1$ for some $k \in \mathbb{Z}$.

The twist along γ is now

$$\tau_\gamma([R,f]) = k \cdot \ell_\gamma([R,f]) \pm \mathrm{d}(p_1,p_2) \in \mathbb{R}$$

where

- p_1 and p_2 are the points where α_1 and α_2 hit δ.
- The signs are determined by the orientation of R in the following way. The orientation of R gives a notion of "twisting to the left" along δ. Left twists are counted positively and right twists negatively.

12.4. Exercises

Exercise 12.1. Show that the topology induced on Teichmüller space by the Teichmüller metric is the same as that induced by Beltrami coefficients.

Exercise 12.2. (a) Prove that every hyperbolic isometry of $\mathbb{H}^2$ is conjugate to one of the form

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \quad \lambda > 1.$$

(b) Prove that every parabolic isometry of $\mathbb{H}^2$ is conjugate to

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

(c) Prove Lemma 12.2.

Exercise 12.3. (a) Show that

$$\mathbb{B} = \left\{ z \in \mathbb{C}; \quad |\mathrm{Im}(z)| < \tfrac{\pi}{2} \ \right\},$$

equipped with the metric

$$ds^2 = \frac{dx^2 + dy^2}{\cos^2(y)}$$

is isometric to $\mathbb{H}^2$. *Hint: this can be done without providing an isometry.*

(b) Show that $\mathbb{R} \subset \mathbb{B}$ is a geodesic.

Exercise 12.4. Let $\mathcal{P}$ be a pants decomposition of a surface S. Show that there exists collection of disjoint simple closed curves Γ so that for each pair of pants P in $S \setminus \mathcal{P}$, $\Gamma \cap \mathcal{P}$ consists of three arcs, each connecting a different pair of boundary components of P.

LECTURE 13

Tecihmüller space is a cell

We finish the proof that Fenchel-Nielsen coordinates give rise to a homeomorphism. We will mainly follow [**Hub06**].

13.1. Fenchel-Nielsen coordinates, part II

Let us prove that twists are continuous:

Lemma 13.1. *Let S and γ be as above. The function*

$$\tau_\gamma : \mathcal{T}(S) \to \mathbb{R}$$

is continuous.

Proof sketch. Suppose that

$$d_T([R, f], [R', f'])$$

is small. This means that the map $f' \circ f^{-1} : R \to R'$ is close to an isometry. Since it maps the curves and arcs used to define $\tau_\gamma([R, f])$ to those used to define $\tau_\gamma([R', f'])$. So, this map lifts to a map $\widetilde{f' \circ f^{-1}} : \mathbb{H}^2 \to \mathbb{H}^2$ that is close to conformal and hence close to an isometry. This means that the numbers $\tau_\gamma([R, f])$ and $\tau_\gamma([R', f'])$ are close. □

13.1.1. Fenchel Nielsen coordinates are homeomorphic. Putting the above together, we obtain a continuous map

$$\mathrm{FN}_\mathcal{P} : \mathcal{T}(S) \to \mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n}$$

defined by

$$\mathrm{FN}_\mathcal{P}([R, f]) = \Big(\ell_\gamma([R, f]), \tau_\gamma([R, f])\Big)_{\gamma \in \mathcal{P}}.$$

For $S = \Sigma_{0,3}$ we will adopt the convention that FN is the constant map.

It turns out that the Fenchel Nielssen map is a homeomorphism:

Theorem 13.2. *Let S be a surface of finite type such that $\chi(S) < 0$ and let $\mathcal{P}$ be a pants decomposition of S. Then the map*

$$\mathrm{FN}_\mathcal{P} : \mathcal{T}(S) \to \mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n},$$

is a homeomorphism.

Proof. Since we have already proved that lengths and twists are continuous, we only need to provide a continuous inverse to the map $\mathrm{FN}_\mathcal{P}$.

Given a vector $(\ell_\gamma, \tau_\gamma)_{\gamma\in\mathcal{P}}$, we can use the gluing construction we discusses in Lectures 5 and 6 in order to produce a hyperbolic surface R. The lengths give us the geometry of the pairs of pants and the gluing along a curve γ is determined by

$$\tau_\gamma^{(0)} = \tau_\gamma + k \cdot \ell_\gamma,$$

where k is such that $\tau_\gamma^{(0)} \in [0, \ell_\gamma)$. Call this surface $R\Big((\ell_\gamma, [\tau_\gamma])_\gamma\Big)$. In particular, by varying the twist τ_γ, we obtain the same surface countably many times.

The question however is what the marking, i.e. the map $f : S \to R\Big((\ell_\gamma, [\tau_\gamma])_\gamma\Big)$, should be. In order to do this, we fix open regular neighborhoods N_γ^S of the curves $\gamma \in \mathcal{P}$ on S so that

$$S \setminus \bigcup_{\gamma\in\mathcal{P}} N_\gamma$$

consists of disjoint pairs of pants $P_1^S, \ldots, P_k^S$. We will once and for all parametrize the annuli

$$N_\gamma^S = \Big(\mathbb{R} / \mathbb{Z}\Big) \times (-1, 1).$$

On $R(\ell_\gamma, [\tau_\gamma])$ we pick such neighborhoods too and obtain neighborhoods N_γ^R and pairs of pants P_i^R. We will assume that

$$N_\gamma^R = \Big\{ x \in R\Big((\ell_\gamma, [\tau_\gamma])_\gamma\Big); \quad \mathrm{d}(x, \gamma) < \varepsilon \Big\}$$

for some ε small enough. Moreover, we assume ε varies continuously as a function of $(\ell_\gamma, [\tau_\gamma])_\gamma$.

In order to build f, we now pick a parametrization

$$N_\gamma^R = \Big(\mathbb{R} / \ell_\gamma \mathbb{Z}\Big) \times (-1, 1)$$

where the subset

$$\Big(\mathbb{R} / \ell_\gamma \mathbb{Z}\Big) \times \{t\} \subset N_\gamma^R$$

is one of the (one or two) components of

$$\Big\{ x \in R\Big((\ell_\gamma, [\tau_\gamma])_\gamma\Big); \quad \mathrm{d}(x, \gamma) = |t| \cdot \varepsilon \Big\},$$

parametrized by a constant multiple (depending on t) of arclength for all $t \in (-1, 1)$.

The map $f_\gamma : N_\gamma^S \to N_\gamma^R$ is now given by

$$f_\gamma(\theta, t) = \left(\ell_\gamma \cdot \theta + \tau_\gamma \cdot \frac{t+1}{2}, t \right).$$

The awkward $(t + 1)/2$ is an artifact of choosing the interval $(-1, 1)$ instead of $(0, 1)$ (the latter would have made some of the previous equations more awkward).

For the complements of the annuli we choose arbitrary homeomorphisms and $f_i^P : P_i^S \to P_i^R$ that smoothly extend the f_γ.

This map is clearly an inverse and since we can make everything depend on the input continuously, it's continuous. □

Remark 13.3. Looking at the proof above, it's a natural question to ask whether we maybe get a fundamental domain for moduli space by only considering $\tau_\gamma \in [0, \ell_\gamma)$.

However, this not the case. To see this, take any $f \in \mathrm{Diff}^+(S, \Sigma)$ (where $S = S_0 \setminus \Sigma$, S_0 is closed and Σ a finite set) that is not homotopic to the identity. Then we get a surface isometric to $R\Big((\ell_\gamma, [\tau_\gamma])_{\gamma \in \mathcal{P}}\Big)$ if we assign the lengths of the curves in $f(\mathcal{P})$ to the curves in $\mathcal{P}$ instead (the isometry will be induced by f).

13.2. Teichmüller spaces of hyperbolic surfaces with boundary

From the perspective of hyperbolic surfaces, the theory above has a natural generalization to surfaces with boundary.

Definition 13.4. Let $\Sigma_{g,b,n}$ be a surface that supports a hyperbolic metric and label its boundary components by $\beta_1, \ldots, \beta_b$. Given $L_1, \ldots, L_b$, the *Teichmüller space* of hyperbolic surfaces diffeomorphic to $\Sigma_{g,b,n}$ with boundary components of length $L_1, \ldots, L_b$ is defined as

$$\mathcal{T}_{g,b,n}(L_1, \ldots, L_b) = \left\{ (R, f); \begin{array}{c} R \text{ a complete hyperbolic surface}, f : \Sigma_{g,b,n} \to R \\ \text{an orientation preserving diffeomorphism so} \\ \text{that } f(\beta_i) \text{ has length } L_i \ \forall i = 1, \ldots, b \end{array} \right\} \Big/ \sim,$$

where $(R, f) \sim (R', f')$ if and only if there exists an isometry $m : R \to R'$ so that

$$(f')^{-1} \circ m \circ f : \Sigma_{g,b,n} \to \Sigma_{g,b,n}$$

is homotopic to the identity.

Note that $\mathrm{MCG}(\Sigma_{g,b,n})$ acts on $\mathcal{T}_{g,b,n}(L_1, \ldots, L_b)$, which leads to:

Definition 13.5. Given $L_1, \ldots, L_b$, the *moduli space* of hyperbolic surfaces diffeomorphic to $\Sigma_{g,b,n}$ with boundary components of length $L_1, \ldots, L_b$ is defined as

$$\mathcal{M}_{g,b,n}(L_1, \ldots, L_b) = \mathcal{T}_{g,b,n}(L_1, \ldots, L_b) / \mathrm{MCG}(\Sigma_{g,b,n}).$$

Note that in this moduli space, the boundary components are still marked.

Tracing the proof of Theorem 13.2, we obtain

Proposition 13.6. *Let $\mathcal{P}$ be a pants decomposition of $\Sigma_{g,b,n}$. The map*

$$\mathrm{FN}_{\mathcal{P}} : \mathcal{T}(\Sigma_{g,b,n}) \to \mathbb{R}_+^{3g-3+b+n} \times \mathbb{R}^{3g-3+b+n}$$

is a bijection.

Proof. Exercise 13.1. □

This means we can equip these moduli and Teichmüller spaces with a topology as well.

13.3. Exercises

Exercise 13.1. Prove Proposition 13.6.

LECTURE 14

Towards a complex structure on Teichmüller space

It turns out Teichmüller space of a surface without boundary is not just homeomorphic to an open set in $\mathbb{C}^{3g-3+n}$, it also also carries a natural complex structure. Our next intermediate goal is to explain where this comes from. The global strategy is to embed Teichmüller space in a space of quadratic differentials via a map called the *Bers embedding*. This goes in two steps. First of all, we identified Teichmüller space with a quotient of a space of Beltrami coefficients. For each Beltrami coefficient μ, we will find a "canonical" quasiconformal map $f = f^\mu$ so that $\mu = \mu_f$. We will then use the Schwarzian derivative to associate a quadratic differential to this map f.

In this lecture, we will discuss some of the ingredients from the above, following [**Hub06, Dum09**].

14.1. Solving the Beltrami equation

We start with the input for the first step of the process we described in Section 14: solving the Beltrami equation. We have:

Proposition 14.1. *Let $D \subset \mathbb{C}$ be a domain. Suppose that $\mu : D \to \mathbb{C}$ is a real analytic function so that*

$$||\mu||_\infty < 1.$$

Then for every $z \in D$ there exists an open neighborhood V of z so that there is a real analytic function $f : V \to \mathbb{C}$ that is a homeomorphism onto its image and

$$\frac{\partial f}{\partial \overline{z}} = \mu \frac{\partial f}{\partial z}$$

on V.

Moreover, if $f_1 : V_1 \to \mathbb{C}$ and $f_2 : V_2 \to \mathbb{C}$ are two such functions then there exists an analytic homeomorphism

$$h : f_1(V_1 \cap V_2) \to f_2(V_1 \cap V_2)$$

so that on $V_1 \cap V_2$: $f_2 = h \circ f_1$.

Proof. The trick is to first think of μ and f as maps with two real inputs, i.e. to write $z = (x, y)$ and then to enlarge our space of inputs by allowing x and y to be complex. We will think of $\mathbb{C}$ embedded in $\mathbb{C}^2$ as $\mathbb{R}^2 \subset \mathbb{C}$.

Choose some neighborhood W of $z_0 = (x_0, y_0)$ so that μ is analytic in W, the Beltrami equation becomes

$$(1 - \mu(x, y))\frac{\partial f(x, y)}{\partial x} + i(1 + \mu(x, y))\frac{\partial f(x, y)}{\partial y} = 0.$$

Now consider the ODE

$$\frac{dy(x)}{dx} = i\frac{1+\mu(x,y(x))}{1-\mu(x,y(x))}.$$

f solves the first equation if and only if f is constant on solutions $y = y(x)$ to the equation above. Indeed,

$$\frac{df(x,y(x))}{dx} = \frac{\partial f(x,y(x))}{\partial x} + \frac{dy(x)}{dx}\frac{\partial f(x,y(x))}{\partial y}.$$

So, this trick turns our PDE into a first order ODE.

This also means that f is uniquely determined by its value on a transversal to the complex lines $y(x)$. So for instance by its values on

$$L = \left\{(x,y) \in W; \quad x = x_0 \right\}.$$

If we let f denote the solution that is equal to y on L, then

$$\frac{\partial f}{\partial y}(x_0,y_0) = 1$$

and

$$\frac{\partial f}{\partial x}(x_0,y_0) = -i\frac{1+\mu(x_0,y_0)}{1-\mu(x_0,y_0)}.$$

Since the latter is not a real number, the inverse function theorem applies and hence f induces a local diffeomorphism between $W \cap \mathbb{R}^2$ and $\mathbb{C}$. Moreover, any other solution satisfies $g(x_0,y) = h(y)$ for some analytic h and we hence obtain $g(x,y) = h \circ f$. $\square$

Remark 14.2. Proposition 14.1 can be extended to L^∞ functions μ (see [**Hub06**, Theorem 4.6.1]).

The proposition above allows us to make the following definition:

Definition 14.3. Let $D \subset \mathbb{C}$ be a domain. Suppose that $\mu : D \to \mathbb{C}$ is a real analytic function of compact support so that

$$||\mu||_\infty < 1.$$

Then the unique $f : D \to \mathbb{C}$ so that

$$\frac{\partial f}{\partial \overline{z}} = \mu \frac{\partial f}{\partial z}$$

and

$$f(z) = z + O\left(|z|^{-1}\right)$$

as $z \to \infty$ will be denoted $f^\mu : D \to \mathbb{C}$.

14.2. The Schwarzian derivative

Now we discuss the Schwarzian derivative. Given a domain $D \subset \widehat{\mathbb{C}}$ and an analytic function $f : D \to \widehat{\mathbb{C}}$ with non vanishing derivatives, the Schwarzian $\mathcal{S}(f)$ measures how far away f is from a Möbius transformation.

14.2.1. The definition. We start with a lemma:

Lemma 14.4. *Let $D \subset \widehat{\mathbb{C}}$ be a domain and $z_0 \in D$. For every analytic map $f : D \to \widehat{\mathbb{C}}$ with non-vanishing derivative there exists a unique element $A \in \mathrm{PSL}(2, \mathbb{C})$ so that*

$$f(z_0) = A(z_0), \quad \frac{df}{dz}(z_0) = \frac{dA}{dz}(z_0) \quad \textit{and} \quad \frac{d^2 f}{dz^2}(z_0) = \frac{d^2 A}{dz^2}(z_0).$$

Proof. Exercise 14.1. □

Looking at this lemma, we see that the third order term of the Taylor expansion of $f - A$ is a good measure of how far f is from a Möbius transformation. That is, we could consider the map

$$D^3(f - A)(z_0) : T_{z_0} D \to T_{f(z_0)} \widehat{\mathbb{C}}.$$

This is naturally a cubic map:

$$D^3(f - A)(z_0)(\lambda v) = \lambda^3 v$$

for all $v \in T_{z_0} D$, $\lambda \in \mathbb{C}$. In order to have something that is completely local, we postcompose with $Df(z_0)^{-1}$. This gives us a quadratic map

$$Df(z_0)^{-1} \circ D^3(f - A)(z_0) : T_{z_0} D \to \mathbb{C}\,.$$

The reason that this is quadratic is that the map

$$Df(z_0) : T_{z_0} D \to T_{f(z_0)} \widehat{\mathbb{C}}$$

induces an isomorphism between the set of quadratic maps

$$\left\{ q : T_{z_0} D \to \mathbb{C}; \quad q(\lambda v) = \lambda^2 q(v), \lambda \in \mathbb{C}, v \in T_{z_0} D \ \right\}$$

and the set of cubic maps

$$\left\{ m : T_{z_0} D \to T_{f(z_0)} \widehat{\mathbb{C}}; \quad m(\lambda v) = \lambda^3 m(v), \lambda \in \mathbb{C}, v \in T_{z_0} D \ \right\}$$

by

$$\Big(Df(z_0) \cdot q \Big)(v) = q(v) \cdot Df(z_0) \cdot v$$

for all $v \in T_{z_0} D$ and all quadratic maps $q : T_{z_0} D \to \mathbb{C}$.

Definition 14.5. Let $D \subset \widehat{\mathbb{C}}$ be a domain and $z_0 \in D$. Moreover, let $f : D \to \widehat{\mathbb{C}}$ be an analytic map with non-vanishing derivative. Finally, let $A \in \mathrm{PSL}(2, \mathbb{C})$ be the unique element so that

$$f(z_0) = A(z_0), \quad \frac{df}{dz}(z_0) = \frac{dA}{dz}(z_0) \quad \text{and} \quad \frac{d^2 f}{dz^2}(z_0) = \frac{d^2 A}{dz^2}(z_0).$$

Then the *Schwarzian derivative* of f at z_0 is given by

$$\mathcal{S}(f)(z_0) := 6 \cdot Df(z_0)^{-1} \circ D^3(f - A)(z_0) : T_{z_0} D \to \mathbb{C}\,.$$

The 6 in the definition above is just a normalization. Such a quadratic map corresponds to a differential form of the form

$$\mathcal{S}(f)(z) = \phi(z) dz^2$$

(assuming f is analytic). This can be made completely explicit:

Lemma 14.6. *Let $D \subset \widehat{\mathbb{C}}$ be a domain and $z_0 \in D$. Moreover, let $f : D \to \widehat{\mathbb{C}}$ be an analytic map with non-vanishing derivative. Then*

$$\mathcal{S}(f)(z) = \left(\frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2 \right) dz^2.$$

Proof. Without loss of generality, we assume that $z = 0$ and $f(0) = 0$. Write

$$f(z) = a_1 z + \frac{a_2}{2} z^2 + \frac{a_3}{6} z^3 + \dots.$$

Since $f(0) = 0$, A is of the form

$$A(z) = \frac{\alpha z}{1 + \beta z} = \alpha z - \alpha\beta z^2 + \alpha\beta^2 z^3 - \dots.$$

This means that

$$\alpha = a_1 \quad \text{and} \quad \beta = -\frac{a_2}{2a_1}.$$

So the difference in the third order term is

$$\frac{a_3}{6} - \frac{a_2^2}{4a_1}.$$

Post-composing with $Df(0)^{-1}$ corresponds to dividing by a_1. So

$$\mathcal{S}(f)(0) = \left(\frac{a_3}{a_1} - \frac{3}{2} \frac{a_2^2}{a_1^2} \right) dz^2,$$

as claimed. □

14.2.2. Basic properties. The following properties of the Schwarzian derivative are immediate:

Proposition 14.7. *Let $D \subset \widehat{\mathbb{C}}$ be a domain and let $f : D \to \widehat{\mathbb{C}}$ be an analytic map with non-vanishing derivative.*

(1) $\mathcal{S}(f) = 0$ on D if and only if $f = A|_D$ for some $A \in \mathrm{PSL}(2, \mathbb{C})$
(2) $\mathcal{S}(A \circ f) = \mathcal{S}(f)$ for all $A \in \mathrm{PSL}(2, \mathbb{C})$.

Intermezzo: Darboux derivatives. There is another interpretation of the Schwarzian derivative. We will not use it in the rest of this course, but since it gives a nice geoemtric interpretation of $\mathcal{S}(f)$, we briefly sketch it here, following [**Dum09**].

Again, let $D \subset \widehat{\mathbb{C}}$ be a domain and $f : D \to \widehat{\mathbb{C}}$ an analytic function with non-vanishing derivative. The map

$$D \to \mathrm{PSL}(2, \mathbb{C})$$

that associates to each point $z \in D$ the unique Möbius transformation that agrees with f up to second order at z is called the *osculation map*. We will denote the value at $z \in D$ by $\mathrm{osc}_z(f)$.

Note that if this map is constant, then f is a Möbius transformation. So in order to see how far f is from a Möbius transformation, we can take a derivative. Note that it follows from the proof of Lemma 14.6 that this is possible. We could take the usual differential geometric derivative, but since the target is a Lie group, there is a way to take a derivative with values in the corresponding Lie algebra. This is what is called the Darboux derivative and goes as follows.

First, recall that given a Lie group G, its Lie algebra $\mathfrak{g}$ is the tangent space T_eG at the identity element $e \in G$. This is an associative algebra with multiplication induced by the Lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. If G is a matrix group, then $\mathfrak{g}$ is naturally a matrix algebra and the Lie bracket on $\mathfrak{g}$ is given by the commutator:

$$[X, Y] = XY - YX.$$

The Lie algebra of $\mathrm{PSL}(2, \mathbb{C})$ is

$$\mathfrak{sl}_2(\mathbb{C}) := \left\{ X \in \mathrm{Mat}_{2\times 2}(\mathbb{C}); \quad \mathrm{tr}\,(X) = 0 \ \right\}.$$

In the usual differential geometric setting, the derivative of a smooth map $F : M \to N$ is a map

$$DF : T_pM \to T_{F(p)}N.$$

In the case where the target is a Lie group, there's a natural way to "forget the underlying points" i.e. to make the differential lie in the Lie algebra of G. This is where the Maurer-Cartan form comes in. Given $g \in G$, the map

$$L_g : G \to G$$

will denote left multiplication with g, i.e.

$$L_g(h) = gh$$

for all $h \in G$. The *Maurer-Cartan form* $\omega_G \in \Lambda^1 G \otimes \mathfrak{g}$ is now given by

$$\omega_G|_g(v) = (DL_g)^{-1} v$$

for all $v \in T_gG$ and $g \in G$. In words, this form identifies the tangent space at a point in the group with its Lie algebra.

The *Darboux derivative* of a smooth map

$$F : M \to G$$

between a manifold M and a Lie group G is the $\mathfrak{g}$-valued 1-form

$$\omega_F := F^* \omega_G \in \Lambda^1(M) \otimes \mathfrak{g}.$$

Now a computation shows that if $\mathcal{S}(f)(z) = \phi(z) dz^2$ then

$$\omega_{\mathrm{osc}_f}(z) = -\frac{1}{2}\,\phi(z) \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} dz.$$

14.3. Exercises

Exercise 14.1. Prove Lemma 14.4. *Hint: most of this is done in the proof of Lemma 14.6.*

LECTURE 15

More about Schwarzians and quadratic differentials

Most of the material below is taken from [**Hub06**].

15.1. Which quadratic differentials do we hit?

A natural question is which quadratic differentials (i.e. expressions locally of the form $q(z) = \phi(z)dz^2$) can be obtained as the Schwarzian of a funcion. We have:

Proposition 15.1. *Let $U \subset \widehat{\mathbb{C}}$ be a simply connected open set and let q be a holomorphic quadratic differential on U. Then for any $z_0 \in U$ and $a_0, a_1, a_2 \in \mathbb{C}$ so that $a_1 \neq 0$ there exists a unique meromorphic function $f : U \to \mathbb{C}$ so that*

$$\mathcal{S}(f) = q, \quad f(z_0) = a_0, \quad \frac{df}{dz}(z_0) = a_1 \quad \textit{and} \quad \frac{d^2 f}{dz^2}(z_0) = a_2.$$

Proof. Existence and uniqueness follows from standard ODE considerations. Write $q(z) = \phi(z)dz^2$ and consider the differential equation

$$\frac{d^2 w}{dz^2} + \frac{\phi}{2} w = 0.$$

Since this is a linear second order differential equation without singularities, it has two linearly independent solutions $w_1, w_2 : U \to \mathbb{C}$. A direct computation shows that there exists a constant $C \in \mathbb{C}$ so that

$$f(z) = C \cdot \frac{w_1(z)}{w_2(z)}$$

satisfies

$$S(f) = q,$$

(Exercise 15.1). Now let A be a Möbius transformation so that $A \circ f$ satisfies the other three conditions. □

Example 15.2. Let us consider an example. Set $q = -2k^2 dz^2$for some $k \in (0, \infty)$ on the unit disk $\mathbb{D} \subset \mathbb{C}$. The differential equation from the proof in Proposition 15.1 becomes

$$\frac{d^2 w}{dz^2} - k^2 w = 0$$

which has solutions

$$w_1(z) = e^{ikz} \quad \text{and} \quad w_2(z) = e^{-ikz}.$$

The ratio

$$f = \frac{e^{ikz}}{e^{-ikz}} = e^{2ikz}$$

is a solution to the equation

$$S(f) = q.$$

Note that this solution is an injective function on $\mathbb{D}$ if and only if $k \leq \pi$.

It turns out this fact that the solution is injective if and only if the quadratic differential is "small" is a general phenomenon. To quantify this, we need to first study quadratic differentials.

15.2. Quadratic differentials

First of all we note that a holomorphic quadratic differential induces natural coordinates near a point where it's not equal to 0. That is, let q be a quadratic differential on compact a Riemann surface R. Furthermore suppose that $x \in R$ has an open neighborhood on which we can write

$$q = q(\zeta)d\zeta^2$$

such that $q(x) \neq 0$. We may assume, by making U smaller if necessary, that $q(\zeta) \neq 0$ for all $\zeta \in U$. This means that on U, q has a well defined holomorphic square root $\sqrt{q(\zeta)}d\zeta$.

We obtain a well defined coordinate on U by setting

$$z(y) = \int_x^y \sqrt{q(\zeta)}d\zeta$$

for all $y \in U$. In this local coordinate, we have

$$q = dz^2.$$

We will call a local coordinate so that q takes this form a *natural coordinate*. Note that if there are two such different coordinates z and z' then locally

$$z' = \pm z + b$$

for some $b \in \mathbb{C}$.

Moreover note that, because translations and flipping the sign preserve the notion of the real and the imaginary direction in $\mathbb{C}$, the quadratic differential induces the notion of a *horizontal* and a *vertical* direction on the surface. Note however that left and right nor up and down are not well defined, because of the possibility of sign changes. Finally, we obtain a local area element $|q|$ by just taking the Euclidean area element in natural coordinates. If $q = q(\zeta)d\zeta^2$ and $\zeta = \xi + i\eta$, where ξ and η are real. Then the area element is given by

$$|q| = |q(\zeta)|\, d\xi d\eta.$$

Note that since the zeroes of holomorphic functions don't accumulate (see eg. [**SS03**, Theorem 3.1.1]), we can find natural coordinates for q near all but finitely many points in R. In particular, the area of q,

$$||q||_1 = \int_R |q|$$

is well defined.

Likewise, if $g \geq 2$ then R has a unique hyperbolic metric inducing its complex structure. If ρ^2 denotes the area element of this hyperbolic metric, then we can define a natural ∞-norm by

$$||q||_\infty = \sup_{z \in R} \frac{|q|(z)}{\rho^2(z)}.$$

Finally, we need a fact from the theory of Riemann surfaces that we will just need to assume, because it would take us too far astray to prove it.

Proposition 15.3. *Let R be a compact Riemann surface of genus $g \geq 1$. The vector space $\mathcal{Q}(R)$ of quadratic differentials on R has complex dimension*

$$\dim_{\mathbb{C}}(\mathcal{Q}(R)) = \begin{cases} 1 & \text{if } g = 1 \\ 3g-3 & \text{otherwise.} \end{cases}$$

This follows from the Riemann-Roch theorem (see [**FK92**, Proposition III.5.2]).

15.3. Nehari's theorem

We now want to prove the claim that an injectove function gives rise to a "small" Schwarzian. We need one more ingredient:

Theorem 15.4 (Area theorem). *Suppose $f : \mathbb{D} \backslash \{0\} \to \mathbb{C}$ is analytic and has the series representation*

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n.$$

Then

$$\sum_{n=0}^{\infty} n\,|a_n|^2 \leq 1.$$

PROOF. Exercise 15.2. □

We are now ready to prove Nehari's theorem:

Theorem 15.5 (Nehari). *Let $f : \mathbb{D} \to \widehat{\mathbb{C}}$ be injective and analytic. Then*

$$||\mathcal{S}(f)||_\infty \leq \frac{3}{2}.$$

PROOF. Since composition with a Möbius transformation does not change the Schwarzian, we may consider the problem near $z = 0$ and assume that $f(0) = \infty$, so f is of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n.$$

We moreover assume that f has constant term 0.

The area theorem implies that $|a_1| \leq 1$. We have

$$f'(z) = -\frac{1}{z^2} + a_1 + O(z), \quad f''(z) = \frac{2}{z^3} + O(1), \quad f'''(z) = -\frac{6}{z^4} + O(1)$$

as $z \to 0$.

$$\begin{aligned}\mathcal{S}(f)(z) &= \left(\frac{-6/z^4 + O(1)}{-1/z^2 + a_1 + O(z)} - \frac{3}{2} \left(\frac{2/z^3 + O(1)}{-1/z^2 + a_1 + O(z)} \right)^2 \right) dz^2 \\ &= \frac{1}{z^2} \frac{6 + O(z^4)}{1 - a_1 z^2 + O(z^3)} - \frac{3}{2z^2} \left(\frac{2 + O(z^3)}{-1 + a_1 z^2 + O(z^3)} \right)^2 \\ &= \frac{1}{z^2} \left(6 + 6a_1 z^2 + O\left(z^3\right)\right) - \frac{3}{2z^2} \left(2 + 2a_1 z^2 + O\left(z^3\right)\right)^2 \\ &= -6a_1 + O(z),\end{aligned}$$

as $z \to 0$. Now we recall that the hyperbolic area element is given by

$$4 \frac{|dz|^2}{(1-z^2)^2)} = \left(4 + O\left(z^2\right)\right) |dz|^2,$$

so

$$||\mathcal{S}(f)||_\infty \leq \frac{|6a_1|}{4} \leq \frac{3}{2}.$$

□

15.4. The Ahlfors-Weill construction, part I

Now we want to go the other way around compared to Nehari's theorem. Suppose $q \in \mathcal{Q}((\mathbb{H}^2)^*)$ is a bounded holomorphic quadratic differential so that

$$||q||_\infty < \frac{1}{2}.$$

Write $q = q(z)dz^2$ and define a Beltrami coefficient on $\widehat{\mathbb{C}}$ by

$$\mu_q(z) := \begin{cases} 2\,\mathrm{Im}(z)^2 q(\overline{z}) & \text{if } z \in \mathbb{H}^2 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $||\mu_q||_\infty < 1$, which means that we can apply the construction from Proposition 14.1 to obtain a function f^{μ_q} that is injective and analytic on $(\mathbb{H}^2)^*$.

Theorem 15.6 (Ahlfors-Weill construction). *Let $q \in \mathcal{Q}((\mathbb{H}^2)^*)$ be so that $||q||_\infty < \frac{1}{2}$. Then $f^{\mu_q}|_{(\mathbb{H}^2)^*} : (\mathbb{H}^2)^* \to \mathbb{C}$ is injective and analytic and we have*

$$\mathcal{S}(f^{\mu_q}|_{(\mathbb{H}^2)^*}) = q.$$

15.5. Exercises

Exercise 15.1. Let $U \subset \widehat{\mathbb{C}}$ be a simply connected open set and let q be a holomorphic quadratic differential on U. Write $q(z) = \phi(z)dz^2$ and consider the differential equation

$$\frac{d^2w}{dz^2} + \frac{\phi}{2}w = 0.$$

Since this is a linear second order differential equation without singularities, it has two linearly independent solutions $w_1, w_2 : U \to \mathbb{C}$. Show that there exists a constant $C \in \mathbb{C}$ so that

$$f(z) = C \cdot \frac{w_1(z)}{w_2(z)}$$

satisfies

$$S(f) = q.$$

Exercise 15.2. In this exercise we prove Theorem 15.4.

(a) Suppose $D \subset \mathbb{C}$ is a domain bounded by a simple closed curve γ. Show that the area of D is given by

$$\text{area}(D) = \pm \int_\gamma x dy = \mp \int_\gamma y dx.$$

Here the sign depends on the orientation of γ. *Hint: Green's theorem.*

(b) Let $f : \mathbb{D} \setminus \{0\}$ be of the form of Theorem 15.4. For $r > 0$, let D_r denote the unique bounded component of $\mathbb{C} \setminus \left\{ f(re^{i\theta});\quad 0 \leq \theta \leq 2\pi \right\}$. Use (a) to show that

$$\text{area}(D_r) = -\frac{1}{2}\text{Re}\left(\int_0^{2\pi} f(re^{-i\theta})\, \overline{re^{-i\theta} f'(re^{-i\theta})}\, d\theta \right).$$

(c) Fill in the expression for f and show that

$$\text{area}(D_r) = -\pi \sum_{n=-1}^{\infty} n r^{2n} \left| a_n \right|^2.$$

(d) Prove Theorem 15.4.

LECTURE 16

A complex structure on Teichmüller space

16.1. The Ahlfors-Weill construction, part II

Theorem 15.6 (Ahlfors-Weill construction)**.** *Let $q \in \mathcal{Q}((\mathbb{H}^2)^*)$ be so that $||q||_\infty < \frac{1}{2}$. Then $f^{\mu_q}|_{(\mathbb{H}^2)^*} : (\mathbb{H}^2)^* \to \mathbb{C}$ is injective and analytic and we have*

$$\mathcal{S}(f^{\mu_q}|_{(\mathbb{H}^2)^*}) = q.$$

Proof sketch. Proposition 15.1 implies that if $f : (\mathbb{H}^2)^* \to \mathbb{C}$ is a solution to the equation

$$\mathcal{S}(f) = q,$$

then any other solution can be obtained from it by composing it with a Möbius transformation. So, let us build f according to the recipe in the proof of Proposition 15.1. That is, we consider the ODE

$$\frac{d^2w(z)}{dz^2} + \frac{q(z)}{2}w(z) = 0,$$

take two linearly independent solutions w_1 and w_2 so that

$$w_1(z)\frac{dw_2(z)}{dz} - w_2(z)\frac{dw_1(z)}{dz} = 1$$

and we let f be the quotient of these on $(\mathbb{H}^2)^*$. We extend f to $\widehat{\mathbb{C}}$ by:

$$f(z) = \begin{cases} \frac{w_1(\overline{z})+(z-\overline{z})w_1'(z)}{w_2(\overline{z})+(z-\overline{z})w_2'(\overline{z})} & \text{if } z \in \mathbb{H}^2 \cup \partial\,\mathbb{H}^2 \\ \frac{w_1(z)}{w_2(\overline{z})} & \text{if } z \in (\mathbb{H}^2)^*. \end{cases}$$

This formulas the following interpretation. A direct computation gives that

$$\mathrm{osc}_f(z)(\zeta) = \frac{w_1(z) + (\zeta - z)w_1'(z)}{w_2(z) + (\zeta - z)w_2'(z)}$$

on $(\mathbb{H}^2)^*$. So we have extended it by setting

$$f(z) = \mathrm{osc}_f(\overline{z})(z)$$

for $z \in \mathbb{H}^2 \cup \partial\,\mathbb{H}^2$.

A direct computation now shows that

$$\frac{\partial f}{\partial \overline{z}} = \mu\frac{\partial f}{\partial z}.$$

Injectivity follows from Proposition 15.1 and real analyticity follows from an argument similar ot the proof of Nehari's theorem (see [**Hub06**, Theorem 6.3.10]). □

16.2. Back to Teichmüller space

Our next goal is to connect the above back up to Teichmüller theory. To make our lives a little easier, we will restrict to closed surfaces of genus $g \geq 2$.

Recall the description

$$\mathcal{T}(S) = B(S)_1 / \mathrm{Diff}_0^+(S),$$

where $B(S)_1$ denotes the space of Beltrami coefficients on S. Since we equipped S with the structure of a Riemann surface in order to do this, we can Identify

$$S = \mathbb{H}^2 / \Gamma$$

for some $\Gamma < \mathrm{PSL}(2,\mathbb{R})$ that acts properly discontinuously and freely on $\mathbb{H}^2$. $B(S)_1$ then gets identified with the space

$$B(\mathbb{H}^2)_1^\Gamma$$

of Beltrami coefficients on $\mathbb{H}^2$ that are invariant under the Γ-action. Given $\mu \in B(\mathbb{H}^2)_1^\Gamma$, we will write $[\mu]$ for its image in $\mathcal{T}(S)$.

Given an element $\mu \in B(\mathbb{H}^2)_1^\Gamma$, we may extend it to the entire complex plane $\mathbb{C}$ by setting

$$\widehat{\mu}(z) = \begin{cases} \mu(z) & \text{if } z \in \mathbb{H}^2 \\ 0 & \text{otherwise.} \end{cases}$$

The reason that this is a sensible thing to do is the following:

Proposition 16.1. *For $\mu \in B(\mathbb{C})_1$, let $f^\mu : \mathbb{C} \to \mathbb{C}$ denote the solution to the equation*

$$\frac{\partial f}{\partial \overline{z}} = \mu \frac{\partial f}{\partial z}$$

that fixes 0 and 1. Then we have

$$[\mu_1] = [\mu_2] \in \mathcal{T}(S)$$

for $\mu_1, \mu_2 \in B(\mathbb{H}^2)_1^\Gamma$ if and only if

$$f^{\widehat{\mu_1}}|_{(\mathbb{H}^2)^*} = f^{\widehat{\mu_2}}|_{(\mathbb{H}^2)^*}.$$

Proof. First of all note that $[\mu_1] = [\mu_2]$ implies that there exists an element $\varphi \in \mathrm{Diff}_0^+(S)$ so that

$$\widetilde{\varphi}^* \mu_2 = \mu_1$$

where $\widetilde{\varphi} : \mathbb{H}^2 \to \mathbb{H}^2$ denotes a lift of $\varphi : S \to S$ that fixes $\overline{\mathbb{R}}$ (here we're really thinking of Beltrami coefficients as forms).

We now claim that

$$\left(f^{\widehat{\mu_2}}\right)^{-1} \circ f^{\widehat{\mu_1}} = \widetilde{\varphi} \quad \text{on } \mathbb{H}^2 .$$

Indeed, by definition (i.e. just by filling in the transformation rule, Proposition 10.3)

$$\left(\left(f^{\widehat{\mu_2}}\right)^{-1} \circ f^{\widehat{\mu_1}}\right)^* \mu_2 = \mu_1$$

on $\mathbb{H}^2$. Proposition 14.1 implies that hence $\left(f^{\widehat{\mu_2}}\right)^{-1} \circ f^{\widehat{\mu_1}}$ and $\widetilde{\varphi}$ differ by a Möbius transformation. Since the both fix 0, 1 and ∞, they must be equal.

That means that $f^{\widehat{\mu_1}}$ and $f^{\widehat{\mu_2}}$ coincide on $\overline{\mathbb{R}}$ and hence on $\left(\mathbb{H}^2\right)^*$, since they are both analytic on $\left(\mathbb{H}^2\right)^*$. ∎

Note that Γ, seen as a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ acts on $(\mathbb{H}^2)^*$ as well. We will denote the $(3g-3)$-dimensional complex vector space of Γ-invariant holomorphic quadratic differentials on $(\mathbb{H}^2)^*$ by $\mathcal{Q}\left((\mathbb{H}^2)^*\right)^\Gamma$.

We are now ready to construct the map from Teichmüller space to a finite dimensional complex vector space. First we define a map on the level of Beltrami coefficients:

Definition 16.2. We define a map

$$\Psi : B(\mathbb{H}^2)_1^\Gamma \to \mathcal{Q}\left((\mathbb{H}^2)^*\right)^\Gamma$$

by

$$\Psi(\mu) = \mathcal{S}(f^{\widehat{\mu}}|_{(\mathbb{H}^2)^*}),$$

for all $\mu \in B(\mathbb{H}^2)_1^\Gamma$.

We now have:

Theorem 16.3 (The Bers embedding)**.** *The map $\Psi : B(\mathbb{H}^2)_1^\Gamma \to \mathcal{Q}\left((\mathbb{H}^2)^*\right)^\Gamma$ induces a map*

$$\mathcal{T}(S) \to \left\{ q \in \mathcal{Q}\left((\mathbb{H}^2)^*\right)^\Gamma;\quad ||q||_\infty < 3/2 \right\}$$

that is a homeomorphism onto its image. Moreover, the complex structure this induces on $\mathcal{T}(S)$ does not depend on the choice of base surface S.

Proof. The fact that Ψ is continuous follows from continuity of all the expressions involved:

- Extending μ to the lower half plane is continuous.
- The map $\mu \mapsto f^\mu$ is continuous.
- Taking the Schwarzian derivative is continuous.

Injectivivty follows immediately from Proposition 16.1. Since we already know $\mathcal{T}(S)$ is homeomorphic to an open subset of $\mathbb{R}^{6g-6}$, invariance of domain tells us that f is a homeomorphism onto its image.

Now suppose S' a Riemann surface that is diffeomorphic to S. Using Fenchel-Nielssen coordingates, we have already seen that this means that $\mathcal{T}(S)$ and $\mathcal{T}(S')$ are homeomorphic. In fact a diffeomorphism $\varphi : S \to S'$ induces an isomorphism

$$\varphi^* : B(\mathbb{H}^2)_1^{\Gamma'} \to B(\mathbb{H}^2)_1^\Gamma.$$

So we obtain a map

$$\psi(\mathcal{T}(S')) \subset \mathcal{Q}((\mathbb{H})^*)^{\Gamma'} \quad \longrightarrow \quad \psi(\mathcal{T}(S)) \subset \mathcal{Q}((\mathbb{H})^*)^\Gamma$$

as follows. Suppose $q \in \psi(\mathcal{T}(S'))$ and choose a $\mu \in B(\mathbb{H}^2)_1^{\Gamma'}$ so that

$$\mathcal{S}(f^\mu|_{(\mathbb{H}^2)^*}) = q.$$

Then we obtain an element in $\psi(\mathcal{T}(S))$ of the form

$$\mathcal{S}(f^{\varphi^*\mu}|_{(\mathbb{H}^2)^*}) \in \mathcal{Q}((\mathbb{H})^*)^\Gamma.$$

All the maps involved are analytic (the fact that pulling back a Beltrami coefficient by an orientation preserving diffeomorphism is analytic follows from Proposition 10.3), so this gives a holomorphic map. $\square$

Given a Riemann surface $R = \mathbb{H}^2/\Gamma$, we also obtain a *conjugate* Riemann surface

$$R^* = \left(\mathbb{H}^2\right)^*/\Gamma.$$

We have an isomorphism

$$\mathcal{Q}\left((\mathbb{H}^2)^*\right)^\Gamma \simeq \mathcal{Q}\left(R^*\right)$$

Looking at the proof above, we get local coordinates near $[R, f] \in \mathcal{T}(S)$ in $\mathcal{Q}\left(R^*\right)$. This also means that:

Proposition 16.4. *The Bers embedding induces an identification*

$$T_{[R,f]}\,\mathcal{T}(S) \simeq \mathcal{Q}(R^*).$$

It turns out that the cotangent space to Teichmüller space is naturally given by quadratic differentials, i.e.

$$T^*_{[R,f]}\,\mathcal{T}(S) \simeq \mathcal{Q}(R).$$

This comes out of a pairing

$$\mathcal{Q}(R^*) \times \mathcal{Q}(R) \to \mathbb{C}\,.$$

Given by

$$(p, q) \mapsto \int_R \frac{p(\overline{z})q(z)}{\lambda^2(z)}$$

where $\lambda^2(z)$ denotes the hyperbolic area element on R.

Proposition 16.5. *The pairing above induces an identification*

$$T^*_{[R,f]}\,\mathcal{T}(S) \simeq \mathcal{Q}(R).$$

LECTURE 17

Symplectic geometry

17.1. The Weil-Petersson form

Recall that we found an identification cotangent space

$$T^*_{[R,f]}\,\mathcal{T}(S) \simeq \mathcal{Q}(R)$$

for all $[R,f] \in \mathcal{T}(S)$.

This space also comes with a natural inner product:

Definition 17.1 (The Weil-Petersson inner product)**.** Let R be a Riemann surface. Given $p, q \in \mathcal{Q}(R)$. We define their Weil Petersson inner product by

$$\langle p, q\rangle = \int_R \frac{\overline{p}q}{\lambda^2},$$

where λ^2 denotes the hyperbolic area element on R. ω_{WP} will denote the form on $T_{[R,f]}\,\mathcal{T}(S)$ that is dual to $\mathrm{Im}\langle\cdot,\cdot\rangle$.

It turns out this equips Teichmüller space with the structre of a Kähler manifold:

Theorem 17.2. *The Weil-Petersson metric is Kähler, i.e. $d\omega_{\mathrm{WP}} = 0$.*

We will not prove this and refer to [**Hub06**, Theorem 7.7.2] for a proof.

Note that ω_{WP} is a symplectic form. That is,

(1) ω_{WP} is bilinear
(2) ω_{WP} is non-degenerate, i.e. if $v \in T_{[R,f]}\,\mathcal{T}(S)$ is so that
$$\omega_{\mathrm{WP}}(v,w) = 0 \quad \text{for all } w \in T_{[R,f]}\,\mathcal{T}(S)$$
then $v = 0$.
(3) ω_{WP} is anti-symmetric, i.e.
$$\omega_{\mathrm{WP}}(v,w) = -\omega_{\mathrm{WP}}(w,v)$$
for all $v, w \in T_{[R,f]}\,\mathcal{T}(S)$.

(see Exercise 17.1).

17.2. Symplectic manifolds

In short, Teichmüller space comes with the structure of a symplectic manifold. So, let us briefly discuss symplectic manifolds. For a more comprehensive introduction, we refer to [**Hec14**], from which most of the material below has been taken.

Definition 17.3. A *symplectic form* on a smooth manifold M is a smooth 2-form $\omega \in \Omega^2(M)$ that is closed and non-degenerate, i.e.

- $d\omega = 0$ and
- $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate bilinear form.

A smooth manifold equipped with a symplectic form is called a *symplectic manifold*.

Note that when we compare the definition above to the list of properties of a symplectic form on a vector space, we see two differences. First of all, there is no anti-symmetry requirement. The reason for this is that 2-forms are automatically anti-symmetric, which makes the requirement superfluous. The second addition is that the form is required to be closed. This comes from the requirement that ω does not change under *Hamiltonian flows*.

17.2.1. Why are symplectic forms required to be closed?

Definition 17.4. Let (M, ω) be a symplectic manifold and let $H : M \to \mathbb{R}$ be a smooth function. The *Hamiltonian vector field* associated to H is the unique vector $v_H : M \to TM$ so that

$$dH = -\omega(v_H, \cdot).$$

Note that this uses the fact that ω is non-degenerate.

Recall that a vector field $v : M \to TM$ induces a flow on M. That is, for every $p \in M$ we consider the differential equation

$$\gamma_p(0) = p, \quad \frac{d\gamma_p(t)}{dt} = v(\gamma_p(t)) \in T_pM.$$

This is a first order differential equation that can be solved on some maximal open interval $I_p \subset \mathbb{R}$ around 0. This defines a *flow*

$$\phi_t^v(p) = \gamma_p(t), \quad t \in I_p, p \in M.$$

The flow associated to the Hamiltonian vector v_H field is called the *Hamiltonian flow* associated to H.

In order to make sense of "staying constant along the flow of a vector field", we need to be able to differentiate along the flow lines. This is where the Lie derivative comes in.

Definition 17.5. Given a vector field v and a form $\alpha \in \Omega^k(M)$, the *Lie derivative* of α along v is given by

$$\mathcal{L}_v \, \alpha = \frac{d}{dt} \left((\phi_t^v)^* \alpha\right)_{t=0} \in \Omega^k(M).$$

Note that for a function, we have

$$\mathcal{L}_v \, f = v(f) = df(v).$$

In particular

$$\mathcal{L}_{v_H} \, H = dH(v_H) = -\omega(v_H, v_H) = 0,$$

by antisymmetry. So H is a *constant of motion* along its Hamiltonian flow.

The formula $\mathcal{L}_v \, f = v(f) = df(v)$ turns out to generalize to something called Cartan's formula. Given a vector field $v : M \to TM$, we define

$$\imath_v : \Omega^k(M) \to \Omega^{k-1}(M)$$

by

$$\imath_v\alpha(w_1,\ldots,w_{k-1}) = \alpha(v,w_1,\ldots,w_{k-1}).$$

Cartan's formula now reads:

Lemma 17.6 (Cartan's formula)**.** *Let $v : M \to TM$ be a vector field and let $\alpha \in \Omega^k(M)$. Then*

$$\mathcal{L}_v\,\alpha = d(\imath_v\alpha) + \imath_v d\alpha.$$

PROOF. Exercise 17.2. □

Now we get to the reason why we assume our form to be closed:

Lemma 17.7. *Let $\omega \in \Omega^2(M)$ be closed. Then*

$$\mathcal{L}_{v_H}\,\omega = 0$$

for all smooth $H : M \to \mathbb{R}$.

PROOF. Using Lemma 17.6, we obtain

$$\mathcal{L}_{v_H}\,\omega = d(\imath_{v_H}\omega) + \imath_{v_H}d\omega = -d(dH) = 0.$$

□

17.2.2. Examples.

Example 17.8. The classical example is the manifold $M = \mathbb{R}^{2n}$ equipped with coordinates $(q_1,\ldots,q_n,p_1,\ldots,p_n)$ and the symplectic form

$$\omega = \sum_{j=1}^n dp_j \wedge dq_j.$$

Note that

$$\imath_{\partial/\partial p_j}\omega = dq_j \quad \text{and} \quad \imath_{\partial/\partial q_j}\omega = -dp_j.$$

If $H : \mathbb{R}^{2n} \to \mathbb{R}$ is a smooth function, then

$$dH = \sum_{j=1}^n \frac{\partial H}{\partial q_j}dq_j + \frac{\partial H}{\partial p_j}dp_j.$$

Using the three expressions above, this means that

$$v_H = \sum_{j=1}^n \frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j}\frac{\partial}{\partial p_j}.$$

So the Hamiltonian flow associated to H is determined by the equations

$$\frac{dq_j(t)}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j(t)}{dt} = -\frac{\partial H}{\partial q_j}.$$

These are Hamilton's equations of motion and this is also the reason for the sign convention in the definition of the Hamiltonian vector field.

For instance, for the classical Hamiltonian

$$H(q,p) = K(p) + V(q),$$

where q should be thought of as the position of some object of mass m, p its momentum, K its kinetic energy

$$K(p) = \frac{m}{2}\sum_{j=1}^{n} p_j^2$$

and V is some potential, Hamilton's equation turns into

$$m\frac{dq(t)}{dt} = p(t), \quad \frac{dp}{dt} = -\nabla V.$$

So if we take a second derivative on the right hand side, this turns into

$$F(q) = m\frac{d^2q(t)}{dt^2},$$

Newton's second law of motion for the conservative force field $F(q) = -\nabla V(q)$.

Example 17.9 (Cotangent bundles)**.** Let M be an n-dimensional manifold. Its cotangent bundle T^*M is a smooth $2n$-manifold. Consider the projection map

$$\pi : T^*M \to M.$$

This is a smooth map, so it has a derivative

$$D_\xi\pi : T_\xi(T^*M) \to T_{\pi(\xi)}M$$

at every point $\xi \in T^*M$. From this we obtain a linear form

$$\theta_\xi : T_\xi(T^*M) \to \mathbb{R}$$

by

$$\theta_\xi(v) = \xi(D_\xi\pi \cdot v)$$

for all $v \in T_\xi(T^*M)$.

So, this gives rise to an element $\theta \in \Omega^1(T^*M)$ and a closed (even exact) two form

$$\omega = d\theta \in \Omega^2(M).$$

All we need to do is chack that ω is non-degenerate. To this end, let us write out a local expression for it. Take local coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, x_n)$ so that

$$\pi(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n) = (x_1, \ldots, x_n).$$

So

$$\theta = \sum_{j=1}^{n} \xi_j dx_j$$

and hence

$$\omega = d\theta = \sum_{j=1}^{n} d\xi_j \wedge dx_j,$$

so locally our form is just the symplectic form from the previous example. Since non-degeneracy is a local condition, this means ω is non-degenerate and hence a symplectic form. ω is sometimes called the *canonical symplectic form* on T^*M.

We note that symplectic manifolds are always even-dimensional. This for instance follows from the following.

Lemma 17.10. *In a symplectic vector space* (V, ω) *one can choose a* symplectic basis*. I.e. a basis* $(e_1, \dots, e_n, f_1, \dots, f_n)$*, so that*

$$\omega(e_j, e_k) = \omega(f_j, f_k) = 0, \quad \omega(f_j, e_k) = \omega(e_j, f_k) = \delta_{jk}.$$

In particular symplectic vector spaces have even dimension.

Proof. Choose any non-zero vector $e_1 \in V$. Because ω is non-degenerate, there exists a vector $f_1 \in V$ so that

$$\omega(e_1, f_1) = 1$$

Set $U = \mathbb{R}\, e_1 \oplus \mathbb{R}\, f_1$. This is a direct sum, because $\omega(e_1, e_1) = 0$. Write

$$V = U \oplus \left\{ w \in V; \quad \omega(e_1, w) = \omega(f_1, w) = 0 \ \right\},$$

this is a direct sum because ω is non-degenerate on U an V. We can now proceed by induction. □

We finish this section with a result we won't prove. It states that symplectic manifold locally all look the same. This is sort of a manifold version of the lemma above. Note that this is very different from Riemannian manifolds: different metrics can have wildly different curvature tensors.

Theorem 17.11 (Darboux)**.** *If* (M, ω) *is a symplectic manifold then around each point* $p \in M$*, we can find a local parametrization* $\varphi : U \to M$ *with coordinates*

$$(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$$

so that

$$\varphi^* \omega = \sum_j d\xi_j \wedge dx_j$$

on U*.*

The coordinates in this theorem are often called *Darboux coordinates.*

17.3. Exercises

Exercise 17.1. Let V be a complex vector space and let $h : V \times V \to \mathbb{C}$ be a hermitian inner product and set $\omega(v, w) = \mathrm{Im}(h(v, w))$. Show that ω is a symplectic form.

Exercise 17.2.
(a) Prove Lemma 17.6 for 1-forms.
(b) Prove that if Lemma 17.6 holds for forms α and β, then it also holds for $\alpha \wedge \beta$ and conclude the proof of the lemma.

LECTURE 18

Wolpert's magical formula

The Weil-Petersson symplectic form was obtained in a rather complicated way. In order to work with it, it's useful to find Darboux coordinates for it. Wolpert proved that that Fenchel-Nielsen coordinates are global Darboux coordinates for ω_{WP}. In what follows, we will discuss this proof, following [**Hub06**].

We start with the theorem:

Theorem 18.1 (Wolpert's magical formula)**.** *Let S be a closed orientable surface of genus $g \geq 2$ and let $\mathcal{P}$ be a pants decomposition of S. Then*

$$2\omega_{\mathrm{WP}} = \sum_{\gamma \in \mathcal{P}} d\ell_\gamma \wedge d\tau_\gamma.$$

We will prove this theorem in multiple steps. The first step is a result from the hyperbolic geometry of surfaces called the collar theorem, due to Keen [**Kee74**]. This theorem is used throughout the study of hyperbolic surfaces. Define a function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\eta(\ell) = \frac{1}{2}\log\left(\frac{\cosh(\ell/2)+1}{\cosh(\ell/2)-1}\right)$$

for all $\ell \in \mathbb{R}_+$. The collar theorem now reads:

Theorem 18.2 (Collar theorem)**.** *Let X be a complete hyperbolic surface and let $\Gamma = (\gamma_n)_n$ be a collection of disjoint simple closed geodesics. Then the collars*

$$C(\gamma_i) = \left\{ x \in X; \quad \mathrm{d}(x,\gamma_i) < \eta(\ell(\gamma_i)) \ \right\}$$

are homeomorphic to annuli and pairwise disjoint.

Proof. Consider the geodesics γ_i and γ_j in the collection. Since they are disjoint, we can find a pants decomposition of X that includes them both. It can be shown (Exercise 18.1) that $C(\gamma_i)$ is contained in the (one or two) pairs of pants bounded by γ_i. So the only thing we need to check is the case where γ_i and γ_j are both boundary components of the same pair of pants (Exercise 18.1). □

The second and longest step in the proof of Wolpert's theorem is:

Proposition 18.3. *Let $\xi \in T_{[R,f]}\mathcal{T}(S)$ then*

$$\omega_{\mathrm{WP}}\left(\frac{\partial}{\partial \tau_\gamma}, \xi\right) = \frac{1}{2} d\ell_\gamma(\xi).$$

In other words, $\partial/\partial\tau_\gamma$ is the Hamiltonian vector field for $\frac{1}{2}\ell_\gamma$.

Proof sketch. Let us first set up some notation. Let

$$\pi_\gamma : R_\gamma \to R$$

denote the infinite cover corresponding to γ. That is, we consider the unique closed geodesic on R that is homotopic to $f(\gamma)$, identify this with an element $g \in \pi_1(R)$ and set $R_\gamma = \widetilde{R}/\langle g\rangle$. We will denote the corresponding geodesic with $\gamma \subset R$ as well.

We will use the band model again and make the identification

$$R_\gamma = \mathbb{B}\,/\ell_\gamma([R,f])\,\mathbb{Z}\,.$$

Moreover, we will denote by $C \subset R$ and $\widetilde{C} \subset R_\gamma$ the collars around $\gamma \subset R$ and the central curve $\widetilde{\gamma} \subset R_\gamma$ given to us by the collar theorem. Finally, let $\Omega_\gamma \subset R_\gamma$ denote a fundamental domain for the quotient $\pi_\gamma : R_\gamma \to R$.

The proof now goes by building a quadratic differential q_γ that "represents" $\partial/\partial\tau_\gamma$. Here represents is in quotation marks, because the space of quadratic differentials is the *co*tangent space and not the tangent space.

Note that the quadratic differential dz^2 on $\mathbb{B}$ is invariant under translations, which means that it descends to R_γ. We are going to push this forward to a quadratic differential on R, using π_γ. This goes as follows. Suppose $U \subset R$ is simply connected and open and $(U, \zeta : U \to \mathbb{C})$ is a coordinate patch. Then

$$\pi_\gamma^{-1}(U) \subset R_\gamma$$

is a union of disjoint sets $U_i \subset R$ so that $\pi_\gamma : U_i \to U$ is a diffeomorphism. So we obtain local coordinates

$$\zeta_i : U_i \to \mathbb{C}$$

by

$$\zeta_i := \zeta \circ \pi_\gamma|_{U_i}.$$

So if we have a quadratic differential $q \in \mathcal{Q}(R_\gamma)$ then on U_i it can be written as

$$q(\zeta_i) = \varphi_i(\zeta_i) d\zeta_i^2.$$

We now set

$$\Big((\pi_\gamma)_* q\Big)|_U = \sum_i \varphi_i d\zeta^2.$$

This allows us to define

$$q_\gamma = (\pi_\gamma)_* dz^2 \in \mathcal{Q}(R).$$

Any tangent vector ξ is represented by a quadratic differential $p_\xi \in \mathcal{Q}(R^*)$. Our goal is to show that

$$\omega_{\mathrm{WP}}\left(\frac{\partial}{\partial\tau_\gamma}, \xi\right) = \frac{1}{\pi}\mathrm{Re}\int_R \frac{q_\gamma \overline{p_\xi}}{\lambda^2},$$

and

$$\langle d\ell_\gamma, \xi\rangle = \frac{2}{\pi}\mathrm{Re}\int_R \frac{q_\gamma \overline{p_\xi}}{\lambda^2},$$

where λ^2 denotes the hyperbolic area element on R. Note that this would prove the proposition. This goes in multiple steps:

(1) First we identify $\partial/\partial\tau_\gamma$ with a Beltrami form.

(2) Then we use some linear algebra to argue how the Weil-Petersson form can be evaluates on a Beltrami differential.
(3) Then we evaluate both sides of the first of these identities, using a Fourier series.
(4) Finally, we prove the second identity.

Step 1. So first, we will associate a Beltrami coefficient to a twist deformation. To this end, write h for the half-width of C, so

$$h = \eta\left(\ell_\gamma([R, f])\right).$$

Suppose R_t is the surface obtained by performing a twist of magnitude t along γ on R. Define a map

$$f_t : R \to R_t$$

that is constant on $R \setminus A$ and

$$f_t(x+iy) = \begin{cases} x + iy + \frac{t}{2h}(y-h) & \text{if } y > 0 \\ x + iy - \frac{t}{2h}(y-h) & \text{if } y < 0 \end{cases}$$

for $x + iy \in A$. Here we parameterized A using a fundamental domain for A in $\mathbb{B}$ that is symmetric around $\mathbb{R} \subset \mathbb{B}$. The corresponding annulus in R_t is parametrized in a discontinuous way: we use the same parametrization as on R, which below γ is shifted by t. Figure 1 shows what this map does.

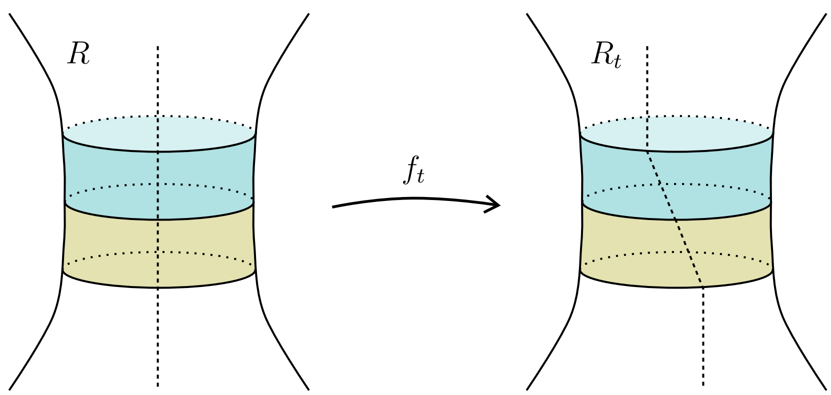

FIGURE 1. The map $f_t; R \to R_t$.

$f_t : R \to R_t$ is a diffeomorphism away from ∂A. So, we may take the Beltrami coefficient of f away from ∂ and obtain

$$\mu_t = \frac{\partial f_t / \partial \overline{z}}{\partial f_t / \partial z} = \begin{cases} \frac{-t}{4hi+t} & \text{on } A \\ 0 & \text{on } R \setminus A. \end{cases}$$

Thinking of this as a Beltrami form, i.e. writing

$$\mu_t = \frac{-t}{4hi+t} \frac{d\overline{z}}{dz}$$

on A, we have

$$\left(\frac{d}{dt}\mu_t\right)_{t=0} = \frac{i}{4h}\frac{d\overline{z}}{dz}.$$

on A. Of course this is not quite a smooth Beltrami form on R, but an approximation argument can be used in order to resolve this.

Step 2: Using the embedding $\mathcal{T}(S) \hookrightarrow B(S)_1/\mathrm{Diff}_0^+(S)$, this form also represents a tangent vector, exactly the tangent to our twist deformation. In order to see what the Weil-Petersson pairing does to a tangent vector of this form, we need to trace some definitions.

A hermitian inner product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ on a complex vector space induces an isomorphism $\Phi : V \to V^*$ by

$$\Big(\Phi(v)\Big)(w) = \langle v, w\rangle$$

So the Weil-Petersson pairing induces an isomorphism

$$\mathcal{Q}(R) \to T_{[R,f]}\mathcal{T}(S)$$

by

$$q \mapsto \frac{\overline{q}}{\lambda^2}$$

where λ^2 is the hyperbolic area element on R. The right hand side is naturally a Beltrami form on R (i.e. a form of type $(-1,1)$). The inner product with a Beltrami form is then given by

$$\left\langle \frac{\overline{q}}{\lambda^2}, \mu \right\rangle = \int_R q\mu.$$

So

$$\omega_{\mathrm{WP}}\left(\left(\frac{d}{dt}\mu_t\right)_{t=0}, \frac{\overline{p}}{\lambda^2}\right) = \mathrm{Im}\frac{i}{4h}\int_C \overline{p} = \frac{1}{4h}\mathrm{Re}\int_C \overline{p}.$$

Here we filled in $\overline{p}/\lambda^2$, because, using the fact that the Weil-Petersson form is non-degenerate, we may assume that our Beltrami coefficient is of this form.

Step 3: In order to evaluate this last expression, we lift p to R_γ. I.e. we set $\widetilde{p} = (\pi_\gamma)^* p$. Since it's periodic with period $\ell = \ell_\gamma([R,f])$, we may develop it into a Fourier series. We set

$$\widetilde{p} = \frac{1}{\ell}\sum_{k=-\infty}^{\infty} b_k e^{2\pi i k z/\ell} dz^2.$$

So for all $-\pi/2 < y < \pi/2$ we have

$$\int_0^\ell \widetilde{p}(x+iy)dx = b_0.$$

So we compute

$$\begin{aligned}\omega_{\mathrm{WP}}\left(\left(\frac{d}{dt}\mu_t\right)_{t=0},\frac{\overline{p}}{\lambda^2}\right) &= \frac{1}{4h}\mathrm{Re}\int_C \overline{p} \\ &= \frac{1}{4h}\mathrm{Re}\int_{\widetilde{C}} \overline{\widetilde{p}}(z)\,|dz|^2 \\ &= \frac{1}{4h}\mathrm{Re}\int_{-h}^{h}\int_0^l \widetilde{p}(x,y)dxdy \\ &= \frac{1}{2}\mathrm{Re}(b_0)\end{aligned}$$

On the other hand

$$\begin{aligned}\frac{1}{\pi}\mathrm{Re}\int_R \frac{q_\gamma\overline{p}}{\lambda^2} &= \frac{1}{\pi}\mathrm{Re}\int_{\Omega_\gamma}(\widetilde{q}(z)dz^2)(\overline{\widetilde{p}}(z)d\overline{z}^2)\left(\frac{\cos^2(y)}{|dz|^2}\right) \\ &= \int_{R_\gamma} dz^2(\overline{\widetilde{p}}(z)d\overline{z}^2)\left(\frac{\cos^2(y)}{|dz|^2}\right) \\ &= \int_{-\pi/2}^{\pi/2}\int_0^l \overline{\widetilde{p}}(x+iy)dx\cos^2(y)dy = \frac{1}{2}\mathrm{Re}(b_0).\end{aligned}$$

Step 4: Now we need to show that

$$\langle d\ell_\gamma,\xi\rangle = \frac{2}{\pi}\mathrm{Re}\int_R\frac{q_\gamma\overline{p_\xi}}{\lambda^2}.$$

We have seen that we can parametrize the set of hyperbolic annuli up to isometry by the length of their core curve. This gives us a function

$$L:\mathcal{T}(A)\to\mathbb{R}$$

for any annulus A. We claim that this function is analytic and as such, we obtain a linear functional

$$dL: T_A\,\mathcal{T}(A)\to\mathbb{R}$$

By a similar lifting element to the above, we have

$$dL(\pi_\gamma^*\mu) = \langle d\ell_\gamma,\mu\rangle$$

for all Beltrami forms μ. If we can show that this has the same kernel as

$$\xi\mapsto\mathrm{Re}\int_{R_\gamma}\frac{q_\gamma\overline{p_\xi}}{\lambda^2},$$

then these two functionals must be multiples of each other.

So, we need to identify the kernels of both. In order to do this, we interpret Beltrami forms on A as antilinear maps $TA\to TA$, given by

$$\mu(z)\frac{d\overline{z}}{dz}w(z)\frac{\partial}{\partial z} = \mu(z)\overline{w(z)}\frac{\partial}{\partial z},$$

coming from applying $d\overline{z}$ to $w(z)\frac{\partial}{\partial z}$.

We claim that these kernels are exactly all Beltrami forms of the form $\partial\xi/\partial\overline{z}$, where $\xi : A \to TA$ is a vector field so that $\xi|_{\partial A}$ is tangent to ∂A. This follows from careful analysis. For details, see [**Hub06**]. □

The third step is:

Proposition 18.4.

$$\mathcal{L}_{\partial/\partial\tau_\gamma}\,\omega_{\text{WP}} = 0$$

Proof. This is direct from Proposition 18.3, Theorem 17.2 and Lemma 17.7. □

Now we can prove the theorem.

Proof of Theorem 18.1. Write

$$\omega_{\text{WP}} = \sum_{\alpha,\beta\in\mathcal{P}} a_{\alpha\beta} d\ell_\alpha \wedge d\ell_\beta + b_{\alpha\beta} d\ell_\alpha \wedge d\tau_\beta + c_{\alpha\beta} d\tau_\alpha \wedge d\tau_\beta.$$

Our first claim is that these coefficients are constant under twists. To this end, we use that

$$\mathcal{L}_v\,\psi(w_1, w_2) = \frac{\partial}{\partial v}\psi(w_1, w_2) - \frac{\partial}{\partial v}\psi([v, w_1], w_2) - \frac{\partial}{\partial v}\psi(w_1, [v, w_2])$$

for all vector fields v, w_1, w_2 and 2-forms ψ (Exercise 18.2).

We have for instance

$$\frac{\partial}{\partial\tau_\gamma} a_{\alpha\beta} = \frac{\partial}{\partial\tau_\gamma}\left(\omega_{\text{WP}}\left(\frac{\partial}{\partial\ell_\alpha}, \frac{\partial}{\partial\ell_\beta}\right)\right) = \mathcal{L}_{\partial/\partial\tau_\gamma}\,\omega_{\text{WP}}\left(\frac{\partial}{\partial\ell_\alpha}, \frac{\partial}{\partial\ell_\beta}\right) = 0,$$

where we have used that the brackets of the tangent vectors vanish and Proposition 18.4.

This means that in order to compute the coefficients $a_{\alpha\beta}, b_{\alpha\beta}, c_{\alpha\beta}$, we may twist so that all the twist-coordinates of our surface are 0. A surface R of which all the twists are 0 admits an involution $\sigma : R \to R$, that flips the two hexagons in each pair of pants. Since this involution is anti-holomorphic, ω_{WP} is odd under this form. $d\ell_\alpha$ is even under σ and $d\tau_\alpha$ is odd. So the only odd terms in ω_{WP} are the terms of the form $d\ell_\alpha \wedge d\tau_\beta$. So $a_{\alpha\beta} = c_{\alpha\beta} = 0$. Proposition 18.3 implies that

$$b_{\alpha\beta} = \omega_{\text{WP}}\left(\frac{\partial}{\partial\ell_\alpha}, \frac{\partial}{\partial\tau_\beta}\right) = \begin{cases} -\frac{1}{2} & \text{if } \alpha = \beta \\ 0 & \text{otherwise.} \end{cases}$$

□

18.1. Exercises

Exercise 18.1. Complete the proof of Theorem 18.2.

Exercise 18.2. Let M be a manifold, $v, w_1, w_2 : M \to TM$ vector fields and $\psi \in \Omega^2(M)$. Prove that

$$\mathcal{L}_v \, \psi(w_1, w_2) = \frac{\partial}{\partial v}\psi([v, w_1], w_2) + \frac{\partial}{\partial v}\psi(w_1, [v, w_2]).$$

LECTURE 19

Integrating geometric functions on Moduli space

The main goal for the rest of this course is to derive Mirzakhani's recurrences for Weil-Petersson volumes of moduli spaces. We will mainly follow [**Mir07a**].

19.1. The Weil-Petersson volume form on moduli space

The Weil-Petersson symplectic form descends to $\mathcal{M}_g$, one way to see this is from Wolpert's formula. Indeed

$$\varphi^* d\ell_\alpha = d\ell_{\varphi^{-1}(\alpha)} \quad \text{and} \quad \varphi^* d\tau_\alpha = d\tau_{\varphi^{-1}(\alpha)}$$

for all simple closed curves α on Σ_g and $\varphi \in \mathrm{MCG}(\Sigma_g)$. Since Wolpert's magical formula (Theorem 18.1) holds for any pants decomposition, it follows that

$$\varphi^* \omega_{\mathrm{WP}} = \omega_{\mathrm{WP}}$$

for all $\varphi \in \mathrm{MCG}(\Sigma_g)$ and hence that ω_{WP} descends to $\mathcal{M}_g$. We are actually skipping over a little issue when we say this: $\mathcal{M}_g$ is *not* a manifold, it is only an orbifold. In particular, it does not have a well-defined tangent space at each point. There are multiple ways out of this, that we will discuss when the issue comes up.

Since ω_{WP} descends, so does the volume form

$$d\,\mathrm{vol}_{\mathrm{WP}} = \frac{2^{3g-3}}{(3g-3)!} \wedge^{3g-3} \omega_{\mathrm{WP}}$$

that it induces [1]. Note that this is just the standard Euclidean volume form in Fenchel-Nielsen coordinates.

The main question we want to answer is:

$$\text{What is } V_g := \mathrm{vol}_{\mathrm{WP}}(\mathcal{M}_g)\ ?$$

One way to turn this into a well defined question is to set

$$V_g = \mathrm{vol}_{\mathrm{WP}}(\mathcal{F}_g),$$

where $\mathcal{F}_g$ is a Borel fundamental domain[2] for the action of $\mathrm{MCG}(\Sigma_g)$ on $\mathcal{T}_g$. Again, because of the $\mathrm{MCG}(\Sigma_g)$-invariance of ω_{WP}, this is well-defined.

First of all, it is not at all clear that this is finite, because $\mathcal{M}_g$ is not compact (Exercise 19.1). However, this follows from Wolpert's magical formula together with a theorem by Bers:

Corollary 19.1. *Let $g \geq 2$. Then*

$$V_g < \infty.$$

[1] The factor in front is just a normalization that will make the recursion come out nicer.

[2] A fundamental domain that is also a Borel set.

Proof sketch. Bers [**Ber74**] proved that there exists a constant[3] $B_g > 0$ so that every closed orientable surface of genus g admits a pants decomposition in which all curves have length at most B_g.

There are finitely many $\mathrm{MCG}(\Sigma_g)$-orbits of (topological) pants decompositions of Σ_g (Exercise 19.2). Let $\{\mathcal{P}_1, \ldots, \mathcal{P}_k\}$ be a set of representatives of these orbits (one representative per orbit). Then

$$\bigcup_{i=1}^{k} \left\{ X \in \mathcal{T}_g; \quad \ell_\alpha(X) \leq B_g, 0 \leq \tau_\alpha(X) \leq \ell_\alpha(X) \forall \alpha \in \mathcal{P}_i \right\}$$

contains a fundamental domain for the mapping class group. On the other hand, it is finite a collection of bounded sets in $\mathcal{T}_g$ and hence has finite volume. □

Since there are estimates on B_g, the proof above can even be improved in order to bound V_g from above. For us, it just indicates that the quest for the value of V_g is not pointless.

19.2. Symplectic forms on moduli spaces of surfaces with boundary

The most natural thing to try would of course be to find a nice fundamental domain for the $\mathrm{MCG}(\Sigma_g)$-action on $\mathcal{T}_g$ and try to integrate $d\,\mathrm{vol}_{\mathrm{WP}}$ over it. However, it turns out that describing such a fundamental domain is a really hard problem. So, instead we are going to determine a recurrence for volumes of moduli spaces of surfaces with boundary.

To this end, we will define the Weil-Petersson form on such moduli spaces too. Recall that we defined these spaces in Section 13.2. We will simplify our notation slightly, in order to be consistent with the notation in [**Mir07a**]. That is, we will write $\mathcal{T}_{g,n}(L_1, \ldots, L_n)$ and $\mathcal{M}_{g,n}(L_1, \ldots, L_n)$ instead, where all the postive entries in the vector $(L_1, \ldots, L_n)$ correspond to boundary components of that length and $L_i = 0$ indicates that the entry corresponds to a puncture.

Definition 19.2. Let $g, n \in \mathbb{N}$ be so that $\chi(\Sigma_{g,n}) < 0$ and let $\mathcal{P}$ be a pants decomposition of $\Sigma_{g,n}$. Moreover, let $L_1, \ldots, L_n \in \mathbb{R}_{\geq 0}$. The *Weil-Petersson symplectic form* on $\mathcal{T}_{g,n}(L_1, \ldots, L_n)$ is given by

$$\omega_{\mathrm{WP}} = \frac{1}{2} \sum_{\alpha \in \mathcal{P}} d\ell_\alpha \wedge d\tau_\alpha.$$

The associated volume form is

$$d\,\mathrm{vol}_{\mathrm{WP}} = \frac{2^{3g+n-3}}{(3g+n-3)!} \wedge^{3g+n-3} \omega_{\mathrm{WP}}.$$

This is a $\mathrm{MCG}(\Sigma_{g,n})$-invariant form and hence descends to $\mathcal{M}_{g,n}(L_1, \ldots, L_n)$. For us, this form seems to come out of thin air, we just define it by analogy to the case of closed surfaces. In reality, this is not the case. The spaces $\mathcal{T}_{g,n}(L_1, \ldots, L_n)$ come with

[3] What the value of B_g is, is an open problem. It is not even clear at which rate it grows as a function of g. The best known lower bound on this rate is $\approx \sqrt{g}$ [**Bus10**, Chapter 5] and the best known upper bound is $\approx g$ [**Par14**]

a natural symplectic form called the *Goldman symplectic form* [**Gol84**]. It's a theorem by Wolpert [**Wol82**] that this form can be expressed as above in Fenchel-Nielsen. In particular, it follows that it coincedes with the Weil-Petersson symplectic form when $n = 0$ or $L_1 = L_2 = \ldots = L_n = 0$.

19.3. Level sets of length functions

As we mentioned in the previous section, we are going to set up a recurrence for Weil-Petersson volumes of moduli spaces of surfaces with boundary.

In order to do this, we make use of the following observation. Consider an essential simple closed curve $\alpha \subset \Sigma_{g,n}$. Then from hyperbolic surface $X \in \mathcal{T}_{g,n}(L_1, \ldots, L_n)$ so that $\ell_\alpha(X) = t$, we obtain a hyperbolic surface $X' \in \mathcal{T}(\Sigma_{g,n} \setminus \alpha, L_1, \ldots, L_n, t, t)$ by cutting X open along α.

Our goal is to use this operation to build maps between moduli spaces. Of course, there are several issues in doing this, among them the fact that level sets of length functions in moduli spaces do not make any sense. The goal of this section is to make this work.

First of all, in order to make set of level sets of length functions, we define the following cover of moduli space:

Definition 19.3. Let $g, n \in \mathbb{N}$ be so that $\chi(\Sigma_{g,n}) < 0$ and let $L_1, \ldots, L_n \in \mathbb{R}_{\geq 0}$. Moreover, let $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ be a collection of distinct homotopy classes of essential simple closed curves on $\Sigma_{g,n}$. Define

$$\mathcal{M}_{g,n}(L_1, \ldots, L_n)^\Gamma = \mathcal{T}_{g,n}(L_1, \ldots, L_n) / \operatorname{Stab}_{\mathrm{MCG}(\Sigma_{g,n})}(\Gamma).$$

Here

$$\operatorname{Stab}_{\mathrm{MCG}(\Sigma_{g,n})}(\Gamma) = \bigcap_{i=1}^{k} \operatorname{Stab}_{\mathrm{MCG}(\Sigma_{g,n})}(\gamma_i).$$

Furthermore, we will write

$$\pi^\Gamma : \mathcal{M}_{g,n}(L_1, \ldots, L_n)^\Gamma \to \mathcal{M}_{g,n}(L_1, \ldots, L_n)$$

for the projection map.

In words: in $\mathcal{M}_{g,n}(L_1, \ldots, L_n)^\Gamma$, the boundary components *and* the curves in Γ are still marked. Informally, we can also write

$$\mathcal{M}_{g,n}(L_1, \ldots, L_n)^\Gamma = \left\{ (X, \eta); \begin{array}{c} X \in \mathcal{M}_{g,n}(L_1, \ldots, L_n), \\ \eta \in \mathcal{M}(\Sigma_{g,n}) \cdot \Gamma \text{ realized by closed geodesics} \end{array} \right\}.$$

Note that since ω_{WP} is $\mathrm{MCG}(\Sigma_{g,n})$ invariant, $\mathcal{M}_{g,n}(L_1, \ldots, L_n)^\Gamma$ also comes with a symplectic form. Given $a \in \mathbb{R}_+^k$, we will also define

$$\mathcal{M}_{g,n}(L_1, \ldots, L_n)^\Gamma[a] := \left\{ (X, \eta) \in \mathcal{M}_{g,n}(L_1, \ldots, L_n)^\Gamma; \;\; \ell_{\eta_i}(X) = a_i, \; i = 1, \ldots, k \; \right\}.$$

These level sets come with k natural Hamiltonian flows

$$\phi_i^t : \mathcal{M}_{g,n}(L_1, \ldots, L_n)^\Gamma[a] \to \mathcal{M}_{g,n}(L_1, \ldots, L_n)^\Gamma[a],$$

obtained by twisting around the i^{th} curve in Γ. The fact that this flow is Hamiltonian is direct from Definition 19.2. Note that this flow induces an action of a k-dimensional torus $\mathbb{T}^k$ on $\mathcal{M}_{g,n}(L_1,\ldots,L_n)^\Gamma[a]$. Let us write

$$\mathcal{M}_{g,n}(L_1,\ldots,L_n)^\Gamma[a]^* = \mathcal{M}_{g,n}(L_1,\ldots,L_n)^\Gamma[a]/\mathbb{T}^k.$$

Since the flows are Hamiltonian, they preserve the symplectic form and hence $\mathcal{M}_{g,n}(L_1,\ldots,L_n)^\Gamma[a]^*$ comes with a symplectic form too.

Now note that we have a natural map

$$\mathcal{M}(\Sigma_{g,n}\setminus\Gamma, L_1,\ldots,L_n,a,a)\to\mathcal{M}_{g,n}(L_1,\ldots,L_n)^\Gamma[a]^*$$

obtained by "gluing Γ together". The reason that a appears twice is that every curve in Γ gives rise to two boundary components. This map is

$$\left[\mathrm{Stab}_{\mathrm{MCG}(\Sigma_{g,n})}(\Gamma):\bigcap_{i=1}^k \mathrm{Stab}^+_{\mathrm{MCG}(\Sigma_{g,n})}(\gamma_i)\right]\text{ - to - 1},$$

because of the fact that boundary components are marked. Here $\mathrm{Stab}^+_{\mathrm{MCG}(\Sigma_{g,n})}(\gamma_i)$ denotes the subgroup of the mapping class group that also preserves the left and right hand side of γ_i, or equivalently, an orientation on it.

So, the identification we spoke about in the beginning of this section is:

Lemma 19.4. *The map*

$$\mathcal{M}(\Sigma_{g,n}\setminus\Gamma, L_1,\ldots,L_n,a,a)\to\mathcal{M}_{g,n}(L_1,\ldots,L_n)^\Gamma[a]^*$$

locally preserves $d\,\mathrm{vol}_{\mathrm{WP}}$.

PROOF. This is direct from Definition 19.2. □

19.4. Geometric functions

In this section we will define a specific type of functions on $\mathcal{M}_{g,n}(L_1,\ldots,L_n)$. These will be called *geometric functions*.

Definition 19.5 (Geometric functions)**.** Let $g,n\in\mathbb{N}$ be so that $\chi(\Sigma_{g,n})<0$ and let $L_1,\ldots,L_n\in\mathbb{R}_{\geq 0}$. Moreover, let $\Gamma=\{\gamma_1,\ldots,\gamma_k\}$ be a collection of distinct homotopy classes of essential simple closed curves on $\Sigma_{g,n}$. Finally, let $F:\mathbb{R}_+^k\to\mathbb{R}$ be a function. Then we define

$$F^\Gamma:\mathcal{M}_{g,n}(L_1,\ldots,L_n)\to\mathbb{R}$$

by

$$F^\Gamma(X)=\sum_{(\alpha_1,\ldots,\alpha_k)\in\mathrm{MCG}(\Sigma_{g,n})\cdot\Gamma}F(\ell_{\alpha_1}(X),\ldots,\ell_{\alpha_k}(X)).$$

Note that, even though the individual length functions are not well defined on $\mathcal{M}_{g,n}(L_1,\ldots,L_n)$, the sum above is.

Example 19.6. Consider an non-separating simple closed curve α on $\Sigma_{g,n}$ (like the curve in Figure 1). Moreover, let $\chi_{[a,b]} : \mathbb{R}_+ \to \mathbb{R}$ denote the characteristic function of the interval $[a, b]$. That is,

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in \mathbb{R}$.

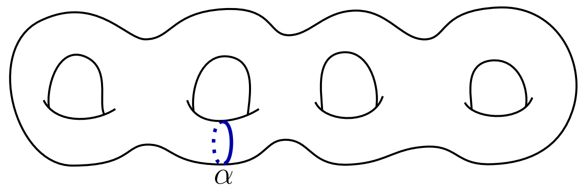

FIGURE 1. A non-separating simple closed curve α

First of all note that the orbit $\mathrm{MCG}(\Sigma_{g,n}) \cdot \alpha$ is the set of all non-separating simple closed curves on $\Sigma_{g,n}$. So

$$\chi^{\alpha}_{[a,b]}(X) = \sum_{\substack{\gamma \text{ a non-separating simple} \\ \text{closed geodesic on } X}} \chi_{[a,b]}(\ell_\gamma(X)) = \# \left\{ \begin{array}{c} \text{non-separating simple} \\ \text{closed geodesics on } X \text{ with} \\ \text{length in } [a,b] \end{array} \right\}$$

for all $X \in \mathcal{M}_{g,n}(L_1, \ldots, L_n)$.

19.5. Integration, part I

We are now ready to prove Mirzakhani's integration formula for geometric functions. We first introduce some terminology. Let $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ be a collection of distinct homotopy classes of essential simple closed curves on $\Sigma_{g,n}$. Then the *symmetry group* of Γ is the group

$$\mathrm{Sym}(\Gamma) = \mathrm{Stab}_{\mathrm{MCG}(\Sigma_{g,n})}(\Gamma) \Big/ \bigcap_{i=1}^{k} \mathrm{Stab}^{+}_{\mathrm{MCG}(\Sigma_{g,n})}(\gamma_i).$$

Moreover, we set

$$M(\Gamma) = \#\{\text{connected components of } \Sigma_{g,n} \setminus \Gamma \text{ that are homeomorphic to } \Sigma_{1,1}\}.$$

We then have

Theorem 19.7 (Mirzakhani's integration formula)**.** *Let $g, n \in \mathbb{N}$ be so that $\chi(\Sigma_{g,n}) < 0$ and let $L \in \mathbb{R}^k_{\geq 0}$. Moreover, let $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ be a collection of distinct homotopy classes of essential simple closed curves on $\Sigma_{g,n}$. Finally, let $F : \mathbb{R}^k_+ \to \mathbb{R}$ be integrable.*

Then

$$\int_{\mathcal{M}_{g,n}(L)} F^{\Gamma}(X) d\,\mathrm{vol}_{\mathrm{WP}}(X)$$

$$= C_{\Gamma} \int_{\mathbb{R}_+^k} F(x)\,\mathrm{vol}_{\mathrm{WP}}(\mathcal{M}(\Sigma_{g,n} \setminus \Gamma, L, x, x) \cdot x_1 \cdot x_2 \cdots x_k dx_1 \cdots dx_k,$$

where C_{Γ} *is a constant. If* $(g,n) \notin \{(1,1),(2,0)\}$ *then*

$$C_{\Gamma} = \frac{1}{2^{M(\Gamma)} \# \mathrm{Sym}(\Gamma)}.$$

19.6. Exercises

Exercise 19.1. Prove that $\mathcal{M}_g$ is not compact. *Hint: consider sequences of hyperbolic surfaces with shorter and shorter closed geodesics on them.*

Exercise 19.2. Show that there are finitely many $\mathrm{MCG}(\Sigma_g)$-orbits of (topological) pants decompositions of Σ_g. *Hint: find a bijection between* $\mathrm{MCG}(\Sigma_g)$*-orbits of pants decompositions of* Σ_g *and trivalent graphs on* $2g-2$ *vertices.*

LECTURE 20

Integration and the McShane-Mirzakhani identity

20.1. Integration, part II

Our first goal is to prove Mirzakhani's integration formula:

Theorem 19.7 (Mirzakhani's integration formula). *Let $g, n \in \mathbb{N}$ be so that $\chi(\Sigma_{g,n}) < 0$ and let $L \in \mathbb{R}^k_{\geq 0}$. Moreover, let $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ be a collection of distinct homotopy classes of essential simple closed curves on $\Sigma_{g,n}$. Finally, let $F : \mathbb{R}^k_+ \to \mathbb{R}$ be integrable. Then*

$$\int_{\mathcal{M}_{g,n}(L)} F^\Gamma(X) d\,\mathrm{vol}_{\mathrm{WP}}(X)$$

$$= C_\Gamma \int_{\mathbb{R}^k_+} F(x)\,\mathrm{vol}_{\mathrm{WP}}(\mathcal{M}(\Sigma_{g,n} \setminus \Gamma, L, x, x) \cdot x_1 \cdot x_2 \cdots x_k dx_1 \cdots dx_k,$$

where

$$\mathrm{vol}_{\mathrm{WP}}(\mathcal{M}_{0,3}(L_1, L_2, L_3)) = 1$$

for all $L_1, L_2, L_3 \geq 0$ and C_Γ is a constant. If $(g, n) \notin \{(1,1), (2,0)\}$ then

$$C_\Gamma = \frac{1}{2^{M(\Gamma)} \#\mathrm{Sym}(\Gamma)}.$$

Proof. First of all, let us define a function

$$\widehat{F} : \mathcal{M}_{g,n}(L)^\Gamma \to \mathbb{R}$$

by

$$\widehat{F}(X, \eta) = F(\ell_{\eta_1}(X), \ldots, \ell_{\eta_k}(X))$$

for all $(X, \eta) \in \mathcal{M}_{g,n}(L)$. We then have

$$F^\Gamma(X) = \sum_{(X,\eta) \in (\pi^\Gamma)^{-1}(X)} \widehat{F}(X, \eta)$$

and hence

$$\begin{aligned}\int\limits_{\mathcal{M}_{g,n}(L)} F^\Gamma(X) d\operatorname{vol}_{\mathrm{WP}}(X) &= \int\limits_{\mathcal{M}_{g,n}(L)} \sum_{(X,\eta)\in(\pi^\Gamma)^{-1}(X)} \widehat{F}(X,\eta) d\operatorname{vol}_{\mathrm{WP}}(X) \\ &= \int\limits_{\mathcal{M}_{g,n}(L)^\Gamma} \widehat{F}(X,\eta) d\operatorname{vol}_{\mathrm{WP}}(X,\eta) \\ &= \int_{\mathbb{R}_+^k} \int\limits_{\mathcal{M}_{g,n}(L)^\Gamma[a]} \widehat{F}(a_1,\dots a_k) d\operatorname{vol}_{\mathrm{WP}}(X,\eta) da_1\cdots da_k \\ &= \int_{\mathbb{R}_+^k} \widehat{F}(a_1,\dots a_k) \operatorname{vol}_{\mathrm{WP}}(\mathcal{M}_{g,n}(L)^\Gamma[a]) da_1\cdots da_k.\end{aligned}$$

Now we claim that

$$\operatorname{vol}_{\mathrm{WP}}(\mathcal{M}_{g,n}(L)^\Gamma[a]) = 2^{-M(\Gamma)} a_1\cdots a_k \cdot \operatorname{vol}_{\mathrm{WP}}(\mathcal{M}_{g,n}(L)^\Gamma[a]^*).$$

The reason for this is that generically,

$$\varphi_i^t(X,\eta) \neq (X,\eta) \text{ for all } 0 < t < \ell_{\gamma_i}(X).$$

On the other hand, this does describe the whole fibre of the quotient map

$$\mathcal{M}_{g,n}(L)^\Gamma[a] \to \mathcal{M}_{g,n}(L)^\Gamma[a]^*.$$

The only exception is if γ_i cuts off a one-holed torus. Every one holed torus has an order 2 symmetry. As such we only need to go up to $a_i/2$ in order to describe the whole fibre. Note that if $\Sigma \setminus \Gamma$ consists of pairs of pants, then the only deformations of the elements in $\mathcal{M}_{g,n}(L)^\Gamma[a]$ are twists along the curves in Γ. As such, we need to set

$$\operatorname{vol}_{\mathrm{WP}}(\mathcal{M}_{0,3}(L_1,L_2,L_3) = 1$$

in order to make our claim work.

All in all we get

$$\begin{aligned}&\int\limits_{\mathcal{M}_{g,n}(L)} F^\Gamma(X) d\operatorname{vol}_{\mathrm{WP}}(X) \\ &\quad = 2^{-M(\Gamma)} \int_{\mathbb{R}_+^k} \widehat{F}(a_1,\dots a_k) \operatorname{vol}_{\mathrm{WP}}(\mathcal{M}_{g,n}(L)^\Gamma[a]^*) a_1\cdots a_k \cdot da_1\cdots da_k \\ &= \frac{1}{2^{M(\Gamma)}\operatorname{Sym}(\Gamma)} \int_{\mathbb{R}_+^k} \widehat{F}(a_1,\dots a_k) \operatorname{vol}_{\mathrm{WP}}(\mathcal{M}(\Sigma_{g,n}\setminus\Gamma, L, a, a)) a_1\cdots a_k \cdot da_1\cdots da_k,\end{aligned}$$

using Lemma 19.4. □

20.2. The McShane-Mirzakhani identity

The previous section tells us how to integrate geometric functions over $\mathcal{M}_{g,n}(L)$. However, in order to compute the volume of $\mathcal{M}_{g,n}(L)$, we need to integrate a constant function over $\mathcal{M}_{g,n}(L)$. So, we need to find a constant function that can be expressed as a geometric function, i.e. an identity that holds for *all* hyperbolic surfaces in $\mathcal{M}_{g,n}(L)$.

First, we define two functions $\mathcal{D}, \mathcal{R} : \mathbb{R}^3_{\geq 0} \to \mathbb{R}$ by

$$\mathcal{D}(x,y,z) = 2\log\left(\frac{e^{\frac{x}{2}} + e^{\frac{y+z}{2}}}{e^{-\frac{x}{2}} + e^{\frac{y+z}{2}}}\right) \quad \text{and} \quad \mathcal{R}(x,y,z) = x - \log\left(\frac{\cosh(\frac{y}{2}) + \cosh(\frac{x+z}{2})}{\cosh(\frac{y}{2}) + \cosh(\frac{x-z}{2})}\right),$$

for all $(x,y,z) \in \mathbb{R}^3_{\geq 0}$. Moreover, since the boundary components are marked in everything that we do, we will give them names: $\beta_1, \ldots, \beta_n$. Finally, we define sets of (tuples of) curves on $\Sigma_{g,n}$:

$$\mathcal{F}_0 = \left\{ \{\alpha_1, \alpha_2\}; \begin{array}{c} \alpha_i \text{ a homotopy class of essential simple closed curves} \\ \text{on } \Sigma_{g,n},\ i = 1,2, \alpha_1 \neq \alpha_2 \text{ s.t. } \beta_1, \alpha_1 \text{ and } \alpha_2 \text{ bound a} \\ \text{pair of pants together} \end{array} \right\}$$

and

$$\mathcal{F}_j = \left\{ \alpha; \begin{array}{c} \alpha \text{ a homotopy class of essential simple closed curves on } \Sigma_{g,n}, \\ \text{s.t. } \beta_1, \beta_j \text{ and } \alpha \text{ bound a pair of pants together} \end{array} \right\}.$$

Note that these sets consist of finitely many $\mathrm{MCG}(\Sigma_{g,n})$-orbits of (tuples of) curves. Figures 1 and 2 show examples of elements of these sets.

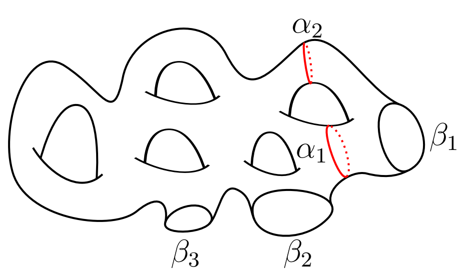

FIGURE 1. A pair $(\alpha_1, \alpha_2) \in \mathcal{F}_0$

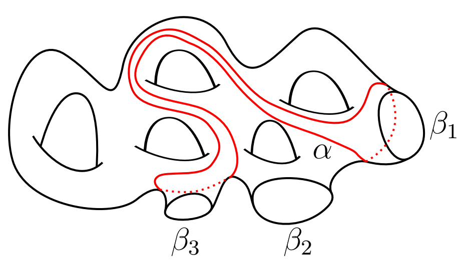

FIGURE 2. An element $\alpha \in \mathcal{F}_3$

The identity that we will prove is:

Theorem 20.1 (McShane-Mirzakhani identity)**.** *Let $g, n \in \mathbb{N}$ so that $n > 0$ and $3g + n - 3 > 0$, $L \in \mathbb{R}^n_{\geq 0}$ and $X \in \mathcal{M}_{g,n}(L)$. Then*

$$\sum_{\{\alpha_1,\alpha_2\}\in\mathcal{F}_0} \mathcal{D}(L_1, \ell_\alpha(X), \ell_\beta(X)) + \sum_{j=2}^{n} \sum_{\alpha\in\mathcal{F}_j} \mathcal{R}(L_1, L_j, \ell_\alpha(X)) = L_1.$$

Before we prove this theorem, we note that it leads to a recursive formula for the volumes of moduli spaces of surfaces with boundary. Let us write

$$V_{g,n}(L) = \mathrm{vol}_{\mathrm{WP}}(\mathcal{M}_{g,n}(L))$$

Corollary 20.2 (Volume recursion)**.** *We have*

$$\begin{aligned} L_1 \cdot V_{g,n}(L) = \int_0^\infty \int_0^\infty \mathcal{D}(L_1, x, y) \cdot V_{g-1,n+1}(L_2, \ldots, L_n, x, y) \cdot x \cdot y \; dxdy \\ + \sum_{\substack{J_1 \sqcup J_2 = \{2,\ldots,n\} \\ g_1+g_2=g}} \int_0^\infty \int_0^\infty \mathcal{D}(L_1, x, y) \cdot V_{g_1,|J_1|+1}(L_{J_1}, x) \cdot V_{g_2,|J_2|+1}(L_{J_2}, x) \cdot x \cdot y \; dxdy \\ + \sum_{j=2}^{n} \int_0^\infty \mathcal{R}(L_1, L_j, x) \cdot V_{g,n-1}(L_2, \ldots, \widehat{L_j}, \ldots, L_n, y) \cdot x \; dx, \end{aligned}$$

where L_{J_i} denotes the vector $(L_j)_{j \in J_i}$.

Proof. The only thing to worry about is that the first sum in Theorem 20.1 is over unordered pairs, whereas mapping class group orbits are sums over ordered pairs. So, we need to include a factor 2. On the other hand $C_\Gamma = \frac{1}{2}$, so these factors cancel each other. ☐

20.2.1. The idea of the proof. The idea of the proof is to decompose β_1 as follows

$$\beta_1 = E \sqcup \left(\bigsqcup_{(\alpha_1,\alpha_2) \in \mathcal{F}_0} I_{0,\alpha_1,\alpha_2} \right) \sqcup \left(\bigsqcup_{j=2}^{n} \bigsqcup_{\alpha \in \mathcal{F}_j} I_{j,\alpha} \right)$$

where the sets I_{0,α_1,α_2}, $I_{j,\alpha}$ are (unions of) intervals and E is a set of measure 0. So what we then get is

$$L_1 = \sum_{(\alpha_1,\alpha_2) \in \mathcal{F}_0} \ell(I_{0,\alpha_1,\alpha_2}) + \sum_{j=2}^{n} \sum_{\alpha \in \mathcal{F}_j} \ell(I_{j,\alpha}). \tag{3}$$

The unions of intervals and E are constructed by the following process. For $p \in \beta_1$, let ν_p denote the inward pointing unit tangent vector orthogonal to β_1. This allows us to define a geodesic

$$\gamma_p : [0, T_p] :\to X$$

that satisfies

$$\gamma_p(0) = p \quad \text{and} \quad \frac{d\gamma(t)}{dt}|_{t=0} = \nu_p.$$

Here $T_p \in \mathbb{R}_+ \cup \{\infty\}$ is the largest possible time for which this geodesic is defined (it might hit the boundary of the surface again).

Now there are multiple options:

(1) $T_p = \infty$ and γ_p is simple. In that case, $p \in E$,

(2) $T_p < \infty$, γ_p is simple and $\gamma_p(T_p) \in \beta_1$. In that case, we consider a closed regular neighborhood N of

$$\gamma_p([0, T_p]) \cup \beta_1.$$

The boundary of N consists of two simple closed curves α_1 and α_2 that cut out a pair of pants together with β_1. In this case, $p \in I_{0,\alpha_1,\alpha_2}$.

(3) $T_p < \infty$, γ_p is simple and $\gamma_p(T_p) \in \beta_j$ for some $j \neq 1$. In that case, consider a closed regular neighborhood N of

$$\gamma_p([0, T_p]) \cup \beta_1 \cup \beta_j.$$

The boundary of N consists of a simple closed curve α that cuts out a pair of pants together with β_1 and β_j. In this case, $p \in I_{j,\alpha}$.

(4) γ_p is not simple. In this case, let t_p denote the time at which γ_p first intersects itself. Let N denote a regular neighborhood of

$$\gamma_p([0, t_p]) \cup \beta_1$$

The boundary of N consists of two simple closed curves α_1 and α_2 that cut out a pair of pants together with β_1. In this case, $p \in I_{0,\alpha_1,\alpha_2}$.

Figures 3 and 4 show the third and fourth case above.

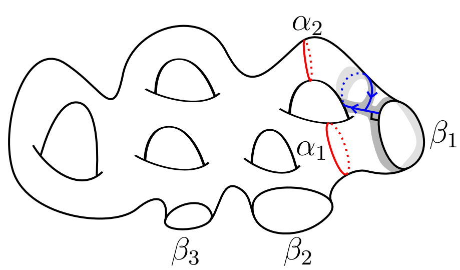

FIGURE 3. A pair $(\alpha_1, \alpha_2) \in \mathcal{F}_0$

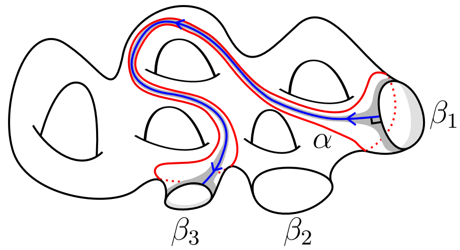

FIGURE 4. An element $\alpha \in \mathcal{F}_3$

Intuitively, the reason that the sets I_{0,α_1,α_2} and $I_{j,\alpha}$ are unions of intervals is that if we move the base point $p \in \beta_1$ for the geodesic a little bit to a new point p', then γ_p and $\gamma_{p'}$ stay parallel long enough so that the process leads to the same set of curves.

So, proving the theorem consists of two parts:

(1) Proving that the lengths of the intervals I_{0,α_1,α_2} and $I_{j,\alpha}$ are given by the functions $\mathcal{D}$ and $\mathcal{R}$ respectively. That is, we have to determine the contribution of each pair of pants.
(2) Proving that E has measure 0.

20.2.2. Geodesics on a pair of pants. Our first goal is to determine the lengths of the intervals described above. To this end, consider a hyperbolic pair of pants P with boundary components $\beta_1, \beta_2, \beta_3$ of lengths x_1, x_2 and x_3 respectively. Moreover, let $\widetilde{P} \subset \mathbb{H}^2$ denote the Riemannian universal cover of P. Figure 5 shows an example.

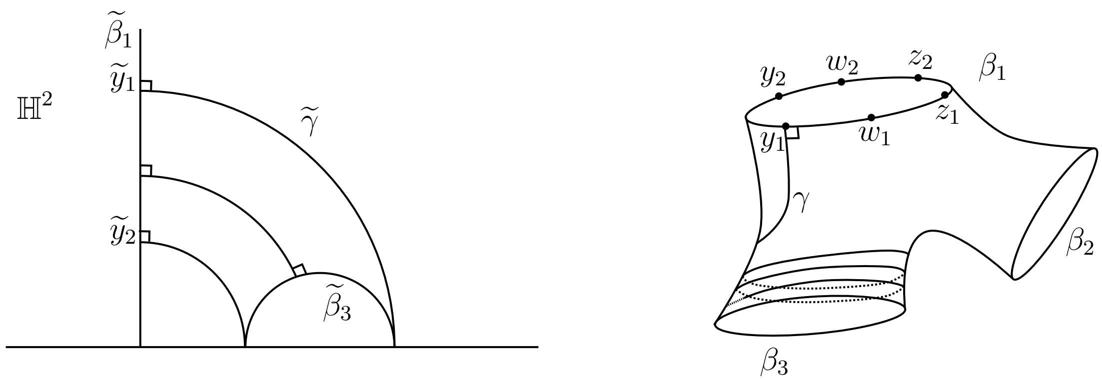

FIGURE 5. A pair of pants and a part of its universal cover

Now consider lifts $\widetilde{\beta}_1$ and $\widetilde{\beta}_3$ of β_1 and β_3 that realize the distance between β_1 and β_3.

The orthogonal projection of $\widetilde{\beta}_3$ to $\widetilde{\beta}_1$ (drawn on the left in Figure 5) determines two points $\widetilde{y}_1$ and $\widetilde{y}_2$ on $\widetilde{\beta}_1$: the endpoints of this projection. The geodesic $\widetilde{\gamma}$ that runs between $\widetilde{y}_1$ and the corresponding endpoint of $\widetilde{\beta}$ is asymptotic to $\widetilde{\beta}_3$ and hence projects to a simple geodesic based at some point $y_1 \in \beta_1$ that spirals around β_3.

In total, this gets us four points on β_1: y_1 and y_2 coming from the projection of $\widetilde{\beta}_3$ to $\widetilde{\beta}_1$ and z_1 and z_2 coming from the projection of $\widetilde{\beta}_2$ to $\widetilde{\beta}_1$.

We define two more points $w_1, w_2 \in \beta_1$ to be the points where the unique orthogonal between β_1 and itself is realized.

We now claim the following:

Lemma 20.3. *Using the notation from above:*

(1) $\mathcal{R}(x_1, x_2, x_3)$ *is the length of the subsegment of* β_1 *between* y_1 *and* y_2 *and containing* w_1 *and* w_2*.*
(2) $\mathcal{D}(x_1, x_2, x_3)$ *is the sum of the lengths of the segments between* y_i *and* z_i *containing* w_i*,* $i = 1, 2$*.*

PROOF. Exercise 20.1. □

20.2.3. The set of simple geodesics. Now it's time to formalize the decomposition of the boundary component and the identification with embedded pairs of pants. First, let us once and for all fix a surface $X \in \mathcal{M}_{g,n}(L)$. We write

$$E(X) = \bigcup_{\substack{\gamma \text{ a complete simple geodesic} \\ \text{on } X \text{ that meets } \partial X \\ \text{orthogonally once or twice}}} \gamma \quad \subset X$$

and

$$E_i = E(X) \cap \beta_i.$$

Our first claim is that these sets have measure 0:

Lemma 20.4. *E_i has measure 0 for all $i = 1, \ldots, n$.*

PROOF SKETCH. We first claim that $E(X)$ has measure 0 as a subset of X. This follows from a result due to Birman and Series [**BS85**]. Birman and Series prove that the set of all simple geodesics on a closed hyperbolic surface Y has measure 0. If we double X along its boundary, then we obtain a closed surface. Moreover, all the geodesics in $E(X)$ remain simple geodesics (note that this uses that they are orthogonal to the boundary). So $E(X)$ has measure 0 in Y and hence in X.

Now consider a collar neighborhood

$$U_r^{(i)} = \left\{ y \in Y; \quad \mathrm{d}(y, \beta_i) < r \ \right\}$$

of β_i in Y. We have

$$\mu_2(E(X) \cap U_i) \sim 2r \cdot \mu_1(E_i)$$

as $r \to 0$, where μ_2 denotes the area measure on Y and μ_1 the length measure on β_i. Since the left hand side is equal to 0, we must have

$$\mu_1(E_i) = 0.$$

□

20.2.4. The proof. Now we are ready to prove the identity:

PROOF OF THEOREM 20.1. Lemma 20.4 implies that we just need to figure out how much length each pair of pants contributes to L_1.

So, consider a pair of pants formed by β_1, β_j and some simple closed geodesic α. Again look at Figure 5 with $\beta_2 = \beta_j$ and $\beta_3 = \alpha$. Then $p \in I_{j,\alpha}$ if and only if p lies between y_1 and y_2, on the side that also contains w_1 and w_2.

Likewise for I_{0,α_1,α_2}, we consider Figure 5 with $\beta_2 = \beta_1$ and $\beta_3 = \alpha_2$. Then $p \in I_{0,\alpha_1,\alpha_2}$ if and only if p lies either between y_1 and z_1, on the side that contains w_1 or between y_2 and z_2, on the side that contains w_2. □

20.3. Exercises

Exercise 20.1. Prove Lemma 20.3.

LECTURE 21

Applications of Mirzakhani's volume recursion

Corollary 20.2 has the following applications:

(1) The volume of $\mathcal{M}_{g,n}(L)$ is a polynomial in L [**Mir07a**].
(2) A new proof of the Witten conjecture on the intersection theory of $\mathcal{M}_g$ [**Mir07b**].
(3) The number of simple closed curves of length at most L on a hyperbolic surface of finite type is asymptotic to a polynomial in L [**Mir08**].
(4) The large genus asymptote of the Weil-Petersson volume of $\mathcal{M}_g$ [**MZ15**].
(5) Because the Weil-Petersson volume of $\mathcal{M}_g$ is finite, we can use the volume form to pick a point in $\mathcal{M}_g$ at random. That is, we define a probability measure by

$$\mathrm{Prob}_g[A] := \frac{\mathrm{vol}_{\mathrm{WP}}(A)}{\mathrm{vol}_{\mathrm{WP}}(\mathcal{M}_g)}, \quad A \subseteq \mathcal{M}_g \text{ measurable.}$$

In [**Mir13**], Mirzakhani studied the shape of a "typical" hyperbolic surface of large genus.

In this final lecture, we will discuss the first and third application. This lecture is intended as an outlook and nothing, except the computation of $V_{1,1}$ below, is part of the exam material.

21.1. Weil-Petersson volumes are polynomial in the boundary lengths

We start with an example, taken from [**Mir07a**]. Using the methods from the previous lecture, we can explicitly compute the volume of $\mathcal{M}_{1,1}$. To this end we need the classical McShane identity:

Theorem 21.1 (McShane identity)**.** [**McS98**] *Let X be a once-puntured torus, equipped with a complete hyperbolic metric. Then*

$$\sum_{\substack{\gamma \text{ a simple closed} \\ \text{geodesic on} X}} \frac{1}{1+\exp(\ell_\gamma(X))} = \frac{1}{2}.$$

This theorem can be derived from Theorem 20.1. However, historically, this was the version that was proved first, with a very similar proof.

In order to calculate $V_{1,1} = V_{1,1}(0)$, we proceed as in the proof of Corollary 20.2. So, we apply Theorem 19.7 and get:

$$\frac{1}{2}V_{1,1} = \int_0^\infty \frac{1}{1+\exp(x)} V_{0,3}(x,x,0)\, x\, dx = \int_0^\infty \frac{x}{1+\exp(x)} dx = \frac{\pi^2}{12}.$$

So

$$V_{1,1} = \frac{\pi^2}{6}.$$

In [**Mir07a**], the above is proved without appealing directly to Theorem 19.7. Instead, Mirzakhani proves this with a simplified version of the proof of that theorem.

The general statement of the polynomiality of Weil-Petersson volumes is [**Mir07a**, Theorem 6.1]:

Theorem 21.2 (Mirzakhani)**.** *The function $V_{g,n}(L)$ is a polynomial in $L_1^2, \ldots, L_n^2$ of the form*

$$V_{g,n}(L) = \sum_{\substack{d_1,\ldots,d_n \geq 0 \\ d_1+\ldots+d_n \leq 3g-3+n}} C_{d_1,\ldots,d_n} \cdot L_1^{2d_1} \cdots L_n^{2d_n}$$

where

$$C_{d_1,\ldots,d_n} \in \pi^{6g-6+2n-2d_1-\ldots-2d_n}\, \mathbb{Q}_{>0},$$

for all $d_1, \ldots, d_n \geq 0$ so that $\sum_i d_i \leq 3g-3+n$.

Proof idea. The proof is inductive and based on a recurrence derived from Corollary 20.2. □

21.2. The number of short simple closed curves

For a hyperbolic surface X and $L > 0$, set

$$c_X(L) = \# \left\{ \begin{array}{c} \gamma \text{ a closed geodesic on } X \\ \text{of length } \leq L \end{array} \right\},$$

where we do not distinguish between geodesics γ and γ' if one can be obtained by reparametrizing the other, but do distinguish between two geodesics with opposite orientations.

A classical result, originally due to Huber [**Hub56**], states that on a closed hyperbolic surface[1]

$$c_X(L) \sim \frac{e^L}{L} \quad \text{as } L \to \infty.$$

This is a somewhat surprising result, because it is completely independent of the geometry and topology of X.

Mirzakhani used her methods to study the number of *simple* closed geodesics. We set

$$s_X(L) = \# \left\{ \begin{array}{c} \gamma \text{ a simple closed geodesic} \\ \text{on } X \text{ of length } \leq L \end{array} \right\}.$$

In the function s_X, we will *not* distinguish between different orientations on our closed geodesics. This is just because this fits better with our methods. In any event, the difference between the two functions is a factor 2.

It turns out that the number of simple geodesics grows much slower than all geodesics [**Mir08**, Theorem 1.1]:

Theorem 21.3. *Let $X \in \mathcal{M}_{g,n}$ be a hyperbolic surface. Then there exists a constant $b_X > 0$ so that*

$$s_X(L) \sim b_X \cdot L^{6g-6+2n} \quad \textit{as } L \to \infty.$$

The proof of this theorem is based on the convergence of a sequence of measures on the space of measured geodesic laminations on $\Sigma_{g,n}$.

[1] Recall that the notation "$f(x) \sim g(x)$ as $x \to \infty$" means that $\lim_{x\to\infty} f(x)/g(x) = 1$.

21.2.1. Measured laminations. We will very briefly sketch what measured laminations and their key properties are. For a more background, we refer to [**CB88, PH92, AL17**].

A *lamination* on a hyperbolic surface X is a closed subset that is a disjoint union of complete simple geodesics that do not intersect each other.

One example of a geodesic lamination is a collection of disjoint simple closed geodesics (a *multicurve*), like for instance a pants decomposition. Figure 1 shows another example.

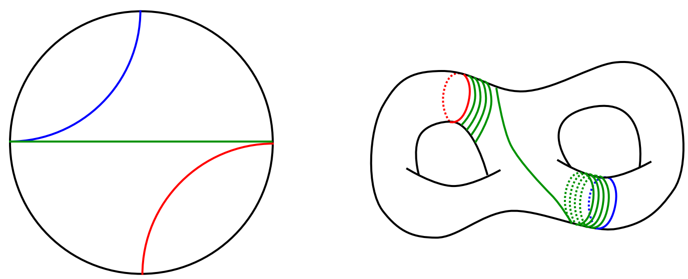

FIGURE 1. A lamination consisting of two simple closed geodesics and one geodesic arc. The left hand side is consists of three lifts of the components to $\mathbb{H}^2$

A *measured geodesic lamination* on X is a geodesic lamination on X, together with an assignment

$$\alpha \mapsto \lambda_\alpha$$

that assigns a Radon measure λ_α so each arc α that is transverse to the lamination. This assignment has to satisfy the following conditions:

- Subarcs: If $\alpha' \subset \alpha$ is a subarc, then $\lambda_{\alpha'}$ is the restriction of λ_α to α'
- Sliding: If α and α' are homotopic via a homotopy F_t so that $F_1 : \alpha \to \alpha'$ is a homeomorphism and $F_t(\alpha)$ is transverse to the leaves of the lamination for all t, then

$$\lambda_{\alpha'} = (F_1)_* \lambda_\alpha.$$

Note that it follows that the support of the measure λ_α is contained in the intersection of α with the lamination.

The set of measured laminations $\mathcal{ML}(X)$ can be topologized using weak star convergence. It turns out that this topology does not depend on the geometry of X. So we will often write $\mathcal{ML}_{g,n}$ for the space of measured laminations. There are coordinates, called Dehn-Thurston coordinates, that are somewhat similar to Fenchel-Nielsen coordinates and that provide a homeomorphism

$$\mathcal{ML}_{g,n} \to \mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n}.$$

Unlike Fenchel-Nielssen coordinates, these coordinates do however not give $\mathcal{ML}_{g,n}$ the structure of a smooth manifold.

Note that $\mathbb{R}_+$ acts on $\mathcal{ML}_{g,n}$, by multiplying the measures with a scalar. $\mathrm{MCG}(\Sigma_{g,n})$ acts on $\mathcal{ML}_{g,n}$ as well.

Every multicurve induces a measured lamination. The lamination is a multicurve and the measure assigned to an arc α is the sum Dirac masses on the intersections of α and the multicurve. We will denote this element of $\mathcal{ML}_{g,n}$ by

$$\sum_i \gamma_i \in \mathcal{ML}_{g,n},$$

where $\{\gamma_i\}_i$ are the geodesics on X that form the multicurve.

For us, the most important example of a measured lamination is a weighted multicurve, i.e. we scale the measures on arcs transverse to γ_i with a number $t_i \in \mathbb{R}_+$. These measured laminations will be denoted

$$\sum_i t_i\gamma_i \in \mathcal{ML}_{g,n}.$$

It turns out that the set of such measures is actually dense in $\mathcal{ML}_{g,n}$.

The length function on geodesics can naturally be extended to weighted multicurves, by setting

$$\ell(\sum_i t_i\gamma_i) = \sum_i t_i\ell(\gamma_i).$$

It turns out that ℓ extends to a continuous function

$$\ell : \mathcal{ML}_{g,n} \times \mathcal{T}_{g,n} \to \mathbb{R}_+$$

that satisfies:

- For any simple closed curve γ
$$\ell(\gamma, X) = \ell_\gamma(X).$$
- For all $t \in \mathbb{R}_+$, $\lambda \in \mathcal{ML}_{g,n}$, $X \in \mathcal{T}_{g,n}$
$$\ell(t\lambda, X) = t\ell(\lambda, X).$$
- For all $\varphi \in \mathrm{MCG}(\Sigma_{g,n})$, $\lambda \in \mathcal{ML}_{g,n}$
$$\ell(\varphi\lambda, \varphi X) = \ell(\lambda, X)$$

Finally, $\mathcal{ML}_{g,n}$ can be equipped with a $\mathrm{MCG}(\Sigma_{g,n})$-invariant measure μ_{Th} called the *Thurston measure*. This measure satisfies

$$\mu_{\mathrm{Th}}(tU) = t^{6g-6+2n}\mu_{\mathrm{Th}}(U)$$

and [**Mas85**]:

Theorem 21.4 (Masur)**.** *Suppose ν is a measure on $\mathcal{ML}_{g,n}$ that is absolutely continuous*[2] *with respect to μ_{Th} and $\mathrm{MCG}(\Sigma_{g,n})$-invariant. Then ν is a multiple of μ_{Th}.*

[2]recall that a measure λ is absolutely continuous with repsect to another measure ν if $\nu(A) = 0$ implies that $\lambda(A) = 0$ for all measurable sets A.

21.2.2. The proof. The proof of Theorem 21.3 now goes as follows. Once and for all fix a simple closed curve γ on $\Sigma_{g,n}$ and for all $L > 0$, define a measure $\mu_{L,\gamma}$ on $\mathcal{ML}_{g,n}$ by:

$$\mu_{L,\gamma} = \frac{1}{L^{6g-6+2n}} \sum_{\alpha \in \mathrm{MCG}(\Sigma_{g,n})\cdot\gamma} \delta_{\frac{1}{L}\gamma},$$

where for $\lambda \in \mathcal{ML}_{g,n}$, δ_λ denotes the Dirac mass on λ.

For $X \in \mathcal{T}_{g,n}$, write

$$B_X = \left\{ \lambda \in \mathcal{ML}_{g,n}; \quad \ell(\lambda, X) \leq 1 \right\}$$

We have

$$\begin{aligned} \mu_{L,\gamma}(B_X) = \frac{1}{L^{6g-6+2n}} \sum_{\alpha \in \mathrm{MCG}(\Sigma_{g,n})\cdot\gamma} \mathbf{1}_{\frac{1}{L}\gamma \text{ has length } \leq 1} \\ = \frac{1}{L^{6g-6+2n}} \sum_{\alpha \in \mathrm{MCG}(\Sigma_{g,n})\cdot\gamma} \mathbf{1}_{\gamma \text{ has length } \leq L} \\ =: \frac{1}{L^{6g-6+2n}}\ s_X(L,\gamma). \end{aligned}$$

There are finitely many mapping class group orbits of simple closed curves on $\Sigma_{g,n}$, which means that $s_X(L)$ splits as a finite sum of measures of the form above.

So, if we can prove that, as $L \to \infty$ the measures $\mu_{L,\gamma}$ converge to fixed measures μ_γ, we would get

$$s_X(L,\gamma) \sim \mu_\gamma(B_X) L^{6g-6+2n}$$

as $L \to \infty$. Then, since $s_X(L)$ is a finite sum of the $s_X(L,\gamma)$'s, we would be done.

The proof of this goes as follows.

Step 1: One proves that there exists a constant $C(X,\gamma) > 0$ so that

$$\mu_{L,\gamma}(B_X) = \frac{s_X(L,\gamma)}{L^{6g-6+2n}} \leq C(X,\gamma).$$

Since any compact set K lies in $L \cdot B_X$ for L large enough, this proves that for any compact set K,

$$\mu_{L,\gamma}(K)$$

is bounded as a function of L and as such (by Banach-Alaoglu) there is a subsequential limit

$$\mu_{L_n,\gamma} \to \nu.$$

Step 2: We need to prove that all such limits are multiples of the Thurston measure. It is not so hard to see that the limit is $\mathrm{MCG}(\Sigma_{g,n})$-invariant. So, we need to show that it is absolutely continuous with respect to μ_{Th}, so that we can invoke Masur's theorem (Theorem 21.4). For this, we refer to [**Mir08**].

Step 3: Now that we know all our subsequential limits are multiples of the Thurston measure, we need to prove that *which* multiple it is, does not depend on the subsequence. This is where Weil-Petersson volumes come in.

Let $(L_k)_k$ be so that

$$\mu_{L_k,\gamma} \to R_{(L_k)_k} \mu_{\mathrm{Th}},$$

as $n \to \infty$. We have

$$\begin{aligned}
R_{(L_k)_k} \int_{\mathcal{M}_{g,n}} \mu_{\mathrm{Th}}(B_X) d\,\mathrm{vol}_{\mathrm{WP}}(X) &= \int_{\mathcal{M}_{g,n}} \lim_{k\to\infty} \mu_{L_k,\gamma}(B_X) d\,\mathrm{vol}_{\mathrm{WP}}(X) \\
&= \int_{\mathcal{M}_{g,n}} \lim_{k\to\infty} \frac{s_X(L_k,\gamma)}{L_n^{6g-6+2n}} d\,\mathrm{vol}_{\mathrm{WP}}(X) \\
&= \lim_{n\to\infty} \int_{\mathcal{M}_{g,n}} \frac{s_X(L_k,\gamma)}{L_n^{6g-6+2n}} d\,\mathrm{vol}_{\mathrm{WP}}(X) \\
&= \lim_{k\to\infty} \int_0^{L_k} \frac{1}{L_k^{6g-6+2n}} \mathrm{vol}_{\mathrm{WP}}(\mathcal{M}(\Sigma \setminus \gamma, x, x))\; x\; d\,\mathrm{vol}_{\mathrm{WP}}(X).
\end{aligned}$$

Now we use that $\mathrm{vol}_{\mathrm{WP}}(\mathcal{M}(\Sigma \setminus \gamma, x, x))$ is a polynomial (Theorem 21.2) in x and see that the right hand side converges to a constant that is independent of the sequence L_k. The integral

$$\int_{\mathcal{M}_{g,n}} \mu_{\mathrm{Th}}(B_X) d\,\mathrm{vol}_{\mathrm{WP}}(X)$$

also doesn't depend on the sequence, so neither can $R_{(L_k)_k}$, which finishes the proof.

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