8.1. (This is left over from the previous sheet) A line bundle $L \to M$ on a complex manifold is *positive* if it admits a Hermitian metric with positive curvature form. Let M be a closed (compact without boundary) complex manifold which admits a positive line bundle. Let $E \to M$ be a holomorphic vector bundle of rank at least two. Show that the bundle $N \to M$ whose fibre over x equals the projectivization of the fibre of E over x admits a positive line bundle.

8.2. Let M be a closed Kähler manifold and let E, E' holomorphic vector bundles over M of rank k, k' at least two.

(a) Show that the direct sum $E \oplus E'$ is a holomorphic vector bundle.

(b) Show that the bundle $N \to M$ whose fibre over x equals the product $\mathbb{P}(E_x) \times \mathbb{P}(E'_x)$ (the direct product of the projectivizations of the fibres of E, E' at x) is a closed complex manifold.

(c) Show that the manifold N as in (b) admits a Kähler metric. Can one find uncountably many pairwise non-isometric Kähler metrics?

8.3. Let X and Y be complex manifolds and let $f : X \to Y$ be a holomorphic map.

(a) Prove that if α is a (p, q)-form on Y then $f^*\alpha$ is a (p, q)-form on X.

(b) Use this to show that the map

$$f^* : H^{p,q}_{\bar{\partial}}(Y) \to H^{p,q}_{\bar{\partial}}(X)$$

defined by

$$f^*[\alpha] = [f^*\alpha]$$

is a homomorphism.

8.4. Define $\Delta^* \subset \mathbb{C}$ by

$$\Delta^* = \{z \in \mathbb{C};\ 0 < |z| < 1\}.$$

Show that for all $p \geq 0$, $q \geq 1$ we have

$$H^{p,q}_{\bar{\partial}}(\Delta^*) = 0.$$