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# Hodge Structures, Coniveau and Algebraic Cycles

Claire Voisin*

**Abstract**

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## 0 Introduction

Let $C$ be a smooth connected projective curve over $\mathbb{C}$ or, from the viewpoint of analytic geometry, a compact Riemann surface. Then if $g$ is the genus of $C$, the space $H^{1,0}(C) = H^0(C, \Omega_C)$ of holomorphic differentials on $C$ is contained in the Betti cohomology $H^1_B(C, \mathbb{C})$ of $C$ (that is, of the corresponding Riemann surface).

*Institut de Mathématiques de Jussieu, Équipe Topologie et géométrie algébriques, 4, place Jussieu 75252 Paris Cedex 05 France. E-mail: claire.voisin@imj-prg.fr

Indeed, a holomorphic 1-form $\eta$ which is exact has to be identically 0 since then $\int_C \iota\eta \wedge \overline{\eta} = 0$ while the form $\iota\eta \wedge \overline{\eta}$ is positive on $C$ away from the zeroes of $\eta$. Furthermore, one has the following decomposition of the cohomology of $C$ with complex coefficients:

$$H^1_B(C,\mathbb{C}) = H^{1,0}(C) \oplus \overline{H^{1,0}(C)},$$

and it follows that the image of $H^1_B(C,\mathbb{Z})$ in

$$H^1_B(C,\mathbb{C})/H^{1,0}(C) \cong H^{0,1}(C) := \overline{H^{1,0}(C)} = H^1(C,\mathcal{O}_C)$$

is a lattice, called the lattice of periods. The quotient

$$J(C) = H^{0,1}(C)/H^1_B(C,\mathbb{Z}) \tag{1}$$

is a complex torus, the Jacobian of $C$, and it plays a fundamental role in the study of curves and their moduli. Abel's theorem allows to understand $J(C)$ as the group of 0-cycles of $C$ modulo linear equivalence. Here a 0-cycle is a formal combination $\sum_i n_i x_i$, $x_i \in C$, $n_i \in \mathbb{Z}$. The degree of $z$ is the integer $\sum_i n_i$. First of all, by Serre duality, $H^1(C,\mathcal{O}_C)$ is dual to $H^0(C,\Omega_C)$ and by Poincaré duality, $H^1_B(C,\mathbb{Z}) \cong H_{1,B}(C,\mathbb{Z})$, which allows to rewrite (1) as $J(C) = H^0(C,\Omega_C)^*/H_{1,B}(C,\mathbb{Z})$. One has the Abel map

$$AJ_C : \mathcal{Z}_0(C)_0 \to J(C)$$

defined on the group of 0-cycles of degree 0, which to $z$ associates the linear form $\int_\gamma \in H^0(C,\Omega_C)^*$ modulo the lattice of periods, for any choice of path $\gamma$ on $C$ such that $\partial\gamma = z$.

**Theorem 0.1.** *(See [2, Chapter I, Section 3]) A* 0*-cycle* $z$ *of degree* 0 *is the divisor* $\operatorname{div}\phi$ *of a nonzero rational function on* $C$ *if and only if* $AJ(z) = 0$ *in* $J(C)$.

What makes this two centuries old result fascinating is the fact that it identifies two objects of completely different nature: on the left hand side, we have the group $\mathrm{CH}_0(C)_{hom}$ of 0-cycles of degree 0 modulo linear (or rational) equivalence, while on the right hand side, we have a complex torus defined via uniformization, whose tangent space is an algebraic datum, but whose integral homology $H_1(J(C),\mathbb{Z}) = H_{1,B}(C,\mathbb{Z})$ is purely transcendental.

The subject of Hodge theory and algebraic cycles is the study of the higher dimensional/degree analogue of the interplay between Chow groups of a projective complex manifold $X$ (built from algebraic cycles on $X$) and the Hodge structures on the cohomology of $X$, that is, the data of the Betti cohomology of $X$ equipped with the Hodge filtration, the later coming in fact from algebraic geometry. In the 70's, Mumford [20] and Griffiths [14] discovered several sorts of pathologies showing that Chow groups are not in general good objects of algebraic geometry (eg extensions of finitely generated groups by abelian varieties). Mumford discovered that the presence of higher degree holomorphic forms forces the group $\mathrm{CH}_0(X)$ to be enormous, while Griffiths discovered that there exist in intermediate dimensions and codimensions cycles homologous to zero but not algebraically equivalent to zero, and Clemens [9] even proved that the groups

$\mathrm{Griff}^k(X) = \mathcal{Z}^k(X)_{hom}/\mathcal{Z}^k(X)_{alg}$, although countable by definition, can be infinitely generated, even modulo torsion. Here the subgroup $\mathcal{Z}^k(X)_{alg}$ of codimension $k$ cycles algebraically equivalent to 0 is generated by cycles of the form $\mathcal{Z}_b - \mathcal{Z}_{b'}$, where $B$ is connected, $\mathcal{Z} \subset B \times X$ is a closed codimension $k$ subvariety flat over $B$, and $b$, $b'$ are two points of $B$. It turns out, as we will explain in Section 2.4, that the nontriviality of the Griffiths group is the main obstacle to the solution of a number of conjectures on Chow groups that we will survey here. The subject of algebraic cycles grew up since these major advances to include and understand these pathologies, and this has motivated on the complex geometry side the Bloch-Beilinson conjectures predicting that Chow groups are governed by Hodge structures (the converse is well-known by Bloch-Srinivas [7], see also [19], [23], who gave a vast generalization of Mumford's original theorem), via a certain filtration whose graded piece are controlled by Hodge structures on $X$ of adequate degrees and niveau. It is important to note that in these developments, a beautiful formula as in Theorem 0.1 does not exist anymore, although a replacement has been proposed by Beilinson: His idea is that the first graded piece is given by the cycle class, hence should identify to the group of Hodge classes (assuming the Hodge conjecture). Hodge classes of degree $2k$ on a smooth projective variety $X$ can be seen as morphisms of Hodge structures from the trivial Hodge structure to $H_B^{2k}(X, \mathbb{Q})$ (see Section 1.1.1). The next graded piece should be controled by Abel-Jacobi invariants, so that it should be a subgroup of the Griffiths intermediate Jacobian of the adequate degree. The intermediate Jacobian has an interpretation as $\mathrm{Ext}^1$'s in the category of mixed Hodge structures between the trivial Hodge structure and the adequate odd degree Hodge structure. The expected interpretation of the higher graded pieces is in terms of higher Ext's in a category which remains to be constructed. Indeed, there are no higher Ext's in the category of mixed Hodge structures (see [13]).

The purpose of this paper is to describe more precisely the objects mentioned above (Chow groups, Hodge structures), and the main conjectures concerning them (Bloch-Beilinson's conjectures, generalized Bloch conjecture, Hodge and Grothendieck-Hodge conjectures). We will also explain some recent progress made in [31], [32] on the generalized Bloch conjecture in the case of very general complete intersections in a variety with trivial Chow groups. Our results, whose proofs are of an elementary nature, tell that for these varieties, the geometric coniveau determines the vanishing of Chow groups of cycles of small dimension homologous to 0 as predicted by the generalized Bloch conjecture:

**Theorem 0.2.** *Let $Y$ be a smooth projective variety over $\mathbb{C}$ with trivial Chow groups. Let $E \to Y$ be a very ample vector bundle and let $X \subset Y$ be the zero set of a very general section of $E$. Then if $X$ is strongly of geometric coniveau $\geq c$, that is, the cohomology class of the corrected diagonal of $X$ is the cohomology class of a cycle supported on $W \times X$, with $W \subset X$ closed algebraic of codimension $\geq c$, one has $\mathrm{CH}_i(Y)_{hom} = 0$ for $i < c$.*

Here, the corrected diagonal of $X$ is the diagonal of $X$ corrected by decomposable cycles coming from $Y \times Y$, and having the property that its cohomology class is the projector onto the vanishing cohomology of $X$.

# 1 Hodge structures

## 1.1 Hodge structures

Let $X$ be a smooth projective variety. The cohomology groups $H^k_B(X,\mathbb{C})$ are on one hand isomorphic to $H^k_B(X,\mathbb{Q})\otimes\mathbb{C}$ and on the other hand, can be written as the direct sum (Hodge decomposition)

$$H^k_B(X,\mathbb{C}) = \oplus_{p+q=k} H^{p,q}(X), \tag{2}$$

where $H^{p,q}(X)\subset H^{p+q}_B(X,\mathbb{C})$ is the space of classes representable by a closed form of type $(p,q)$. In particular, $H^{p,q}(X)\subset H^k_B(X,\mathbb{C})$ is complex conjugate to $H^{q,p}(X)\subset H^k_B(X,\mathbb{C})$ (Hodge symmetry). These data provide us with a *weight $k$ (effective) Hodge structure*, that is, a $\mathbb{Q}$-vector space $L$, with a decomposition

$$L_{\mathbb{C}} := L\otimes\mathbb{C} = \oplus_{p+q=k} L^{p,q}$$

with $\overline{L^{p,q}} = L^{q,p}$. Here "effective" means that $L^{p,q}=0$ if $p<0$ or $q<0$, but this condition is not part of the definition of a Hodge structure.

Hodge structures form a category, where the morphisms of Hodge structures between $L$ (of weight $k$) and $L'$ (of weight $k'=k+2r$) are the morphisms $\phi : L\to L'$ of $\mathbb{Q}$-vector spaces such that

$$\phi_{\mathbb{C}}(L^{p,q})\subset L'^{p+r,q+r}$$

for any $p$, $q$. A Hodge substructure $L'\subset L$ is a $\mathbb{Q}$-vector subspace having an induced Hodge decomposition

$$L'_{\mathbb{C}} = \oplus_{p+q=k} L'^{p,q}$$

where $L'^{p,q} = L'_{\mathbb{C}}\cap L^{p,q}$. A Hodge structure is said to be trivial if it is of even weight $2k$ and $L_{\mathbb{C}} = L^{k,k}$. If a Hodge structure $(L, L^{p,q})$ is nontrivial, most $\mathbb{Q}$-vector subspaces $L'\subset L$ are not a Hodge substructure, and a general Hodge structure with the same Hodge numbers $l^{p,q} := \dim L^{p,q}$ will contain no nontrivial Hodge substructures. The image of a morphism of Hodge structures $\phi : L\to L'$ is a Hodge substructure of $L'$, and its kernel is a Hodge substructure of $L$. One can put a quotient Hodge structure on the cokernel $\operatorname{Coker}\phi$. Thus, Hodge structures form an abelian category. This category is not however semi-simple. It turns out that Hodge structures coming from algebraic geometry admit polarizations, and that the category of polarized Hodge structures is semi-simple. Note that the Hodge structures should rather be said polarizable, since in general the polarization is not canonical, and that the morphisms of polarized Hodge structures are simply the morphisms of the corresponding Hodge structures.

In the case of $H^1_B(C,\mathbb{Q})$ which we mentioned in the introduction, the polarization is canonical. It is given by the intersection pairing $(\,,\,)$ on $H^1_B(C,\mathbb{Q})$. This pairing is skew-symmetric and satisfies the following Riemann relations:

$$(\alpha,\beta)=0 \text{ for } \alpha,\ \beta\in H^{1,0}(C).$$

$$\iota(\alpha, \overline{\alpha}) > 0 \text{ for } 0 \neq \alpha \in H^{1,0}(C).$$

The higher degree/dimension version of these relations is a bilinear pairing $Q$ on a Hodge structure of weight $k$ which is symmetric if $k$ is even and skew-symmetric if $k$ is odd and satisfies the so-called Hodge-Riemann bilinear relations. They say the following:

1. (First Hodge-Riemann bilinear relations) $Q(H^{p,q}, \overline{H^{p',q'}}) = 0$ if $(p,q) \neq (p',q')$.
2. (Second Hodge-Riemann bilinear relations) $\epsilon_k(-1)^q \iota^k Q(\alpha, \overline{\alpha}) > 0$ if $0 \neq \alpha \in H^{p,q}$. Here $\epsilon_k = \pm 1$ is a sign which depends only on $k$.

To construct such a pairing on $H^k_B(X,\mathbb{Q})$ for any smooth projective variety $X$, we need to choose the class $l = c_1(\mathcal{L}) \in H^2_B(X,\mathbb{Q})$ of an ample line bundle $\mathcal{L}$ on $X$. Assuming $k \leq n = \dim X$, one has a pairing

$$(\alpha, \beta)_l = \int_X l^{n-k} \smile \alpha \smile \beta$$

on $H^k_B(X,\mathbb{Q})$, which is not yet a polarization of the Hodge structure on $H^k_B(X,\mathbb{Q})$. What we need to do is to use the Lefschetz decomposition

$$H^k_B(X,\mathbb{Q}) = \oplus_{2r \leq k} l^r \smile H^{k-2r}_{B,prim}(X,\mathbb{Q}), \tag{3}$$

which is orthogonal for $(\ ,\ )_l$, and then define the pairing $(\alpha,\beta)_{l,pol}$ as being the one for which (3) is also an orthogonal decomposition, and which is equal to $(-1)^r(\ ,\ )_l$ on the piece $l^r \cup H^{k-2r}_{B,prim}$ of the Lefschetz decomposition (3). This complicated construction makes subordinated to the Lefschetz standard conjecture a number of conjectures on algebraic cycles. The importance of the existence of polarizations lies in the following result:

**Lemma 1.1.** *Let $L$ be a polarized Hodge structure and $L' \subset L$ be a Hodge substructure. Then there exists a Hodge substructure $L'' \subset L$ such that*

$$L \cong L' \oplus L''.$$

*Thus the category of polarized Hodge structures is semi-simple.*

*Proof.* Indeed, choosing a polarization $Q$ on $L$, we define $L''$ as the orthogonal complement of $L'$ with respect to $Q$. The first Hodge-Riemann relations guarantee that $L''$ is a Hodge substructure of $L$. The second Hodge-Riemann relations imply that $Q_{|L'}$ is non-degenerate, so that $L''$ and $L'$ are in direct sum. ☐

### 1.1.1 Hodge classes and morphisms of Hodge structures

The natural morphisms of Hodge structures we encounter in algebraic geometry are the following:

**Pull-back.** Let $\phi : X \to Y$ be a morphism of smooth complex projective varieties. Then $\phi^* : H^k_B(Y,\mathbb{Q}) \to H^k_B(X,\mathbb{Q})$ is a morphism of Hodge structures for any $k$.

**Push-forward (or Gysin) morphisms**. Let $\phi : X \to Y$ be a morphism of smooth complex projective varieties. Then $\phi_* : H^k_B(X, \mathbb{Q}) \to H^{k+2r}_B(Y, \mathbb{Q})$, where $r := \dim Y - \dim X$, is a morphism of Hodge structures for any $k$.

**Cup-product.** Let us start with the following definition.

**Definition 1.2.** *A Hodge class in a Hodge structure $L$ of weight $2k$ is an element of $L$ which is also in the middle piece $L^{k,k}$ of the Hodge decomposition of $L$.*

Let now $X$ be a smooth projective variety. Recall that $H^{p,q}(X)$ is defined as the set of classes representable by a closed form of type $(p, q)$. It immediately follows that $H^{p,q}(X) \smile H^{p',q'}(X) \subset H^{p+p',q+q'}(X)$. Hence we get:

**Lemma 1.3.** *The cup-product map $\alpha \smile: H^k_B(X, \mathbb{Q}) \to H^{k+2r}_B(X, \mathbb{Q})$ by a Hodge class $\alpha$ on $X$ is a morphism of Hodge structures for any $k$.*

Another very important relationship between Hodge classes and morphisms of Hodge structures is the following: Recall that if $X$ and $Y$ are smooth projective varieties, with $\dim X = m$, we have by Künneth decomposition and Poincaré duality

$$\begin{aligned} H^{2k}_B(X \times Y, \mathbb{Q}) &= \oplus_{r+s=2k} H^r_B(X, \mathbb{Q}) \otimes H^s_B(Y, \mathbb{Q}) \\ &\cong \oplus_r \mathrm{Hom}\,(H^r(X, \mathbb{Q}), H^{r+2k-2m}(Y, \mathbb{Q})). \end{aligned}$$

These isomorphisms are isomorphisms of Hodge structures.

**Lemma 1.4.** *A class $\alpha \in H^{2k}_B(X \times Y, \mathbb{Q})$ is a Hodge class if and only if each morphism*

$$\alpha_r : H^r_B(X, \mathbb{Q}) \to H^{r+2k-2m}_B(Y, \mathbb{Q})$$

*is a morphism of Hodge structures.*

## 1.2 Chow groups and cycle classes

Let $X$ be an algebraic variety over a field $K$. One defines $\mathrm{CH}_i(X)$ to be the quotient of the free abelian group $\mathcal{Z}_i(X)$ generated by the irreducible closed algebraic subsets $Z \subset X$ of dimension $i$ (defined over $K$ and irreducible over $K$) by the subgroup $\mathcal{Z}_i(X)_{rat}$ generated by the cycles $n_*(\mathrm{div}\,\phi)$, for any irreducible subvariety $W \subset X$ of dimension $i + 1$, where $n : \widetilde{W} \to X$ is the normalization map of $W$ followed by the inclusion of $W$ in $X$, and any nonzero rational function $\phi \in K(W)^*$. Here $n_*$ is the morphism $\mathcal{Z}_i(\widetilde{W}) \to \mathcal{Z}_i(X)$ which is defined more generally for any proper morphism $n : Y \to X$ on generators $Z$ of $\mathcal{Z}_i(Y)$ by the following rule: $n_*(Z) = \deg\,(Z/n(Z))\, n(Z)$ if $n : Z \to n(Z)$ is generically finite and $n_*(Z) = 0$ if $\dim n(Z) < i$.

This definition is the higher dimensional generalization of the group

$$\mathrm{CH}_0(C) := \mathcal{Z}_0(C)/\textit{linear equivalence}$$

appearing implicitly in the introduction. When $X$ is of pure dimension $n$, one writes $\mathrm{CH}^i(X) := \mathrm{CH}_{n-i}(X)$. When $C$ is a curve, the group $\mathrm{CH}_0(C)$ is thus also $\mathrm{CH}^1(C)$. In any dimension, the group $\mathrm{CH}^1(X)$ has the following interpretation:

**Lemma 1.5.** *If $X$ is locally factorial of dimension $n$, one has* $\mathrm{CH}_{n-1}(X) = \mathrm{CH}^1(X) = \mathrm{Pic}\,X$.

The local factoriality guarantees that irreducible closed algebraic subsets $D \subset X$ of dimension $n-1$ are Cartier divisors (locally defined by one equation defined up to multiplication by an invertible function), thus providing a locally free sheaf $\mathcal{O}_X(D)$ of rank 1, defined as the dual of the ideal sheaf $\mathcal{I}_D$. In the other direction, one associates to $\mathcal{L} \in \mathrm{Pic}\,X$ the divisor of any nonzero rational section of $\mathcal{L}$.

Coming back to the case where $K = \mathbb{C}$ and $X$ is projective, the group $\mathrm{CH}^1(X)$ is very well understood in this case thanks to to the GAGA principle [29] which says that

$$\mathrm{Pic}\,X = \mathrm{Pic}\,X^{an},$$

that is, holomorphic line bundles and algebraic line bundles are the same objects, and to the exponential exact sequence of sheaves on $X^{an}$:

$$0 \to \mathbb{Z} \stackrel{2\iota\pi}{\to} \mathcal{O}_{X^{an}} \stackrel{\exp}{\to} \mathcal{O}^*_{X^{an}} \to 1. \tag{4}$$

Using the identification $\mathrm{Pic}\,X^{an} = H^1(X^{an}, \mathcal{O}^*_{X^{an}})$, one gets the long exact sequence

$$H^1_B(X,\mathbb{Z}) \stackrel{2\iota\pi}{\to} H^1(X^{an}, \mathcal{O}_{X^{an}}) \stackrel{\exp}{\to} \mathrm{Pic}\,X^{an} \to H^2_B(X,\mathbb{Z}) \stackrel{2\iota\pi}{\to} H^2(X^{an}, \mathcal{O}_{X^{an}}). \tag{5}$$

The third map $c_1 : \mathrm{CH}^1(X) = \mathrm{Pic}\,X = \mathrm{Pic}\,X^{an} \to H^2_B(X,\mathbb{Z})$ in (5) is the Betti cycle class for codimension 1 cycles. Using the fact that the group of integral Hodge classes in $H^2_B(X,\mathbb{Z})$ is exactly the kernel of the fourth map $H^2_B(X,\mathbb{Z}) \to H^2(X^{an}, \mathcal{O}_{X^{an}})$, the exponential exact sequence not only describes $\mathrm{CH}^1(X)$ but also proves the Lefschetz $(1,1)$-theorem saying that integral Hodge classes of degree 2 are classes of codimension 1 cycles.

Despite the remarkable developments of algebraic $K$-theory, the theory of algebraic cycles misses an analogue of the exponential exact sequence allowing to understand higher codimensional cycles.

### 1.2.1 Cycle class

Let us describe various cycle classes in the setting of analytic geometry. We will first construct cycle classes for smooth closed algebraic subvarieties $Z \subset X$, where $X$ is a smooth projective complex variety. In the Betti cohomology setting, such a $Z$ has a cohomology class $[Z] \in H^{2i}(X,\mathbb{Z})$, where $i = \mathrm{codim}\,Z$, defined as the Poincaré dual of the fundamental homology class $[Z]_{hom} := i_{Z*}([Z]_{fund}) \in H_{2n-2i,B}(X,\mathbb{Z})$, where $i_Z$ is the inclusion of $Z$ in $X$ and $[Z]_{fund} \in H_{2n-2i,B}(Z,\mathbb{Z})$ is the fundamental homology class of the compact oriented real manifold underlying $Z$.

If $Z$ is not smooth but $X$ is projective, we can also define $[Z] \in H^{2i}_B(X,\mathbb{Z})$ thanks to the resolution of singularities: indeed, we can introduce a resolution of singularities $\widetilde{Z} \to Z$ and define similarly as above $[Z] \in H^{2i}_B(X,\mathbb{Z})$ as the Poincaré dual of the homology class $i_{Z*}([\widetilde{Z}]_{fund}) \in H_{2n-2i,B}(X,\mathbb{Z})$. As two resolutions are dominated by a third which admits a degree 1 morphism to both of them, the resulting class does not depend on the resolution.

In the quasi-projective case, if $Z \subset X$ is any closed algebraic subset of codimension $i$, where $X$ is smooth quasiprojective, we can choose a smooth projective completion $\overline{X}$ of $X$ and extend $Z$ to a cycle $\overline{Z}$. Then we define $[Z]$ as $[\overline{Z}]_{|X} \in H^{2i}_B(X, \mathbb{Z})$ and check this does not depend on the choice of the extensions. It is not hard to check that the cycle class $[Z] = \sum_j n_j [Z_j] \in H^{2i}_B(X, \mathbb{Z})$ of a cycle $Z = \sum_j n_j Z_j$ depends only of the class of $Z$ in $\mathrm{CH}^i(X)$. This provides the cycle class map

$$cl : \mathrm{CH}^i(X) \to H^{2i}_B(X, \mathbb{Z})$$

in Betti cohomology.

The above cycle class is purely topological. It does not use the fact that $Z$ and $X$ are complex analytic spaces. We now turn to refined cycle classes which contain an information depending on the complex structure. Here we assume $X$ to be smooth projective. Griffiths [14] defined the Abel-Jacobi map

$$\Phi_X : \mathrm{CH}^i(X)_{hom} := \mathrm{Ker}\, cl \to J^{2i-1}(X),$$

where the intermediate Jacobian $J^{2i-1}(X)$ is a complex torus defined as

$$J^{2i-1}(X) := H^{2i-1}_B(X, \mathbb{C}) / F^i H^{2i-1}_B(X, \mathbb{C}) \oplus H^{2i-1}_B(X, \mathbb{Z}),$$

where the Hodge filtration $F^{\cdot}$ is deduced from the Hodge decomposition by

$$F^k H^l_B(X, \mathbb{C}) = \oplus_{p \geq k} H^{p,l-p}(X).$$

Using Poincaré duality, this intermediate Jacobian can also be seen as

$$J^{2i-1}(X) = F^{n-i+1} H^{2n-2i+1}_B(X, \mathbb{C})^* / H_{2n-2i+1,B}(X, \mathbb{Z}).$$

The Abel-Jacobi map is defined as follows: if $z$ is a codimension $i$ cycle homologous to 0, let $\Gamma$ be a real contour of dimension $2n - 2i + 1$ such that $\partial\Gamma = z$. Then there is a well-defined linear form

$$\int_\Gamma \in F^{n-i+1} H^{2n-2i+1}_B(X, \mathbb{C})^*$$

which to the class of a closed form $\eta$ of type $(2n-2i+1, 0) + \ldots + (2n-2i+1, 2n-2i)$ associates $\int_\Gamma \eta$. The point is that if such an $\eta$ is exact, then it can be written as $d\eta'$ where $\eta'$ is a differential form of type $(2n-2i, 0) + \ldots + (2n-2i+1, 2n-2i-1)$, hence integrates to 0 against $z$. The Abel-Jacobi map $\Phi_X$ maps $z$ to $\int_\Gamma$ modulo the period lattice $H_{2n-2i+1,B}(X, \mathbb{Z})$ mod. torsion $\subset F^{n-i+1} H^{2n-2i+1}_B(X, \mathbb{C})^*$.

Note that there is a global cycle class called the Deligne cycle class, which takes value in the Deligne cohomology group

$$H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i)) := \mathbb{H}^{2i}(X, \mathbb{Z}_{\mathcal{D}}(i))$$

where the Deligne complex $\mathbb{Z}_{\mathcal{D}}(i)$ is the complex

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{d} \ldots \to \Omega^{i-1}_X \to 0$$

of sheaves on $X^{an}$, where $\mathbb{Z}$ is put in degree 0. The Deligne complex is an extension

$$0 \to \Omega_X^{\cdot \leq i-1}[-1]\mathbb{Z}_{\mathcal{D}}(i) \to \mathbb{Z} \to 0,$$

from which one deduces by taking the associated long exact sequence sequence that the Deligne cohomology group $H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i))$ is an extension

$$0 \to J^{2i-1}(X) \to H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i)) \to \mathrm{Hdg}^{2i}(X, \mathbb{Z}) \to 0.$$

All these statements use in a crucial way the fact that the complex analytic space $X^{an}$ is a compact Kähler manifold.

### 1.2.2 Functoriality and intersection

Chow groups have the following functoriality properties: If $\phi : X \to Y$ is a proper morphism, then the morphism $\phi_* : \mathcal{Z}_i(X) \to \mathcal{Z}_i(Y)$ defined in the previous section factors through rational equivalence and induces

$$\phi_* : \mathrm{CH}_i(X) \to \mathrm{CH}_i(Y).$$

If $\phi : X \to Y$ is a flat morphism of relative dimension $r$, the group morphism $\phi^* : \mathcal{Z}_i(Y) \to \mathcal{Z}_{i+r}(X)$ is defined on generators by

$$\phi^*(Z) = c(\phi^{-1}(Z)),$$

where $c$ is the Hilbert-Chow map which to a subscheme $\mathcal{Z} \subset X$ of dimension $i + r$ associates the $i + r$-cycle $Z$ of its irreducible components $Z_i$ counted with multiplicity given by the length of $\mathcal{O}_{\mathcal{Z}, Z_i}$. It factors through rational equivalence and induces

$$\phi^* : \mathrm{CH}_i(Y) \to \mathrm{CH}_{i+r}(X).$$

Flatness is a very restrictive assumption. Fortunately, one also has a pull-back morphism $\phi^* : \mathrm{CH}^i(Y) \to \mathrm{CH}^i(X)$ when $\phi$ is the inclusion of a lci subscheme. It follows that if $Y$ is smooth, one has a pull-back morphism $\phi^* : \mathrm{CH}^i(Y) \to \mathrm{CH}^i(X)$, defined as the composite of the flat pull-back morphism

$$pr_2^* : \mathrm{CH}^i(Y) \to \mathrm{CH}^i(X \times Y)$$

and of the restriction map

$$i_\Gamma^* : \mathrm{CH}^i(X \times Y) \to \mathrm{CH}^i(X),$$

where $i_\Gamma$ is the inclusion $(Id_X, \phi)$ of the graph of $\phi$.

In particular, we get an intersection theory on $\mathrm{CH}^*(Y)$ when $Y$ is smooth, given by

$$Z \cdot Z' = i_\Delta^*(Z \times Z'),$$

where $i_\Delta : Y \to Y \times Y$ is the diagonal inclusion.

There are various compatibility properties between the morphisms $\phi^*$, $\psi_*$ and the intersection product. The most important ones are the following:

1. For smooth $Y$ and $X$, $\phi^* : \mathrm{CH}^i(Y) \to \mathrm{CH}^i(X)$ is compatible with the intersection product.

2. (*Projection formula.*) If $X$, $Y$ are smooth and $\phi : X \to Y$ is proper, $\phi_*(\phi^* z \cdot z') = z \cdot \phi_* z'$, for any $z \in \mathrm{CH}(Y)$, $z' \in \mathrm{CH}(X)$.

3. If $\phi : X \to Y$ is flat and $\psi : Y' \to Y$ is proper, let

$$\psi' : X' := X \times_Y Y' \to X,\ \phi' : X' \to Y'$$

be the natural morphisms appearing in the following Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "\psi'"] \arrow[d, "\phi'"] & X \arrow[d, "\phi"] \\ Y' \arrow[r, "\psi"] & Y \end{tikzcd}.$$

Then $\psi'$ is proper, $\phi'$ is flat, and one has

$$\psi'_* \circ \phi'^* = \phi^* \circ \psi_* : \mathrm{CH}(Y') \to \mathrm{CH}(X).$$

**Definition 1.6.** *A correspondence between $X$ and $Y$ is a cycle $Z \in \mathrm{CH}(X \times Y)$.*

Assuming $X$ and $Y$ are smooth and projective, such a correspondence induces morphisms

$$Z_* : \mathrm{CH}(X) \to \mathrm{CH}(Y),$$
$$Z^* : \mathrm{CH}(Y) \to \mathrm{CH}(X),$$

defined by

$$Z_*(\alpha) = pr_{2*}(pr_1^* \alpha \cdot Z),\ Z^*(\alpha) = pr_{1*}(pr_2^* \alpha \cdot Z).$$

If one wants to keep track of the grading, one needs to introduce the dimensions $m$, $n$ of $X$, resp. $Y$, and the codimension $k$ of $Z$. Then $Z_*$ maps $\mathrm{CH}^l(X)$ to $\mathrm{CH}^{k+l-m}(Y)$ and $Z^*$ maps $\mathrm{CH}^l(Y)$ to $\mathrm{CH}^{k+l-n}(X)$.

Correspondences $Z \in \mathrm{CH}(X \times Y)$, $Z' \in \mathrm{CH}(Y \times W)$ between smooth projective varieties can be composed by the rule

$$Z' \circ Z = pr_{XW*}(pr_{XY}^* Z \cdot pr_{YW}^* Z') \in \mathrm{CH}(X \times W),$$

where $pr_{\bullet\bullet}$ denotes the projection from $X \times Y \times Z$ on the corresponding product of two of its factors.

The three properties stated above are all that is needed formally to prove the following result concerning the composition of correspondences.

**Proposition 1.7.** *If $X$, $Y$, $W$ are smooth and projective, and $Z \in \mathrm{CH}(X \times Y)$, $Z' \in \mathrm{CH}(Y \times W)$ are correspondences, one has*

$$(Z' \circ Z)_* = Z'_* \circ Z_* : \mathrm{CH}(X) \to \mathrm{CH}(W).$$

## 1.3 Coniveau

Let $X$ be a smooth complex projective variety.

**Definition 1.8.** *A Betti cohomology class $\alpha \in H^*_B(X, \mathbb{Q})$ is said to be of geometric coniveau $c$ if there exists a closed algebraic subset $Y \subset X$ of codimension $c$ such that $\alpha$ vanishes in $H^*_B(X \setminus Y, \mathbb{Q})$.*

The fundamental results of Deligne [10] concerning mixed Hodge structures have in particular the following consequence:

**Theorem 1.9.** *Let $\alpha \in H^*_B(X, \mathbb{Q})$ vanish in $H^*_B(X \setminus Y, \mathbb{Q})$, where $Y$ is of pure codimension $c$. Let $j : \widetilde{Y} \to X$ be a desingularization of $Y$. Then there exists a cohomology class $\beta \in H^{*-2c}_B(\widetilde{Y}, \mathbb{Q})$ such that*

$$\alpha = j_*\beta \text{ in } H^*_B(X, \mathbb{Q}).$$

Let us introduce the notion of Hodge coniveau of a weight $k$ Hodge structure $(L, L^{p,q})$. The Hodge structure is given by the Hodge decomposition

$$L = L^{k,0} \oplus \ldots \oplus L^{0,k}.$$

We will say that $L$ has Hodge coniveau $c$ if the Hodge decomposition takes the form

$$L = L^{k-c,c} \oplus \ldots \oplus L^{c,k-c}$$

with $L^{k-c,c} \neq 0$.

**Corollary 1.10.** *The subgroup $H^k_C(X, \mathbb{Q})_c$ of cohomology classes of geometric coniveau $\geq c$ is a Hodge substructure of $H^k_B(X, \mathbb{Q})$, which is of Hodge coniveau $\geq c$.*

*Proof.* Indeed, it suffices to show that for any cohomology class $\alpha$ of geometric coniveau $\geq c$, there exists a Hodge substructure $L_\alpha$ of $H^k_B(X, \mathbb{Q})$, of Hodge coniveau $\geq c$, which contains $\alpha$ and is contained in $H^k_B(X, \mathbb{Q})_c$. We apply Theorem 1.9 and put $L_\alpha := \mathrm{Im}\,(j_* : H^{k-2c}_B(\widetilde{Y}, \mathbb{Q}) \to H^k_B(X, \mathbb{Q}))$. This is a Hodge substructure of $H^k_B(X, \mathbb{Q})$ (see Section 1.1) and it is of Hodge coniveau $\geq c$ since $j_*$ shifts the Hodge decomposition by a bidegree $(c, c)$.

□

The generalized Hodge conjecture due to Grothendieck [15] states the following:

**Conjecture 1.11.** *Let $L \subset H^k_B(X, \mathbb{Q})$ be a Hodge substructure of Hodge coniveau $\geq c$. Then $L$ is contained in $H^k_B(X, \mathbb{Q})_c$.*

The Hodge conjecture itself is the particular case of Conjecture 1.11 where $k = 2k'$ and $c = k'$. Then a Hodge substructure of Hodge coniveau $c = k'$ consists of Hodge classes and the generalized Hodge conjecture then exactly predicts that it is contained in the image of a Gysin morphism $j_* : H^0_B(\widetilde{Y}, \mathbb{Q}) \to H^{2k'}_B(X, \mathbb{Q})$, that is, in the $\mathbb{Q}$-vector space generated by the classes of the irreducible components of $Y = j(\widetilde{Y})$.

We will say that a variety $X$ is of geometric coniveau $\geq c$ if the Betti cohomology of $X$ decomposes as

$$H^*_B(X,\mathbb{Q}) = H^*_B(X,\mathbb{Q})_{alg} + H^*_B(X,\mathbb{Q})_c.$$

The generalized Hodge conjecture predicts that this is equivalent to the condition $H^{p,q}(X) = 0$ for $p > c$ (or $q > c$) and $p \neq q$. We will use later on the following notion whose importance will be explained in Section 2.2.

**Definition 1.12.** *The variety $X$ is strongly of geometric coniveau $\geq c$ if there exists a decomposition of the cohomology class $[\Delta_X] \in H^{2n}_B(X\times X,\mathbb{Q})$ which is as follows:*

$$[\Delta_X] = [Z_1] + [Z_2] \in H^{2n}_B(X\times X,\mathbb{Q}), \tag{6}$$

*where $Z_1$ is a decomposable cycle in $X\times X$, that is a combination with rational coefficients of product $W_i \times W_j$, and the cycle $Z_2$ is supported in $Y\times X$ for some closed algebraic subset $Y \subset X$ of codimension $\geq c$.*

**Proposition 1.13.** *(i) If $X$ is strongly of geometric coniveau $\geq c$, it is of geometric coniveau $\geq c$.*

*(ii) Assuming the Hodge conjecture, if $X$ is of geometric coniveau $\geq c$, it is strongly of geometric coniveau $\geq c$.*

*Proof.* (i) If $X$ is strongly of geometric coniveau $c$, then letting both sides of (6) act on $H^*_B(X,\mathbb{Q})$, we get for any $\alpha \in H^*_B(X,\mathbb{Q})$

$$\alpha = [\Delta_X]^*\alpha = [Z_1]^*\alpha + [Z_2]^*\alpha$$

where $[Z_1]^*\alpha$ is a combination of cycles classes and $[Z_2]^*\alpha$ vanishes away of $Y$ hence is of geometric coniveau $\geq c$.

(ii) If we know that $X$ has geometric coniveau $\geq c$, and furthermore the Hodge conjecture holds, then we have

$$H^*_B(X,\mathbb{Q}) = H^*_B(X,\mathbb{Q})_{alg} + H^*_B(X,\mathbb{Q})_c = H^*_B(X,\mathbb{Q})_{alg} \oplus H^*_B(X,\mathbb{Q})_c^{\perp alg},$$

because the intersection pairing is nondegenerate on $H^*_B(X,\mathbb{Q})_{alg}$ by the second Hodge-Riemann bilinear relations and the fact that $H^*_B(X,\mathbb{Q})_{alg}$ is stable under the Lefschetz decomposition if the Hodge conjecture is true. It follows that we can write the class $[\Delta_X]$ as the sum of two projectors, namely an element $A_1$ in $H^*_B(X,\mathbb{Q})_{alg}\otimes H^*_B(X,\mathbb{Q})_{alg}$, which is the class of a decomposable cycle, and an element $A_2$ which is a Hodge class in $H^*_B(X,\mathbb{Q})_c \otimes H^*_B(X,\mathbb{Q}) \subset H^{2n}_B(X\times X,\mathbb{Q})$. Now we write $H^*_B(X,\mathbb{Q})_c = \mathrm{Im}\,(j_* : H^{*-2c}_B(\widetilde{Y},\mathbb{Q}) \to H^*_B(X,\mathbb{Q}))$ and we use the semi-simplicity Lemma 1.1 which together with Lemma 1.4 implies that there is a Hodge class $B$ on $\widetilde{Y}\times X$ such that $A_2 = (j, Id_X)_*B$ in $H^{2n}_B(X\times X,\mathbb{Q})$. Then if the Hodge conjecture holds, the class $B$ is algebraic on $\widetilde{Y}\times X$, that is $B = [W]$ and thus we have $A_2 = [(j, Id_X)_*W]$, where the cycle $(j, Id_X)_*W$ is supported on $Y\times X$, with $Y = j(\widetilde{Y})$. ∎

Notice that for surfaces, as was already observed by Bloch in [5], the Lefschetz theorem on $(1,1)$-classes implies that a smooth complex projective surface $X$ has $p_g = q = 0$ if and only if it is strongly of coniveau $\geq 1$ and then the cohomology class of its diagonal is decomposable.

# 2 Bloch-Beilinson conjectures

## 2.1 Mumford-Roitman's theorem

The original result proved by Mumford concerned 0-cycles on surfaces:

**Theorem 2.1.** *(Mumford [20]) Let $X$ be a smooth complex projective surface. Then if $X$ has some nonzero holomorphic 2-form, the kernel of the Abel or rather Albanese map (see Introduction or Section 1.2.1)*

$$\mathrm{alb}_X : \mathrm{CH}_0(X)_{hom} \to \mathrm{Alb}\, X$$

*is nontrivial, and even infinite dimensional.*

Infinite dimensionality here can be made as precise as one wants. In fact, as shows Mumford's proof, under the same assumption the general 0-cycle $z \in X^{(k)}$ is isolated in its orbit under rational equivalence. Hence "morally" the map $X^{(k)} \to \mathrm{CH}_0(X)$ has an image of dimension $2k = \dim X^{(k)}$ for any $k$. In particular, this map is not surjective onto the set of 0-cycles of degree $k$ modulo rational equivalence for any $k$.

Subsequent work by Roitman [26] has proved that finite dimensionality is equivalent to the fact that for some curve $C \subset X$, the map $\mathrm{CH}_0(C) \to \mathrm{CH}_0(X)$ is surjective. Admitting this, Mumford's theorem is then generalized in the following form due to Roitman [26]:

**Theorem 2.2.** *Let $X$ be a smooth projective complex variety and let $Y \subset X$ be a closed algebraic subset of dimension $\leq r$. Then if the map*

$$\mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$$

*is surjective, one has $H^0(X, \Omega_X^k) = 0$ for any $k > r$.*

## 2.2 Bloch-Srinivas theorem and coniveau

In the paper [7], Bloch and Srinivas gave a completely new proof of Theorem 2.2, which has been the starting point of further developments on Chow groups and coniveau. The result of Bloch and Srinivas is the following "decomposition of the diagonal" principle. We already encountered the decomposition of the diagonal in its cohomological form in Section 1.3; we insist on the fact that the Bloch-Srinivas decomposition holds in $\mathrm{CH}(X \times X)_{\mathbb{Q}}$.

**Theorem 2.3.** *Under the assumption of Theorem 2.2, there is a decomposition*

$$\Delta_X = Z_1 + Z_2 \text{ in } \mathrm{CH}(X \times X)_{\mathbb{Q}}, \tag{7}$$

*where $Z_1$ is a cycle with $\mathbb{Q}$-coefficients supported in $X \times Y$, and $Z_2$ is a cycle with $\mathbb{Q}$-coefficients supported in $D \times X$ for some closed proper algebraic subset $D \subset X$.*

Note that, conversely, if such a decomposition exists, then $\mathrm{CH}_0(X)_{\mathbb{Q}}$ is supported on $Y$, since letting both sides of (7) act on $\mathrm{CH}_0(X)_{\mathbb{Q}}$, one gets

$$z = Z_{1*}z \text{ in } \mathrm{CH}_0(X)_{\mathbb{Q}}$$

for any $z \in \mathrm{CH}_0(X)_{\mathbb{Q}}$ and of course $Z_{1*}z$ is supported on $Y$. Theorem 2.3 implies in the case where $Y$ is just one point $y$ the following corollary (which is called the decomposition of the diagonal):

**Corollary 2.4.** *If $\mathrm{CH}_0(X) = \mathbb{Z}$ (that is, all points of $X$ are rationally equivalent), then*

$$\Delta_X = X \times x + Z \text{ in } \mathrm{CH}(X \times X)_{\mathbb{Q}}, \tag{8}$$

*where $x$ is any point of $X$, and $Z$ is a cycle with $\mathbb{Q}$-coefficients supported in $D \times X$ for some closed proper algebraic subset $D \subset X$.*

Let us now show how Theorem 2.3 implies Theorem 2.2.

*Proof of Theorem 2.2.* Under the assumption of Theorem 2.2, we have the decomposition of the diagonal (7), which by taking cohomology classes gives the decomposition

$$[\Delta_X] = [Z_1] + [Z_2] \text{ in } H^{2n}(X \times X, \mathbb{Q}), \tag{9}$$

where $n = \dim X$, $Z_1$ is a cycle with $\mathbb{Q}$-coefficients supported in $X \times Y$, and $Z_2$ is a cycle with $\mathbb{Q}$-coefficients supported in $D \times X$ for some closed proper algebraic subset $D \subset X$.

We now let both sides of (9) act on cohomology of $X$, and particularly on the subspaces $H^{i,0}(X) \subset H^i(X, \mathbb{C})$ and get for any $\alpha \in H^{i,0}(X)$ :

$$[\Delta_X]^*\alpha = \alpha = [Z_1]^*\alpha + [Z_2]^*\alpha \text{ in } H^{i,0}(X). \tag{10}$$

As $Z_2$ is supported on $D \times X$, $[Z_2]^*\alpha$ is supported on $D$, and because $[Z_2]^*\alpha$ is a holomorphic, that is, belongs to $H^{i,0}(X)$, this easily implies that $[Z_2]^*\alpha = 0$. On the other hand, if $i > r = \dim Y$ one has $[Z_1]^*\alpha = 0$ since $\alpha$ vanishes under restriction to $Y$. Thus $\alpha = 0$ by (10). □

Theorem 2.3 or rather its corollary 2.4 has been generalized in [23] where the following decomposition theorem up to codimension $c$ is proved.

**Theorem 2.5.** *Let $X$ be smooth projective over $\mathbb{C}$ and assume that the cycle class map*

$$cl : \mathrm{CH}_i(X, \mathbb{Q}) \to H^{2n-2i}(X, \mathbb{Q})$$

*is injective for $i < c$. Then there is a decomposition*

$$\Delta_X = Z_1 + Z_2 \text{ in } \mathrm{CH}^n(X \times X)_{\mathbb{Q}}, \tag{11}$$

*where $Z_1$ is a decomposable cycle and $Z_2$ is a cycle with $\mathbb{Q}$-coefficients supported in $D \times X$ for some closed proper algebraic subset $D \subset X$ of codimension $\geq c$.*

Bloch-Srinivas Corollary 2.4 is the case $c = 1$. Note that Theorem 2.5 is optimal since conversely, if (11) holds, letting both sides act on $z \in \mathrm{CH}_i(X)_{\mathbb{Q}}$, we get

$$z = \Delta_{X*}z = Z_{1*}z + Z_{2*}z \text{ in } \mathrm{CH}(X)_{\mathbb{Q}}.$$

We now observe that for the decomposable cycle $Z_1$, we have $Z_{1*}z = 0$ if $z$ is cohomologous to 0 (indeed, for a product cycle $T = T_1 \times T_2$, we have $T_*z = (\deg T_1 \cdot z)\, T_2$ which vanishes if $z$ is cohomologous to 0). On the other hand, since $\operatorname{codim} D \geq c$ and $Z_2$ is supported on $D \times X$, we have $Z_{2*}z = 0$ if $i < c$. So finally we conclude that $z = 0$ if $z \in \mathrm{CH}_i(X)_{\mathbb{Q},hom}$ and $i < c$.

As before we get the following corollary which has been proved in one or another form by Laterveer [18], Lewis [19], Schoen [28]:

**Corollary 2.6.** *Under the assumptions of Theorem 2.5, $X$ is of geometric coniveau $\geq c$.*

*Proof.* Indeed, by taking cycle classes in (11), we get that $X$ is strongly of geometric coniveau $c$ so we only have to apply Proposition 1.13, (i). ☐

## 2.3 Bloch and Bloch-Beilinson conjectures on Chow groups

In [5], Bloch conjectured the converse to Mumford's theorem 2.1:

**Conjecture 2.7.** *(Bloch's conjecture on surfaces with $p_g = 0$) Let $X$ be a smooth projective complex surface with $H^{2,0}(X) = 0$. Then*

$$alb_X : \mathrm{CH}_0(X)_0 \to Alb(X)$$

*is injective.*

This conjecture is known to be true for surfaces which are not of general type [6] and for surfaces of general type, it has been proved for some surfaces ([16], [3], [6], [24]) and some families of surfaces ([34], [33]). As we will see below, this conjecture, although very challenging, is presumably less important than Conjecture 2.13, since one can imagine proving it by particular methods for each class of surfaces of general type with $q = p_g = 0$. Indeed, if one considers only minimal models, which is allowed since the $\mathrm{CH}_0$ group is birationally invariant, there are only finitely many deformation types of surfaces of general type with $p_g = q = 0$. The next conjecture is a converse to Corollary 2.6.

**Conjecture 2.8.** *Let $X$ be a smooth projective complex of dimension $n$. Assume $X$ is of geometric coniveau $\geq c$. Then the cycle class map*

$$cl : \mathrm{CH}_i(X)_{\mathbb{Q}} \to H_B^{2n-2i}(X, \mathbb{Q})$$

*is injective for $i < c$. Equivalently, $\mathrm{CH}_i(X)_{hom,\mathbb{Q}} = 0$ for $i < c$.*

Let us state the following slightly weaker variant of this conjecture:

**Conjecture 2.9.** *Let $X$ be smooth projective complex of dimension $n$. Assume $X$ is of strongly of geometric coniveau $\geq c$. Then $\mathrm{CH}_i(X)_{hom,\mathbb{Q}} = 0$ for $i < c$.*

In this last formulation, we just replaced "of geometric coniveau $\geq c$" by "strongly of geometric coniveau $\geq c$" (we refer to Proposition 1.13 for the comparison between the two notions). As already noticed, in the surface case, the two notions are equivalent and both conjectures are equivalent to Bloch's conjecture 2.7. We will explain in Section 3.2 the proof of the following theorem, which solves 2.9 for very general complete intersections in a variety with trivial Chow groups. Let us first introduce the following definition:

**Definition 2.10.** *We say that a smooth projective variety $X$ has trivial Chow groups if the cycle class map $cl$ is injective on $\mathrm{CH}_i(X)_{\mathbb{Q}}$ for all $i$.*

**Theorem 2.11.** *(Cf. [32]) Let $Y$ be a smooth projective variety with trivial Chow groups and $\mathcal{L}$ a very ample line bundle on $Y$. Then if $X \in |\mathcal{L}|$ is a very general member, $X$ satisfies Conjecture 2.9, assuming the vanishing cohomology of $X$ is nontrivial.*

**Remark 2.12.** If we combine Conjecture 2.8 and Conjecture 1.11, we get the most ambitious conjecture saying that if the Hodge coniveau of $X$ is at least $c$ in the sense that $H^*(X,\mathbb{Q}) = H^*(X,\mathbb{Q})_{alg} + L$ where $L$ is a sum of Hodge structures of Hodge coniveau at least $c$, then the cycle class map

$$cl : \mathrm{CH}_i(X) \to H_B^{2n-2i}(X,\mathbb{Q})$$

is injective for $i < c$.

In [5], Bloch made another conjecture concerning Chow groups of surfaces, namely he observes that the group $\mathrm{CH}_0(X)$, where $X$ is a smooth connected projective surface has a natural filtration given by

$$\begin{aligned} F^0\mathrm{CH}_0(X) &= \mathrm{CH}_0(X), \\ F^1\mathrm{CH}_0(X) &= \mathrm{CH}_0(X)_0, \text{ (the group of cycles of degree 0)}, \\ F^2\mathrm{CH}_0(X) &= \mathrm{Ker}\,(\mathrm{CH}_0(X)_0 \stackrel{alb_X}{\to} Alb(X)). \end{aligned} \tag{12}$$

This filtration is respected by correspondences of surfaces, and Mumford's theorem can be phrased by saying that if $H^{2,0}(X) \neq 0$, then $F^2\mathrm{CH}_0(X) \neq 0$. The general Bloch conjecture on 0-cycles on surfaces is then

**Conjecture 2.13.** *(Bloch [5]) Let $\Gamma \in \mathrm{CH}^2(X \times Y)$ be a correspondence between two surfaces $X$ and $Y$. Then if $[\Gamma]^* : H^{2,0}(Y) \to H^{2,0}(X)$ is 0, so is $\Gamma_* : F^2\mathrm{CH}_0(X) \to F^2\mathrm{CH}_0(Y)$.*

**Remark 2.14.** As we can easily see, there is no reason to impose in the above conjecture the condition that $X$ is a surface. In fact the conjecture in the surface case easily implies the conjecture for any smooth projective variety $X$. However it is crucial that $Y$ is a surface.

We can now put this conjecture in the framework of the Bloch-Beilinson conjectures on filtrations on Chow groups. Indeed, we can observe that the filtration

(12) has the property that it is respected by correspondences of codimension 2 and that for $i = 0$, 1, the induced map on graded pieces

$$\Gamma_* : Gr_F^0\mathrm{CH}_0(X) \to Gr_F^0\mathrm{CH}_0(Y), \text{ resp } \Gamma_* : Gr_F^1\mathrm{CH}_0(X) \to Gr_F^1\mathrm{CH}_0(Y)$$

vanishes if and only if the maps $[\Gamma]^* : H_B^0(Y,\mathbb{Q}) \to H_B^0(X,\mathbb{Q})$, resp. $[\Gamma]^* : H_B^1(Y,\mathbb{Q}) \to H_B^1(X,\mathbb{Q})$ vanish, while for $i = 2$, the Bloch conjecture 2.13 says that $\Gamma_* : Gr_F^2\mathrm{CH}_0(X) \to Gr_F^2\mathrm{CH}_0(Y)$ vanishes if and only if $[\Gamma]^* : H^{2,0}(Y) \to H^{2,0}(X)$ vanishes. The last condition is equivalent to saying that $[\Gamma]^* : H_B^2(Y,\mathbb{Q})_{tr} \to H_B^2(X,\mathbb{Q})_{tr}$ vanishes, where for any surface $S$, we denote $H_B^2(S,\mathbb{Q})_{tr} := H_B^2(S,\mathbb{Q})_{alg}^{\perp}$. Indeed, by the Lefschetz theorem on $(1,1)$-classes, $H_B^2(S,\mathbb{Q})_{tr}$ is the smallest Hodge substructure of $H_B^2(S,\mathbb{Q})$ which contains $H^{2,0}(S)$ and thus a morphism of Hodge structures $H_B^2(Y,\mathbb{Q})_{tr} \to H_B^2(X,\mathbb{Q})_{tr}$ vanishes if and only if it vanishes on the $(2,0)$-parts of the Hodge decomposition.

These observations in the case of surfaces have been generalized to the following conjecture (Bloch-Beilinson conjectures on filtrations on Chow groups):

**Conjecture 2.15.** *For any smooth complex projective variety $X$, thee exists a descending filtration $F^\bullet$ on $\mathrm{CH}^i(X)_\mathbb{Q}$, with the following properties:*

1. $F^0\mathrm{CH}(X)_\mathbb{Q} = \mathrm{CH}(X)_\mathbb{Q},\ F^1\mathrm{CH}(X)_\mathbb{Q} = \mathrm{CH}(X)_{\mathbb{Q},hom}$.
2. *The filtration is functorial with respect to correspondences: If $\Gamma \in \mathrm{CH}(X\times Y)$ is a correspondence,*
$$\Gamma_*(F^l\mathrm{CH}(X)_\mathbb{Q}) \subset F^l\mathrm{CH}(Y)_\mathbb{Q}.$$
3. *One has $F^i\mathrm{CH}(X)_\mathbb{Q} \cdot F^j\mathrm{CH}(X)_\mathbb{Q} \subset F^{i+j}\mathrm{CH}(X)_\mathbb{Q}$.*
4. *One has $F^i\mathrm{CH}^l(X)_\mathbb{Q} = 0$ for $i > l$.*

Note that if $\Gamma \in \mathrm{CH}(X \times Y)$ is a correspondence homologous to 0, then $\Gamma \in F^1\mathrm{CH}(X \times Y)_\mathbb{Q}$ and thus for any $z \in F^i\mathrm{CH}(X)_\mathbb{Q}$,

$$\Gamma_* z = pr_{2*}(pr_1^* z \cdot z) \in F^{i+1}\mathrm{CH}(Y)_\mathbb{Q}$$

by axioms 2 and 3. In other words, the morphisms $\Gamma_*:Gr_i^F\mathrm{CH}(X)_\mathbb{Q}\to Gr_i^F\mathrm{CH}(Y)_\mathbb{Q}$ must be controled by the cohomology class $[\Gamma]$.

We will explain in next section how Conjecture 2.15 implies Conjecture 2.9.

## 2.4 Nilpotence conjectures

Let $X$ be a smooth projective variety. According to section 1.2.2, $\mathrm{CH}(X \times X)_\mathbb{Q}$ has a ring structure given by composition of correspondences. This ring structure is compatible by Proposition 1.7 with the map

$$\mathrm{CH}(X \times X)_\mathbb{Q} \to \mathrm{End}\,\mathrm{CH}(X)_\mathbb{Q}.$$

The following nilpotence conjecture is a weak version of the smash nilpotence conjecture of Voevodsky:

**Conjecture 2.16.** *Let $X$ be a smooth projective variety. Then a correspondence $Z \in \mathrm{CH}(X \times X)_{\mathbb{Q},hom}$ is nilpotent: There exists $N \in \mathbb{N}^*$ such that $Z^{\circ N} = 0$ in $\mathrm{CH}(X \times X)_{\mathbb{Q}}$.*

Let us explain the following chain of implications:

$$\text{Conjecture 2.15} \Rightarrow \text{Conjecture 2.16} \Rightarrow \text{Conjecture 2.9.}$$

**Lemma 2.17.** *Let $X$ be smooth projective over $\mathbb{C}$. Conjecture 2.15 for the powers of $X$ and correspondences between them implies Conjecture 2.16 for $X$.*

*Proof.* Indeed, let $Z \in \mathrm{CH}(X \times X)_{\mathbb{Q},hom}$. Then according to Axiom 1 of Conjecture 2.15, one has $Z \in F^1\mathrm{CH}(X \times X)_{\mathbb{Q}}$. According to Axioms 3 and 2 of Conjecture 2.15, one then has $Z^{\circ N} \in F^N\mathrm{CH}(X \times X)_{\mathbb{Q}}$ for any integer $N$. Finally, Axiom 4 gives that $F^N\mathrm{CH}(X \times X)_{\mathbb{Q}} = 0$ for large $N$, so that $Z^{\circ N} = 0$ in $\mathrm{CH}(X \times X)_{\mathbb{Q}}$ for large $N$. □

**Lemma 2.18.** *Let $X$ be smooth projective over $\mathbb{C}$. Conjecture 2.16 for $X$ implies Conjecture 2.9 for $X$.*

*Proof.* Assume Conjecture 2.16 and let $X$ be strongly of coniveau $\geq c$. This means that there is a cohomological decomposition of the diagonal of $X$ which takes the form:

$$[\Delta_X] = [Z_1] + [Z_2] \text{ in } H_B^{2n}(X \times X, \mathbb{Q}), \tag{13}$$

where $Z_1$ is a decomposable cycle and $Z_2$ is a cycle in $X \times X$ which is supported on $D \times X$, $D \subset X$ being closed algebraic of codimension at least $c$. We rewrite (13) as

$$[\Gamma] = 0 \text{ in } H_B^{2n}(X \times X, \mathbb{Q}),$$

where $\Gamma := \Delta_X - Z_1 - Z_2$. Conjecture 2.16 now tells us that $\Gamma$ is nilpotent in $\mathrm{CH}(X \times X)_{\mathbb{Q}}$, and thus

$$\Gamma_* \in \operatorname{End}\mathrm{CH}(X)_{\mathbb{Q}}$$

is nilpotent. On the other hand, we claim that $\Gamma_*$ acts as identity on $\mathrm{CH}_i(X)_{\mathbb{Q},hom}$ for $i < c$. Indeed, $\Delta_X$ acts as identity on $\mathrm{CH}_i(X)_{\mathbb{Q}}$ for all $i$, $Z_{1*}$ acts as 0 on $\mathrm{CH}_i(X)_{\mathbb{Q},hom}$ for all $i$ because $Z_1$ is decomposable, and $Z_{2*}$ acts as 0 on $\mathrm{CH}_i(X)_{\mathbb{Q}}$ for $i < c$ because $Z_2$ is supported on $D \times X$ and $\operatorname{codim} D \geq c$. This proves the claim and we conclude that $\mathrm{CH}_i(X)_{hom,\mathbb{Q}} = 0$ for $i < c$. □

To conclude this section, let us state two results concerning the nilpotence conjecture:

**Theorem 2.19.** *(Voevodsky [30], Voisin [35]) Let $Z \in \mathrm{CH}(X \times X)$ be a correspondence algebraically equivalent to* 0*. Then $Z$ is nilpotent.*

This reduces the nilpotence conjecture to cycles in the Griffiths group of cycles homologous to 0 modulo algebraic equivalence. Via Lemma 2.18, this also reduces Conjecture 2.9 to the study of the Griffiths group of $X \times X$.

The next result is due to Kimura [17]. It proves the nilpotence conjecture for varieties with finite dimensional motive (in Kimura sense). We refer to [17] or [1] for this notion. Products of curves are finite-dimensional in Kimura sense, and so are all varieties dominated by products of curves.

**Theorem 2.20.** *(Kimura [17]) Let $X$ be a variety which is finite dimensional, for example a variety which is dominated by a product of curves. Then any self-correspondence of $X$ which is cohomologous to 0 is nilpotent.*

**Corollary 2.21.** *(Kimura [17]) Let $X$ be a smooth projective surface which satisfies $q(X) = p_g(X) = 0$ and is rationally dominated by a product of curves. Then $\mathrm{CH}_0(X) = \mathbb{Z}$ (X satisfies Bloch's conjecture 2.7).*

*Proof.* Indeed, the conditions $q(X) = p_g(X) = 0$ imply by the Lefschetz theorem on $(1, 1)$-classes that $X$ is strongly of geometric coniveau $\geq 1$. By Theorem 2.20, $X$ satisfies the nilpotence conjecture and by Lemma 2.18, we conclude that $X$ satisfies Conjecture 2.9, hence $\mathrm{CH}_0(X)_{\mathbb{Q}} = \mathbb{Q}$. The fact that $\mathrm{CH}_0(X) = \mathbb{Z}$ is then a consequence of Roitman's theorem [25] saying that if $H^1_B(X, \mathbb{Z}) = 0$, $\mathrm{CH}_0(X)$ has no torsion.

□

# 3 Griffiths group, families and algebraic cycles

## 3.1 Griffiths and Nori's theorems

Let $X$ be a smooth hypersurface in $\mathbb{P}^n$. Then if $n \geq 4$, $H_{2,B}(X, \mathbb{Z}) = \mathbb{Z}$ by Lefschetz theorem on hyperplane sections. Here the isomorphism is given by the degree, that is, the intersection number with the hyperplane section. If $X$ is a Calabi-Yau hypersurface, $X$ contains lines, which are of degree 1, hence all have the same class in $H_{2,B}(X, \mathbb{Z})$. When $n = 4$, the general hypersurface $X$ contains finitely many lines. Griffiths proved in [14] the following result:

**Theorem 3.1.** *Let $X \subset \mathbb{P}^4$ be a very general quintic threefold. Then the difference $l - l'$ of two lines in $X$ is a 1-cycle which is not algebraically equivalent to 0, even modulo torsion.*

The proof involves Griffiths' Abel-Jacobi map, and the fact that the intermediate Jacobian

$$J^3(X) := H^3_B(X, \mathbb{C})/F^2H^3_B(X, \mathbb{C}) \oplus H^3_B(X, \mathbb{Z}),\ F^2H^3_B(X, \mathbb{C}) := H^{3,0}(X) \oplus H^{2,1}(X),$$

a complex torus which replaces the Jacobian of a curve, does not contain for very general $X$ any nontrivial subtorus whose tangent space is contained in $H^{1,2}(X)$ (here we identify canonically the tangent space of $J^3(X)$ to $H^{1,2}(X) \oplus H^{0,3}(X)$). On the other hand, Griffiths proves that the Abel-Jacobi map

$$\Phi_X : \mathrm{CH}^2(X)_{hom} \to J(X)$$

(see Section 1.2.1) maps cycles algebraically equivalent to 0 to the maximal subtorus of $J^3(X)$ whose tangent space is contained in $H^{1,2}(X)$. This shows that for $X$ very general as above, the Abel-Jacobi map factors through the Griffiths group $\mathrm{CH}^2(X)_{hom}/\sim alg$. The end of the argument is contained in the following proposition:

**Proposition 3.2.** *[14] Let $X$ be a general quintic threefold. Then for two different lines $l, l'$ in $X$, $\Phi_X(l-l')$ is not a torsion point in $J^3(X)$.*

**Remark 3.3.** It has been proved only recently (see [8], [27]) that the vanishing locus of a normal function is algebraic. In our case, we have a normal function defined on the space parameterizing the triples $t = (X, l, l')$ where $X$ is a smooth quintic threefold and $l$, $l'$ are two distinct lines in $X$. The normal function is the section of the family of intermediate Jacobians $J^3(X_t)$ given by $\nu(t) = \Phi_X(l-l') \in J^3(X)$. In any case, Theorem 3.1 will only hold for very general $X$ because the previous step works only for very general $X$.

Griffiths' theorem 3.1 proves the nontriviality of the Griffiths group using the Abel-Jacobi invariant. In the case of cycles of codimension $> 2$, Nori proves the existence of cycles annihilated by the Abel-Jacobi map and not algebraically equivalent to 0 modulo torsion.

**Theorem 3.4.** *(Nori [21]) Let $X$ be the very general complete intersection of two sufficiently large degree hypersurfaces in a smooth projective variety $Y$ of dimension $2n \geq 6$. Let $Z \in \mathrm{CH}^n(Y)$ be a cycle such that $[Z] \neq 0$ in $H_B^{2n}(Y, \mathbb{Q})$. Then $Z_{|X}$ is not algebraically equivalent to 0 modulo torsion.*

Consider the case where $Y$ is an even dimensional quadric in $\mathbb{P}^{2n+1}$ and let $Z$ be the difference of two rullings in $Y$. Then the restriction $Z_{|X}$ is cohomologous to 0 and Abel-Jacobi equivalent to 0, since by Lefschetz theorem on hyperplane sections, the cohomology group $H_B^{2n}(X, \mathbb{Q})$ is isomorphic to $\mathbb{Q}$ and generated by $c_1(\mathcal{O}_X(1))^n$, and the cohomology group $H_B^{2n-1}(X, \mathbb{Q})$ vanishes. This gives the desired example. On the other hand, Nori made the following conjecture for algebraic cycles of codimension 2:

**Conjecture 3.5.** *Let $Z$ be a codimension 2 cycle on a smooth projective complex variety $X$. Then if $Z$ is cohomologous to 0 and annihilated by the Abel-Jacobi map, $Z$ is algebraically equivalent to 0.*

This conjecture is very important. In fact we have the following result saying that it implies Bloch's conjecture 2.7 on surfaces with $p_g = 0$.

**Lemma 3.6.** *Let $S$ be a smooth complex projective surface with $p_g(S) = 0$. Then $F^2\mathrm{CH}_0(S) = 0$ if Nori's conjecture is satisfied by $S \times S$.*

*Proof.* Let us assume for simplicity that $q(S) = 0$ and let $X = S \times S$. Then by our assumption, the cohomology of $S$ is generated by classes of algebraic cycles, and by Künneth decomposition, so is the cohomology of $X$. Thus we can write

$$[\Delta_S] = [S \times s] + [s \times S] + \sum_i \alpha_i [C_i \times C_j] \text{ in } H^4(X, \mathbb{Q}).$$

As the intermediate Jacobian $J^3(X)$ is 0, the cycle

$$Z := \Delta_S - S \times s - s \times S - \sum_i \alpha_i C_i \times C_j$$

on $X$ is annihilated by the Abel-Jacobi map hence is algebraically equivalent to 0 if Nori's conjecture is satisfied by $X$. By the nilpotence theorem 2.19, $Z$ is then nilpotent in $\mathrm{CH}^2(X)_{\mathbb{Q}}$. But then $Z_* : \mathrm{CH}_0(S)_{\mathbb{Q}} \to \mathrm{CH}_0(S)_{\mathbb{Q}}$ is also nilpotent. As it acts as the identity on $\mathrm{CH}_0(S)_{hom,\mathbb{Q}}$, one concludes that $\mathrm{CH}_0(S)_{hom,\mathbb{Q}} = 0$. □

We conclude this section by a remark on Chow groups of the universal family which will have an important development in next section. Let $\mathcal{X} \to B$ be the universal family of smooth degree 5 hypersurfaces in $\mathbb{P}^4$. Here $B \subset \mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5)))$ is the Zariski open set parametrizing smooth hypersurfaces and

$$\mathcal{X} \subset B \times \mathbb{P}^4,\ \mathcal{X} = \{(f,x),\ f(x) = 0\} \tag{14}$$

is the universal hypersurface. The above mentioned results of Griffiths show that for some generically finite cover $B'$ of $B$, the pulled-back family $\mathcal{X}' = B' \times_B \mathcal{X}$ acquires some interesting codimension 2 cycle. Note that it is essential here to allow a generically finite base-change. Indeed, we have the following lemma:

**Lemma 3.7.** *Let $U \subset B$ be a dense Zariski open set. Then the natural map*

$$p_2^* : \mathrm{CH}^*(\mathbb{P}^4)_{\mathbb{Q}} \to \mathrm{CH}^*(\mathcal{X}_U)_{\mathbb{Q}}$$

*is surjective.*

*Proof.* Looking at the definition (14), we see that $\mathcal{X}_U$ is Zariski open in

$$\overline{\mathcal{X}} := \{(f,x),\ f(x) = 0\} \subset \mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5))) \times \mathbb{P}^4\}. \tag{15}$$

The localization exact sequence (see [12, 1.8]) says that the restriction map $\mathrm{CH}(\overline{\mathcal{X}}) \to \mathrm{CH}(\mathcal{X}_U)$ is surjective. On the other hand, it is clear from (15) that via the second projection $p_2 : \overline{\mathcal{X}} \to \mathbb{P}^4$, $\overline{\mathcal{X}}$ is a projective bundle $\mathbb{P}$ over $\mathbb{P}^4$, with line bundle $\mathcal{O}_{\mathbb{P}}(1)$ given by

$$p_1^*\mathcal{H},\ \mathcal{H} := \mathcal{O}_{\mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5)))}(1).$$

Denoting $h = c_1(\mathcal{H})$, the computation of the Chow groups of a projective bundle then shows that $\mathrm{CH}(\overline{\mathcal{X}})$ is generated over $p_2^*(\mathrm{CH}^*(\mathbb{P}^4))$ by the powers of $p_1^*h$. Thus the same property holds on $\mathcal{X}_U$. On the other hand, as $B \subset \mathbb{P}(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5)))$ is the complement of a hypersurface (the discriminant hypersurface) whose class is a multiple of $h$, one has

$$h^i_{|B} = 0 \text{ in } \mathrm{CH}^i(B)_{\mathbb{Q}}$$

for $i > 0$, and this vanishing holds a fortiori on $U$ and thus $(p_1^*h)^i$ vanishes in $\mathrm{CH}(\mathcal{X}_U)_{\mathbb{Q}}$ for $i > 0$, which finishes the proof.

□

## 3.2 The generalized Bloch conjecture for very general complete intersections

Our aim in this section is to describe the proof of Theorem 2.11. Let $Y$ be a smooth projective variety with trivial Chow groups (see Definition 2.10). We know by Lewis' work [19] that the Betti cohomology of $Y$ with $\mathbb{Q}$-coefficients is then algebraic (in particular it vanishes in odd degrees). Let now $X \stackrel{j}{\hookrightarrow} Y$ be an ample hypersurface. By Lefschetz theorem on hyperplane sections, we have

$$H_B^*(X, \mathbb{Q}) = H_B^*(Y, \mathbb{Q})_{|X} \oplus H_B^n(X, \mathbb{Q})_{van},$$

where $n = \dim X$ and

$$H_B^n(X, \mathbb{Q})_{van} := \mathrm{Ker}\,(j_* : H_B^n(X, \mathbb{Q}) \to H_B^{n+2}(Y, \mathbb{Q})).$$

This sum is an orthogonal sum with respect to the intersection pairing on $X$. It follows that the cohomology class of the diagonal of $X$ decomposes as

$$[\Delta_X] = \pi_{alg} + \pi_{van}$$

where $\pi_{alg} \in \mathrm{End}\, H_B^*(X, \mathbb{Q})$ is the orthogonal projector onto the first summand $H_B^*(Y, \mathbb{Q})_{|X}$ and $\pi_{van} \in \mathrm{End}\, H_B^*(X, \mathbb{Q})$ is the orthogonal projector onto the second summand. We have $\pi_{alg} \in H_B^*(Y, \mathbb{Q})_{|X} \otimes H_B^*(Y, \mathbb{Q})_{|X}$ and since the cohomology of $Y$ is algebraic, it follows that $\pi_{alg}$ is the class of an algebraic cycle $\Delta_{X,alg}$ on $X \times X$, which is the restriction of an algebraic cycle on $Y \times Y$. This cycle is not uniquely determined, but it can be constructed explicitly from the intersection matrix of the intersection pairing $(\ ,\ )_X$ on $H^*(Y, \mathbb{Q})_{|X}$. We will denote $\Delta_{X,van} := \Delta_X - \Delta_{X,alg}$. This is an algebraic cycle on $X \times X$. As $\Delta_{X,alg}$ is the class of a decomposable cycle and the cohomology class of $\Delta_{X,van}$ is a projector on (part of) $H^n(X, \mathbb{Q})$, the condition that $X$ is strongly of geometric coniveau $\geq c$ is equivalent to the fact that $[\Delta_{X,van}] = [Z] \in H_B^{2n}(X \times X, \mathbb{Q})$ for some algebraic cycle $Z$ on $X \times X$ supported on $D \times X$, where $D \subset X$ is of codimension $\geq c$. Theorem 2.11 says the following:

**Theorem 3.8.** *Assume the line bundle $\mathcal{L} = \mathcal{O}_Y(X)$ is very ample and that smooth members of $|\mathcal{L}|$ have nontrivial vanishing cohomology. Then if a very general member $X$ of $|\mathcal{L}|$ is strongly of geometric coniveau $\geq c$, one has*

$$\mathrm{Ker}\,(cl : \mathrm{CH}_i(X)_{\mathbb{Q}} \to H_B^{2n-2i}(X, \mathbb{Q})) = 0.$$

To start the proof, we first claim that it suffices to prove that the cycle $\Delta_{X,van} - Z \in \mathrm{CH}^n(X \times X)_{\mathbb{Q}}$ is the restriction of a cycle $\Gamma$ on $Y \times Y$. Indeed, if we have

$$\Delta_{X,van} - Z = \Gamma_{|X \times X} \in \mathrm{CH}^n(X \times X)_{\mathbb{Q}} \tag{16}$$

for some cycle $\Gamma$ on $Y \times Y$, then as well

$$\Delta_X - Z = \Gamma'_{|X \times X} \in \mathrm{CH}^n(X \times X)_{\mathbb{Q}} \tag{17}$$

for some cycle $\Gamma'$ on $Y \times Y$ since $\Delta_X - \Delta_{X,van}$ is the restriction of a cycle on $Y \times Y$. Now we let both sides of (17) act on $\mathrm{CH}_i(X)_{\mathbb{Q}}$ and we observe that $Z_*$ acts as 0 on $\mathrm{CH}_i(X)_{\mathbb{Q}}$ for $i < c$, since $Z$ is supported on $D \times X$, with $\operatorname{codim} D \geq c$. On the other hand, for any codimension $n$ cycle $\Gamma'$ on $Y \times Y$ and for any cycle $z \in \mathrm{CH}_i(X)_{\mathbb{Q}}$, one has

$$(\Gamma'_{|X \times X})_* z = (\Gamma'_*(j_* z))_{|X} \text{ in } \mathrm{CH}_i(X)_{\mathbb{Q}},$$

where $j : X \to Y$ is the inclusion map. Hence we conclude from (17) that for $i < c$,

$$z = \Delta_{X*} z = (\Gamma'_*(j_* z))_{|X} \text{ in } \mathrm{CH}_i(X)_{\mathbb{Q}}. \tag{18}$$

If now $z$ is cohomologous to 0, $j_* z$ is cohomologous to 0 on $Y$ and since $Y$ has trivial Chow groups, it is rationally equivalent on $Y$. Hence the right hand side is 0, and we proved that $\mathrm{CH}_i(X)_{\mathbb{Q},hom} = 0$ for $i < c$. The claim is proved.

The proof of Theorem 3.8 now rests on the following proposition 3.9: Let $\mathcal{X} \to B$ be the universal family of smooth hypersurfaces in $|\mathcal{L}|$. Let $\mathcal{X} \times_B \mathcal{X} \to B$ be the fibered self-product. The relative diagonal $\Delta_{\mathcal{X}/B} \in \mathrm{CH}^n(\mathcal{X} \times_B \mathcal{X})_{\mathbb{Q}}$ splits as

$$\Delta_{\mathcal{X}/B} = \Delta_{van/B} + \Delta_{alg/B}, \tag{19}$$

where $\Delta_{alg/B} \in \mathrm{CH}^n(\mathcal{X} \times_B \mathcal{X})_{\mathbb{Q}}$ comes from a cycle on $Y \times Y$ and $\Delta_{van/B}$ is defined by formula (19).

**Proposition 3.9.** *Assume that the very general fiber $X = \mathcal{X}_b$ is strongly of geometric coniveau coniveau $\geq c$. Then there exists a codimension $c$ closed algebraic subset $\mathcal{W} \subset \mathcal{X}$ and a cycle $\mathcal{Z}$ of $\mathcal{X} \times_B \mathcal{X}$ supported on $\mathcal{W} \times_B \mathcal{X}$ such that for any $b \in B$,*

$$[\Delta_{\mathcal{X}_b,van}] = [\mathcal{Z}_b] \text{ in } H^{2n}_B(\mathcal{X}_b \times \mathcal{X}_b, \mathbb{Q}).$$

We refer to [31] for the proof of this proposition. The crucial point here is the fact that the cycle $\mathcal{Z}$ exists over the base $B$ itself while most "spreading statements" for algebraic cycles necessitate to work on a generically finite cover of the base. The proof of Theorem 3.8 then concludes with the following proposition.

**Proposition 3.10.** *Assume the line bundle $\mathcal{L} = \mathcal{O}_Y(X)$ is very ample and that smooth members of $|\mathcal{L}|$ have nontrivial vanishing cohomology. Then for any $b \in B$, there is a cycle $\Gamma \in \mathrm{CH}(Y \times Y)$ such that*

$$\Delta_{\mathcal{X}_b,van} - \mathcal{Z}_b = \Gamma_{|\mathcal{X}_b \times \mathcal{X}_b} \in \mathrm{CH}^n(\mathcal{X}_b \times \mathcal{X}_b)_{\mathbb{Q}}. \tag{20}$$

According to the claim above, Proposition 3.10 suffices to conclude the proof of Theorem 3.8.

The proof of Proposition 3.10 is similar to the proof of Lemma 3.7. Indeed it plays on the description of the Chow groups of $\mathcal{X} \times_B \mathcal{X}$ using the natural map $\mathcal{X} \times_B \mathcal{X} \to Y \times Y$. Actually, in order to give this map a nice structure, we need to blow-up the diagonals on both sides. Then using the fact that the line bundle $\mathcal{L}$ is very ample, we find that the morphism

$$\widetilde{\mathcal{X}\times_B\mathcal{X}}_\Delta\to\widetilde{Y\times Y}_\Delta$$

represents $\widetilde{\mathcal{X}\times_B\mathcal{X}}_\Delta$ a Zariski open set in a projective bundle over $\widetilde{Y\times Y}_\Delta$.

## 3.3 Nilpotence results for surfaces

We conclude with a result proved in [33] concerning the nilpotence conjecture 2.16 for surfaces. Our assumptions are the following: Let $\mathcal{S}\to B$ be a smooth family of projective surfaces. We will assume that the fibered product $\mathcal{S}\times_B\mathcal{S}$ has a rationally connected smooth projective completion $\overline{\mathcal{S}\times_B\mathcal{S}}$. In fact, the property which is of interest for us is the much weaker property that cycles of codimension 2 on $\overline{\mathcal{S}\times_B\mathcal{S}}$ which are cohomologous to 0 are algebraically equivalent to 0. That the first property (or only the weaker property that $\mathrm{CH}_0(\overline{\mathcal{S}\times_B\mathcal{S}})=\mathbb{Z}$) implies the second one is a result due to Bloch and Srinivas [7]. Our result is the following:

**Theorem 3.11.** *(Voisin [33]) Let $\mathcal{S}\to B$ be a family satisfying one of the conditions discussed above. Assume the fibers $\mathcal{S}_b$ are regular, that is $q(\mathcal{S}_b)=0$. Let $\mathcal{Z}\in\mathrm{CH}^2(\mathcal{S}\times_B\mathcal{S})$ and assume that $\mathcal{Z}_b\in\mathrm{CH}^2(\mathcal{S}_b\times\mathcal{S}_b)$ is cohomologous to $0$ for any $b\in B$. Then $\mathcal{Z}_b$ is nilpotent for any $b\in B$.*

This result is used in [33] to prove new cases of the Bloch conjecture 2.7:

**Corollary 3.12.** *Let $\mathcal{S}\to B$ be a family as in Theorem 3.11. Then if the fibers $\mathcal{S}_b$ satisfy $q=p_g=0$, they have $\mathrm{CH}_0(\mathcal{S}_b)_{hom}=0$.*

*Proof.* Let $\mathcal{L}$ be a relatively ample line bundle on $\mathcal{S}$. Then if $d=\deg c_1(\mathcal{L}_{|\mathcal{S}_b})^2$, the codimension 2 cycle

$$\Delta_{\mathcal{S}/B}-\frac{1}{d}pr_2^*\mathcal{L}^2-\frac{1}{d}pr_1^*\mathcal{L}^2$$

has the property that its restriction to $\mathcal{S}_b$ has its cohomology class decomposable, that is, a combination of classes of products of curves on $\mathcal{S}_b$. One thus concludes by Proposition 3.9 that there exist a divisor $\mathcal{D}\subset\mathcal{S}$ and a cycle $\mathcal{Z}_1\in\mathrm{CH}^2(\mathcal{S}\times_B\mathcal{S})_{\mathbb{Q}}$ supported on $\mathcal{D}\times_B\mathcal{S}$ such that the cycle

$$\mathcal{Z}:=\Delta_{\mathcal{S}/B}-\frac{1}{d}pr_2^*\mathcal{L}^2-\frac{1}{d}pr_1^*\mathcal{L}^2-\mathcal{Z}_1$$

has the property that the restriction $\mathcal{Z}_b$ is cohomologous to 0 for any $b\in B$. We can then apply Theorem 3.11 and conclude that $\mathcal{Z}_b$ is nilpotent in $\mathrm{CH}^2(\mathcal{S}_b\times\mathcal{S}_b)_{\mathbb{Q}}$ for any $b\in B$. As $\mathcal{Z}_b$ differs from $\Delta_{\mathcal{S}_b}$ by a decomposable cycle, $\mathcal{Z}_b$ acts as identity on $\mathrm{CH}_0(\mathcal{S}_b)_{hom,\mathbb{Q}}$, hence $\mathrm{CH}_0(\mathcal{S}_b)_{hom,\mathbb{Q}}=0$. One then concludes by Roitman's theorem [25]. □

Let us say a word about the proof of Theorem 3.11. Examining the spectral sequence of the map $\pi:\mathcal{S}\times_B\mathcal{S}\to B$ (or rather, the corresponding continuous map of analytic spaces), we first prove that under our assumptions, there are codimension 2 cycles $\mathcal{Z}_1$, $\mathcal{Z}_2$ on $\mathcal{S}$ such that the corrected cycle

$$\mathcal{Z}' = \mathcal{Z} - pr_1^*\mathcal{Z}_1 - pr_2^*\mathcal{Z}_2$$

is cohomologous to 0 on $\mathcal{S}\times_B\mathcal{S}$. Note that the original cycle $\mathcal{Z}$ satisfied the weaker property that its class $[\mathcal{Z}] \in H^4_B(\mathcal{S}\times_B\mathcal{S},\mathbb{Q})$ vanishes in the first Leray quotient $H^0(B, R^4\pi_*\mathbb{Q})$. We then observe that it suffices to prove the nilpotence property for $\mathcal{Z}'$ and we now use the following proposition.

**Proposition 3.13.** *Let $U \subset X$ be a Zariski open set in a smooth projective variety and let $Z \in \mathrm{CH}^2(U)_{\mathbb{Q}}$ be a codimension 2 cycle which is cohomologous to 0. Then there exists a cycle $\overline{Z} \in \mathrm{CH}^2(X)_{\mathbb{Q}}$ which is cohomologous to 0 on $X$ such that $Z_{|U} = Z$.*

We apply this proposition to $U = \mathcal{S}\times_B\mathcal{S}$, $X = \overline{\mathcal{S}\times_B\mathcal{S}}$ and $Z = \mathcal{Z}'$. We thus conclude that there is a codimension 2 cycle $\overline{\mathcal{Z}'} \in \mathrm{CH}^2(\overline{\mathcal{S}\times_B\mathcal{S}})_{\mathbb{Q}}$ which is cohomologous to 0 and such that

$$\overline{\mathcal{Z}'}_{|\mathcal{S}\times_B\mathcal{S}} = \mathcal{Z}' \text{ in } \mathrm{CH}^2(\mathcal{S}\times_B\mathcal{S})_{\mathbb{Q}}.$$

We now use the fact that codimension 2 cycles cohomologous to 0 on $\overline{\mathcal{S}\times_B\mathcal{S}}$ are algebraically equivalent to 0. Thus $\overline{\mathcal{Z}'}$ is algebraically equivalent to 0, and so is its restriction $\mathcal{Z}'_b = \mathcal{Z}'_{|\mathcal{S}_b\times\mathcal{S}_b}$. By Theorem 2.19, $\mathcal{Z}'_b$ is nilpotent.

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