4ECM Stockholm 2004

Recent Progresses in Kähler and Complex Algebraic Geometry

Claire Voisin

0. Introduction

On a complex vector space V, a Hermitian bilinear form h is decomposed into real and imaginary parts as

$$h = g - i\omega,$$

where g is a symmetric real bilinear form and ω is a real 2-form which is of type $(1,1)$ for the complex structure on V. Here the notion of (complex valued) form of type (p,q) on V is the following: the space $V^* \otimes \mathbb{C}$ of complex valued forms on V splits as a direct sum of $V^{*1,0} \oplus V^{*0,1}$, where $V^{*1,0}$ is the space of $\mathbb{C}$-linear forms and $V^{*0,1}$ is its complex conjugate. Then the forms of type (p,q) are generated by the $\alpha_1 \wedge \cdots \wedge \alpha_p \wedge \beta_1 \wedge \cdots \wedge \beta_q$, where $\alpha_i \in V^{*1,0}$ and $\beta_j \in V^{*0,1}$.

The correspondence $h \mapsto \omega$ is a bijection between Hermitian bilinear forms and real forms of type $(1,1)$ on V. Thus the notion of (semi)-positivity for Hermitian bilinear forms provides a corresponding notion of (semi)-positivity for real forms of type $(1,1)$. Note that when h is positive definite, ω is non-degenerate, i.e., $\omega^n \neq 0$, $n = rk_{\mathbb{C}} V$.

On a complex manifold X, the tangent space $T_{X,x}$ at any point is endowed with a complex structure, and the above correspondence induces a bijective correspondence between Hermitian bilinear forms on T_X, and real 2-forms of type $(1,1)$ on X, that is of type $(1,1)$ on $T_{X,x}$ for any $x \in X$. In particular, if h is a Hermitian metric on T_X, one can write

$$h = g - i\omega,$$

where g is a Riemannian metric (compatible with the complex structure), and ω is a positive real $(1,1)$-form.

Definition 0.1. The metric h is said to be Kähler if furthermore the 2-form ω is closed.

Since ω is non-degenerate, it will provide in particular a symplectic structure on the Kähler manifold X, thus putting Kähler geometry at the intersection of symplectic geometry and complex geometry. The work of Gromov [20] made the relation between symplectic and Kähler geometry stronger: a symplectic manifold (X, ω) can be endowed with a compatible almost complex structure (i.e., a complex structure on each tangent space $T_{X,x}$, varying in a smooth way), which is well defined up to deformations. Here "compatible" means that ω_x has to be of type $(1,1)$ and positive on each $T_{X,x}$ for the given complex structure.

Assuming X is compact, the Kähler assumption has for main differential-topological consequence the Hodge decomposition theorem.

Theorem 0.2. *If X is Kähler compact, the de Rham cohomology spaces*

$$H^k(X,\mathbb{C}) := \{\textit{closed complex valued } k\textit{-forms}\}/\{\textit{exact ones}\}$$

splits as

$$H^k(X,\mathbb{C}) = \oplus_{p+q=k} H^{p,q}(X), \tag{0.1}$$

where $H^{p,q}(X)$ is the space of classes admitting a representative which is a closed form of type (p,q) (that is of type (p,q) at any point).

Note that by the definition of $H^{p,q}(X)$, one has

$$\overline{H^{p,q}(X)} = H^{q,p}(X),$$

a property which is called Hodge symmetry. The data of the decomposition (0.1), together with the rational (integral) structure of $H^k(X,\mathbb{C})$, that is the isomorphism

$$H^k(X,\mathbb{C}) = H^k(X,\mathbb{Q}) \otimes \mathbb{C},\ (H^k(X,\mathbb{C}) = H^k(X,\mathbb{Z}) \otimes \mathbb{C}),$$

is exactly what is called a rational (integral) Hodge structure of weight k (see [19], [10], [36] I, 7.1.1).

A deeper consequence of Hodge theory is the formality theorem [11], which says that the rational homotopy type of a compact Kähler manifold is determined by its rational cohomology ring, but we won't use it here.

Let us now turn to complex projective manifolds. They are defined as the complex submanifolds of $\mathbb{P}^N(\mathbb{C})$. A classical theorem due to Chow, later generalized by Serre [29], says that they are also the smooth algebraic subvarieties of projective space.

The projective space is Kähler, and by restriction of any Kähler metric, it follows that complex projective manifolds are Kähler.

The projective space also carries a holomorphic line bundle $L = \mathcal{O}(1)$, whose holomorphic sections identify to linear forms on $\mathbb{C}^{N+1}$. Namely, L is defined as the dual of the tautological sub-line bundle whose fiber at $u \in \mathbb{P}^N(\mathbb{C})$ is the line generated by u in $\mathbb{C}^{N+1}$. If $X \subset \mathbb{P}^N(\mathbb{C})$ is a complex submanifold, the induced holomorphic line bundle admit as holomorphic sections the restrictions $\sigma_0, \ldots, \sigma_{N+1}$ of the linear forms on $\mathbb{C}^{N+1}$, and these sections have the property that for any point $x \in X$, at least one of these does not vanish on the fiber L_x, and that the map

$$x \mapsto (\sigma_0(x), \ldots, \sigma_N(x)),$$

(a non-zero $N+1$-tuple which is well defined up to a multiplicative coefficient, according to the trivialization of L_x chosen), is holomorphic and provides the initial holomorphic embedding to $\mathbb{P}^N(\mathbb{C})$.

Definition 0.3. A line bundle on a compact complex manifold is said to be very ample if its holomorphic sections provide as above an embedding to projective space. It is said to be ample if some power $L^{\otimes k}$ is very ample.

Since the pioneering work of Kodaira [23], line bundles in complex projective geometry can be considered to have as Kähler analogues real $(1,1)$-classes in Kähler compact geometry.

We survey in this paper classical and recent results which underline both the similarities and the differences between Kähler and complex projective geometries.

The first section is devoted to the results by Demailly and his collaborators showing a complete similarity between various notions of positivity for line bundles on projective manifolds and for real closed forms of type $(1,1)$ on Kähler compact manifolds.

Sections 2 and 3 show in contrast strong differences between these geometries. On the analytic side, we show that the Hodge conjecture cannot be possibly extended to Kähler compact manifolds: Hodge classes are not necessarily generated over $\mathbb{Q}$ by Chern classes of coherent sheaves. We also show that coherent sheaves on compact Kähler manifolds do not necessarily admit locally free resolutions, while the existence of locally free resolutions in algebraic geometry plays a key role in the proof of central theorems (see, e.g., [30], [2]).

On the topological side, we show that there exist Kähler compact manifolds which do not have the homotopy type of, and a fortiori cannot be deformed to, complex projective manifolds.

It is interesting to note that these differences appear only in higher dimensions. Any compact complex curve is projective (hence Kähler). Any compact Kähler surface has small deformations which are projective (a result due to Kodaira). The Hodge conjecture is true for degree 2 classes on complex manifolds in the form of the Lefschetz theorem on $(1,1)$-classes, and coherent sheaves on compact complex surfaces admit finite locally free resolutions [28].

1. Positivity properties of line bundles and $(1,1)$-classes

1.1. Line bundles and their Chern forms and currents. Let L be a holomorphic line bundle on a complex manifold and h be an Hermitian metric on L. On small open sets U of X, we can choose non-zero holomorphic sections σ_U trivializing L. The function $h(\sigma_U)$ is thus positive where defined and we can define the real $(1,1)$-form

$$\omega_{L,h,U} := \frac{1}{2i\pi}\partial\overline{\partial}\log\, h(\sigma_U).$$

It is immediate to see that this form does not depend on the choice of σ_U (this follows from the vanishing $\partial\overline{\partial}\log \mid g \mid^2 = 0$ for g an invertible holomorphic function), so that the $\omega_{L,h,U}$ coincide on the overlaps and we have in fact a globally defined $(1,1)$-form $\omega_{L,h}$ called the Chern form of (L,h).

If h is changed to $e^u h$, for some real function u, $\omega_{L,h}$ is changed to $\omega_{L,h} + \frac{1}{2i\pi}\partial\overline{\partial}u$, from which it follows that the forms $\omega_{L,h}$ determine a class $c(L)$ in the space $H^{1,1}_{\partial\overline{\partial}}(X,\mathbb{R})$, defined as the quotient of the space of d-closed real forms of $(1,1)$-type by the space consisting of $i\partial\overline{\partial}f$, f a real functions on X.

Note that since the later space consists of d-exact forms, there is a natural map

$$H^{1,1}_{\partial\overline{\partial}}(X,\mathbb{R}) \to H^2(X,\mathbb{R}),$$

which is an isomorphism onto the subspace

$$H^{1,1}_{\mathbb{R}}(X) := H^{1,1}(X) \cap H^2(X,\mathbb{R})$$

when X is Kähler, by the $\partial\overline{\partial}$-lemma (cf. [36] I, 6.1.3).

The image of $c(L)$ under this map is the real Chern class $c_1(L)$, which is a topological invariant of L.

As we shall see in next sections, a deep use is made by analysts of a singular version of this construction. Namely, introduce singular metrics on L, which are locally of the form

$$h_{\text{sing}} = e^{\phi}h,$$

where h is a smooth metric, and ϕ is an integrable function. Then one can define locally the closed current $T_{L,h_{\text{sing}}}$ by the formula

$$T_{L,h_{\text{sing}}} = \omega_{L,h} + \frac{1}{2i\pi}\partial\overline{\partial}\log\phi.$$

This is a real closed current of type $(1,1)$, that is a linear form on the space of compactly supported forms of degree $2n-2$ on X, $n = \dim X$, which is real on real forms, and vanishes on forms of type $(p,q) \neq (n-1,n-1)$.

Positivity or semi-positivity of $(1,1)$-forms makes sense as explained in the introduction. Similarly, positivity of $(1,1)$-currents is defined as follows:

Definition 1.1. A current T is said to be positive if $T(\alpha) \geq 0$, for any $(n-1,n-1)$-form α which can be locally written as

$$\alpha = (i\alpha_1 \wedge \overline{\alpha}_1) \wedge \cdots \wedge (i\alpha_{n-1} \wedge \overline{\alpha}_{n-1}),$$

where the α_i's are of type $(1,0)$, and more generally, on any combination of such forms with coefficients given by non-negative real functions.

A typical example of a positive $(1,1)$-current is the current of integration on an analytic hypersurface of X.

There are on the other hand two different notions of positivity for line bundles: that of ampleness, (see the introduction), and that of effectivity, where L is said to be effective if there is a non-zero holomorphic section of L on X. This last notion is in fact better behaved if one introduces the notion of pseudo-effectiveness:

Definition 1.2. (see [13]) A line bundle L on X is said to be pseudo-effective if its class $c(L)$ is in the closure of the set of classes $c(L')$, for L' effective.

These two notions of positivity are strongly different. Indeed, an effective line bundle may become negative after restriction to the zero locus of one of its sections, hence may be very far from ample. The typical example is

$$L = \mathcal{O}_X(E_p) := \mathcal{I}_{E_p}^{-1}.$$

Here $\tau : X_p \to X$ is the blow-up of a point $p \in X$, and E_p is the exceptional divisor. Its ideal sheaf $\mathcal{I}_{E_p}$ is a holomorphic line bundle, whose inverse admits a canonical section whose 0-divisor is E_p. On the other hand $L_{|E_p}$ is negative.

It turns out that these two notions correspond respectively to the notions of positivity for $(1,1)$-forms and $(1,1)$-currents:

Lemma 1.3. *If L is ample on X, there exists an Hermitian metric h on L whose Chern form $\omega_{L,h}$ is positive.*

This follows from the corresponding statement for projective space. The Fubini-Study Kähler form on projective space is the Chern form of an adequate metric on the line bundle $\mathcal{O}(1)$.

Lemma 1.4. *If L is pseudo-effective, there exists a singular Hermitian metric h_{sing} on L such that the associated Chern current $T_{L,h_{\text{sing}}}$ is positive.*

When a multiple of L is effective, let σ be a non-zero section of $L^{\otimes m}$. The metric on L will be defined as $h_m^{\frac{1}{m}}$, where h_m is the singular Hermitian metric on L_m for which $h_m(\sigma) = 1$. The associated current is easily shown to be $\frac{1}{m}\int_D$, where D is the divisor of σ.

The converse statements are central in complex algebraic geometry.

Theorem 1.5. (*Kodaira* [23]) *A line bundle L on a compact complex manifold X is ample if and only if it admits a metric h, such that $\omega_{L,h}$ is a positive $(1,1)$-form.*

Theorem 1.6. (*Demailly* [13]) *A line bundle on a projective complex manifold X is pseudo-effective if and only if it admits a singular Hermitian metric whose associated Chern current is positive.*

Kodaira's theorem has been extended by Siu [31], [32] to the semi-positive case.

Theorem 1.7. *Let L be a line bundle on a compact complex manifold X, which admits a Hermitian metric whose Chern form is semi-positive, and satisfies $\int_X c_1(L)^n > 0$, where $n = dim\, X$. Then $h^0(X, L^{\otimes m})$ grows like m^n with m and X is Moishezon.*

(Recall that a Moishezon manifold is a compact complex manifold which is birationally equivalent to a projective manifold.)

The assumptions in the above theorems are not of an algebraic nature. The following result, in contrast, gives a purely algebraic criterion for ampleness of line bundles:

Theorem 1.8. *(Nakai-Moishezon criterion) A line bundle on a (complex) projective manifold X is ample if and only if, for any subvariety $Y \subset X$ of dimension p, one has*

$$\int_Y c_1(L)^p > 0.$$

The proof is by induction on the dimension, using Riemann-Roch theorem and Serre's vanishing theorem. Note that, unlike Kodaira's theorem, one has to assume first that X is projective.

One might ask what happens when the inequalities in the Nakai-Moishezon criterion become large. The line bundles which satisfy the conditions

$$\int_Y c_1(L)^p \geq 0, \text{ for any } Y \subset X \text{ of dimension } p$$

are called nef (numerically effective). Applying the Nakai-Moishezon criterion, and assuming X is projective, one sees that their Chern classes lie in the closure of the ample cone generated by the Chern classes of ample line bundles. It is unfortunately not true that we can extend Lemma 1.3 to this case, allowing $\omega_{L,h}$ to be semi-positive (see, e.g., [15] for a counterexample).

To conclude, let us mention the Kleiman-Seshadri criterion which says that ampleness can be tested on closed complex curves $C \subset X$ only:

Theorem 1.9. *Let X be projective and L be a line bundle on X. Then L is ample if and only if its first class $c_1(L)$ belongs to the interior of the subset of $H^{1,1}_{\mathbb{R}}(X)$ defined by the equations*

$$\int_C \alpha \geq 0, \forall C \subset X.$$

1.2. Cones of curves, divisors and $(1,1)$-classes. Let us now assume for simplicity that X is Kähler. For a class α in $H^{1,1}_{\mathbb{R}}(X)$, we want to define various notions of positivity, extending the ones introduced in the context of line bundles. One important point is that there might be no proper closed analytic subset of positive dimension in X, so that positivity cannot a priori be tested by integration over analytic subsets.

Definition 1.10. A class $\alpha \in H^{1,1}_{\mathbb{R}}(X)$ is Kähler if it can be represented (in de Rham cohomology) by a a Kähler form.

Definition 1.11. A class $\alpha \in H^{1,1}_{\mathbb{R}}(X)$ is pseudo-effective if it can be represented by a real closed positive current of type $(1,1)$.

Definition 1.12. A class $\alpha \in H^{1,1}_{\mathbb{R}}(X)$ is numerically effective if for any $\epsilon > 0$, it can be represented by a closed real $(1,1)$-form $\tilde{\alpha}_\epsilon$ such that

$$\alpha_\epsilon + \epsilon\omega \geq 0$$

as a real $(1,1)$-form.

Here ω is a given Kähler form. As shows the example mentioned in the previous section, this does not imply that α can be represented by a semi-positive $(1,1)$-form.

Definition 1.13. A Kähler current T is a real current of type $(1,1)$ such that for some $\epsilon > 0$, $T - \epsilon\omega > 0$ as $(1,1)$-current.

The set of Kähler classes is an open cone in $H^{1,1}_{\mathbb{R}}(X)$, called the Kähler cone. The set of pseudo-effective classes is a closed cone, called the pseudo-effective cone, which obviously contains the Kähler cone. It is immediate from the definitions that the closure of the Kähler cone is the numerically effective cone consisting of numerically effective classes, and that the set of classes of Kähler currents is the interior of the pseudo-effective cone.

If X is complex projective, it is natural to restrict these definitions to the $\mathbb{Q}$ or $\mathbb{R}$-vector space $NS(X)$ generated by Chern classes of line bundles, called the rational (or real) Néron-Severi group. Due to the hard Lefschetz theorem, this space is dual via Poincaré duality to the $\mathbb{Q}$ (or $\mathbb{R}$) vector subspace of $H^{n-1,n-1}_{\mathbb{R}}(X)$ generated by cohomology classes $[C]$ of closed complex curves in X.

It is clear from Kodaira's embedding theorem that L is numerically effective in the sense of the previous section, if and only if its first Chern class $c_1(L)$ is numerically in the sense of Definition 1.12.

A consequence of Kleiman-Seshadri criterion for ampleness is then the following:

Theorem 1.14. *Let X be projective and L be a holomorphic line bundle on X. Then $c_1(L)$ is numerically effective if and only if $\int_C c_1(L) \geq 0$, for any complex curve $C \subset X$.*

Finally, we have the following easy fact concerning pseudo-effective line bundles (see previous section): for a curve $C \subset X$, consider the Hilbert scheme $\mathcal{M}$ parametrizing deformations of C in X. There is a universal subscheme

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{p} & X \\ q\downarrow & & \\ \mathcal{M} & & . \end{array}$$

We have then

Lemma 1.15. *If for generic $m \in \mathcal{M}$, the curve C_m is irreducible, and the map p is surjective, then for any pseudo-effective line bundle L, we have $\int_C c_1(L) \geq 0$.*

The last inequality turns out to be also true more generally for pseudo-effective classes.

The proof of the Lemma is as follows. It suffices to show it for line bundles L such that $L^{\otimes k}$ is effective for some $k > 0$. Next let σ be a section of $L^{\otimes k}$ and D be its divisor. Then by the properties above, the generic curve C_m has no component contained in D. It follows that the intersection number

$C_m \cdot D \geq 0$. But this is equal to $k \int_C c_1(L)$ since C and C_m are homologous and $[D] = kc_1(L)$.

1.3. Analytic characterizations of the Kähler and pseudo-effective cones. To complete the parallel between the positivity properties of line bundles and that of $(1,1)$-classes, and to have a good picture of how positivity can be tested by restriction to subvarieties, there are two missing statements in the previous sections, which are

(1) A characterization of the Kähler cone analogous to the characterization of the ample cone given by Nakai-Moishezon criterion.
(2) A converse to Lemma 1.15, providing a characterization of the pseudo-effective cone for projective varieties.

These are precisely the two recent theorems proved by Demailly and his collaborators.

Theorem 1.16. (*Demailly-Paun* [14]) *Let X be a compact Kähler manifold. Then the Kähler cone of X is a connected component of the subset of $H^{1,1}_{\mathbb{R}}(X)$ defined by the equations*

$$\int_Y \alpha^p > 0,\ Y \subset X, \dim Y = p. \tag{1.1}$$

Remark 1.17. It is not clear whether this set is open or not.

Remark 1.18. In contrast to what happens in the projective situation, that is in the Nakai-Moishezon criterion, the Kähler cone cannot be in general equal to the whole subset defined above. Indeed, consider the case of a general complex torus T. Then T does not contain any positive dimensional proper analytic subset. So we just get the inequality $\int_T \alpha^n > 0$. On the other hand, the space $H^{1,1}_{\mathbb{R}}(T)$ identifies to the space of Hermitian forms on $\mathbb{C}^n$, while the Kähler cone identifies to the set of positive Hermitian forms. Since $\int_T \alpha^n$ identifies to the discriminant of the Hermitian form in an adequate basis of $\mathbb{C}^n$, Theorem 1.16 just says that positive Hermitian forms are a component of the set of Hermitian forms with positive discriminant.

Theorem 1.16 had been proved before by Campana and Peternell for X projective and $\alpha \in NS(X)_{\mathbb{R}}$. An extension of this result to the case of classes $\alpha \in H^{1,1}_{\mathbb{R}}(X) \subset H^2(X, \mathbb{R})$ which become rational when pulled-back to the universal cover of X was proved by Eyssidieux ([17]). More importantly, it had been established before by Lamari [24] and independently by Buchdahl [5], [6] in the case of surfaces.

The proof of Theorem 1.16 starts as follows: one wants to show that the Kähler cone is both open and closed in the set defined by the inequalities (1.1). It is clearly open. Next consider a class α which is in the closure of the Kähler cone and satisfies these inequalities. So α is numerically effective (see Definition 1.12) and $\int \alpha^n > 0$.

The first step is then to show:

Theorem 1.19. [14] *If a real* $(1,1)$ *class* α *is numerically effective and satisfies the condition* $\int \alpha^n > 0$*, then* α *is representable by a Kähler current* (*see Definition* 1.13).

The second step is then an induction step, which makes use of earlier results of Paun:

Theorem 1.20. [27] *Let* X *be a complex analytic space and* $\alpha \in H^{1,1}_{\partial\bar\partial}(X)$ *be a real class which is representable by a Kähler current. Then if for any proper closed analytic subset* Y *of* X*, the restriction* $\alpha_{|Y}$ *is a Kähler class,* α *is a Kähler class.*

Note the shift here from complex manifolds to analytic spaces, necessary in order to make an induction argument. □

Next, we have the following theorem, due to Boucksom, Demailly, Paun and Peternell giving a numerical characterization of the pseudo-effective cone:

Theorem 1.21. [3] *Let* X *be a projective manifold. Then the pseudo-effective cone consisting of pseudo-effective classes* (*cf. definition* 1.11) $\beta \in NS(X)_{\mathbb{R}}$*, is equal to the set*

$$\{\alpha \in H^{1,1}_{\mathbb{R}}(X),\ \int_C \alpha \geq 0\},$$

for all curves $C \subset X$ *satisfying the assumptions of Lemma* 1.15.

The proof of Theorem 1.21 provides another, a priori smaller, set of inequalities characterizing the pseudo-effective cone. Namely, there is the notion of moving intersection of pseudo-effective classes, which is the analytic analogue of the "intersection of the moving part" of an effective divisor. The paper proves that the pseudo-effective cone is equal to the set

$$\{\alpha \in NS(X)_{\mathbb{R}},\ <\alpha, \beta_m^{n-1}> \geq 0\},$$

where β runs through the set of pseudo-effective divisors, and

$$\beta_m^{n-1} \in H^{n-1,n-1}(X)$$

is the $n-1$th moving intersection of β. (The bracket here is the intersection pairing between $H^{1,1}(X)$ and $H^{n-1,n-1}(X)$.)

The proof uses the following: the pseudo-effective cone is certainly contained in the one defined by the above inequalities. So, to show they are equal, it suffices to show that if a pseudo-effective class in $NS(X)_{\mathbb{R}}$ is in the interior of the cone defined by the above inequalities, it is also in the interior of the pseudo-effective cone. This is proved eventually using a criterion due to Boucksom ([4]) characterizing the interior of the pseudo-effective cone as the set of pseudo-effective classes $\beta \in NS(X)_{\mathbb{R}}$ having a positive moving self-intersection: $\beta_m^n > 0$. □

2. Hodge classes and analytic geometry

2.1. Constructions of Hodge classes. Let X be a compact complex manifold of dimension n, and k be an integer $\leq n$.

Definition 2.1. The space $Hdg^{2k}(X)$ of degree $2k$ rational Hodge classes is the set of classes $\alpha \in H^{2k}(X, \mathbb{Q})$ which can be represented in de Rham cohomology by a closed form of type (k, k).

It can be shown that this is equivalent to be representable by a closed current of type (k, k). When X is Kähler, classes representable by a closed form of type (k, k) are exactly the elements of the space $H^{k,k}(X) \subset H^{2k}(X, \mathbb{C})$ (see Introduction), so that in that case

$$Hdg^{2k}(X) = H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X).$$

There are three standard ways of constructing Hodge classes (in fact integral ones).

– **The class of an analytic subset.** Let $Z \subset X$be a closed analytic subset of codimension k. Then there is a closed analytic subset

$$Z_{\mathrm{sing}} \subset Z \subset X$$

which is of codimension $k + 1$, such that $Z \setminus Z_{\mathrm{sing}} \subset X \setminus Z_{\mathrm{sing}}$ is a complex submanifold of codimension k. Thus we have a class

$$[Z \setminus Z_{\mathrm{sing}}] \in H^{2k}(X \setminus Z_{\mathrm{sing}}, \mathbb{Z}),$$

and the isomorphism $H^{2k}(X, \mathbb{Z}) \cong H^{2k}(X \setminus Z_{\mathrm{sing}}, \mathbb{Z})$, which comes from the fact that the real codimension of Z_{sing} is $\geq 2k+2$, provides us with the desired class $[Z] \in H^{2k}(X, \mathbb{Z})$. This is a Hodge class, as a consequence of Lelong's theorem, which says that the current of integration over Z

$$\int_Z = \int_{Z \setminus Z_{\mathrm{sing}}}$$

is well defined and closed. It is then immediate to check that it represents the class $[Z]$, and since it is of type (k, k), this concludes the proof.

– **Chern classes of holomorphic vector bundles.** If E is a complex vector bundle on a topological manifold X, we have the rational Chern classes $c_i(E) \in H^{2i}(X, \mathbb{Q})$. (Note that the Chern classes are usually defined as integral cohomology classes, $c_i \in H^{2i}(X, \mathbb{Z})$, but in this text, the notation c_i will be used for the rational ones.) If E is now a holomorphic vector bundle on a complex manifold X, the Chern classes of E are Hodge classes.

This follows indeed from Chern-Weil theory, which provides de Rham representatives of $c_i(E)$ as follows: If ∇ is a complex connection on E, with curvature operator $R_\nabla \in A^2_X \otimes \mathrm{End}\, E$, then a representative of $c_k(E)$ is given by the degree $2k$ closed form

$$\sigma_k(\frac{i}{2\pi} R_\nabla),$$

where σ_k is the polynomial invariant under conjugation on the space of matrices, which to a matrix associates the kth symmetric function of its eigenvalues.

Now, if E is a holomorphic vector bundle on X, there exists a complex connection ∇ on E such that R_∇ is of type $(1,1)$, that is $R_\nabla \in A_X^{1,1} \otimes End\, E$. (Given a Hermitian metric h on E, one can take the so-called Chern connection, which is compatible with h, and has the property that its $(0,1)$-part is equal to the $\overline{\partial}$-operator of E.) This implies that $\sigma_k(\frac{i}{2\pi}R_\nabla) \in A^{k,k}(X)$, and shows that $c_k(E)$ is Hodge.

– **Chern classes of coherent sheaves.** Coherent sheaves $\mathcal{F}$ on a complex manifold X are sheaves of $\mathcal{O}_X$-modules which are locally presented as quotients

$$\mathcal{O}_X^r \stackrel{\phi}{\rightarrow} \mathcal{O}_X^s \rightarrow \mathcal{F} \rightarrow 0,$$

where ϕ is a matrix of holomorphic functions.

If X is a smooth projective complex manifold, it is known that coherent sheaves are algebraic and admit a finite locally free resolution

$$0 \rightarrow \mathcal{F}_n \rightarrow \cdots \rightarrow \mathcal{F}_0 \rightarrow \mathcal{F} \rightarrow 0,$$

where the $\mathcal{F}_i$ are locally free, i.e., locally isomorphic to some $\mathcal{O}_X^s$. Such a locally free sheaf of $\mathcal{O}_X$ is the sheaf of sections of a holomorphic vector bundle F_i on X, and we can define the Chern classes of $\mathcal{F}$ by the Whitney formula:

$$c(\mathcal{F}) := \Pi_l c(\mathcal{F}_l)^{\epsilon_l}.$$

Here the total Chern class $c(\mathcal{F})$ determines the Chern classes $c_i(\mathcal{F})$ by the formula $c(\mathcal{F}) = 1 + c_1(\mathcal{F}) + \cdots + c_n(\mathcal{F})$, and we put by definition $c(\mathcal{F}_l) := c(F_l)$. (Here $\epsilon_l = (-1)^l$, and the series can be inverted because the cohomology ring is nilpotent in degree > 0.) The Whitney formula and the case of holomorphic bundles imply that the Chern classes $c_i(\mathcal{F})$ are Hodge classes.

On a general compact complex manifold (and even Kähler), such a finite locally free resolution does not exist in general (see section 2.3). In order to define the $c_i(\mathcal{F})$, one can use a finite locally free resolution

$$0 \rightarrow \mathcal{F}_n \rightarrow \cdots \rightarrow \mathcal{F}_0 \rightarrow \mathcal{F} \otimes \mathcal{H}_X \rightarrow 0,$$

of $\mathcal{F} \otimes \mathcal{H}_X$ by sheaves of locally free $\mathcal{H}_X$-modules, where $\mathcal{H}_X$ is the sheaf of real analytic complex functions. The $\mathcal{F}_l$ are then the sheaves of real analytic sections of some complex vector bundles F_l of real analytic class, and one can then define (using again the definition $c(\mathcal{F}_l) = c(F_l)$)

$$c(\mathcal{F}) = c(\mathcal{F} \otimes \mathcal{H}_X),\ c(\mathcal{F} \otimes \mathcal{H}_X)) = \Pi_l c(\mathcal{F}_l)^{\epsilon_l}.$$

This defines unambiguously the Chern classes of $\mathcal{F}$, and some further work allows to show that these classes are Hodge classes.

2.2. The projective case. The three constructions described above provide us with three subspaces of $Hdg^{2k}(X)$, namely the $\mathbb{Q}$-vector space generated by the classes $[Z]$, $Z \subset X$ of codimension k, the $\mathbb{Q}$-vector space generated by the Chern classes $c_k(E)$, for all holomorphic vector bundles E on X, and the $\mathbb{Q}$-vector space generated by the Chern classes $c_k(\mathcal{F})$, for all coherent sheaves $\mathcal{F}$ on X.

It is always the case that the first space is contained in the last one. Indeed, if $Z \subset X$ is a closed analytic subset of codimension k, one can consider its ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$. It is a coherent sheaf, and one has the relation (cf. [2], [18] p. 298):

$$c_k(\mathcal{I}_Z) = (-1)^k (k-1)! [Z]. \tag{2.1}$$

In the projective situation, one has furthermore the following result:

Theorem 2.2. *If X is a smooth projective complex variety, these three subspaces of $Hdg^{2k}(X)$ coincide.*

That the second and third space coincide follows from the above mentioned fact that coherent sheaves admit finite locally free resolutions.

That the Chern classes of a holomorphic vector bundle E are integral combinations of classes of subvarieties follows from the following fact: if L is an ample line bundle on X, the sheaf vector bundles $E \otimes L^{\otimes k}$ is generated by global holomorphic sections, for large k. It follows that $E \otimes L^{\otimes k}$ is the pull-back via a holomorphic map

$$\Phi : X \to G$$

of the tautological quotient vector bundle Q on G, where G is the Grassmannian of codimension r subspaces of $H^0(X, E)$, $r = rank\, E$. (Indeed the map Φ is the map which to $x \in X$ associates the subspace $V_x \subset H^0(X, E)$ consisting of sections vanishing at x.)

So the Chern classes of $E \otimes L^{\otimes k}$ are the pull-back via Φ of those of Q, and one uses then the fact that the cohomology of G is generated by classes of algebraic subvarieties.

Finally, the Hodge conjecture predicts the following:

Conjecture 2.3. *(Hodge) If X is smooth projective complex, the Hodge classes of X are generated over $\mathbb{Q}$ by classes $[Z]$ of algebraic subvarieties of X (or equivalently by Chern classes of holomorphic vector bundles or coherent sheaves).*

The conjecture is known to be true for degree 2 Hodge classes (it is then known as the Lefschetz theorem on $(1,1)$-classes). It is in this case an easy consequence of the exponential exact sequence

$$0 \to \mathbb{Z} \stackrel{2i\pi}{\to} \mathcal{O}_X \stackrel{\exp}{\to} \mathcal{O}_X^* \to 0,$$

and the fact that the space $H^1(X, \mathcal{O}_X^*)$ identifies to the set of isomorphism classes of holomorphic line bundles on X. □

Note that this proof shows that for X compact Kähler, degree 2 integral Hodge classes are of the form $c_1(L)$, for L a holomorphic line bundle on X.

The Hodge conjecture is also true for degree $2n-2$ Hodge classes. This is a consequence of the above and of the hard Lefschetz theorem, a particular case of which will say the following: let L be an ample line bundle on X. Then cup-product with the class $c_1(L)^{n-2}$ induces an isomorphism

$$\cup c_1(L)^{n-2} : H^2(X,\mathbb{Q}) \cong H^{2n-2}(X,\mathbb{Q}).$$

One can show that this induces an isomorphism on Hodge classes:

$$\cup c_1(L)^{n-2} : Hdg^2(X,\mathbb{Q}) \cong Hdg^{2n-2}(X,\mathbb{Q}). \qquad \square$$

Note that this proof already fails in the general Kähler case, since in general there will not be anymore a Hodge class α of degree 2 inducing a Lefschetz isomorphism $\cup\alpha^{n-2}$ as above.

2.3. The Kähler case. In the Kähler case, it was classically known that the construction of Hodge classes via analytic subsets and via holomorphic vector bundles may not generate the same subspace of $Hdg(X)$. There are examples of Chern classes of holomorphic vector bundles on a compact Kähler manifold X which are not in $\mathbb{Q}$-vector space generated by classes of analytic subsets. Namely take for X a complex torus which has $Hdg^2(X) \cong \mathbb{Q}$ generated by $c_1(L)$, where $c_1(L)$ is represented by a real $(1,1)$-form on $\mathbb{C}^n$ which is non degenerate but neither positive nor negative. (We use here the fact that for a torus $X = \mathbb{C}^n/\Gamma$, the space $H^{1,1}(X)$ identifies naturally to the space of real $(1,1)$-forms with constant coefficients on $\mathbb{C}^n$.) Then such a torus contains no complex hypersurface, because such an hypersurface $D \subset X$ is the zero set of a holomorphic section σ_D of a line bundle L_D on X, and $c_1(L_D)$ is represented by a semi-positive non zero $(1,1)$-form on $\mathbb{C}^n$. $\square$

Next, we proved in [34] that on compact Kähler manifolds of dimension ≥ 3, Chern classes of coherent sheaves may generate a subspace of $Hdg(X)$ which is strictly larger than the space generated by Chern classes of vector bundles.

Theorem 2.4. (*Voisin* [34]) *Let X be a compact Kähler manifold which satisfies the assumptions*

$$Hdg^2(X) = Hdg^4(X) = 0.$$

Then any holomorphic vector bundle E on X satisfies the property

$$c_i(E) = 0,\ \forall i > 0.$$

Note that a general complex torus of dimension ≥ 3 satisfies these assumptions.

On the other hand, let X be as in Theorem 2.4, and let $x \in X$. Then the Hodge class $[x] \in Hdg^{2n}(X)$ is non zero, and by (2.1) this is up to a coefficient the Chern class of the coherent sheaf $\mathcal{I}_x$. This provides the announced example, since in this case no non-zero Hodge classes comes from Chern classes of holomorphic vector bundles. $\square$

Note that this result also implies the following:

Corollary 2.5. *There exist compact Kähler manifolds X and coherent sheaves on them which do not admit a locally free resolution by sheaves of $\mathcal{O}_X$-modules.*

Indeed, consider the above example: if the coherent sheaf $\mathcal{I}_x$ admitted a locally free resolution

$$0 \to \mathcal{F}_n \to \cdots \to \mathcal{F}_0 \to \mathcal{I}_x,$$

Whitney formula would give:

$$c(\mathcal{I}_x) = \Pi_l c(\mathcal{F}_l^{\epsilon_l}.$$

But the theorem says that the right-hand side is equal to 1, while the left-hand side has the non zero term $c_n(\mathcal{I}_x)$ proportional to $[x]$ in top degree. □

Remark 2.6. The existence of locally free resolutions has been proved for coherent sheaves on compact complex surfaces by Schuster [28].

Theorem 2.4 is a consequence of the Bando-Siu extension of Uhlenbeck-Yau theorem to reflexive sheaves $\mathcal{F}$ on compact Kähler manifolds. A reflexive sheaf is a sheaf which has the Hartogs extension property that any section defined away from a codimension 2 closed analytic subset extends. Equivalently, $\mathcal{F}$ should be equal to its bidual.

Theorem 2.7. (*Bando-Siu* [1]) *Let $\mathcal{F}$ be a reflexive coherent on compact Kähler manifold X. Assume $\mathcal{F}$ is stable with respect to some Kähler form ω. Then $\mathcal{F}$ admits a Hermite-Einstein metric relative to ω.*

It follows that if furthermore $c_1(\mathcal{F}) = 0 = c_2(\mathcal{F})$, then $\mathcal{F}$ is locally free and admits a flat holomorphic connection, so that $c_i(\mathcal{F}) = 0$, $i > 0$.

We shall explain later on the notions of stability and Hermite-Einstein metrics in the easier context of locally free sheaves.

From the above results, one concludes that the only possible way to extend the Hodge conjecture to the Kähler case would be the following:

Question 2.8. Are the Hodge classes on a compact Kähler manifold generated by Chern classes of coherent sheaves?

This question was answered negatively in [34]:

Theorem 2.9. (*Voisin* [34]) *Let X be a compact Kähler manifold of dimension n, with Kähler form ω. Assume the following:*

(1) $Hdg^2(X) = 0$.
(2) $\langle Hdg^4(X), [\omega]^{n-2}\rangle = 0$. (*Here $[\omega] \in H^2(X, \mathbb{R})$ is the de Rham class of ω and the intersection is the Poincaré pairing between $H^4(X)$ and $H^{2n-4}(X)$.*)
(3) *X does not contain proper positive dimensional analytic subset.*

Then any coherent sheaf $\mathcal{F}$ on X satisfies the condition $c_2(\mathcal{F}) = 0$.

On the other hand there exist compact complex manifolds X satisfying the assumptions but have $Hdg^4(X) \neq 0$.

The examples are general 4-dimensional Weil tori. The algebraic Weil tori were proposed as candidates for a counterexample to the Hodge conjecture even in projective geometry. Weil tori are constructed as follows: one starts with a rank $4n$ lattice Γ endowed with an endomorphism I such that $I^2 = -Id_\Gamma$. Let $\Gamma_{\mathbb{C}} := \Gamma \otimes \mathbb{C}$. The torus will be of the form

$$X = \Gamma_{\mathbb{C}}/W \oplus \Gamma,$$

where $W \subset \Gamma_{\mathbb{C}}$ is a rank $2n$ complex vector subspace, which is stable under I, satisfies the property that

$$W \oplus \overline{W} = \Gamma_{\mathbb{C}},$$

and is such that the eigenvalues of I acting on W consist of n eigenvalues equal to i and n eigenvalues equal to $-i$.

The Weil classes on such tori are the degree $2n$ Hodge classes constructed as follows: let $K = \mathbb{Q}[I]$. Then, using the action of I on X, K acts on the space $\Gamma^*_{\mathbb{Q}} = H^1(T, \mathbb{Q})$, and this way $\Gamma^*_{\mathbb{Q}}$ is a K-vector space of rank $2n$. There is a natural trace map

$$\bigwedge_K^{2n} \Gamma^*_{\mathbb{Q}} \to \bigwedge_{\mathbb{Q}}^{2n} \Gamma^*_{\mathbb{Q}} \cong H^{2n}(X, \mathbb{Q}).$$

One shows that the image of this map consists of Hodge classes. (This is a rank 2 $\mathbb{Q}$-vector subspace.)

It was known to Zucker [37] that for general Weil tori, the Weil classes are not in the space generated by classes of analytic subsets. The assumptions of Theorem 2.9 were also essentially checked there.

The proof of Theorem 2.9 uses the Uhlenbeck-Yau Theorem.

Theorem 2.10. (*Uhlenbeck-Yau* [33]) *Let X be a compact complex manifold of dimension n with Kähler form ω. Let E be a holomorphic vector bundle on X, which is stable with respect to ω. Then E admits a Hermite-Einstein metric h relative to ω.*

Here the stability condition is the following: Denote by $\mathcal{E}$ the sheaf of holomorphic sections of E. Then E is ω-stable if for any subsheaf $\mathcal{F} \subset \mathcal{E}$ such that $0 < rk\,\mathcal{F} < rk\,\mathcal{E}$, one has

$$\frac{\langle c_1(\mathcal{F}), [\omega]^{n-1}\rangle}{rk\,\mathcal{F}} < \frac{\langle c_1(\mathcal{E}), [\omega]^{n-1}\rangle}{rk\,\mathcal{E}}.$$

The Hermite-Einstein condition on h is the following. Associated to h is the Chern connection ∇_h, with curvature operator $R_{\nabla_h} \in A^{1,1}_X \otimes End\,E$. Then h is Hermite-Einstein if

$$R_{\nabla_h} = \mu\omega Id_E + R^0_{\nabla_h},$$

where $\mu \in \mathbb{C}$ is determined by the equation

$$\frac{i}{2\pi} rk\,E[\omega]^n \mu = \langle c_1(E), [\omega]^{n-1}\rangle,$$

and the form valued matrix $R^0_{\nabla_h}$ has primitive coefficient. (A 2-form α on X is said to be primitive if $\omega^{n-1} \wedge \alpha = 0$ everywhere on X.)

An important consequence of Uhlenbeck-Yau's theorem is the following:

Corollary 2.11. *If E is ω-stable and satisfies the conditions*

$$c_1(E) = 0, < c_2(E), [\omega]^{n-2} >= 0,$$

then E admits a flat holomorphic connection and thus the rational Chern classes of E vanish, $c_i(E) = 0$, $\forall i > 0$.

The corollary shows that under the assumptions of theorem 2.9, we have $c_2(E) = 0$ for all ω-stable vector bundles on X. Induction on the rank and arguments involving desingularizations of non locally free sheaves give the result for all coherent sheaves on X. □

3. The topology of projective and Kähler manifolds

3.1. Kodaira's theorem on surfaces. Let X be a Kähler compact manifold. Kodaira's embedding theorem gives the following.

Theorem 3.1. (*Kodaira* [23]) *X is projective if and only if X carries a Kähler form ω whose cohomology class is rational, $[\omega] \in H^2(X, \mathbb{Q})$.*

Indeed, by multiplying ω by an integer, we may assume its class is integral. The Lefschetz theorem on $(1,1)$-classes then says that

$$[\omega] = c_1(L),$$

for some line bundle on X. Finally, the isomorphism

$$H^{1,1}_{\partial\overline{\partial}}(X) \cong H^{1,1}_{\mathbb{R}}(X)$$

and the construction of the Chern forms $\omega_{L,h}$ show that for some metric h on L, we have $\omega = \omega_{L,h}$. Kodaira's Theorem 1.5 then says that L is ample. □

Note that, in particular, if X is Kähler and $H^{2,0}(X) = 0$, then X is projective. Indeed in that case $H^{1,1}_{\mathbb{R}}(X) = H^2(X, \mathbb{R})$. Since the Kähler cone is then open in $H^2(X, \mathbb{R})$, it has to contain rational classes, since they are dense in $H^2(X, \mathbb{R})$.

Starting with a Kähler manifold X, one can deform the complex structure. It is known that the small deformations preserve the Kähler property and that the spaces $H^{p,q}$ vary smoothly inside the fixed space $H^{p+q}(X, \mathbb{C})$, which does not depend on the complex structure (see, e.g., [36] I, 9.3.2). Given a family $(X_t)_{t\in B}$ of deformations of the complex structure on X, one can consider the set

$$\cup_{t\in B} H^{1,1}_{\mathbb{R}}(X_t) \subset H^2(X, \mathbb{R}),$$

inside which sits as an open set the union of the Kähler cones $K_t \subset H^{1,1}_{\mathbb{R}}(X_t)$. Assuming the union of the K_t contains an open set of $H^2(X, \mathbb{R})$, then by the same density argument, it must contain a rational class, which means by Kodaira's theorem 3.1 that some X_t is projective.

It turns out that this is precisely what happens in the case of Kähler surfaces.

Theorem 3.2. (*Kodaira* [22]) *A compact Kähler surface admits a (arbitrarily small) deformation which is projective.*

Kodaira's proof was obtained as a consequence of his classification of surfaces. A more direct proof was given recently by Buchdahl [7], in the case of unobstructed surfaces. His proof uses the following criterion, valid in any dimension, for X to admit small projective deformations.

Proposition 3.3. *Assume an unobstructed compact Kähler manifold X has a Kähler class*

$$\omega \in H^{1,1}_{\mathbb{R}}(X) \subset H^{1,1}(X) \cong H^1(X, \Omega_X)$$

satisfying the following condition: the interior product (combined with cup-product in cohomology)

$$\lrcorner\omega : H^1(X, T_X) \to H^2(X, \mathcal{O}_X) \tag{3.1}$$

is surjective. Then X admits arbitrarily small deformations which are projective.

Let us explain the relation between this criterion with the previous argument: The space $H^1(X, T_X)$ is the space of first-order deformations of the complex structure up to isomorphisms. Assume that there is an actual family of deformations $(X_t)_{t\in B}$ of the complex structure on $X \cong X_0$ such that the tangent space $T_{B,0}$ identifies to $H^1(X, T_X)$, by the Kodaira-Spencer map which to a tangent vector to B at 0 associates the corresponding infinitesimal deformation of X_0. Then the surjectivity of the map (3.1) means exactly that the natural map

$$\sqcup_{t\in B} H^{1,1}_{\mathbb{R}}(X_t) \to H^2(X, \mathbb{R})$$

has a surjective differential at the point ω (see [36] II, 5.3.4).

This certainly implies that, even after shrinking B, the union of the Kähler cones of the X_t contains an open set of $H^2(X, \mathbb{R})$, so that we are reduced to the previous situation.

However, in general, the family $(X_t)_{t\in B}$ does not exist (there are usually obstructed first-order deformations, which do not extend to all higher-orders). This problem limits Buchdahl's proof, which has a more analytic flavor, to the unobstructed case. □

3.2. Higher-dimensional case. In higher dimension, the Kodaira theorem left open the question whether a compact Kähler manifold can be deformed to a projective one, a problem known as the Kodaira problem (see [12]), although it is not clear whether the question was asked by Kodaira himself.

Here we are considering more generally large deformation, that is, we say that X is a deformation of X' if there exist connected analytic spaces

$$\mathcal{X},\ B,$$

a smooth proper holomorphic map

$$\phi : \mathcal{X} \to B,$$

and two points t, $t' \in B$ such that $X_t \cong X$, $X_{t'} \cong X'$. Clearly, if X and X' are deformations of each other, they are diffeomorphic, (although the diffeomorphism between them may not be canonically determined up to isotopy, because of the monodromy group of the fibration given by ϕ). Indeed, this fibration can be trivialized way in a C-infinite way over paths in B, and B is path connected.

So, a fortiori, X and X' are homeomorphic and in particular have the same homotopy type. Hence a weakening of the Kodaira problem asks the following:

Question 3.4. Does any compact Kähler manifold have the homotopy type of a projective complex manifold?

Note that there are no symplectic obstructions, by the work of Donaldson [16], Munoz et al [26] on approximate holomorphic sections of line bundles on symplectic manifolds, which show that any symplectic manifold can be realized as a symplectic submanifold of projective space.

Unfortunately, the answer to this question is negative.

Theorem 3.5. (*Voisin* [35]) *In any dimension* ≥ 4*, there exist compact Kähler manifolds which do not have the homotopy type of complex projective manifolds.*

In any dimension ≥ 6 *there exist simply connected such examples.*

The examples constructed in [35] have the following shape (at least in the non simply-connected case). One considers complex tori T admitting an endomorphism ϕ_T. Later on, we will make an assumption on ϕ_T, but for the moment we just assume that the eigenvalues of ϕ_{T*} acting on the tangent space of T at 0 are all different from 0 or 1.

It follows that inside $T \times T$ the four subtori

$$T_1 := T \times 0,\, T_2 = 0 \times T,$$

$$T_3 = T_{\mathrm{diag}} = \{(x, x),\, x \in T\},\, T_4 = T_{\mathrm{graph}} = \{(x, \phi_T(x)),\, x \in T\}$$

meet pairwise transversally at finitely many points.

We first blow-up the finitely many pairwise intersection points of these tori; then the proper transforms $\widetilde{T}_i$ of the T_i's are smooth and do not meet anymore. So we can blow-up them again. The resulting compact complex manifold is Kähler because the Kähler property is stable under blow-ups.

We prove next that for adequate choice of (T, ϕ_T), the manifold X so constructed does not have the homotopy type of a complex projective manifold. More precisely, let us make the following assumptions on (T, ϕ_T):

(∗) *the dimension* n *of* T *is* ≥ 2 *and the endomorphism* $\phi := \phi_{T*}$ *of* $H_1(T, \mathbb{Z})$ *satisfies the properties that all of its eigenvalues are distinct, none is real, and the Galois group of its characteristic polynomial acts as the symmetric group of* $2n$ *elements on the set of eigenvalues.*

The precise statement is then the following:

Theorem 3.6. *Assume the assumptions (*). If X' is a Kähler compact manifold such that there exists a graded ring isomorphism*

$$\gamma : H^*(X', \mathbb{Z}) \cong H^*(X, \mathbb{Z}),$$

then X' is not projective.

The key point is the notion of polarized Hodge structure. Consider a Hodge structure of weight r, that is, a lattice H and a decomposition

$$H_{\mathbb{C}} = \oplus_{p+q=r} H^{p,q}.$$

Definition 3.7. A polarization of this Hodge structure is a bilinear form

$$q : H \times H \to \mathbb{Z}$$

which is skew-symmetric if r is odd and symmetric otherwise, and satisfies the conditions:

$$q(\alpha, \overline{\beta}) = 0,\ \alpha \in H^{p,q},\ \beta \in H^{p',q'},\ (p,) \neq (p', q').$$

$$(\alpha, \beta) \mapsto i^{p-q} q(\alpha, \overline{\beta})$$

is a positive definite Hermitian form on $H^{p,q}$.

Hodge theory and the Kähler identities show the following (see [36] I, 6.2.3). Let X be a complex projective manifold, and $\eta = c_1(L) \in H^2(X, \mathbb{Z})$ be the first Chern class of an ample line bundle on X. Then defining the primitive cohomology

$$H^r(X, \mathbb{Z})_{\text{prim}} := Ker \cup \eta^{n-r+1} \subset H^r(X, \mathbb{Z},$$

the form

$$q_\eta(\alpha, \beta) = \int_X \alpha \cup \eta^{n-r} \cup \beta$$

defines up to sign a polarization on $H^r(X, \mathbb{Z})_{\text{prim}}$. (Here we are working with cohomology modulo torsion.)

Note that for $r = 1$, we have $H^1(X, \mathbb{Z})_{\text{prim}} = H^1(X, \mathbb{Z})$ and for $r = 2$, we have $H^2(X, \mathbb{Q})_{\text{prim}} \oplus \mathbb{Q}\eta = H^2(X, \mathbb{Q})$.

In other words, the cohomology groups of a projective complex manifolds carry Hodge structures which are compatible with the cup-product, and furthermore, for degree 1 and 2, these Hodge structures can be polarized.

The proof of Theorem 3.6 consists in showing that if we have X', X and γ as stated there, the Hodge structure on $H^1(X', \mathbb{Z})$ (which has to be compatible via the cup-product with the Hodge structures on higher cohomology groups) cannot be polarized.

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