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Hilbert schemes of K3 surfaces, generalized Kummer, and cobordism classes of hyper-Kähler manifolds

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Abstract: We prove that the complex cobordism class of any hyper-Kähler manifold of dimension $2n$ is a unique combination with rational coefficients of classes of products of punctual Hilbert schemes of K3 surfaces. We also prove a similar result using the generalized Kummer varieties instead of punctual Hilbert schemes. As a key step, we establish a closed formula for the top Chern character of their tangent bundles.

Keywords: Chern numbers, hyper-Kähler manifolds.

1. Introduction

The cobordism ring denoted $\mathrm{MU}^*(\mathrm{pt})$ in [27] and Ω^* in [13] has the following easy description (which is not the original Milnor definition), see [25]. In degree i, consider the free abelian group $\mathcal{Z}^i$ generated by i-dimensional compact manifolds M equipped with a stable complex structure α, namely a complex vector bundle structure on the real bundle $T_M \oplus \mathbb{R}^k$, where $\mathbb{R}^k$ is the trivial real vector bundle of rank k on M. It contains the subgroup $\mathcal{Z}^i_b$ generated by boundaries, namely, for any real $i+1$-fold N with boundary equipped with a stable complex structure α, as $T_{N|\partial N} \cong T_{\partial N} \oplus \mathbb{R}$, the stable complex structure on N induces a stable complex structure on the boundary ∂N, defining the boundary $\partial(N, \alpha)$. The group $\mathrm{MU}^*(\mathrm{pt})$ is then defined as the quotient $\mathcal{Z}^i/\mathcal{Z}^i_b$. The ring structure comes from the addition given by the disjoint union, and the product is given by the geometric product. It is proved in [13] that $\mathrm{MU}^*(\mathrm{pt})$ is trivial in odd degree $*$ and torsion free in even degree.

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Furthermore it is also known that the cobordism class of a pair (M, α), with $\dim M = 2i$ is determined by the Chern numbers

$$\int_M P_I(c_l(M, \alpha)),$$

where we use the orientation of M defined by α to compute the integral, the Chern classes $c_l(M, \alpha)$ are those of the complex vector bundle $T_M \oplus \mathbb{R}^k$ equipped with the stable complex structure α, and the P_I generate the space of degree $2i$ weighted homogeneous polynomials in the c_j where $\deg c_j = 2j$. We will in fact work with the $\mathbb{Q}$-vector space $\mathrm{MU}^*(\mathrm{pt}) \otimes \mathbb{Q}$ that we will denote $\mathrm{MU}^*(\mathrm{pt})$ for convenience. Note that, with $\mathbb{Q}$-coefficients, the study of the cobordism ring is much easier, and can be done by the methods of [26].

If we consider hyper-Kähler manifolds of dimension $2n$, or more generally compact complex $2n$-folds X having an everywhere nondegenerate $(2,0)$-form σ_X (not necessarily closed, not necessarily holomorphic), the existence of the isomorphism of complex vector bundles

$$T_X^{1,0} \cong (T_X^{1,0})^*$$

given by σ_X implies that $c_l(X) = 0$ for l odd. It follows that the cobordism classes of such complex manifolds are determined by the Chern numbers

$$\int_X P(c_{2l}(X)),$$

where we use the complex orientation of X to compute the integrals, and the polynomials P generate the space of degree $4n$ weighted homogeneous polynomials in the c_{2l}, where $\deg c_{2l} = 4l$. These polynomials are generated by monomials M_I indexed by partitions I of n, namely to a partition I given by the decomposition $n = n_1 + \ldots + n_k$, one associates the monomial

$$M_I = c_{2n_1} \ldots c_{2n_k}.$$

Starting with a K3 surface S, we can construct in each even dimension $2n$ the following set of symplectic holomorphic manifolds, also indexed by partitions I of n, namely, to a partition I as above one associates

$$S^{[I]} := S^{[n_1]} \times \ldots \times S^{[n_k]}.$$

Similarly, using the generalized Kummer varieties $\mathrm{Kum}_i(A)$ associated with a 2-dimensional complex torus or abelian surface (see [1]) instead of the Hilbert schemes of K3 surfaces, we associate to a partition I as above the symplectic holomorphic $2n$-fold

$$\mathrm{Kum}_I(A) := \mathrm{Kum}_{n_1}(A) \times \ldots \times \mathrm{Kum}_{n_k}(A).$$

The main result of this paper can be formulated as follows.

Theorem 1.1. *(a) The complex cobordism class of any compact complex manifold X with trivial odd Chern classes is a unique combination with rational coefficients of classes $S^{[I]}$, where S is a K3 surface.*

(b) The same result holds if one replaces the varieties $S^{[I]}$ by the varieties $\mathrm{Kum}_I(A)$.

In fact, the theorem that we will prove is even more general. Namely our results apply to any compact complex manifold X or complex cobordism class whose Chern numbers $\int_X M_I(c_1(X), \ldots, c_n(X))$, for any monomial M_I involving nontrivially an odd Chern class, are zero. In this case, we will also say that X has *vanishing odd Chern numbers*. For example, any complex fourfold X with trivial first Chern class has vanishing odd Chern numbers, while it can have $c_3(X) \neq 0$. Similarly, complex n-folds with no nonzero odd degree Chern classes in degree $\leq \frac{n}{2}$ satisfy this property. More generally, the rational subalgebra $MU^*(\mathrm{pt})_{\mathrm{even}}$ of $MU^*(\mathrm{pt})$ consisting of cobordism classes with "trivial odd Chern numbers" in the above sense is a free polynomial algebra over $\mathbb{Q}$ with one generator in each even dimension, and Theorem 1.1 says that the cobordism classes of punctual Hilbert schemes of K3 surfaces, or of the generalized Kummer varieties form a system of generators of this algebra.

Remark 1.2. It is known by [3] that the cobordism class of $S^{[I]}$ for a compact complex surface S depends only on the Chern numbers $\int_S c_2(S)$, $\int_S c_1(S)^2$. Hence we can replace in Theorem 1.1 the K3 surface S by any surface S' with $\int_{S'} c_1(S')^2 = 0$ and $\int_{S'} c_2(S') \neq 0$, for example we can take for S' the blow-up of $\mathbb{P}^2$ in 9 points.

Theorem 1.1 is an analogue for complex manifolds with trivial odd Chern classes of a theorem due to Milnor, stating that the $\mathbb{CP}^r$ provide a multiplicative basis for the algebra $\mathrm{MU}^*(\mathrm{pt})$, that is, the products $\prod_{i_l} \mathbb{CP}^{i_l}$ provide a $\mathbb{Q}$-additive basis for it. Theorem 1.1 even provides two multiplicative bases for $MU^*(\mathrm{pt})_{\mathrm{even}}$, namely the hyper-Kähler $2k$-folds $\mathrm{K3}^{[k]}$ and the hyper-Kähler $2k$-folds $\mathrm{Kum}_k(A)$. This result, together with a number of Chern numbers computations on $\mathrm{K3}^{[k]}$ and $\mathrm{Kum}_k(A)$, raises a number of questions that are presented in Section 5. The general question of what can be the Chern numbers or equivalently the complex cobordism classes of hyper-Kähler manifolds is widely open, although some results are known (see for example [23], [7], [9]). The formalism presented here provides some structure for these numbers and we hope that it can be useful for this study.

We will give a quick proof of Theorem 1.1 (a) in low dimension in Section 2. In higher dimension, we will follow the following strategy, already used by topologists. The Milnor genus of a complex or almost complex manifold of complex dimension m is defined as

$$M(X) = \int_X \mathrm{ch}_m(X), \tag{1}$$

where $\mathrm{ch}(X) = \sum_i \mathrm{ch}_i(X)$ is the Chern character of X (see [8]). As is classical in complex cobordism theory (see [13], [10]) and will be recalled in Section 3, Theorem 1.1 is equivalent to the following result concerning the Milnor genus of $\mathrm{K3}^{[n]}$ and $\mathrm{Kum}_n(A)$.

Theorem 1.3. *(a) The Milnor genus $M(S^{[n]})$ is nonzero for all n.*
(b) The Milnor genus $M(\mathrm{Kum}_n(A))$ is nonzero for all n.

Theorem 1.3 will be proved in Section 4, where an explicit formula for $M(S^{[n]})$ and $M(\mathrm{Kum}_n(A))$ will be established (see Theorems 4.1 and 4.2). In Section 3, which is mostly introductory, we will explain the equivalence between Theorems 1.1 and 1.3. In the last section of the paper, we will present a few natural questions left open by our results.

2. Theorem 1.1 in small dimension

In complex dimensions $2n = 2$, 4 and 6, the group $\mathrm{MU}^{4n}(\mathrm{pt})$ is very simple. Indeed, for $n = 2$, the only class to integrate is c_2. In dimension 4, we get only c_4 and c_2^2. Finally, in dimension 3, we get only c_2^3, c_2c_4, c_6. In all three cases, the space has dimension n, which is not true anymore in higher dimensions (for example, in dimension 8, there is an extra monomial c_4^2). In this situation, there are natural Chern numbers of $S^{[k]}$ that we can use to test the independence of the classes $S^{[k_i]}$ in $\mathrm{MU}^*(\mathrm{pt})_{\mathrm{even}}$, namely the n numbers

$$\chi_k(X) := \chi(X, \Omega_X^k),$$

for $k \leq n$, $2n = \dim X$. We observe that, by Serre duality, the other holomorphic Euler-Poincaré characteristics $\chi(X, \Omega_X^k)$ for $k > n$ do not bring further information. By the Hirzebruch-Riemann-Roch formula, $\chi_k(X)$ is a polynomial of degree $2n$ in the Chern classes of X, hence a combination P_k of monomial Chern numbers of X. Hence, if we are able to prove that the matrix giving the $\chi_k(X)$, $k = 1, 2$, for $X = S \times S$, $S^{[2]}$, then Theorem 1.1 holds in dimension 4. Similarly, the independence of the numbers $\chi_k(X)$ for $k = 1, 2, 3$, and $X = S^3$, $S \times S^{[2]}$, $S^{[3]}$ will imply Theorem 1.1 in dimension 6. This is done in the following

Proposition 2.1. *(1) (*$\dim 4$*) The matrix* $\begin{pmatrix} \chi(\Omega_{S^{[2]}}) & \chi(\Omega_{S\times S}) \\ \chi(\Omega^2_{S^{[2]}}) & \chi(\Omega^2_{S\times S}) \end{pmatrix}$ *has nonzero determinant.*

(2) ($\dim 6$*) The matrix*

$$\begin{pmatrix} \chi(\Omega_{S^{[3]}}) & \chi(\Omega_{S^{[2]}\times S}) & \chi(\Omega_{S^3}) \\ \chi(\Omega^2_{S^{[3]}}) & \chi(\Omega^2_{S^{[2]}\times S}) & \chi(\Omega^2_{S^3}) \\ \chi(\Omega^3_{S^{[3]}}) & \chi(\Omega^3_{S^{[2]}\times S}) & \chi(\Omega^3_{S^3}) \end{pmatrix}$$

has nonzero determinant.

Proof. It is equivalent by Remark 1.2, and in fact easier, to prove the same result for the surface Σ obtained as the blow-up of $\mathbb{P}^2$ in 9 points. Indeed, in this case, the whole cohomology of the Hilbert scheme is of type (p, p) and similarly for their products. Thus we have $\chi(X, \Omega^k_X) = (-1)^k b_{2k}(X)$ for these varieties. The Betti numbers of $\Sigma^{[2]}$ and $\Sigma^{[3]}$ are computed by [2] or [4]. One has

$$b_2(\Sigma) = 10,\ b_2(\Sigma^{[2]}) = 11,\ \ b_4(\Sigma^{[2]}) = 66,$$
$$b_2(\Sigma^{[3]}) = 11,\ \ b_4(\Sigma^{[3]}) = 77,\ \ b_6(\Sigma^{[3]}) = 342.$$

By Künneth decomposition, our matrices are thus $\begin{pmatrix} -11 & -20 \\ 66 & 102 \end{pmatrix}$ in case (1), and this matrix has nonzero determinant and

$$\begin{pmatrix} -11 & -21 & -30 \\ 77 & 177 & 303 \\ -342 & -682 & -1060 \end{pmatrix}$$

in case (2), and this matrix has nonzero determinant. □

3. Reduction to Theorem 1.3

The Chern character $\mathrm{ch}(E)$ of a complex vector bundle of rank r on a topological space X is defined as

$$\mathrm{ch}(E) = \sum_{i=1}^{r} \exp x_i \in H^{2*}(X, \mathbb{Q}),$$

where the x_i are the formal roots of the Chern polynomial of E (see [8]). Its main properties are

(2) $$\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$$

and, when X is a manifold of real dimension k,

(3) $$\mathrm{ch}_i(E) = 0 \text{ for } 2i > k.$$

For a complex manifold X we will use the notation $\mathrm{ch}(X) = \mathrm{ch}(T_X)$. Let X be a compact complex manifold of dimension n which is a product

$$X \cong Y \times W$$

of complex manifolds of respective dimensions n_Y, $n_W < n$. Then, as $T_X = \mathrm{pr}_1^* T_Y \oplus \mathrm{pr}_2^* T_W$, where pr_i denotes the projection on the i-th factor, we get by (2) and (3)

(4) $$\mathrm{ch}_i(X) = \mathrm{pr}_1^*\mathrm{ch}_i(Y) + \mathrm{pr}_2^*\mathrm{ch}_i(W),$$

hence

$$\mathrm{ch}_i(X) = 0 \text{ for } i > \max(n_Y, n_W).$$

The Milnor genus $M(X)$ defined in (1) thus satisfies the following property

Lemma 3.1. *We have $M(X) = 0$ if X is a product of two complex manifolds of dimension smaller than n.*

The formal properties above give the following criterion

Proposition 3.2. *For $i \in \{1, \ldots, n\}$ let X_i be a compact complex manifold of dimension $2i$ with vanishing odd Chern classes: $c_{2l+1}(X_i) = 0$. Then, $\lambda_i := M(X_i)$ is nonzero for any i, if and only if any complex cobordism class of even dimension $\leq 2n$ with vanishing odd Chern numbers can be written uniquely as a rational combination of products*

$$X_I := X_{i_1} \times \ldots \times X_{i_k}, \ \sum_l i_l \leq n.$$

Proof. The "if" follows from Lemma 3.1 which says that ch_{2i} can have a nonzero integral on $X_{i_1} \times \ldots \times X_{i_k}$ only for $I = \{i\}$, that is, when $X_{i_1} \times \ldots \times X_{i_k} = X_i$.

In the other direction, we have to prove that the products $X_{i_1} \times \ldots \times X_{i_k}$ form a basis over $\mathbb{Q}$ of the subring $\mathrm{MU}^*(\mathrm{pt})_{\mathrm{even}}$ of the cobordism ring of classes α with vanishing odd Chern numbers $\int_\alpha M_I(c_i)$, where the monomial M_I involves an odd Chern class. Equivalently, we have to show that for any such class $\alpha \in \mathrm{MU}^{4n}(\mathrm{pt})_{\mathrm{even}}$, there are unique rational coefficients α_I indexed by partitions of n, such that

$$\int_\alpha P(c_2, \dots, c_{2n}) = \sum_I \alpha_I \int_{X_I} P(c_2(X_I), \dots, c_{2n}(X_I))$$

for any degree $2n$ weighted polynomial P in the variables c_{2l}. Instead of using the Chern classes c_{2i} as generators, we can use the Chern characters classes ch_{2i} which are related to the Chern classes by the Newton formulas. We argue by induction on the dimension and conclude that for any $i < n$, there exists a combination

$$Y_i = X_i + \sum_{I, l(I) \geq 2} \alpha_I X_I \in \mathrm{MU}^{4i}(\mathrm{pt}), \tag{5}$$

where, in the above sum, I runs through the partitions $i = \sum_{l=1}^k i_l$ of i and $l(I) := k$, with the following property: for any degree $2i$ monomial $M_K = \mathrm{ch}_2^{k_2} \dots \mathrm{ch}_{2i}^{k_{2i}}$ in the Chern characters ch_l with l even, one has with

$$M_K(Y_i) = 0 \text{ unless } M_K \neq \mathrm{ch}_{2i}, \tag{6}$$

with $M_K(Y_i) = M_K(X_i) + \sum_{I, l(I) \geq 2} \alpha_I \int_{X_I} M_K(\mathrm{ch}_2(X_I), \dots, \mathrm{ch}_{2i}(X_I))$. Furthermore, equation (5), Lemma 3.1 and our assumptions show that $M(Y_i) = \lambda_i \neq 0$. Formulas (4) and (6) then imply that for any product

$$Y_J = \prod_{j_1 + \ldots + j_k = i} Y_{j_l}$$

with $k \geq 2$ (hence all j_s smaller than i), and any monomial M_K as above of weighted degree $2i$, one has $M_K(Y_J) = 0$ for $K \neq J$, $M_K(Y_K) \neq 0$. Finally, we have by assumption $\mathrm{ch}_{2i}(X_i) \neq 0$, so X_i and the Y_J for the partitions J of i such that $l(J) \geq 2$ form a basis of $\mathrm{MU}^{4i}(\mathrm{pt})_{\mathrm{even}}$. □

Remark 3.3. The same criterion (without assumption on the odd Chern classes) was used by topologists to prove that the complex cobordism ring with rational coefficients is generated in degree n by products of projective spaces $\mathbb{P}^{i_l}$ with $\sum_l i_l = n$. It suffices to prove that $M(\mathbb{P}^r) \neq 0$, which is quite easy using the Euler exact sequence which gives

$$\mathrm{ch}(\mathbb{P}^r) = (r+1)\exp(h) - 1,$$

with $h = c_1(\mathcal{O}_{\mathbb{P}^r}(1))$.

We now get in particular

Corollary 3.4. *Theorem 1.3 is equivalent to Theorem 1.1.*

4. Proof of Theorem 1.3

The proof of Theorem 1.3 will use the description of the cohomology of Hilbert schemes of points of surfaces in terms of Nakajima operators. In particular, we will use a result of Li, Qin and Wang [12] which for K-trivial surfaces expresses the operator of multiplication by tautological classes in terms of the Nakajima basis. We refer to [15] for an overview of the main definitions in the subject, and for the conventions that we follow.

We will prove the following closed evaluations, which imply Theorem 1.3.

Theorem 4.1. *For any surface S with $c_1(S) = 0$ in $H^2(S, \mathbb{Q})$, we have for all $n \geq 1$:*

$$\int_{S^{[n]}} \mathrm{ch}_{2n}(S^{[n]}) = (-1)^n \frac{e(S)}{24} \frac{(2n+2)!}{n!^4(2n-1)}$$

where $e(S) = \int_S c_2(S)$ is the topological Euler characteristic of S.

Theorem 4.2. *For any abelian surface A, we have for all $n \geq 1$:*

$$\int_{\mathrm{Kum}_n(A)} \mathrm{ch}_{2n}(\mathrm{Kum}_n(A)) = (-1)^n \frac{(2n+2)!}{n!^4}.$$

4.1. Combinatorial identities

Lemma 4.1. *For $k, n \in \mathbb{N}$, we have the following identities:*

(1) $\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$;
(2) $\sum_{i=0}^{n} i\binom{n}{i}^2 = \frac{n}{2}\binom{2n}{n}$;
(3) $\sum_{i=0}^{n} i^2\binom{n}{i}^2 = \frac{n^3}{2(2n-1)}\binom{2n}{n}$;
(4) $\sum_{i=0}^{k}(-1)^i\binom{n}{i} = (-1)^k\binom{n-1}{k}$;
(5) $\sum_{i=0}^{k}(-1)^i i\binom{n}{i} = (-1)^k n\binom{n-2}{k-1}$.

Proof. For (1), one can compare the degree-n coefficient of the polynomial $(1+x)^{2n}$: the left hand side is obtained using the identity $(1+x)^{2n} = (1+x)^n(1+x)^n$, while the right hand side is simply the binomial coefficient. For (2) and (3), we consider the polynomials $(1+x)^n \cdot \frac{d}{dx}(1+x)^n$ and $(1+x)^n \cdot \left(\frac{d}{dx}\right)^2(1+x)^n$, and follow the same idea as (1).

For (4), we consider the degree-k coefficient of the polynomial $(1-x)^{n-1}$: the right hand side is again just the binomial coefficient, while the left hand side is obtained using the Taylor expansion $(1-x)^{n-1} = \frac{1}{1-x} \cdot (1-x)^n = (1+x+x^2+\cdots) \cdot (1-x)^n$. Similarly, for (5) we consider $-n(1-x)^{n-2} = \frac{1}{1-x} \cdot \frac{d}{dx}(1-x)^n$. $\square$

Proposition 4.2. *We have the following identity*

$$\sum_{l=0}^{n} \sum_{m=0}^{l-1} (-1)^{m+l+1} \frac{l-m}{m!\,l!\,(n-m)!\,(n-l)!} = \frac{n}{2(2n-1)} \frac{(2n)!}{n!^4}.$$

Proof. We rewrite the left hand side using the combinatorial identities from Lemma 4.1

$$\begin{aligned} &\sum_{l=0}^{n} \sum_{m=0}^{l-1} (-1)^{m+l+1} \frac{l-m}{m!l!(n-m)!(n-l)!} \\ &= \frac{1}{n!^2} \sum_{l=0}^{n} \sum_{m=0}^{l-1} (-1)^{m+l+1} (l-m) \binom{n}{m}\binom{n}{l} \\ \text{(take out } l) &= \frac{1}{n!^2} \sum_{l=0}^{n} (-1)^l \binom{n}{l} \left(l \sum_{m=0}^{l-1} (-1)^{m+1} \binom{n}{m} + \sum_{m=0}^{l-1} (-1)^m m \binom{n}{m} \right) \\ \left(\substack{\text{using Lemma 4.1} \\ \text{(4) and (5)}}\right) &= \frac{1}{n!^2} \sum_{l=0}^{n} (-1)^l \binom{n}{l} \left((-1)^l l \binom{n-1}{l-1} + (-1)^{l-1} n \binom{n-2}{l-2} \right) \\ &= \frac{1}{n!^2} \sum_{l=0}^{n} \binom{n}{l} \left(\frac{l^2}{n} \binom{n}{l} - \frac{l^2-l}{n-1} \binom{n}{l} \right) \\ &= \frac{1}{n!^2} \left(\frac{1}{n-1} \sum_{l=0}^{n} l \binom{n}{l}^2 - \frac{1}{n(n-1)} \sum_{l=0}^{n} l^2 \binom{n}{l}^2 \right) \\ \left(\substack{\text{using Lemma 4.1} \\ \text{(2) and (3)}}\right) &= \frac{n}{2(2n-1)} \frac{(2n)!}{n!^4}. \end{aligned}$$

$\square$

4.2. Hilbert schemes of points

Let $\mathcal{Z} \subset S^{[n]} \times S$ be the universal subscheme and let π, π_S be the projections of $S^{[n]} \times S$ to the factors. For any $\gamma \in H^*(S)$ and $d \in \mathbb{Z}$ let

$$\mathfrak{G}_d(\gamma) : H^*(S^{[n]}) \to H^*(S^{[n]})$$

be the operator of multiplication with the class $\pi_*(\mathrm{ch}_d(\mathcal{O}_{\mathcal{Z}} - \mathcal{O}_{S^{[n]}\times S}) \cdot \pi_S^*(\gamma))$.

Let from now on S be a surface with $c_1(S) = 0$ in $H^2(S, \mathbb{Q})$. Then by a result of Li, Qin and Wang [12, Thm.4.6] we have that

$$\mathfrak{G}_d(\gamma) = -\sum_{\substack{|\lambda|=0 \\ \ell(\lambda)=d}} \frac{\mathfrak{q}_\lambda}{\lambda!}(\Delta_*(\gamma)) + \sum_{\substack{|\lambda|=0 \\ \ell(\lambda)=d-2}} \frac{s(\lambda)}{24 \cdot \lambda!}\mathfrak{q}_\lambda(\Delta_*(\gamma \cdot c_2(S))) \tag{7}$$

where $\mathfrak{q}_m(\alpha)$ are the Nakajima Heisenberg operators; the other notations follow [15, Sec.4].[1]

The tangent bundle of the Hilbert scheme can be expressed as a relative Ext sheaf of the universal ideal sheaves [3, Prop.2.2]. This gives an expression for the operator of multiplication with $\mathrm{ch}_k(S^{[n]})$ in terms of the $\mathfrak{G}$'s as follows

$$\mathrm{mult}_{\mathrm{ch}_k(S^{[n]})} = \sum_{i+j=k+2} (-1)^{j+1}\mathfrak{G}_i\mathfrak{G}_j(\Delta) + \frac{e(S)}{12}\sum_{i+j=k}(-1)^{j+1}\mathfrak{G}_i(\mathsf{p})\mathfrak{G}_j(\mathsf{p}) \tag{8}$$

where $k \geq 1$, we let $\mathsf{p} \in H^4(S)$ be the class of a point on S; see also [15, 4.9]. Hence Theorem 4.1 is implied by the following two lemmas:

Lemma 4.3.

$$\sum_{i+j=2n}(-1)^{j+1}\int_{S^{[n]}} \mathfrak{G}_i(\mathsf{p})\mathfrak{G}_j(\mathsf{p})1_{S^{[n]}} = (-1)^{n+1}\frac{(2n)!}{n!^4}$$

Proof. In the Nakajima basis the unit of $H^*(S^{[n]})$ is $\frac{1}{n!}\mathfrak{q}_1(1)^n 1_{S^{[0]}}$ where we let $1_{S^{[0]}}$ denote the unit in the cohomology of $S^{[0]} = \{*\}$ (the subscript $S^{[0]}$ is usually dropped in what follows). We hence have to evaluate

$$\sum_{i+j=2n}(-1)^{j+1}\int_{S^{[n]}}\mathfrak{G}_i(\mathsf{p})\mathfrak{G}_j(\mathsf{p})\frac{1}{n!}\mathfrak{q}_1(1)^n 1 = \tag{9}$$

$$\sum_{i+j=2n}(-1)^{j+1}\int_{S^{[n]}}\left(\sum_{l(\lambda)=i, |\lambda|=0}\frac{\mathfrak{q}_\lambda(\Delta_*(\mathsf{p}))}{\lambda!}\right)\left(\sum_{l(\widetilde{\lambda})=j, |\widetilde{\lambda}|=0}\frac{\mathfrak{q}_{\widetilde{\lambda}}(\Delta_*(\mathsf{p}))}{\widetilde{\lambda}!}\right)\frac{\mathfrak{q}_1(1)^n}{n!}1.$$

The (complex[2]) cohomological degree of a Nakajima cycle $\mathfrak{q}_{k_1}(\gamma_1)\cdots\mathfrak{q}_{k_r}(\gamma_r)1$ lying in $H^*(S^{[n]})$ is $n - r + \sum_i \deg_{\mathbb{C}}(\gamma_i)$. Hence for the integral of such a cycle

[1] There is one exception: our definition for $\mathfrak{G}_d(\gamma)$ agrees with [15] in case $d \geq 1$, while for $d = 0$ we have $\mathfrak{G}_0(\gamma) = -\left(\int_S \gamma\right)\mathrm{id}$ (instead of $\mathfrak{G}_0(\gamma) = 0$ in [15]). The advantage is that (7) holds now for all $d \in \mathbb{Z}$.

[2] The complex degree $\deg_{\mathbb{C}}(\gamma)$ is half the real degree, i.e. $\gamma \in H^{2\deg_{\mathbb{C}}(\gamma)}$.

to be non-zero, we need $k_i = 1$ and $\deg_{\mathbb{C}}(\gamma_i) = 2$ for all i. In particular, the term $\mathfrak{q}_1(1)^n$ appearing in the right hand side of (9) has to be transformed into a multiple of $\mathfrak{q}_1(\mathsf{p})^n$ under the operators $\mathfrak{G}_i(\mathsf{p})\mathfrak{G}_j(\mathsf{p})$. Hence among the $\mathfrak{q}_\lambda$ and $\mathfrak{q}_{\tilde{\lambda}}$ we must have n operators of the form $\mathfrak{q}_{-1}$ and n operators $\mathfrak{q}_1$. Since this accounts for all possible Nakajima operators which can appear, we need that $\lambda = (-1)^a(1)^a$ and $\tilde{\lambda} = (-1)^b(1)^b$ where $i = 2a$ and $j = 2b$. The above expression thus evaluates to

$$\begin{aligned}
&= (-1)\sum_{a+b=n}\frac{1}{a!^2b!^2}\int_{S^{[n]}}\mathfrak{q}_1(\mathsf{p})^a\mathfrak{q}_{-1}(\mathsf{p})^a\mathfrak{q}_1(\mathsf{p})^b\mathfrak{q}_{-1}(\mathsf{p})^b\frac{\mathfrak{q}_1(1)^n}{n!}1\\
&= (-1)\sum_{a+b=n}\frac{1}{a!^2b!^2}(-1)^n\\
&= (-1)^{n+1}\frac{(2n)!}{n!^4}
\end{aligned}$$

where in the last equality we used the first part of Lemma 4.1. □

Lemma 4.4.

$$\sum_{i+j=2n+2}(-1)^{j+1}\int_{S^{[n]}}\mathfrak{G}_i\mathfrak{G}_j(\Delta)1_{S^{[n]}} = e(S)(-1)^n\frac{(2n)!}{n!^4}\left[\frac{n}{12}+\frac{n}{2(2n-1)}\right]$$

Proof. We insert the expansion (7) for $\mathfrak{G}_i$. The contribution from the second term in (7) can be computed by the same methods which were used in Lemma 4.3. The result is $e(S)/24\sum_{a+b=n}(-1)^n 2a/(a!^2b!^2)$. The same applies to the contribution from the second term in $\mathfrak{G}_j$. Inserting this and using part (2) of Lemma 4.1 we find that:

$$\sum_{i+j=2n+2}(-1)^{j+1}\int_{S^{[n]}}\mathfrak{G}_i\mathfrak{G}_j(\Delta)1_{S^{[n]}} = I + e(S)(-1)^n\frac{n^2(2n-1)!}{6\cdot n!^4}$$

where I is the contribution from the first terms in $\mathfrak{G}_i$ and $\mathfrak{G}_j$, that is,

$$I = \sum_{i+j=2n+2}(-1)^{j+1}$$

$$\int_{S^{[n]}}\left(\sum_{l(\lambda)=i,|\lambda|=0}\frac{\mathfrak{q}_\lambda(\Delta_*(\Delta_1))}{\lambda!}\right)\left(\sum_{l(\widetilde{\lambda})=j,|\widetilde{\lambda}|=0}\frac{\mathfrak{q}_{\widetilde{\lambda}}(\Delta_*(\Delta_2))}{\widetilde{\lambda}!}\right)\frac{\mathfrak{q}_1(1)^n}{n!}1$$

where Δ_1, Δ_2 stands for summing over the Künneth factors of the diagonal in $H^*(S\times S)$. With similar reasoning as before (i.e. among the $\mathfrak{q}_\lambda$ and $\mathfrak{q}_{\tilde{\lambda}}$ we need n operators $\mathfrak{q}_1$ and $\mathfrak{q}_{-1}$ each) we now compute:

$$I = \sum_{\ell=1}^{n} \sum_{m=0}^{\ell-1} \int_{S^{[n]}} \frac{\mathfrak{q}_1^{n-m} \mathfrak{q}_{-1}^{n-\ell} \mathfrak{q}_{-(\ell-m)} \mathfrak{q}_{\ell-m} \mathfrak{q}_1^m \mathfrak{q}_{-1}^{\ell}(\Delta)}{b_{m,\ell}} (-1)^{m+\ell} \frac{\mathfrak{q}_1(1)^n}{n!} 1$$

with

$$b_{m,\ell} = \begin{cases} m!\ell!(n-m)!(n-\ell)! & \text{if } m < \ell - 1 \\ \ell!^2 (n-\ell+1)!^2 & \text{if } m = \ell - 1. \end{cases}$$

Commuting the negative Nakajima operators to the right and using the Nakajima commutation relations for cases $m = \ell - 1$ and $m < \ell - 1$ separately, we get

$$I = e(S)(-1)^n \sum_{\ell=1}^{n} \sum_{m=0}^{\ell-1} (-1)^{m+\ell+1} \frac{(\ell - m)}{m!\ell!(n-m)!(n-\ell)!} = \frac{e(S)(-1)^n n}{2(2n-1)} \frac{(2n)!}{n!^4}$$

where we applied Proposition 4.2 in the last step. □

4.3. Generalized Kummer varieties

We first compute the class of $\mathrm{Kum}_n(A)$ in the Nakajima basis of $A^{[n+1]}$.

Lemma 4.5. *In $H^4(A^{[n+1]})$ we have*

$$[\mathrm{Kum}_n(A)] = \mathfrak{G}_2(\alpha)\mathfrak{G}_2(\beta)\mathfrak{G}_2(\gamma)\mathfrak{G}_2(\delta) 1_{A^{[n+1]}}$$

for any $\alpha, \beta, \gamma, \delta \in H^1(A)$ such that $\int_A \alpha\beta\gamma\delta = 1$.

Proof. Let $\sigma : A^{[n+1]} \to A$ be the sum map. We have

$$[\mathrm{Kum}_n(A)] = \sigma^*(\mathsf{p}).$$

Hence it suffices to show that $\sigma^*(\alpha) = G_2(\alpha)$ for any $\alpha \in H^1(A)$, where we let $G_2(\alpha) = \mathfrak{G}_2(\alpha) 1_{A^{[n+1]}}$. Consider $x \in H^3(A, \mathbb{Z}) = H_1(A, \mathbb{Z})$ and let $L(x) = \mathfrak{q}_1(x)\mathfrak{q}_1(\mathsf{p})^n 1$. When x is represented by a singular chain, than $L(x)$ is represented by the chain obtained from the former by adding $n-1$ distinct points to it. This shows that $\sigma_* L(x) = x$, and hence

$$\int_{A^{[n+1]}} \sigma^*(\alpha) \cdot L(x) = \int_A \alpha \cdot \sigma_* L(x) = \int_A \alpha x.$$

On the other hand, a direct calculation using the Nakajima operators also shows $\int_{A^{[n+1]}} G_2(\alpha) \cdot L(x) = \int_A \alpha x$. Since the $L(x)$ generate $H_1(A^{[n+1]})$ this yields the claim. □

Since $[\mathfrak{G}_2(x), \mathfrak{q}_1(y)] = \mathfrak{q}_1(xy)$ for all $x, y \in H^*(S)$ one finds that

$$[\mathrm{Kum}_n(A)] = \sum_{\pi=\{\pi_i\}} \sigma_\pi \frac{1}{(n+1-\ell(\pi))!} \prod_i \mathfrak{q}_1\left(\prod_{x\in\pi_i} x\right) \mathfrak{q}_1(1)^{n+1-\ell(\pi)} 1 \tag{10}$$

with the following notation:

- π runs over all set partitions of $\{\alpha, \beta, \gamma, \delta\}$ with $l(\pi)$ parts,
- $\sigma_\pi \in \{\pm 1\}$ is the sign obtained from bringing $\prod_i \prod_{x\in\pi_i} x$ into the order $\alpha\beta\gamma\delta$,
- in case $n \leq 2$ we sum only over set partitions with $l(\pi) \leq n+1$.

The first terms read:

$$\begin{aligned}[\mathrm{Kum}_n(A)] = \frac{1}{n!}\mathfrak{q}_1(\mathsf{p})\mathfrak{q}_1(1)^n 1 + \frac{1}{(n-1)!}\mathfrak{q}_1(\alpha)\mathfrak{q}_1(\beta\gamma\delta)\mathfrak{q}_1(1)^{n-1}1 \\ + \ldots + \frac{1}{(n-3)!}\mathfrak{q}_1(\alpha)\mathfrak{q}_1(\beta)\mathfrak{q}_1(\gamma)\mathfrak{q}_1(\delta)\mathfrak{q}_1(1)^{n-3}1.\end{aligned}$$

Lemma 4.6.

$$\int_{A^{[n+1]}} \mathfrak{q}_1^{n+1}\mathfrak{q}_{-1}^{n+1}(\Delta)[\mathrm{Kum}_n(A)] = (n+1)^4.$$

Proof. Using Lemma 4.5, equation (10) and the straightforward evaluation

$$\sigma_\pi \int_{A^{[n+1]}} \mathfrak{q}_1^{n+1}\mathfrak{q}_{-1}^{n+1}(\Delta) \prod_i \mathfrak{q}_1\left(\prod_{x\in\pi_i} x\right) \mathfrak{q}_1(1)^{n+1-\ell(\pi)} 1 = (-1)^{n+1}(n+1)!$$

for every π, we find that

$$\begin{aligned}\int_{A^{[n+1]}} \mathfrak{q}_1^{n+1}\mathfrak{q}_{-1}^{n+1}(\Delta)[\mathrm{Kum}_n(A)] = \sum_\pi \frac{(n+1)!(-1)^{n+1}}{(n+1-l(\pi))!} \\ = (-1)^{n+1}(n+1)\Big[1 + 7n + 6n(n-1) + n(n-1)(n-2)\Big] \\ = (-1)^{n+1}(n+1)^4.\end{aligned}$$

□

Proof of Theorem 4.2. We have the exact sequence

$$0 \to T_{\mathrm{Kum}_n(A)} \to T_{A^{[n+1]}}|_{\mathrm{Kum}_n(A)} \to \sigma^*(T_A)|_{\mathrm{Kum}_n(A)} \to 0$$

which together with (7) and (8) (using $e(A) = 0$) shows that

$$\int_{\mathrm{Kum}_n(A)} \mathrm{ch}_{2n}(T_{\mathrm{Kum}_n(A)}) = \int_{A^{[n+1]}} \mathrm{ch}_{2n}(T_{A^{[n+1]}}) \cap [\mathrm{Kum}_n(A)]$$
$$= \sum_{i+j=2n+2} (-1)^{j+1} \int_{A^{[n+1]}} \mathfrak{G}_i \mathfrak{G}_j(\Delta)[\mathrm{Kum}_n(A)].$$

Consider the expansion $\mathfrak{G}_i\mathfrak{G}_j(\Delta) = \sum_{l(\lambda)=i, l(\tilde{\lambda})=j} \mathfrak{q}_\lambda \mathfrak{q}_{\tilde{\lambda}}(\Delta)/(\lambda!\tilde{\lambda}!)$. Since $\mathfrak{q}_{\tilde{\lambda}}$ acts on (10) which consists only of terms of the form $\prod_i \mathfrak{q}_1(x_i)\, 1$, for a summand to contribute, $\tilde{\lambda}$ can only have negative parts equal to -1. Assume $\tilde{\lambda}$ has a positive part $k > 1$. Then λ has to have a corresponding negative part $-k$, and these two parts have to interact when commuting all negative Nakajima operators to the right. However, this will yield the term

$$[\mathfrak{q}_{-k}, \mathfrak{q}_k]\mathfrak{q}_{\lambda'}\mathfrak{q}_{\tilde{\lambda}'}(\pi_{12*}(\Delta_{12}\Delta_{12\cdots(l(\lambda)+l(\tilde{\lambda}))})) = -k\mathfrak{q}_{\lambda'}\mathfrak{q}_{\tilde{\lambda}'}(c_2(A)\Delta) = 0$$

where π_{12} is the projection away from the first two factors and $\lambda', \tilde{\lambda}'$ are the partitions $\lambda, \tilde{\lambda}$ without the parts $k, -k$. We conclude that only the summands with $\lambda = (-1)^a(1)^a$ and $\tilde{\lambda} = (-1)^b(1)^b$ where $i = 2a$ and $j = 2b$ can contribute to the integral. Moreover, applying a similar argument we have $\mathfrak{q}_1^a\mathfrak{q}_{-1}^a\mathfrak{q}_1^b\mathfrak{q}_{-1}^b(\Delta) = \mathfrak{q}_1^{a+b}\mathfrak{q}_{-1}^{a+b}(\Delta)$.

We thus find the following expression:

$$= \sum_{a+b=n+1} \frac{(-1)}{a!^2 b!^2} \int_{A^{[n+1]}} \mathfrak{q}_1^{n+1}\mathfrak{q}_{-1}^{n+1}(\Delta)[\mathrm{Kum}_n(A)]$$
$$= \sum_{a+b=n+1} \frac{(-1)}{a!^2 b!^2}(-1)^{n+1}(n+1)^4$$
$$= (-1)^n \frac{(2n+2)!}{n!^4}$$

where we used the first part of Lemma 4.1. □

The computations above can be generalized to arbitrary products of Chern characters. The following qualitative result is almost immediate:

Proposition 4.7. *Let $n \geq 1$. For any partition $n = k_1 + k_2 + \ldots + k_r$ we have*

$$(-1)^n \int_{\mathrm{Kum}_n(A)} \mathrm{ch}_{2k_1}(\mathrm{Kum}_n(A)) \cdots \mathrm{ch}_{2k_r}(\mathrm{Kum}_n(A)) > 0.$$

Proof. Let $n-1 = k_1 + \ldots + k_r$ be a partition of $n-1$. Then

$$\begin{aligned}
&\int_{\mathrm{Kum}_{n-1}(A)} \mathrm{ch}_{2k_1}(\mathrm{Kum}_{n-1}(A)) \cdots \mathrm{ch}_{2k_r}(\mathrm{Kum}_{n-1}(A)) \\
&= \sum_{\substack{i_1+j_1=2k_1+2 \\ \cdots \\ i_r+j_r=2k_r+2}} (-1)^{j_1+\ldots+j_r+r} \int_{A^{[n]}} \mathfrak{G}_{i_1}\mathfrak{G}_{j_1}(\Delta)\cdots\mathfrak{G}_{i_r}\mathfrak{G}_{j_r}(\Delta)[\mathrm{Kum}_{n-1}(A)]
\end{aligned}$$

We express the $\mathfrak{G}_d$ in terms of Nakajima operators via (7), which produces a sum consisting of summands with precisely

$$\sum_{s=1}^{r} i_s + j_s = \sum_s (2k_s + 2) = 2n + 2(r-1)$$

Nakajima factors acting on the class of $\mathrm{Kum}_{n-1}(A)$. When commuting all negative Nakajima operators to the right, we see that for a term to contribute there have to be at least $r-1$ Nakajima interactions between these $2n+2(r-1)$ factors. Moreover, since $e(A) = 0$ (compare the proof of Theorem 4.2) only the following is allowed:

(a) There can be no Nakajima interactions between factors belonging to the same $\mathfrak{G}_{i_s}\mathfrak{G}_{j_s}(\Delta)$.
(b) There can be at most one Nakajima interaction between factors belonging to $\mathfrak{G}_{i_s}\mathfrak{G}_{j_s}(\Delta)$ and $\mathfrak{G}_{i_{s'}}\mathfrak{G}_{j_{s'}}(\Delta)$ for $s \neq s'$.

This shows that there can be at most $r-1$ Nakajima interactions. The total sign contribution from these Nakajima interactions is $(-1)^{r-1}$ and the outcome will be a multiple of the operator $\mathfrak{q}_1^n\mathfrak{q}_{-1}^n(\Delta)$. By Lemma 4.6 the degree of $\mathfrak{q}_1^n\mathfrak{q}_{-1}^n(\Delta)[\mathrm{Kum}_{n-1}(A)]$ yields a sign of $(-1)^n$. Since there always is at least one summand that contributes with a non-zero value, the claim now follows as soon as we can prove that $j_1 + \ldots + j_r$ is even.

If $\lambda = \big(\ldots(-2)^{l_2}(-1)^{l_1}(1)^{l_1}(2)^{l_2}\ldots\big)$ is a generalized partition of size $|\lambda| = \sum_i il_i = 0$, then by considering this equality mod 2 we get that the number of odd parts $l_{\mathrm{odd}} := \sum_j l_{2j+1}$ is even, and hence that $l(\lambda)$ is equal to the number of even parts $l_{\mathrm{even}}(\lambda) := \sum_j l_{2j}$ modulo 2. Let $\lambda_s, \tilde{\lambda}_s$ be the generalized partitions appearing in a given summand of $\mathfrak{G}_{i_s}\mathfrak{G}_{j_s}$. We see

$$(-1)^{j_1+\ldots+j_r} = (-1)^{l_{\mathrm{even}}(\lambda_1)+\ldots+l_{\mathrm{even}}(\lambda_r)}.$$

Moreover, since $i_s + j_s$ is even, for every s we have $l_{\mathrm{even}}(\lambda_s) + l_{\mathrm{even}}(\tilde{\lambda}_s)$ is even. This shows that there is always an even number of even Nakajima factors in $\mathfrak{G}_{i_s}\mathfrak{G}_{j_s}(\Delta)$. Let m be the number of $s \in \{1, \dots, r\}$ such that there exists even Nakajima factors in $\mathfrak{G}_{i_s}\mathfrak{G}_{j_s}(\Delta)$. Since all even Nakajima factors have to interact with each other, we see that there are at least m Nakajima interactions between these m factors, This implies that either (a) or (b) above is violated, and the corresponding contribution vanishes. Hence for any non-zero summand contributing to the Chern character number, all Nakajima factors are odd, so we have $j_s \equiv 0(2)$ and therefore $(-1)^{j_1+\dots+j_r}$ even. $\square$

Remark 4.8. Arbitrary Chern character numbers of $\mathrm{Kum}_n(A)$ can be computed in a parallel manner, however the expressions become more complicated. For example, the double Chern character numbers of the generalized Kummer for $0 < k < n$ are given as

$$\int_{\mathrm{Kum}_n(A)} \mathrm{ch}_{2k}(\mathrm{Kum}_n(A))\mathrm{ch}_{2n-2k}(\mathrm{Kum}_n(A)) = 4(-1)^n(n+1)^4$$

$$(2k+1)!(2n-2k+1)!\sum_{i=0}^{k} \frac{2i+1}{((k-i)!(k+i+1)!(n-k-i)!(n-k+i+1)!)^2}.$$

For $k = 1$ one gets

$$\int_{\mathrm{Kum}_n(A)} \mathrm{ch}_2(\mathrm{Kum}_n(A))\mathrm{ch}_{2n-2}(\mathrm{Kum}_n(A)) = (-1)^n \frac{(2n)!}{n!^4}\left(4n(n+1)^2(n^2+n+1)\right).$$

5. Remarks and open questions

A first obvious question is the following

Question 5.1. *Compute $M(\Sigma^{[n]})$ for any smooth projective surface Σ.*

More precisely, it is a consequence of [3] that we have a formula

$$M(\Sigma^{[n]}) = \alpha_n \int_\Sigma c_1(\Sigma)^2 + \beta_n \int_\Sigma c_2(\Sigma), \tag{11}$$

so the question is to compute α_n and β_n. Formula (11) follows from the main result of [3] which says that $M(\Sigma^{[n]})$ depends only on $\int_\Sigma c_1(\Sigma)^2$ and $\int_\Sigma c_2(\Sigma)$ and from the formula

$$\Sigma^{[n]} = \sqcup_{k+l=n} \Sigma_1^{[k]} \times \Sigma_2^{[l]}$$

when $\Sigma = \Sigma_1 \sqcup \Sigma_2$, which by Lemma 3.1 gives

$$M(\Sigma^{[n]}) = M(\Sigma_1^{[n]}) + M(\Sigma_2^{[n]}),$$

proving that $M(\Sigma^{[n]})$ is a linear function of $\int_\Sigma c_1(\Sigma)^2$ and $\int_\Sigma c_2(\Sigma)$. Equation (11) suggests that another approach to Theorem 1.3 would be by computing the Milnor genus of $\Sigma^{[n]}$ for two conveniently chosen surfaces Σ, in the spirit of [28].

Theorem 4.1 shows that $\beta_n = (-1)^n \frac{(2n+2)!}{24(2n-1)(n!)^4}$. The Milnor genus of $(\mathbb{P}^2)^{[n]}$ can be numerically computed using Bott's residue formula for small values of n, so we get the following list of α_n.

n	α_n
1	$1/2$
2	$-5/12$
3	$91/540$
4	$-67/1680$
5	$5599/907200$
6	$-8047/11975040$
7	$295381/5448643200$
8	$-17616097/5230697472000$
9	$797006281/4801780279296000$
10	$-404188861/60822550204416000$
11	$15479922001/70250045486100480000$
12	$-8942373821/1454175941562279936000$

Turning to hyper-Kähler geometry, an obvious open question, that was our original motivation for formulating Theorem 1.1, is

Question 5.2. *What are the constraints on the complex cobordism classes of hyper-Kähler manifolds?*

In view of Theorem 1.1, we can rephrase this question in terms of inequalities or equalities between the coefficients $\alpha_I(X)$ (resp. β_I) given by Theorem 1.1, expressing the class of X as a combination of classes of the $S^{[I]}$ (resp. $\mathrm{Kum}_I(A)$). One obvious restriction is the affine relation given by the fact that $\chi(X, \mathcal{O}_X) = n+1$ for X hyper-Kähler of dimension $2n$. Using the Hirzebruch-Riemann-Roch formula, this gives a relation between the Chern numbers of X, but we can express it more simply using the α_I since $\chi(S^{[I]}, \mathcal{O}_{S^{[I]}}) = (n_1+1)\dots(n_k+1)$ for the partition I of n given by $n = n_1 + \dots + n_k$. The relation is thus

$$n+1 = \sum_I \alpha_I (n_1+1)\dots(n_k+1) \tag{12}$$

and similarly for the β_I. For example, in dimension 4, the Hirzebruch-Riemann-Roch formula provides the relation (see [22])

$$3 = \frac{1}{240}\left(\int_X c_2(X)^2 - \frac{1}{3}\int_X c_4(X)\right), \tag{13}$$

while in our setting, it writes

$$3\alpha_2 + 4\alpha_{1,1} = 3.$$

In the case of dimension 4 we have two topological models, the Hilbert scheme $S^{[2]}$ and the generalized Kummer variety $\mathrm{Kum}_2(A)$ and they clearly have independent classes, since otherwise by (12) their classes would be equal, hence also their topological Euler-Poincaré characteristic c_4, which is not the case. In dimension 6, we have 3 topological models, namely $S^{[3]}$, $\mathrm{Kum}_3(A)$ and OG6 constructed in [20], and their classes are linearly independent, as proves the following computation. The Chern numbers c_2^3, c_2c_4, c_6 of $\mathrm{K3}^{[3]}$ are computed in [3], those of $\mathrm{Kum}_3(A)$ are computed in [17], and those of OG6 are computed in [14]. Thanks to these works, the matrix of Chern numbers for these three varieties takes the form (where the first line indicates the Chern numbers of $\mathrm{K3}^{[3]}$, the second line those of $\mathrm{Kum}_3(A)$, and the third line those of OG6):

$$\begin{pmatrix} 36800 & 14720 & 3200 \\ 30208 & 6784 & 448 \\ 30720 & 7680 & 1920 \end{pmatrix}.$$

The determinant of this matrix is nonzero, proving the independence of the three classes. Thus, up to dimension 6, the classes of hyper-Kähler manifolds generate the affine space defined by (12). It is likely that there are linear relations in higher dimension.

Other contraints are given by inequalities. For example, the class c_2 has positivity properties related to the existence of Kähler-Einstein metrics. Positivity results for some Chern numbers have been also obtained by Jiang [9] who proves that the coefficients of the Riemann-Roch polynomial of a line bundle L on X, expressed as a polynomial in $q(L)$, has positive coefficients. It is proved in [16] that for an adequate normalization of the Beauville-Bogomolov form q, these coefficients are given by Chern numbers of X (depending only on the dimension). In dimension 4, work of Guan [7] gives inequalities on $\int_X c_4(X)$ that come from the study of the cohomology algebra of X. In higher dimension $2n$, work of [5] also predicts bounds on Betti numbers which in turn gives conjectural bounds on the topological Euler-Poincaré characteristic $\int_X c_{2n}(X)$. It would be very interesting to have an idea of the convex set generated by classes of hyper-Kähler manifolds. Let us now mention three specific questions in this direction.

(a) **The numbers $\chi(X,\Omega_X^i)$.** In the case of the varieties $S^{[n]}$ and $\mathrm{Kum}_n(A)$, we have the following result.

Lemma 5.3. *Let S be a K3 surface. Then the numbers $(-1)^i\chi(S^{[n]},\Omega^i_{S^{[n]}})$ are increasing in the range $0\leq i\leq n$.*

Similarly, for n fixed, the numbers $(-1)^i\chi(\mathrm{Kum}_n(A),\Omega^i_{\mathrm{Kum}_n(A)})$ are increasing.

Proof. We argue as in Section 2. As these numbers are Chern numbers by the Hirzebruch-Riemann-Roch formula, we can replace by [3] the K3 surface S by the disjoint union Σ of two copies of $\mathbb{P}^2$ blown-up in 9 points. Then $(-1)^i\chi(\Sigma^{[n]},\Omega^i_{\Sigma^{[n]}})=b_{2i}(\Sigma^{[n]})$ so the statement is that $b_{2i}(\Sigma^{[n]})$ is increasing in the range $0\leq i\leq n$ and this follows from the hard Lefschetz theorem since $\dim\Sigma^{[n]}=2n$.

For the second statement, the numbers $(-1)^i\chi(\mathrm{Kum}_n(A),\Omega^i_{\mathrm{Kum}_n(A)})$ are computed in [6] which gives the following formula

$$\sum_i(-1)^i\chi(\mathrm{Kum}_n(A),\Omega^i_{\mathrm{Kum}_n(A)})y^i=n\sum_{d|n}d^3(1+y+\ldots+y^{n/d-1})^2y^{n-n/d}.$$

It immediately follows that these numbers are increasing in the range $0\leq i\leq n$. $\square$

We also computed these numbers for OG6 and OG10 and got

$$(-1)^i\chi(\mathrm{OG6},\Omega^i_{\mathrm{OG6}})=\ 4,\ 24,\ 348,\ 1168$$

respectively for $i=0,\,1,\,2,\,3$ and

$$(-1)^i\chi(\mathrm{OG10},\Omega^i_{\mathrm{OG10}})=6,\ 111,\ 1062,\ 7173,\ 33534,\ 93132,$$

respectively for $i=0,\,1,\,2,\,3,\,4,\,5$. In the two cases, these numbers are increasing. This raises the following question.

Question 5.4. *Is it true that the numbers $(-1)^i\chi(X,\Omega^i_X)$ are increasing in the range $0\leq i\leq n$ for any hyper-Kähler manifold X of dimension $2n$?*

Remark 5.5. If $i\leq n$, the cup-product map by σ_X^{n-i} gives an isomorphism $\Omega^i_X\cong\Omega_X^{2n-i}$, hence induces an isomorphism $\sigma_X^{n-i}:H^j(X,\Omega^i_X)\cong H^j(X,\Omega^{2n-i}_X)$ for any integer j. It follows that the wedge-product by σ_X is injective on

$H^j(X, \Omega_X^i)$ for $i < n$. This implies that, if X has no odd degree cohomology, one has $(-1)^i\chi(X, \Omega_X^i) \leq (-1)^i\chi(X, \Omega_X^{i+2})$ for $i < n$, which gives a partial answer to Question 5.4. (We thank one of the referees on this paper for this remark.)

Remark 5.6. One has $\chi_{\text{top}}(X) = \sum_i (-1)^i \chi(X, \Omega_X^i)$, using the fact that the holomorphic de Rham complex gives a resolution of the constant sheaf $\mathbb{C}$ on X. Using the isomorphisms $\Omega_X^i \cong \Omega_X^{2n-i}$ above, we can rewrite this as

$$\chi_{\text{top}}(X) = 2\sum_{i=0}^{i=n-1}(-1)^i\chi(X, \Omega_X^i) + (-1)^n\chi(X, \Omega_X^n).$$

If Question 5.4 had an affirmative answer, each term in the above sum would be $\geq \chi(X, \mathcal{O}_X) = n+1$ and we would thus have the inequality $\chi_{\text{top}}(X) \geq (2n+1)(n+1)$. It is not even known in general if $\chi_{\text{top}}(X) \geq 0$, but the inequality $\chi_{\text{top}}(X) = \int_X c_{2n}(X) \geq 0$ was already conjectured (see Question 5.10).

(b) **Chern character numbers**. Theorems 4.1 and 4.2 prove that the two numbers $(-1)^n \int_{S^{[n]}} \mathrm{ch}_{2n}(S^{[n]})$ and $(-1)^n \int_{\mathrm{Kum}_n(A)} \mathrm{ch}_{2n}(\mathrm{Kum}_n(A))$ are positive for any n.

This suggests the following question.

Question 5.7. *Is it true that $(-1)^n M(X) = (-1)^n \int_X \mathrm{ch}_{2n}(X)$ is positive for any hyper-Kähler manifold X of dimension $2n$?*

The following lemma gives an affirmative answer in dimension 4.

Lemma 5.8. *Let X be a hyper-Kähler fourfold. Then $M(X) = \int_X \mathrm{ch}_4(X) > 0$.*

Proof. We have $\mathrm{ch}_4(X) = \frac{1}{24}(2c_2^2(X) - 4c_4(X))$ so the statement is equivalent to $\int_X (c_2^2(X) - 2c_4(X)) > 0$. Formula (13) gives us $\int_X c_2(X)^2 = 720 + \frac{1}{3}\int_X c_4(X)$, so the desired inequality is equivalent to

$$\int_X c_4(X) = \chi_{\text{top}}(X) < \frac{9 \cdot 240}{5} = 432. \tag{14}$$

Inequality (14) now follows from work of Salamon [23] and Guan [7]. By [23], $b_3(X) + b_4(X) = 46 + 10b_2(X)$, hence $\chi_{\text{top}}(X) = b_4(X) - 2b_3(X) + 2b_2(X) + 2 \leq 48 + 12b_2(X)$. Guan proves that $b_2(X) \leq 23$, so we get

$$\chi_{\text{top}}(X) \leq 48 + 12 \cdot 23 = 324,$$

proving (14). □

Proposition 4.7 shows that $(-1)^n \int_{\mathrm{Kum}_n(A)} \mathrm{ch}_{2k_1} \dots . \mathrm{ch}_{2k_r} > 0$ for any choice of partition $n = \sum_i k_i$. This suggests the following question

Question 5.9. *Is it true that $(-1)^n \int_X \mathrm{ch}_{2k_1}(X) \cdots \mathrm{ch}_{2k_r}(X)$ is positive for any hyper-Kähler manifold X of dimension $2n$ and any choice of partition $n = \sum_i k_i$?*

(c) **Positivity of monomial Chern numbers**. We recall here for completeness that positivity properties had been observed already in [18], [24] for the monomial Chern numbers $\int_X c_{2k_1}(X) \dots c_{2k_r}(X)$ of known hyper-Kähler manifolds. The following question was asked in [18]

Question 5.10. *Is it true that $\int_X c_{2k_1}(X) \dots . c_{2k_r}(X)$ is positive for any hyper-Kähler manifold X of dimension $2n$ and any choice of partition $n = \sum_i k_i$?*

We note that, in the case of dimension 4, it is still unknown that $e(X) = \int_X c_4(X) > 0$. The questions (b) and (c) look very similar but they lead to very different convexity inequalities and, in dimension 4, the two inequalities $\int_X c_4(X) > 0$ (conjectured above) and $\int_X \mathrm{ch}_4(X) > 0$ proved in Lemma 5.8 imply together the finiteness of the complex cobordism classes of hyper-Kähler fourfolds.

We finish with two questions more specifically related to our results, concerning the comparison of the two systems of linear generators $S^{[I]}$ and $\mathrm{Kum}_I(A)$. It would be interesting to know more about the matrix comparing these two systems of linear generators in each dimension.

Question 5.11. *Is there a geometric way of understanding and computing this matrix?*

Another intriguing fact concerns the shape of the coefficients of these matrices. Since the Chern numbers of $S^{[k]}$ and $\mathrm{Kum}_k(A)$ are known for small values of k, and the Chern numbers of a product $X \times Y$ can be expressed in terms of Chern numbers of X and Y, one gets consequently the Chern numbers of $S^{[I]}$ and $\mathrm{Kum}_I(A)$ for all partitions I of k. One may then study the linear relations among the classes of these manifolds. Below is the explicit expression giving the class of $S^{[k]}$ as a $\mathbb{Q}$-linear combination of the classes of $\mathrm{Kum}_I(A)$ for $k \leq 5$.

(15)

$$S^{[2]} = \frac{1}{3}\mathrm{Kum}_2(A) + \frac{1}{2}\mathrm{Kum}_{1,1}(A)$$

$$S^{[3]} = \frac{1}{5}\mathrm{Kum}_3(A) + \frac{14}{45}\mathrm{Kum}_{2,1}(A) + \frac{1}{6}\mathrm{Kum}_{1,1,1}(A)$$

$$
\begin{aligned}
S^{[4]} &= \frac{1}{7}\mathrm{Kum}_4(A) + \frac{7}{40}\mathrm{Kum}_{3,1}(A) + \frac{1}{21}\mathrm{Kum}_{2,2}(A) \\
&\quad + \frac{47}{315}\mathrm{Kum}_{2,1,1}(A) + \frac{1}{24}\mathrm{Kum}_{1,1,1,1}(A) \\
S^{[5]} &= \frac{1}{9}\mathrm{Kum}_5(A) + \frac{62}{525}\mathrm{Kum}_{4,1}(A) + \frac{4}{75}\mathrm{Kum}_{3,2}(A) + \frac{49}{600}\mathrm{Kum}_{3,1,1}(A) \\
&\quad + \frac{23}{525}\mathrm{Kum}_{2,2,1}(A) + \frac{151}{3150}\mathrm{Kum}_{2,1,1,1}(A) + \frac{1}{120}\mathrm{Kum}_{1,1,1,1,1}(A).
\end{aligned}
$$

The leading coefficient being $\frac{1}{2k-1}$ can be explained by the difference in the expression of Milnor genus for the two infinite series, since the other terms are products and do not contribute to the Milnor genus.

Similarly, we computed the class of $\mathrm{Kum}_k(A)$ as a $\mathbb{Q}$-linear combination of the classes of $S^{[I]}$ for $k \leq 5$.

$$
\begin{aligned}
\mathrm{Kum}_2(A) &= 3S^{[2]} - \frac{3}{2}S^{[1,1]} \\
\mathrm{Kum}_3(A) &= 5S^{[3]} - \frac{14}{3}S^{[2,1]} + \frac{3}{2}S^{[1,1,1]} \\
\mathrm{Kum}_4(A) &= 7S^{[4]} - \frac{49}{8}S^{[3,1]} - 3S^{[2,2]} + \frac{67}{12}S^{[2,1,1]} - \frac{21}{16}S^{[1,1,1,1]} \\
\mathrm{Kum}_5(A) &= 9S^{[5]} - \frac{186}{25}S^{[4,1]} - \frac{36}{5}S^{[3,2]} + \frac{1287}{200}S^{[3,1,1]} \\
&\quad + \frac{159}{25}S^{[2,2,1]} - \frac{577}{100}S^{[2,1,1,1]} + \frac{423}{400}S^{[1,1,1,1,1]}.
\end{aligned} \tag{16}
$$

Equations (15) strongly suggest the following question.

Question 5.12. *Is it true that for any n, the class of $S^{[n]}$ is a linear combination with positive coefficients of the classes of $\mathrm{Kum}_I(A)$?*

There are only two known hyper-Kähler manifolds which do not belong to the two infinite series discussed above, namely the 6-dimensional and 10-dimensional O'Grady manifolds OG6 and OG10 (see [20], [21]). Their cobordism classes are expressed as follows in the generalized Kummer basis (showing in particular that not any hyper-Kähler manifold has its class in the convex cone generated by products of generalized Kummer varieties).

$$
\begin{aligned}
\mathrm{OG6} &= \frac{6}{5}\mathrm{Kum}_3(A) - \frac{16}{45}\mathrm{Kum}_{2,1}(A) + \frac{1}{6}\mathrm{Kum}_{1,1,1}(A), \\
\mathrm{OG10} &= \frac{25}{168}\mathrm{Kum}_5(A) + \frac{67}{700}\mathrm{Kum}_{4,1}(A) + \frac{3}{700}\mathrm{Kum}_{3,2}(A) + \frac{163}{1600}\mathrm{Kum}_{3,1,1}(A) \\
&\quad + \frac{2617}{37800}\mathrm{Kum}_{2,2,1}(A) + \frac{493}{12600}\mathrm{Kum}_{2,1,1,1}(A) + \frac{17}{1920}\mathrm{Kum}_{1,1,1,1,1}(A).
\end{aligned}
$$

Our last observation is the following. There is a mysterious link (in fact related to mirror symmetry) between hyper-Kähler manifolds of dimension $2n$ and rational homology projective space $\mathbb{CP}^n$. It appears for example in [11] where it is proved that the dual complex of the central fiber of a maximally unipotent dlt degeneration of a hyper-Kähler $2n$-fold is a rational homology projective space $\mathbb{CP}^n$. There is another mysterious and more precise link between $\mathrm{K3}^{[n]}$ and projective space $\mathbb{P}^n$, which comes from the study of the Riemann-Roch polynomials. Indeed, one has the following result that can be formulated using the Chern numbers of X by [16]. (This result is proved by looking at the natural Lagrangian fibration of a variety $S^{[n]}$ where S is a K3 surface equipped with an elliptic fibration.)

Theorem 5.13. *[3, Lem.5.1] Let X be a hyper-Kähler manifold of $\mathrm{K3}^{[n]}$-deformation type and q be its Beauville-Bogomolov form. Then for any line bundle L on X with $q(c_1(L)) = 2k$, one has $\chi(X, L) = \chi(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k+1)) = h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k+1))$.*

The formalism used in the present paper proposes a further analogy between $\mathrm{K3}^{[n]}$ and $\mathbb{P}^n$. Namely the classical complex cobordism gives the projective spaces $\mathbb{P}^n$ as multiplicative rational generators of $\mathrm{MU}^*(\mathrm{pt})$ while we proved that the $\mathrm{K3}^{[n]}$ are multiplicative rational generators of $\mathrm{MU}^*(\mathrm{pt})_{\mathrm{even}}$.

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