

Universally defined cycles I

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To the memory of Jacob Murre

Abstract

We introduce and study the notion of universally defined cycles of smooth varieties of dimension d , and prove that they are given by polynomials in the Chern classes. A similar result is proved for universally defined cycles on products of smooth varieties. We also state a conjectural explicit form for universally defined cycles on powers of smooth varieties, and provide some steps towards establishing it.

1 Introduction

The Franchetta conjecture, which has now at least two proofs (see [8], [5]) asserts that for any line bundle $\mathcal{C}$ on the universal curve $\mathcal{C} \rightarrow U$, where U is a Zariski open set of the moduli space of curves of genus $g \geq 2$, the restriction of $\mathcal{C}$ to the fibers $\mathcal{C}_b$, for any point b of U , is a multiple of the canonical bundle.

If we want to prove a similar result for higher dimensional manifolds, we are faced to several difficulties. The main problem is the following: there is no moduli space of manifolds of given dimension. Even if we restrict to canonically polarized manifolds, where such moduli spaces exist, some components consist of just one point (rigid manifolds) so the above statement certainly fails.

This paper studies the notion of *universally defined cycles* introduced in [12] and establishes in this setting an analogue of the Franchetta conjecture for higher dimensional manifolds. In the case of curves, the analogue of the notion we introduce would be roughly the data of a line bundle $\mathcal{L}_g$ on the “universal curve over $\mathcal{M}_g$ ” for each g , satisfying some mild compatibility properties relating the curves of various genera, using the degenerations of curves of genus g to curves of genus g' with $g' < g$. This notion has some similarity with the functorial Chow groups considered by Mumford [10]. Then the conclusion would be that the restriction of these line bundles to fibers (that is curves of genus g defined over a field) is a rational multiple (independent of g) of the canonical bundle.

The precise definition is as follows. Here Chow groups are taken with $\mathbb{Q}$ -coefficients.

Definition 1.1. *A universally defined cycle on smooth complex varieties of dimension n is the data of a cycle $\mathcal{Z}(\pi) \in \text{CH}(\mathcal{X})$ for each smooth morphism $\pi : \mathcal{X} \rightarrow B$ of relative dimension n , where $\mathcal{X}$ and B are smooth quasi-projective. These data should satisfy the following axioms:*

(i) *If $r : B' \rightarrow B$ is a morphism, with B' smooth, and $r' : \mathcal{X}' := \mathcal{X} \times_B B' \rightarrow \mathcal{X}$, $\pi' : \mathcal{X}' \rightarrow B'$ are the two natural morphisms,*

$$\mathcal{Z}(\pi') = r'^* \mathcal{Z}(\pi) \text{ in } \text{CH}(\mathcal{X}').$$

(ii) *If $\mathcal{X}' \subset \mathcal{X}$ is a Zariski open subset, and $\pi' : \mathcal{X}' \rightarrow B$ is the composition of the inclusion and of the morphism $\pi : \mathcal{X} \rightarrow B$,*

$$\mathcal{Z}(\pi') = \mathcal{Z}(\pi)|_{\mathcal{X}'} \text{ in } \text{CH}(\mathcal{X}').$$

Remark 1.2. 1) We could a priori work over any field K and smooth algebraic varieties $\mathcal{X}$, B over K . It might be however that our results need more assumptions on K .

2) We will also use the notation $\mathcal{Z}(\mathcal{X})$ although the cycle depends on the morphism π . When B is a point, that is, X is a variety defined over a field, we will systematically use the notation $\mathcal{Z}(X) \in \text{CH}(X)$.

3) In the definition above, smoothness of the morphism π is essential, as well as the absence of any projectivity assumption on π .

Example 1.3. The Chern classes of the relative tangent bundle $T_\pi := T_{\mathcal{X}/B}$ are universally defined, and more generally any fixed polynomial in the relative Chern classes is universally defined.

Our first main result in this paper is the following:

Theorem 1.4. *Let $\mathcal{Z}$ be a universally defined cycle on varieties of dimension d . Then there exists a polynomial P with rational coefficients in d variables $c_1, \dots, c_d$, such that for any smooth quasi-projective variety X over $\mathbb{C}$, $\mathcal{Z}(X) = P(c_1(X), \dots, c_d(X))$ in $\text{CH}(X)$.*

Remark 1.5. Our definition involves families $\mathcal{X} \rightarrow B$ and the conclusion concerns only the constant families $X \rightarrow \text{pt}$. One may wonder if one could weaken the notion of universally defined cycle by working only with varieties defined over a field (including function fields), the axiom (i) being replaced by Fulton specialization. We have not been able to prove Theorem 1.4 with these weaker data.

Remark 1.6. As we will see in Theorem 1.7, the polynomial P is in fact uniquely determined by $\mathcal{Z}$, assuming it has weighted degree $\leq d$ in the variables c_i (where the degree of c_i is i).

Let us give a sketch of the proof for curves (the proof is much easier in this case): The idea is the following: Any smooth curve C can be imbedded in $\mathbb{P}^3$, say by a morphism $i : C \rightarrow \mathbb{P}^3$, and, choosing a large enough positive integer d , there is a curve C_d which is the complete intersection of two surfaces of degree d , containing $i(C)$ as a component. The universal (quasi-projective) smooth complete intersection curve of two surfaces of degree d in $\mathbb{P}^3$ has very simple Chow groups. So for the smooth part of C_d as above, there exists a coefficient α_d which is a rational number such that $\mathcal{Z}(C_{d,\text{reg}}) = \alpha_d c_1(K_{C_{d,\text{reg}}})$ in $\text{CH}^1(C_{d,\text{reg}})$. Restricting this equality to $i(C) \setminus (C \cap \text{Sing } C_d)$, the localization exact sequence shows that $\mathcal{Z}(C) = \alpha_d K_C + z'$ in $\text{CH}^1(C)$, where z' is supported on the intersection $C \cap \text{Sing } C_d$. Testing this equality on curves for which there is no relations in $\text{CH}^1(C_{d,\text{reg}})$ between K_C and points in $C \cap \text{Sing } C_d$, this implies that α_d does not depend on d . Finally, an extra trick (eg a monodromy argument on the points of $C \cap \text{Sing } C_d$) shows that the cycle z' has to be 0 for any curve C .

Let us emphasize the following point (see also Remark 1.2, 3)): the reason why we get in the case of curves a stronger result than the Franchetta conjecture, (since our conclusion is that we get the same multiple of the canonical bundle for all genera) is due to the fact that we work with quasi-projective varieties, without properness assumption on the morphism. As it appears in the above sketch of proof, this allows to hide in axiom (ii) above many degenerations (where we remove the singular points of the singular fibers), and this is why we can compare what happens for various genera.

Theorem 1.4 has a natural generalization concerning universally defined cycles on products of varieties $X_1 \times \dots \times X_l$ of respective fixed dimensions $d_1, \dots, d_l$. We mean by this the assignment of a cycle $\mathcal{Z}(\mathcal{X}_1, \dots, \mathcal{X}_l) \in \text{CH}(\mathcal{X}_1 \times \dots \times \mathcal{X}_l)$ for each data of l smooth morphisms $\mathcal{X}_1 \rightarrow B_1, \dots, \mathcal{X}_l \rightarrow B_l$ to smooth quasi-projective basis B_i , satisfying compatibility conditions as in Definition 1.1 (see Definition 2.13 for more detail). We will prove

Theorem 1.7. *Let $\mathcal{Z}$ be a universally defined cycle on products of r smooth varieties of dimension $d_1, \dots, d_r$. Then there exists a uniquely defined polynomial $P_{\mathcal{Z}}$ with rational coefficients in the variables $c_{1,1}, \dots, c_{d_1,1}, \dots, c_{1,r}, \dots, c_{d_r,r}$, of weighted degree $\leq d_i$ in each set of variables $c_{1,i}, \dots, c_{d_i,i}$ (the weight of $c_{j,i}$ being j), such that for any given smooth varieties $X_1, \dots, X_r$ over $\mathbb{C}$, of respective dimensions $d_1, \dots, d_r$, the following equality*

$$\mathcal{Z}(X_1, \dots, X_r) = P_{\mathcal{Z}}(p_1^* c_1(X_1), \dots, p_1^* c_{d_1}(X_1), \dots, p_r^* c_1(X_r), \dots, p_r^* c_{d_r}(X_r)) \quad (1)$$

holds in $\mathrm{CH}(X_1 \times \dots \times X_r)$, where p_i denotes the projection from $X_1 \times \dots \times X_r$ to X_i .

Theorem 1.7 is motivated by the following conjecture 1.8, for which we first need to introduce universally defined cycles on k -th powers of n -dimensional algebraic varieties. We refer to Section 3 for the precise definition, but let us just say that such a universally defined cycle $\mathcal{Z}$ is the assignment of a cycle $\mathcal{Z}(\mathcal{X}) \in \mathrm{CH}(\mathcal{X}^{k/B})$ for any smooth morphism $\mathcal{X} \rightarrow B$, where B is also smooth, satisfying the same axioms as in Definition 1.1. A typical example is given by the diagonal $\Delta_{I,\mathcal{X}}$ determined by any partition I of $\{1, \dots, k\}$ (see Section 3 for the precise definition). The present paper presents the first steps toward proving the following statement, which, when $k = 1$, is Theorem 1.4.

Conjecture 1.8. *For any universally defined cycle on k -th powers of varieties of dimension d , there exists a unique collection of polynomials P_I indexed by partitions I of $\{1, \dots, k\}$, with the following properties: denoting by $l(I)$ the length of the partition $I = \{I_1, \dots, I_{l(I)}\}$, P_I is a polynomial in the variables $c_{1,1}, \dots, c_{d,1}, \dots, c_{1,l(I)}, \dots, c_{d,l(I)}$, which is of weighted degree $\leq d$ in each set of variables $c_{1,s}, \dots, c_{d,s}$, and, for each smooth variety X of dimension d over $\mathbb{C}$, the equality*

$$\mathcal{Z}(X) = \sum_I \Delta_{I,X*} P_I(\mathrm{pr}_1^* c_1(X), \dots, \mathrm{pr}_1^* c_d(X), \dots, \mathrm{pr}_{l(I)}^* c_1(X), \dots, \mathrm{pr}_{l(I)}^* c_d(X)) \quad (2)$$

holds in $\mathrm{CH}(X^k)$.

In formula (2), we see $\Delta_{I,X}$ as a proper morphism from $X^{l(I)}$ to X^k .

In the third section of the paper, we will establish the uniqueness statement in Conjecture 1.8 (Cf. Theorem 3.4).

Conjecture 1.8 had been unwisely announced as a forthcoming theorem in the paper [12] where we first introduced the formalism of universally defined cycles, but its proof is still incomplete. It would have many consequences that are described in [14] and also in [12]. The first consequence would be an easy alternative proof of the Ellingsrud-Göttsche-Lehn theorem [3]. The proofs of [12, Section 5] concerning the Chow rings of Hilbert schemes of $K3$ surfaces depend on this statement and are therefore incomplete. In particular, as observed by Maulik, Conjecture 1.8 would have major applications to the Beauville-Voisin conjecture on the Chow ring of Hilbert schemes of $K3$ surfaces (see [1], [13]), obtained by proving the Lehn commutation relations (see [6]) in the Chow setting. In the paper [7] by Maulik and Negut, these relations have been given a direct proof.

Let us say that a universally defined cycle on powers of varieties is standard if it takes the form (2). Conjecture 1.8 says that any universally defined cycle on powers of varieties is standard. In Section 3.1, we establish the following result, which provides a strong motivation for Conjecture 1.8:

Theorem 1.9. *Let $\mathcal{Z}$ be a universally defined cycle on powers of d -dimensional smooth varieties, which is standard. If $\mathcal{Z}(X)$ is cohomologous to 0 for any smooth d -dimensional variety X defined over $\mathbb{C}$, $\mathcal{Z}(X) = 0$ in $\mathrm{CH}(X)$ for any such X .*

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2 Universally defined cycles on products

2.1 Proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4. It is a particular case of Theorem 1.7 which will be proved in the next section and whose proof needs extra ingredients. The proof

of Theorem 1.4 will use several preparatory results. The following Proposition 2.1 will allow to reduce Theorem 1.4 to the case of complete intersections, which is studied in Proposition 2.4. Let X be a variety and $E \rightarrow X$ be a rank k vector bundle on X generated by a finite dimensional space W of sections. Let $G = \mathbf{G}(k, W)$ be the Grassmannian of k -dimensional vector subspaces of W . Let $\mathcal{C} \subset G \times \mathbb{P}(E^*)$ be the set of pairs

$$([V], e), V \subset W, e \in \mathbb{P}(E_x^*),$$

such that

$$e(\sigma_x) = 0 \quad \forall \sigma \in V,$$

and let $\mathcal{C}_X$ be its image in $G \times X$ via the natural projection $\pi : \mathbb{P}(E^*) \rightarrow X$. Let us denote by $\mathcal{Q}$ the rank k quotient bundle on G , so that $\mathcal{Q}^*$ is the rank k subbundle on G with fiber V over the point $[V]$, and by p_X, p_G the projections from $G \times X$ to X and G respectively. The hypersurface $\mathcal{C}_X$ is thus the universal determinantal hypersurface associated to the universal evaluation morphism

$$p_G^* \mathcal{Q}^* \rightarrow p_X^* E$$

and $n := \pi|_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}_X$ is a natural desingularization of $\mathcal{C}_X$.

Proposition 2.1. *If $\dim W$ is large compared to $\dim X$, the pull-back and restriction composite map $\mathrm{CH}(X) \xrightarrow{p_X^*} \mathrm{CH}(G \times X) \rightarrow \mathrm{CH}((G \times X) \setminus \mathcal{C}_X)$ is injective.*

Proof. Let $h \in \mathrm{CH}(\mathbb{P}(E^*))$ be the first Chern class of the line bundle $\mathcal{O}_{\mathbb{P}(E^*)}(1)$. We observe that $\mathcal{C}$ is the zero-set of the tautological section of $\mathcal{Q} \boxtimes \mathcal{O}_{\mathbb{P}(E^*)}(1)$ on $G \times \mathbb{P}(E^*)$. Furthermore, as W is generated by sections, $\mathcal{C}$ is smooth of the right codimension k : more precisely, via the projection to $\mathbb{P}(E^*)$, $\mathcal{C}$ is fibered over $\mathbb{P}(E^*)$ in Grassmannians $G(k, W^{\perp e})$ for hyperplanes $W^{\perp e} \subset W$, hence it is smooth and furthermore the restriction map $\mathrm{CH}(G \times \mathbb{P}(E^*)) \rightarrow \mathrm{CH}(\mathcal{C})$ is surjective. We thus conclude that the class C of $\mathcal{C}$ in $\mathrm{CH}(G \times \mathbb{P}(E^*))$ is $c_k(\mathcal{Q} \boxtimes \mathcal{O}_{\mathbb{P}(E^*)}(1))$ and that, denoting j the inclusion of $\mathcal{C}$ in $G \times \mathbb{P}(E^*)$, $\mathrm{Im}(j_* : \mathrm{CH}^*(\mathcal{C}) \rightarrow \mathrm{CH}^{*+k}(G \times \mathbb{P}(E^*)))$ is equal to $\mathrm{Im}(C : \mathrm{CH}^*(G \times \mathbb{P}(E^*)) \rightarrow \mathrm{CH}^{*+k}(G \times \mathbb{P}(E^*)))$. Finally, as we are working with rational coefficients, the natural map $n_* : \mathrm{CH}_*(\mathcal{C}) \rightarrow \mathrm{CH}_*(\mathcal{C}_X)$ is surjective, and thus we conclude, using the localization exact sequence, that $\mathrm{Ker}(\mathrm{CH}(G \times X) \rightarrow \mathrm{CH}((G \times X) \setminus \mathcal{C}_X))$ is equal to

$$\mathrm{Im}(\pi_* \circ C : \mathrm{CH}^*(G \times \mathbb{P}(E^*)) \rightarrow \mathrm{CH}(G \times X)^{*+1}). \quad (3)$$

The space $\mathrm{CH}^*(G \times \mathbb{P}(E^*))$ is generated over the $\mathbb{Q}$ -algebra $\mathrm{CH}^*(G \times X)$ by the classes h^j , for $j = 0, \dots, k-1$. As π_* is a $\mathrm{CH}^*(G \times X)$ -linear morphism by the projection formula, we conclude that $\mathrm{Im}(\pi_* \circ C)$ is generated over $\mathrm{CH}^*(G \times X)$ by the classes $\pi_*(C \cdot h^j)$. Proposition 2.1 thus follows from the following

Claim 2.2. *If $\alpha \in \mathrm{CH}(X) \subset \mathrm{CH}(X \times G)$ has the property that α belongs to the ideal generated by the elements $\pi_*(C \cdot h^j)$, $j = 0, \dots, k-1$, then $\alpha = 0$.*

Let us now prove the claim. We use the same notation $c_i(\mathcal{Q})$ and h for the pull-back of these classes to $G \times \mathbb{P}(E^*)$. Then $C = \sum_{i=0}^{i=k} c_i(\mathcal{Q}) h^{k-i}$ and it follows that

$$\pi_*(C \cdot h^j) = \sum_{i=0}^{i=k} c_i(\mathcal{Q}) s_{j-i+1}(E^*) \quad (4)$$

for $j = 0, \dots, k-1$, where $s_i(E^*)$ denotes the i -th Segre class of E^* . In particular,

$$\pi_*(C \cdot h^j) = c_{j+1}(\mathcal{Q}) + P_j, \quad (5)$$

where P_j is a polynomial in the variables $c_i(\mathcal{Q})$ with $i \leq j$, with coefficients in $\mathrm{CH}(X)$, and $P_0 \in \mathrm{CH}(X)$.

As $\dim W \gg \dim X$ and we can obviously restrict ourselves to considering cycles of degree $\leq \dim X$, there are no nontrivial relations in these degrees between the classes $c_j(\mathcal{Q})$ for $j > 0$. In other words, we can do as if $\mathrm{CH}(G)$ were the polynomial ring with generators $Y_1 = c_1(\mathcal{Q}), \dots, Y_k = c_k(\mathcal{Q})$ and thus $\mathrm{CH}(G \times X) = \mathrm{CH}(X)[Y_1, \dots, Y_k]$. We know by (5) that $\pi_*(C \cdot h^j) = Y_{j+1} + P_j$, where P_j is a polynomial in the Y_i with $i \leq j$. We apply then the following elementary lemma to $R = \mathrm{CH}(X)$, $a = \alpha$:

Lemma 2.3. *Let R be a unitary commutative ring and let $G_j = Y_j + P_j(Y_1, \dots, Y_{j-1}) \in R[Y_1, \dots, Y_k]$, for $j = 1, \dots, k$. Then for any relation $\sum_j F_j G_j = a$, with $F_j \in R[Y_1, \dots, Y_k]$ and $a \in R$, one has $a = 0$.*

Proof. Indeed, from the structure of the polynomials G_j , we construct inductively $r_1, \dots, r_k \in R$ such that $G_j(r_1, \dots, r_k) = 0$ for all j , by the formulas $r_j = -P_j(r_1, \dots, r_{j-1})$, $r_0 = P_0$. Then $\sum_j F_j(r.) G_j(r.) = 0 = a$. $\square$

Claim 2.2 is now proved and the proof of Proposition 2.1 is now finished. $\square$

The proof of Theorem 1.4 will now be based on the following Proposition 2.4, which is a stronger version of Theorem 1.4 for complete intersections. For a given integer d , consider the Grassmannian $\mathbf{G}(d, N)$, where $N \geq 2d + 1$. From now on, the integer N will be fixed, large enough compared to d . More precisely, we choose N in such a way that any d -dimensional smooth variety can be embedded in $\mathbf{G}(d, N)$. For any given integer l , we consider the universal family of d -dimensional complete intersections of type $(l, \dots, l)$ in $\mathbf{G}(d, N)$. It is constructed as follows: Consider the Grassmannian $\mathbf{G}(k_d, V_{d,l})$ parameterizing k_d -dimensional vector subspaces of $V_{d,l} := H^0(\mathbf{G}(d, N), \mathcal{L}^{\otimes l})$, where $\mathcal{L}$ is the Plücker line bundle and $k_d := \dim \mathbf{G}(d, N) - d = d(N - d - 1)$. The number k_d is the codimension of a d -dimensional subvariety of $\mathbf{G}(d, N)$. A d -dimensional complete intersection in $\mathbf{G}(d, N)$ of type $(l, \dots, l)$ is thus defined by a vector subspace $V \subset V_{d,l}$ of dimension k_d . Consider the following variety $\mathcal{X}_{d,N,l} \subset \mathbf{G}(k_d, V_{d,l}) \times \mathbf{G}(d, N)$

$$\mathcal{X}_{d,N,l} = \{([V], x) \in \mathbf{G}(k_d, V_{d,l}) \times \mathbf{G}(d, N), v(x) = 0 \forall v \in V\}. \quad (6)$$

Note that $\mathcal{X}_{d,N,l}$ is smooth and irreducible of codimension k_d in $\mathbf{G}(k_d, V_{d,l}) \times \mathbf{G}(d, N)$, being fibered over $\mathbf{G}(d, N)$, via the second projection pr_2 , into Grassmannians $\mathbf{G}(k_d, V'_{d,l})$ for some hyperplanes $V'_{d,l} \subset V_{d,l}$. Let $\mathcal{X}_{d,N,l}^0 \subset \mathcal{X}_{d,N,l}$ be the dense Zariski open set where $\pi_{d,N,l} := \mathrm{pr}_1 : \mathcal{X}_{d,N,l} \rightarrow \mathbf{G}(k_d, V_{d,l})$ is smooth. The fibers of the restriction

$$\pi_{d,N,l}^0 : \mathcal{X}_{d,N,l}^0 \rightarrow \mathbf{G}(k_d, V_{d,l}) \quad (7)$$

are thus d -dimensional smooth *quasi-projective* complete intersections. We will use the notation $c_i(\pi_{d,N,l}^0) := c_i(\mathcal{X}_{d,N,l}^0 / \mathbf{G}(k_d, V_{d,l})) \in \mathrm{CH}(\mathcal{X}_{d,N,l}^0)$. We denote below $B := \mathbf{G}(k_d, V_{d,l})$.

Proposition 2.4. *Let $\mathcal{Z}$ be a universally defined cycle of degree $\leq d$ on varieties of dimension d . Then there exists a uniquely defined polynomial $P \in \mathbb{Q}[c_1, \dots, c_d]$, of weighted degree $\leq d$ in the variables c_j (of weighted degree j), such that for any l large enough,*

$$\mathcal{Z}(\pi_{d,N,l}^0) = P(c_1(\pi_{d,N,l}^0), \dots, c_d(\pi_{d,N,l}^0)) \quad (8)$$

in $\mathrm{CH}(\mathcal{X}_{d,N,l}^0)$,

Remark 2.5. The result is very easy after restriction to the general fiber of $\pi_{d,N,l}^0$ or equivalently $\pi_{d,N,l}$. However we actually want the equality to hold in $\mathrm{CH}^*(\mathcal{X}_{d,N,l}^0)$, which is much bigger than the Chow ring of the general fiber of $\pi_{d,N,l}^0$ because of the classes coming from the base. Indeed, we are working with cycles of codimension $* \leq d$, and the complement of the Zariski open set $\mathcal{X}_{d,N,l}^0 \subset \mathcal{X}_{d,N,l}$ is $d+1$, so in these degrees, $\mathrm{CH}^*(\mathcal{X}_{d,N,l}) \cong \mathrm{CH}^*(\mathcal{X}_{d,N,l}^0)$.

We first show how Proposition 2.4 implies Theorem 1.4.

Proof of Theorem 1.4. Let $\mathcal{Z}$ be a universally defined cycle on varieties of dimension d . As noticed in Remark 2.14, we can assume $\mathcal{Z}$ of codimension $\leq d$, as otherwise the statement is trivial. Let then P be the polynomial given by Proposition 2.4. It suffices to show that for any smooth variety X of dimension d , $\mathcal{Z}(X) = P(c_i(X))$ in $\text{CH}(X)$. We embed X in $\mathbf{G}(d, N)$. For large l , X is schematically defined by degree l equations in $\mathbf{G}(d, N)$. Let $V_{d,l,X} \subset V_{d,l}$ be the space of sections of $\mathcal{L}^{\otimes l}$ vanishing along X . Then by differentiation along X , we get a natural surjective morphism

$$V_{d,l,X} \otimes \mathcal{O}_X \rightarrow N_{X/\mathbf{G}(d,N)}^*(l), \quad (9)$$

which makes $V_{d,l,X}$ into a generating space of sections of the vector bundle $N_{X/\mathbf{G}(d,N)}^*(l)$. Let $\mathcal{C}_X \subset \mathbf{G}(k_d, V_{d,l,X}) \times X$ be the corresponding degeneracy locus as discussed before Proposition 2.1. The points $([V], x)$ of $\mathcal{C}_X$ are the couples where the differentiation morphism $V_{d,l,X} \rightarrow N_{X/\mathbf{G}(d,N)}^*(l)|_x$ of (9) is not an isomorphism at the point x , or equivalently, where the variety defined by the polynomials in V is not locally isomorphic to X . The complement $(\mathbf{G}(k_d, V_{d,l,X}) \times X) \setminus \mathcal{C}_X$ is thus open in a family of smooth complete intersections in $\mathbf{G}(d, N)$. More precisely, consider the inclusion $i : \mathbf{G}(k_d, V_{d,l,X}) \subset \mathbf{G}(k_d, V_{d,l})$. Then we have an open immersion

$$(\mathbf{G}(k_d, V_{d,l,X}) \times X) \setminus \mathcal{C}_X \subset i^* \mathcal{X}_{d,N,l}^0$$

over $\mathbf{G}(k_d, V_{d,l,X})$, where $i^* \mathcal{X}_{d,N,l}^0$ is the universal family $\mathcal{X}_{d,N,l}^0$ of (7), base-changed via i to $\mathbf{G}(k_d, V_{d,l,X})$.

Let p_X denote the projection from $(\mathbf{G}(k_d, V_{d,l,X}) \times X) \setminus \mathcal{C}_X$ to X . Applying axioms (i) and (ii) of Definition 2.13, we conclude that

$$p_X^* \mathcal{Z}(X) = \mathcal{Z}(\pi_{d,N,l}^0|_{(\mathbf{G}(k_d, V_{d,l,X}) \times X) \setminus \mathcal{C}_X})$$

in $\text{CH}((\mathbf{G}(k_d, V_{d,l,X}) \times X) \setminus \mathcal{C}_X)$. We now use Proposition 2.4 and observe that

$$c_i(\pi_{d,N,l}^0|_{(\mathbf{G}(k_d, V_{d,l,X}) \times X) \setminus \mathcal{C}_X}) = c_i(T_{(\mathbf{G}(k_d, V_{d,l,X}) \times X) \setminus \mathcal{C}_X / \mathbf{G}(k_d, V_{d,l,X})}) = p_X^* c_i(X). \quad (10)$$

We then conclude that

$$p_X^* \mathcal{Z}(X) = p_X^* P(c_i(X)) \text{ in } \text{CH}((\mathbf{G}(k_d, V_{d,l,X}) \times X) \setminus \mathcal{C}_X). \quad (11)$$

By Proposition 2.1, the morphism p_X^* is injective and we thus conclude that

$$\mathcal{Z}(X) = P(c_i(X)) \text{ in } \text{CH}(X).$$

□

Remark 2.6. What has been used in a crucial way in this proof is the fact that the relative Chern classes are universally defined, hence, for a morphism given by a projection $X \times B \rightarrow B$, they are pulled-back from X , and this remains true after restriction to an open set of $X \times B$ (this is formula (10)).

We now start the proof of Proposition 2.4. We will need the following intermediate result:

Proposition 2.7. *There exists a smooth projective (not irreducible) variety X_0 , all of whose components are of dimension d , and a morphism $f : X_0 \rightarrow \mathbf{G}(d, N)$, which satisfy the property that there are no nonzero cohomological polynomial relations $P(c_i(X_0), c'_j) = 0$ in $H^{2*}(X_0, \mathbb{Q})$ in degree $* \leq d$, where the c'_j 's are the pull-backs to X_0 of the classes $c_j(\mathbb{Q}) \in H^{2j}(\mathbf{G}(d, N), \mathbb{Q})$.*

Remark 2.8. It is clear that it suffices to prove the statement for polynomials of maximal degree $* = d$.

For the proof of this proposition, we first establish the following statement.

Lemma 2.9. *There exist abelian varieties A_i of dimension d , and morphisms*

$$f_i : A_i \rightarrow \mathbf{G}(d, N) \times \mathbf{G}(d, N)$$

such that, denoting

$$c_{j,1} := \mathrm{pr}_1^* c_j(\mathcal{Q}), c_{j,2} := \mathrm{pr}_2^* c_j(\mathcal{Q}) \in H^{2j}(\mathbf{G}(d, N) \times \mathbf{G}(d, N), \mathbb{Q}),$$

there is no nonzero polynomial P of weighted degree d in the variables $(x_{j,1}, x_{j',2})$, such that the relation

$$P(f_i^* c_{j,1}, f_i^* c_{j',2}) = 0 \text{ in } H^{2d}(A_i, \mathbb{Q}), \quad (12)$$

is satisfied for all i .

Proof. According to [9], [11], there are no nonzero linear relations between the Chern numbers of products

$$\mathbb{P}^I := \mathbb{P}^{r_1} \times \dots \times \mathbb{P}^{r_{l(I)}}$$

of projective spaces indexed by partitions $I = \{r_1 \leq \dots \leq r_s\}$ of d , where $\sum_s r_s = d$. Each $\mathbb{P}^I$ can be imbedded into $\mathbf{G}(d, N)$ by taking an adequate set of global sections of its tangent bundle $T_{\mathbb{P}^I}$. We denote by $g_I : \mathbb{P}^I \rightarrow \mathbf{G}(d, N)$ such an embedding. Then $g_I^* \mathcal{Q} = T_{\mathbb{P}^I}$ so there is no nonzero degree d polynomial relation between the classes $g_I^* c_j(\mathcal{Q})$ which holds in $H^{2d}(\mathbb{P}^I, \mathbb{Q})$ for all I . Fixing two partitions I, J of d , we know that the degree $2d$ cohomology $H^{2d}(\mathbb{P}^I \times \mathbb{P}^J, \mathbb{Q})$ is generated by classes of products of projective spaces $i_{K,I,J} : \mathbb{P}^K \subset \mathbb{P}^I \times \mathbb{P}^J$. It follows now from Poincaré duality that there are no nonzero degree d polynomial relations between the Chern classes $i_{K,I,J}^*((g_I, g_J)^* c_{j,1}(\mathcal{Q})), (g_I, g_J)^* c_{j',2}(\mathcal{Q}))$, holding for all $i_{K,I,J}$. Hence we almost proved Lemma 2.9, except that our varieties $\mathbb{P}^K$ mapped to $\mathbf{G}(d, N) \times \mathbf{G}(d, N)$ via $(g_I, g_J) \circ i_{K,I,J}$ are not abelian varieties, but products of projective spaces. As any product $\mathbb{P}^K$ of projective spaces is dominated by an abelian variety $r_K : A_K \rightarrow \mathbb{P}^K$ of the same dimension, we can replace the morphisms $(g_I, g_J) \circ i_{K,I,J} : \mathbb{P}^K \rightarrow \mathbf{G}(d, N) \times \mathbf{G}(d, N)$ by $(g_I, g_J) \circ i_{K,I,J} \circ r_K : A_K \rightarrow \mathbf{G}(d, N) \times \mathbf{G}(d, N)$, and the proof is finished. $\square$

Proof of Proposition 2.7. Consider the abelian varieties A_i and the morphisms

$$f_i = (f_{i,1}, f_{i,2}) : A_i \rightarrow \mathbf{G}(d, N) \times \mathbf{G}(d, N)$$

of Lemma 2.9. First of all, we can assume by an easy twist argument, that in Lemma 2.9, each $\mathcal{Q}_{1,i} := f_{i,1}^* \mathcal{Q}$ is a very ample vector bundle on A_i . We now define $B_i \subset \mathbb{P}(\mathcal{Q}_{1,i})$ to be a general complete intersection of $d-1$ hypersurfaces in $|\mathcal{O}_{\mathbb{P}(\mathcal{Q}_{1,i})}(m)|$, where m is taken large enough, independent of i . Using the fact that the tangent bundle of A_i is trivial, the Chern classes of the tangent bundle of $\mathbb{P}(\mathcal{Q}_{1,i})$ equal those of the relative tangent bundle $T_{\mathbb{P}(\mathcal{Q}_{1,i})/A_i}$. The relative tangent bundle of

$$\pi_i : \mathbb{P}(\mathcal{Q}_{1,i}) \rightarrow A_i$$

is described by the Euler exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{Q}_{1,i})} \rightarrow \pi_i^* \mathcal{Q}_{1,i}^* \otimes \mathcal{O}_{\mathbb{P}(\mathcal{Q}_{1,i})}(1) \rightarrow T_{\mathbb{P}(\mathcal{Q}_{1,i})/A_i} \rightarrow 0,$$

so that

$$c(T_{\mathbb{P}(\mathcal{Q}_{1,i})/A_i}) = c(\pi_i^* \mathcal{Q}_{1,i}^* \otimes \mathcal{O}_{\mathbb{P}(\mathcal{Q}_{1,i})}(1)) \text{ in } H^{2*}(\mathcal{Q}_{1,i}, \mathbb{Q}).$$

Combined with the normal bundle sequence of B_i in $\mathbb{P}(\mathcal{Q}_{1,i})$, we get the following formula for the total Chern class of B_i (where $h_i := c_1(\mathcal{O}_{\mathbb{P}(\mathcal{Q}_{1,i})}(1))|_{B_i}$):

$$c(B_i) = (1 - mh_i + \dots + (-1)^d m^d h_i^d)^{d-1} c(\pi_i^* \mathcal{Q}_{1,i}^* \otimes \mathcal{O}_{\mathbb{P}(\mathcal{Q}_{1,i})}(1)). \quad (13)$$

Denoting by $\pi'_i : B_i \rightarrow A_i$ the restriction of π_i to B_i , we have

$$\pi'_{i*} h_i^l = (-1)^l m^{d-1} s_l(\mathcal{Q}_{1,i}), \quad (14)$$

and we conclude from (13), (14) and the projection formula that for any polynomial $P(x_l, y_{l'})$ of weighted degree d in the variables x_l of degree l and $y_{l'}$ of degree l' , there is a polynomial $P'(x_l, y_{l'})$ of the same weighted degree such that, for all i ,

$$\pi'_{i*}(P(c_l(B_i), c_{l'}(\pi'^{*}_i \mathcal{Q}_{2,i}))) = P'(c_l(\mathcal{Q}_{1,i}), c_{l'}(\mathcal{Q}_{2,i})) \text{ in } H^{2*}(A_i, \mathbb{Q}).$$

Furthermore, if m is taken large enough, the linear map $P \rightarrow P'$ is a bijection of the space of weighted homogeneous polynomials of the given degree. The disjoint union X_0 of the B_i , equipped with the vector bundles $\pi'^{*}_i \mathcal{Q}_{2,i}$, thus satisfy the conclusion of Proposition 2.7. $\square$

Proof of Proposition 2.4. Let $\mathcal{Z}$ be a universally defined cycle of codimension $\leq d$ on varieties of dimension d . Recall that $\mathcal{X}_{d,N,l}^0$ is the universal smooth d -dimensional complete intersection of type $(l, \dots, l)$ in $\mathbf{G}(d, N)$. It is Zariski open in the variety $\mathcal{X}_{d,N,l} \subset \mathbf{G}(k_d, V_{d,l}) \times \mathbf{G}(d, N)$ introduced in (6). Via the second projection $\text{pr}_{\mathbf{G}(d,N)} : \mathcal{X}_{d,N,l} \rightarrow \mathbf{G}(d, N)$, $\mathcal{X}_{d,N,l}$ is a fibration into Grassmannians $\mathbf{G}(k_d, V_{d,l}^{\perp e})$, where $V_{d,l}^{\perp e} \subset V_{d,l}$ is the hyperplane of sections vanishing at $e \in \mathbf{G}(d, N)$, hence we conclude, with the notation

$$B := \mathbf{G}(k_d, V_{d,l}),$$

that $\text{CH}(B) \otimes \text{CH}(\mathbf{G}(d, N)) = \text{CH}(B \times \mathbf{G}(d, N))$ maps surjectively to $\text{CH}(\mathcal{X}_{d,N,l})$ (and in fact also injectively in the small codimensions we are considering). As the restriction map $\text{CH}(\mathcal{X}_{d,N,l}) \rightarrow \text{CH}(\mathcal{X}_{d,N,l}^0)$ is surjective, the same is true with $\mathcal{X}_{d,N,l}$ replaced by $\mathcal{X}_{d,N,l}^0$. In fact, as the critical locus of $\pi_{d,N,l}$ has codimension $d+1$ in $\mathcal{X}_{d,N,l}$, the last restriction map is injective as well on cycles of codimension $\leq d$, and similarly for the restriction maps $\text{CH}(\mathbf{G}(k_d, V_{d,l})) \rightarrow \text{CH}(\mathbf{G}(k_d, V_{d,l}^{\perp e}))$. It follows that $\mathcal{Z}(\pi_{d,N,l}^0)$ lifts uniquely to an element of $\text{CH}(B \times \mathbf{G}(d, N)) = \text{CH}(B) \otimes \text{CH}(\mathbf{G}(d, N))$. We use now the fact that $\text{CH}(\mathbf{G}(d, N))$ is generated as an algebra by the Chern classes c_i of the rank d quotient bundle $\mathcal{Q}$ to write $\mathcal{Z}(\pi_{d,N,l}^0)$ as a polynomial in the variables $\text{pr}_{\mathbf{G}(d,N)}^* c_i$, with coefficients in $\text{CH}(B)$. In fact, another choice of variables is given by the relative Chern classes $c_i(\mathcal{X}_{d,N,l}/B)$, $i \leq d$ which can be defined as the degree i pieces of the Chern polynomial $c(T_{\mathcal{X}_{d,N,l}})c(\pi_{d,N,l}^* T_B)^{-1}$. Indeed, the normal bundle exact sequence of $\mathcal{X}_{d,N,l}$ in $B \times \mathbf{G}(d, N)$ and Whitney formula provide for l large enough an invertible change of variables between the classes c_i and $c_i(\mathcal{X}_{d,N,l}/B)$, with coefficients in $\text{CH}(B)$, and of the form $c_i(\mathcal{X}_{d,N,l}/B) = \mu_i c_i + R_i(c_1, \dots, c_{i-1})$ where μ_i is a rational number which is nonzero for l large enough, and R_i is a polynomial with coefficients in $\text{CH}(B)$.

Remark 2.10. The classes $c_i(\mathcal{X}_{N,l}/B)$ as defined above restrict to the relative Chern classes $c_i(\pi_{N,l}^0) \in \text{CH}(\mathcal{X}_{N,l}^0/B)$ by the tangent bundle sequence. The class $c_{d+1}(\mathcal{X}_{N,l}/B)$ can be shown to be equal to the class of the critical locus $\Gamma_{\text{crit}} = \mathcal{X}_{N,l} \setminus \mathcal{X}_{N,l}^0$. The kernel of the restriction map $\text{CH}(\mathcal{X}_{N,l}) \rightarrow \text{CH}(\mathcal{X}_{N,l}^0)$ is the ideal generated by the class of Γ_{crit} .

We thus get a polynomial P_l of d variables x_i , $i = 1, \dots, d$ with coefficients in $\text{CH}(B)$, of weighted degree $\leq d$ in the variables x_i of weighted degree i , such that

$$\mathcal{Z}(\pi_{d,N,l}^0) = P_l(c_1(T_{\mathcal{X}_{d,N,l}^0/B}), \dots, c_d(T_{\mathcal{X}_{d,N,l}^0/B})) \text{ in } \text{CH}(\mathcal{X}_{d,N,l}^0). \quad (15)$$

The main contents of Proposition 2.4 is the fact that P_l has coefficients in $\mathbb{Q}$ (rather than $\text{CH}(B)$), and that it is independent of l . Recall that $B = \mathbf{G}(k_d, V_{d,l})$ so that in degree $* \leq d$ (which is small compared to k_d and N), $\text{CH}^*(B)$ is freely generated by

$$C_i := c_i(\mathcal{Q}_{k_d}),$$

where we can of course consider only the C_j 's with $j \leq d$. We can thus write

$$P_l = P_l(c_i, C_j), \quad (16)$$

where on the right hand side, $P_l(c_i, C_j)$ is now a polynomial with rational coefficients.

We now choose a smooth (nonnecessarily connected) projective algebraic scheme X_0 equidimensional of dimension d satisfying the conclusion of Proposition 2.7. Thus, $\text{CH}^*(X_0)$ contains in degree $* \leq d$ the free polynomial algebra $\mathbb{Q}[c_i, c'_j]_{1 \leq i, j \leq d}$, with $c_i = c_i(X_0)$ and $c'_i = f^* c_i(\mathcal{Q})$ for an embedding $f : X_0 \hookrightarrow G(d, N)$. (We will allow this embedding to vary later on.) Let l be large enough so that $f(X_0)$ is defined by degree l equations. Let $V_{d,l,X_0} \subset V_{d,l}$ be the set of sections of $\mathcal{O}_{G(d,N)}(l)$ vanishing on X_0 . We now make the same construction and computations as in the previous proofs. Denoting $B_{X_0} \subset B$ the Grassmannian $\mathbf{G}(k_d, V_{l,d,X_0})$, we have the inclusion $i : B_{X_0} \hookrightarrow B$, under which the tautological Chern classes C_j of the Grassmannian B restrict to the tautological Chern classes $C_j(B_{X_0})$ of the Grassmannian B_{X_0} , and the inclusion

$$(B_{X_0} \times X_0) \setminus \mathcal{C}_{X_0} \subset i^* \mathcal{X}_{d,N,l}^0 \quad (17)$$

over B_{X_0} . We then deduce from (15) and (16), using the axioms of Definition 2.13, that

$$P_l(c_i(X_0), C_j) = p_{X_0}^*(\mathcal{Z}(X_0)) \text{ in } \text{CH}((B_{X_0} \times X_0) \setminus \mathcal{C}_{X_0}), \quad (18)$$

where, as before, $p_{X_0} : (B_{X_0} \times X_0) \setminus \mathcal{C}_{X_0} \rightarrow X_0$ is the natural map. Using the analysis of $\text{CH}(\mathcal{C}_{X_0})$ already made in the proof of Proposition 2.1, the image of $\text{CH}(\mathcal{C}_{X_0})$ in $\text{CH}(B_{X_0} \times X_0)$ is the ideal of $\text{CH}(B_{X_0} \times X_0)$ generated over $\text{CH}(B_{X_0} \times X_0)$ by the classes C'_j for $j = 1, \dots, k_d$, where, using the notation $s_i := s_i(N_{X_0/\mathbf{G}(d,N)}(-l))$, we have as in (4)

$$C'_j = C_j + s_1 C_{j-1} + \dots + s_j \in \text{CH}(B_{X_0} \times X_0). \quad (19)$$

Using (19), we get that, modulo the ideal generated by the classes C'_j , hence equivalently, after restriction to $(B_{X_0} \times X_0) \setminus \mathcal{C}_{X_0}$, the classes C_j are given by universal polynomials with $\mathbb{Q}$ -coefficients in the s_i .

Recall now that, in degree $* \leq d$, $\text{CH}^*(X_0 \times B_{X_0})$ contains a weighted polynomial ring in the variables C_k coming from B_{X_0} , and $c_i(X_0), c'_j$ coming from X_0 , and of respective weighted degrees k, i, j . The relations between the classes $c_i(X_0), c'_i$ and s_i are given by the normal bundle sequence of X_0 in $\mathbf{G}(d, N)$, which gives $c(X_0)c(N_{X_0/\mathbf{G}(d,N)}) = c(T_{\mathbf{G}(d,N)})|_{X_0}$ and thus

$$s(N_{X_0/\mathbf{G}(d,N)}) = c(X_0)s(T_{\mathbf{G}(d,N)})|_{X_0}, \quad (20)$$

where s denotes the total Segre class (which is the inverse of the total Chern class). It follows that for fixed l , the s_i 's are given by universal polynomials $U_i(c_i(X_0), c'_j)$ (we also use $c_1(\mathcal{O}_{\mathbf{G}(d,N)}(1)|_{X_0}) = c'_1$ to deal with the twist of the normal bundle).

Combining these observations, we conclude that for some universal polynomials U'_j (depending only on d, k_d, l) in the variables $c_i(X_0), c'_j$,

$$p_{B_{X_0}}^* C_i = p_{X_0}^* U'_i(c_p(X_0), c'_q) \text{ in } \text{CH}((B_{X_0} \times X_0) \setminus \mathcal{C}_{X_0}) \cong \text{CH}(B_{X_0} \times X_0) / \langle C'_j \rangle, \quad (21)$$

which, combined with (18), provides

$$p_{X_0}^*(P_l(c_i(X_0), U'_j(c_p(X_0), c'_q))) = p_{X_0}^*(\mathcal{Z}(X_0)) \text{ in } \text{CH}((B_{X_0} \times X_0) \setminus \mathcal{C}_{X_0}). \quad (22)$$

By proposition 2.1, this relation holds in fact in $\text{CH}(X_0)$, that is,

$$P_l(c_i(X_0), U'_j(c_p(X_0), c'_q)) = \mathcal{Z}(X_0) \text{ in } \text{CH}(X_0). \quad (23)$$

We argue now using the fact that the right hand side of (23) depends only on X_0 while by changing f , the c'_q can be chosen almost freely in the $\mathbb{Q}$ -vector space of weighted degree q homogeneous polynomials in the c'_j 's (this is not completely true because we need to get

an embedding f of X_0 via a vector bundle with Chern classes c'_q and we want $f(X_0)$ to be defined by degree l equations, but we easily see that for l large enough, we can choose the c'_q in the considered subring of $\text{CH}(X_0)$ as free variables with respect to polynomials of bounded degree). Thus the polynomial $P_l(c_i(X_0), U'_j(c_p(X_0), c'_q))$, which is a polynomial with $\mathbb{Q}$ -coefficients in the variables $c_i(X_0)$ and c'_j , has to be constant in the variables c'_q . It remains to see that this actually implies that P_l is constant in the variables C_j . We use for this the following

Lemma 2.11. *The polynomials U'_j take the form $U'_j = \mu_j c'_j + Q_j(c'_1, \dots, c'_{j-1})$ for some universal nonzero constant μ_j (depending only on l, d and N , which are all fixed), where Q_j has coefficients in $\mathbb{Q}[c_i(X_0)]$.*

Proof. Indeed, from the relations $C'_j = \sum_{i=0}^j C_{j-i} s_i(N_{X_0/\mathbf{G}(d,N)}(-l))$ for $j \geq 1$, one deduces, by definition of the Segre classes s_i , that

$$U'_i(c_p(X_0), c'_q) = p_{B_{X_0}}^* C_j = c_j(N_{X_0/\mathbf{G}(d,N)}(-l)) \quad (24)$$

modulo the ideal generated by the C'_j 's for $j \geq 1$. Next, by the exact sequence

$$0 \rightarrow T_{X_0} \rightarrow T_{\mathbf{G}(d,N)|X_0} \rightarrow N_{X_0/\mathbf{G}(d,N)} \rightarrow 0,$$

we get

$$c(N_{X_0/\mathbf{G}(d,N)}) = c(T_{\mathbf{G}(d,N)|X_0}) s(T_{X_0}) \quad (25)$$

with $s_0(T_{X_0}) = 1$ and we observe that the Chern classes $c_j(T_{\mathbf{G}(d,N)})$ satisfy the property

$$c_j(T_{\mathbf{G}(d,N)}) = \nu_j c_j(\mathcal{Q}) + Q_j(c_1(\mathcal{Q}), \dots, c_{j-1}(\mathcal{Q})) \quad (26)$$

for some universal nonzero constant ν_j (depending only on d, N) and polynomials Q_j . Restricting (26) to X_0 , we get

$$c_j(T_{\mathbf{G}(d,N)|X_0}) = \nu_j c'_j + Q_j(c'_1, \dots, c'_{j-1}) \text{ in } \text{CH}(X_0). \quad (27)$$

Finally, we have

$$c_j(N_{X_0/\mathbf{G}(d,N)}(-l)) = c_j(N_{X_0/\mathbf{G}(d,N)}) + \sum_{k \geq 1} (-l)^k c_1 \mathcal{O}(1)^k c_{j-k}(N_{X_0/\mathbf{G}(d,N)}) \quad (28)$$

with $c_1 \mathcal{O}(1) = c'_1$. Combining (25), (26), (27) and (28) gives the desired result. $\square$

We now have the following easy lemma.

Lemma 2.12. *Let $F \in \mathbb{Q}[x_1, \dots, x_d, y_1, \dots, y_d]$ be a polynomial. Let*

$$V_j \in \mathbb{Q}[x_1, \dots, x_d, y_1, \dots, y_d]$$

take the form $V_j = \mu_j y_j + Q_j(x_1, \dots, x_d, y_1, \dots, y_{j-1})$ for some nonzero $\mu_j \in \mathbb{Q}$ and some polynomials Q_j with $\mathbb{Q}$ -coefficients. Then if F satisfies $F(x_1, \dots, x_d, V_1, \dots, V_d) \in \mathbb{Q}[x_1, \dots, x_d] \subset \mathbb{Q}[x_1, \dots, x_d, y_1, \dots, y_d]$, we have $F \in \mathbb{Q}[x_1, \dots, x_d]$.

Proof. In fact, a transformation $x_j \mapsto x_j, y_i \mapsto U_i$ of the form above can be inverted and even induces an automorphism of $\mathbb{Q}[x_1, \dots, x_d, y_1, \dots, y_d]$ over $\mathbb{Q}[x_1, \dots, x_d]$. $\square$

We now conclude the proof of Proposition 2.4. By Lemma 2.11, we can apply Lemma 2.12 to $F = P_l$, $U'_j = V_j$ and we conclude that the polynomial P_l has constant coefficients (so the variables C_i from the base do not appear). It remains to prove that the polynomial P_l is independent of l for large l , thus concluding the proof of Proposition 2.4. Choose l_0 so that P_l as above is well-defined for $l \geq l_0$. Choose $X_0 \subset \mathbf{G}(d, N)$ satisfying the conclusion that there

are no polynomial relations of weighted degree $\leq d$ between the Chern classes $c_i(X_0)$. Now, for any $l \geq l_0$, X_0 is defined by degree l equations and we get a polynomial with constant coefficients P_l satisfying by the same proof as above formula (18), where we know that in the left hand side, the polynomial P_l has constant coefficients, hence equals $p_{X_0}^*(P_l(c_i(X_0)))$ in $\text{CH}((B_{X_0} \times X_0) \setminus \mathcal{C}_{X_0})$. By Proposition 2.1, the map $p_{X_0}^* : \text{CH}(X_0) \rightarrow \text{CH}((B_{X_0} \times X_0) \setminus \mathcal{C}_{X_0})$ is injective, hence we conclude that $P_l(c_i(X_0)) = \mathcal{Z}(X_0) = P_{l_0}(c_i(X_0))$ for $l \geq l_0$, which implies that $P_l = P_{l_0}$. $\square$

2.2 The case of products

This section is devoted to the generalization of Theorem 1.4 to products (we refer to the introduction for the motivation of this generalization). We first spell-out the definition of universally defined cycles for products

Definition 2.13. *A universally defined cycle $\mathcal{Z}$ on products of r varieties of dimensions $d_1, \dots, d_r$ consists in the data, for any given r morphisms*

$$\pi_1 : \mathcal{X}_1 \rightarrow B_1, \dots, \pi_r : \mathcal{X}_r \rightarrow B_r$$

where all varieties are smooth quasiprojective over $\mathbb{C}$ and the morphisms π_i are smooth of relative dimension d_i , of a cycle $\mathcal{Z}(\pi_1, \dots, \pi_r) \in \text{CH}(\mathcal{X}_1 \times \dots \times \mathcal{X}_r)$ satisfying the following axioms:

(i) Base change. *For any base change maps $\gamma_i : B'_i \rightarrow B_i$ with B'_i smooth, with induced morphisms $\gamma'_i : \mathcal{X}'_i \rightarrow \mathcal{X}_i$, $\pi'_i : \mathcal{X}'_i \rightarrow B'_i$, where $\mathcal{X}'_i := \mathcal{X}_i \times_{B_i} B'_i$, one has*

$$\mathcal{Z}(\pi'_1, \dots, \pi'_r) = (\gamma'_1, \dots, \gamma'_r)^* \mathcal{Z}(\pi_1, \dots, \pi_r) \text{ in } \text{CH}(\mathcal{X}'_1 \times \dots \times \mathcal{X}'_r).$$

(ii) Open inclusion : *For any Zariski open sets $\mathcal{U}_i \subset \mathcal{X}_i$ and restricted morphisms $\pi_{i,\mathcal{U}_i} : \mathcal{U}_i \rightarrow B_i$, one has*

$$\mathcal{Z}(\pi_{1,\mathcal{U}_1}, \dots, \pi_{r,\mathcal{U}_r}) = \mathcal{Z}(\pi_1, \dots, \pi_r)|_{\mathcal{U}_1 \times \dots \times \mathcal{U}_r} \text{ in } \text{CH}(\mathcal{U}_1 \times \dots \times \mathcal{U}_r).$$

When the base B is a point, a smooth morphism $\pi : \mathcal{X} \rightarrow B$ is just a smooth variety X over $\mathbb{C}$ and we will use the notation $\mathcal{Z}(X_1, \dots, X_r)$ for $\mathcal{Z}(\pi_1, \dots, \pi_r)$, where π_i are the constant morphisms. When $r = 1$, and $d_1 = d$, we recover the notion of universally defined cycle on varieties of dimension d introduced in Definition 1.1.

We will prove in this section Theorem 1.7 and first make a few comments about the statement.

Remark 2.14. The codimension of the cycle is not mentioned explicitly in the statement of the theorem, but it is hidden in the condition on the weighted degrees of the polynomial P . For cycles of codimension $> \sum_i d_i$, our conclusion just says that they are 0 on products $X_1 \times \dots \times X_r$ of varieties of dimension d_i defined over a field, which is an empty statement.

Remark 2.15. It is possible that the conclusion of Theorem 1.7 actually holds true for the full universally defined cycle $\mathcal{Z}$ on products of families rather than its value on closed points, but we have not been able to prove this. Proposition 2.4 is an indication that a stronger result might hold.

Remark 2.16. The uniqueness statement is easy, as for smooth projective (nonnecessarily connected) schemes X of dimension d , there are no *universal* polynomial relations of weighted degree $\leq d$ between the Chern classes $c_i(X)$ ([9], [11]).

We now turn to the proof of Theorem 1.7. The proof will use the following variant of Theorem 1.4 proved in the previous section, involving Definition 2.17 below. Let Y be a smooth quasiprojective variety over $\mathbb{C}$.

Definition 2.17. *A Y -universally defined cycle on varieties of dimension d is the data, for each smooth morphism $\pi : \mathcal{X} \rightarrow B$, of a cycle $\mathcal{Z}(\pi) \in \text{CH}(\mathcal{X} \times Y)$, satisfying the two axioms (i) and (ii) of Definition 2.13 for base changes $B' \rightarrow B$ and open sets $\mathcal{U} \subset \mathcal{X}$.*

Theorem 2.18. *For any Y -universally defined cycle $\mathcal{Z}$ on varieties of dimension d , there exists a uniquely defined polynomial $P(c_1, \dots, c_d)$ of weighted degree $\leq d$ in the variables c_i (of degree i), with coefficients in $\text{CH}(Y)$, such that for any smooth variety X of dimension d over $\mathbb{C}$,*

$$\mathcal{Z}(X) = P(c_1(X), \dots, c_d(X)) \text{ in } \text{CH}(X \times Y). \quad (29)$$

Remark 2.19. From the structure of the statement, we note that we can assume that the codimension of the cycle $\mathcal{Z}$ is $\leq d + \dim Y$, which we will assume below, since in codimension $t > d + \dim Y$, $\text{CH}^t(X \times Y) = 0$.

Let us first show how the polynomials are determined, as this will be used below. Let M be the dimension of the space of homogeneous polynomials of weighted degree d in the variables $c_1, \dots, c_d$, with basis P_J given by degree d monomials. According to general complex cobordism theory, see [9], [11], we can choose M smooth projective varieties X_i of dimension d , such that the Chern numbers $\int_{X_i} P_J(c_1(X_i), \dots, c_d(X_i))$ give a rank M matrix M_{iJ} . If $\mathcal{Z}$ is a Y -universally defined cycle of the form $\mathcal{Z}(X) = \sum_J \alpha_J c_J$, where $\alpha_J \in \text{CH}(Y)$ and each c_J is a monomial of weighted degree $\leq d$ in the Chern classes, we get for any i , by pushing forward via the second projection $\text{pr}_Y : X_i \times Y \rightarrow Y$:

$$\sum_{|J|=d} M_{iJ} \alpha_J = \text{pr}_{Y*}(\mathcal{Z}(X_i)). \quad (30)$$

By invertibility of the matrix M_{iJ} , we conclude that the coefficients $\alpha_J \in \text{CH}(Y)$ for $|J| = d$ are universal linear combinations of the cycles $\text{pr}_{Y*}(\mathcal{Z}(X_i))$. Similarly, we get the coefficients α_J for other weighted degrees $\deg c_J = k < d$ by considering, for any monomial P_L of degree $d - k$, the Y -universally defined cycle $c_L \cdot \mathcal{Z}$ which to $\pi : \mathcal{X} \rightarrow B$ associates $P_L(c_1(\mathcal{X}/B), \dots, c_d(\mathcal{X}/B)) \cdot \mathcal{Z} \in \text{CH}(\mathcal{X} \times Y)$. Then we get as above that if $\mathcal{Z} = \sum_J \alpha_J c_J$, for any smooth projective X and any L as above

$$\sum_{|J|=k} M_{XJL} \alpha_J = \text{pr}_{Y*}(c_L \cdot \mathcal{Z}(X)), \quad (31)$$

where $M_{XJL} = \int_X c_L(X) c_J(X)$. In order to express α_J using these relations, it thus suffices to exhibit pairs (X_i, L_i) such that the matrix $M_{iJ} := M_{X_i J L_i}$ is nondegenerate, which follows again from general complex cobordism theory.

Proof of the implication Theorem 2.18 $\Rightarrow$ Theorem 1.7. We do this by induction on r . For $r = 1$, Theorem 1.7 is already proved. Assuming Theorem 2.18, and also Theorem 1.7 for $r - 1$, let $\mathcal{Z}$ be a universally defined cycle for products of r smooth varieties of dimensions $d_1, \dots, d_r$. Given $\pi_1 : \mathcal{X}_1 \rightarrow B_1, \dots, \pi_{r-1} : \mathcal{X}_{r-1} \rightarrow B_{r-1}$, let $Y := \mathcal{X}_1 \times \dots \times \mathcal{X}_{r-1}$. The cycle $\mathcal{Z}$ associates to any smooth morphism $\pi_r : \mathcal{X}_r \rightarrow B_r$ of relative dimension d_r , with $\mathcal{X}_r$ and B_r smooth, a cycle

$$\mathcal{Z}(\pi_1, \dots, \pi_r) \in \text{CH}(\mathcal{X}_1 \times \dots \times \mathcal{X}_r) = \text{CH}(Y \times \mathcal{X}_r).$$

This cycle is clearly Y -universal in $\mathcal{X}_r$. Theorem 2.18 says that there exist well defined cycles $\alpha_I \in \text{CH}(Y)$ associated to monomials I of weighted degree $\leq d_r$ such that, for any X_r over $\mathbb{C}$

$$\mathcal{Z}(\mathcal{X}_1, \dots, \mathcal{X}_{r-1}, X_r) = \sum_{I, |I| \leq d_r} \alpha_I c_I(X_r) \quad (32)$$

in $\text{CH}(Y \times X_r) = \text{CH}(\mathcal{X}_1 \times \dots \times \mathcal{X}_{r-1} \times X_r)$. We now have

Lemma 2.20. *The cycles α_I above are universally defined cycles on the products $\mathcal{X}_1 \times \dots \times \mathcal{X}_{r-1}$.*

Proof. We only have to check axioms (i) and (ii) of Definition 2.13. This follows immediately from formulas (30), (31). Indeed, they express the cycles α_I as linear combinations of pushforwards $pr_{\mathcal{X}_1 \times \dots \times \mathcal{X}_{r-1}}^i(c_L \cdot \mathcal{Z})$, where we fix a finite number of X_r^i and $pr_{\mathcal{X}_1 \times \dots \times \mathcal{X}_{r-1}}^i$ is the projection from $\mathcal{X}_1 \times \dots \times \mathcal{X}_{r-1} \times X_r^i$ to $\mathcal{X}_1 \times \dots \times \mathcal{X}_{r-1}$. The fact that $\mathcal{Z}$ satisfies the base change and open inclusion axioms with respect to π_i for $i \leq r-1$ immediately implies that each of these pushforwards satisfies these axioms because pushforward for proper smooth morphisms commutes with base change of the base (see [4, Proposition 1.7]). $\square$

Lemma 2.20 and Theorem 1.7 for $r-1$ thus imply that the cycles $\alpha_I(X_1, \dots, X_r)$ are given for products of $r-1$ smooth varieties over $\mathbb{C}$ by polynomials Q_I with $\mathbb{Q}$ -coefficients in the Chern classes of the X_j , $j \leq r-1$. Combined with (32), this concludes the proof. $\square$

The proof of Theorem 2.18 will now follow the same line of reasoning as the proof of Theorem 1.4, which was the case where Y is a point.

Proposition 2.1 extends in a straightforward way to this case. Let E be a rank k vector bundle over X , and W be a vector space of sections of E generating E at any point. We have the universal determinantal hypersurface $\mathcal{C}_X \subset \mathbf{G}(k, W) \times X$.

Lemma 2.21. *If the dimension of W is large enough (compared to the dimension of $X \times Y$), the pull-back and restriction composite map*

$$p_{X \times Y}^* \text{CH}(X \times Y) \rightarrow \text{CH}(((\mathbf{G}(k, W) \times X) \setminus \mathcal{C}_X) \times Y)$$

is injective.

Proof. This follows indeed from Proposition 2.1 applied to the vector bundle $pr_X^* E$ on $X \times Y$ and its generating space W of sections. $\square$

We will next use a slight elaboration of Proposition 2.4 which is needed to overcome the fact that we had assumed previously that the codimension of the cycle is $\leq d$. The general strategy of the proof is however very similar.

Proof of Theorem 2.18. Let $\mathcal{Z}$ be a Y -universally defined cycle on varieties of dimension d . As before, we introduce the universal d -dimensional smooth complete intersection

$$\pi_{d,N,l}^0 : \mathcal{X}_{d,N,l}^0 \rightarrow B = \mathbf{G}(k_d, V_{d,N,l})$$

of type $l, \dots, l$ in $G(d, N)$. As $\mathcal{X}_{d,N,l}^0$ is Zariski open in $\mathcal{X}_{d,N,l}$ which is fibered into Grassmannians $\mathbf{G}(k_d, V_{d,N,l}^{\perp e})$ over $\mathbf{G}(d, N)$, one has surjections:

$$\text{CH}(B \times \mathbf{G}(d, N)) \otimes \text{CH}(Y) \twoheadrightarrow \text{CH}(\mathcal{X}_{d,N,l} \times Y) \twoheadrightarrow \text{CH}(\mathcal{X}_{d,N,l}^0 \times Y).$$

This provides us with a polynomial $P_l(c_j, C_i)$ with coefficients in $\text{CH}(Y)$, in the variables C_i (coming from B) and c_j (where as before the classes c_j will not correspond to the standard classical classes coming from $\mathbf{G}(d, N)$, but to the Chern classes of the relative tangent bundle of $\pi_{d,N,l}$) with the property that

$$\mathcal{Z}(\mathcal{X}_{d,N,l}^0) = P_l(c_j(T_{\mathcal{X}_{d,N,l}^0/B}), (\pi_{d,N,l}^0)^*(C_i)) \text{ in } \text{CH}(\mathcal{X}_{d,N,l}^0 \times Y). \quad (33)$$

Recall the universal polynomials $U'_i(c_p, c'_q)$ introduced in the course of the proof of Proposition 2.4. They depend only on d, N, l and have the property that for any d -dimensional smooth quasi-projective algebraic subscheme $X \subset \mathbf{G}(d, N)$ defined over a field,

$$c_i(N_{X/\mathbf{G}(d,N)}^*(l)) = U'_i(c_p(X), c'_q) \text{ in } \text{CH}(X), \quad (34)$$

where $c'_q := c_q(\mathcal{Q}|_X)$. We have the following

Lemma 2.22. *For l large enough, the polynomial P_l has the following property: The polynomial P'_l in the variables c_i, c'_j obtained by the formula*

$$P'_l = P_l(c_i, U'_j(c_p, c'_q))$$

belongs to $\text{CH}(Y)[c_i]$ modulo the ideal generated by the monomials of weighted degree $\geq d+1$ in the variables c_i, c'_j 's (of respective degrees i, j). In other words, we have

$$P'_l = Q_l(c_1, \dots, c_d) \bmod \langle c_i, c'_j \rangle_{\geq d+1} \quad (35)$$

for some polynomial Q_l of weighted degree $\leq d$, with coefficients in $\text{CH}(Y)$.

Proof. Let $X_0 \subset \mathbf{G}(d, N)$ be as in Proposition 2.7. For l large enough, X_0 is defined schematically by the space $V_{d,N,l,X_0} \subset V_{d,N,l}$ of degree l equations vanishing on X_0 , and we have a surjective differentiation map $V_{d,N,l,X_0} \otimes \mathcal{O}_{X_0} \rightarrow N_{X_0/\mathbf{G}(d,N)}^*(l)$ with universal degeneracy locus $\mathcal{C}_{X_0} \subset \mathbf{G}(k_d, V_{d,l,X_0}) \times X$. Letting B_{X_0} denote $\mathbf{G}(k_d, V_{d,l,X_0})$, the restricted projection

$$p_{B_{X_0}} : (B_{X_0} \times X_0) \setminus \mathcal{C}_{X_0} \rightarrow B_{X_0}$$

is thus a family of d -dimensional smooth quasiprojective complete intersections of type $l, \dots, l$ in $\mathbf{G}(d, N)$. We thus conclude using the axioms (i) and (ii) in Definition 2.17 that

$$\mathcal{Z}(\pi_{d,N,l}^0)_{|((B_{X_0} \times X_0) \setminus \mathcal{C}_{X_0}) \times Y} = p_{X_0 \times Y}^* \mathcal{Z}(X_0) \quad (36)$$

in $\text{CH}(((B_{X_0} \times X_0) \setminus \mathcal{C}_{X_0}) \times Y)$. Applying (33), the left hand side of (36) equals $P_l(p_{X_0}^* c_i(X_0), p_{B_{X_0}}^* C_j)$, where p_{X_0} is the projection from $((B_{X_0} \times X_0) \setminus \mathcal{C}_{X_0}) \times Y$ to X_0 and $p_{B_{X_0}}$ is now the projection from $((B_{X_0} \times X_0) \setminus \mathcal{C}_{X_0}) \times Y$ to B_{X_0} . Equation (36) holds in $\text{CH}(((B_{X_0} \times X_0) \setminus \mathcal{C}_{X_0}) \times Y)$, where we know by (21) that the relations

$$p_{B_{X_0}}^* C_i = p_{X_0}^* c_i(N_{X_0/\mathbf{G}(d,N)}^*(l)) = p_{X_0}^* U'_i(c_p(X_0), c'_q)$$

hold. (36) thus gives us

$$p_{X_0 \times Y}^* (P_l(c_i(X_0), U'_j(c_p(X_0), c'_q))) = p_{X_0 \times Y}^* \mathcal{Z}(X_0) \quad (37)$$

in $\text{CH}(((B_{X_0} \times X_0) \setminus \mathcal{C}_{X_0}) \times Y)$ and according to Lemma 2.21, this implies

$$P_l(c_i(X_0), U'_j(c_p(X_0), c'_q)) =: P'_l(c_i(X_0), c'_j) = \mathcal{Z}(X_0) \quad (38)$$

in $\text{CH}(X_0 \times Y)$. In (38), the right hand side does not depend on the classes $c'_i \in \text{CH}(X_0)$, while the left hand side is an element of the image of $\text{CH}(Y)[c_i(X_0), c'_j]$ in $\text{CH}(X_0 \times Y)$. The smooth quasiprojective scheme X_0 has been chosen in such a way that the relations in this subring are exactly given by all polynomials of weighted degree $\geq d+1$ in the variables c_i, c'_j 's. Hence we conclude that the polynomial $P'_l(c_i(X_0), c'_j) \in \text{CH}(Y)[c_i(X_0), c'_j]$ is constant in the variables c'_j 's modulo the ideal generated by the monomials of degree $\geq d+1$ in the variables c_i, c'_j 's. $\square$

Remark 2.23. The proof above is slightly different from the proof of Proposition 2.4, but it also proves a slightly weaker statement. With the same arguments as in the previous proof, we could presumably get a more precise statement concerning the polynomials P_l .

We now conclude the proof of Theorem 2.18. Lemma 2.22 provides a polynomial

$$Q_l \in \text{CH}(Y)[c_1, \dots, c_d]$$

of weighted degree $\leq d$ in the c_i 's, and we are going to prove that $Q_l = Q$ is independent of l and that for any smooth quasiprojective variety X of pure dimension d ,

$$\mathcal{Z}(X) = Q(c_i(X)) \text{ in } \text{CH}(X \times Y). \quad (39)$$

We embed X in $\mathbf{G}(d, N)$ and choose l large enough so that the conclusion of Lemma 2.22 holds. Let as before $i : B_X = \mathbf{G}(k_d, V_{d,N,l,X}) \rightarrow B = \mathbf{G}(k_d, V_{d,N,l})$ be the natural inclusion. Then as we already observed, we have the open inclusion over $i(B_X)$

$$(B_X \times X) \setminus \mathcal{C}_X \subset i^* \mathcal{X}_{d,N,l}^0$$

of families of smooth complete intersections of type $(l, \dots, l)$ in $\mathbf{G}(d, N)$. We thus get using the axioms (i) and (ii) of Definition 2.17 the equality

$$p_{X \times Y}^* (\mathcal{Z}(X)) = \mathcal{Z}(\pi_{d,N,l}^0)_{|((B_X \times X) \setminus \mathcal{C}_X) \times Y} \text{ in } \text{CH}(((B_X \times X) \setminus \mathcal{C}_X) \times Y). \quad (40)$$

Using (33) and Lemma 2.22, the right hand side in (40) is given by

$$\mathcal{Z}(\pi_{d,N,l}^0)_{|((B_X \times X) \setminus \mathcal{C}_X) \times Y} = p_{X \times Y}^* Q_l(c_1(X), \dots, c_d(X)). \quad (41)$$

Indeed, in $\text{CH}(((B_X \times X) \setminus \mathcal{C}_X))$ we have

$$p_{B_X}^* C_i = p_X^* (U'_i(c_l(X_0), c'_k)),$$

so we get, by restricting (33) to

$$((B_X \times X) \setminus \mathcal{C}_X) \times Y \subset \mathcal{X}_{d,N,l}^0 \times Y,$$

$$\mathcal{Z}(\pi_{d,N,l}^0)_{|((B_X \times X) \setminus \mathcal{C}_X) \times Y} = P'_l(p_X^* c_i(T_X), p_X^* c'_j) \text{ in } \text{CH}(((B_X \times X) \setminus \mathcal{C}_X) \times Y). \quad (42)$$

Using Lemma 2.22 and the fact that monomials of degree $> d$ in c_i, c'_j vanish on $X \times Y$, we get the desired equality (41). We thus proved that

$$p_{X \times Y}^* (\mathcal{Z}(X)) = p_{X \times Y}^* (Q_l(c_1(X), \dots, c_d(X))) \text{ in } \text{CH}(((B_X \times X) \setminus \mathcal{C}_X) \times Y).$$

By Lemma 2.21, this equality holds already in $\text{CH}(X \times Y)$, proving (39) with a polynomial Q_l which could depend on l . Choosing for X a d -dimensional scheme X_0 such that there are no polynomial relation in $H^{2d}(X_0, \mathbb{Q})$ between the $c_i(X_0)$, the equality $\mathcal{Z}(X_0) = Q_l(c_1(X_0), \dots, c_d(X_0))$ in $\text{CH}(X_0 \times Y)$ for large l shows that Q_l is independent of l , concluding the proof of (39). $\square$

We conclude this section by noting the following variant of Theorem 1.7. Here we consider the data consisting of a smooth variety X of dimension d and vector bundles $E_1, \dots, E_s$ of respective ranks $r_1, \dots, r_s$ on X . A universally defined cycle for such data associates to each smooth morphism $\phi : \mathcal{X} \rightarrow B$ of relative dimension d between smooth projective varieties, and vector bundles $\mathcal{E}_1, \dots, \mathcal{E}_s$ on $\mathcal{X}$ of respective ranks $r_1, \dots, r_s$, a cycle $\mathcal{Z}(\phi, \mathcal{E}_\bullet) \in \text{CH}(\mathcal{X})$ satisfying the axioms (i) and (ii) of Definition 2.13, that is, compatibility with base change and restrictions to Zariski open subsets.

Theorem 2.24. *For any universally defined cycle for varieties of dimension d and s vector bundles of ranks $r_1, \dots, r_s$, there is a uniquely defined polynomial P of weighted degree $\leq d$ in the variables $c_i, c_{l_j, j}$, $1 \leq i \leq d$, $1 \leq j \leq s$, $1 \leq l_j \leq \text{Min}(d, r_j)$, such that for any $X, E_1, \dots, E_s$ as above,*

$$\mathcal{Z}(X, E_\bullet) = P(c_i(X), c_{l_j}(E_j)) \text{ in } \text{CH}(X).$$

Remark 2.25. As in the statement of Theorem 1.7, we can also improve the result by introducing products of k varieties each equipped with vector bundles, and get a similar statement.

The proof of Theorem 2.24 is exactly the same as before, except that we have to work with varieties embedded in products of Grassmannians $\mathbf{G}(d, N) \times \mathbf{G}(r_1, N) \times \dots \times \mathbf{G}(r_s, N)$. The first embedding will provide the Chern classes of X and the i -th other embedding will provide the Chern classes of E_i .

3 Universally defined cycles on powers

We turn in this section to the study of universally defined cycles on powers of d -dimensional smooth varieties, whose precise definition is as follows:

Definition 3.1. *A universally defined cycle $\mathcal{Z}$ on k -th powers of varieties of dimension d is the data, for each smooth morphism $\mathcal{X} \rightarrow B$ of relative dimension d , where B and $\mathcal{X}$ are smooth, of a cycle $\mathcal{Z}(\mathcal{X}) \in \text{CH}(\mathcal{X}^{k/B})$ satisfying the following axioms:*

- (i) *If $f : B' \rightarrow B$ is a base change map, with fibered product $\mathcal{X}' = \mathcal{X} \times_B B' \rightarrow B'$, $f' : \mathcal{X}' \rightarrow \mathcal{X}$, then $\mathcal{Z}(\mathcal{X}') = (f'^k)^* \mathcal{Z}(\mathcal{X})$ in $\text{CH}(\mathcal{X}'^{k/B'})$, where $f'^k : (\mathcal{X}')^{k/B'} \rightarrow \mathcal{X}^{k/B}$ is the natural morphism.*
- (ii) *If $\mathcal{X} \rightarrow B$ is as above, and $\mathcal{U} \subset \mathcal{X}$ is a Zariski open set, then $\mathcal{Z}(\mathcal{U}) = \mathcal{Z}(\mathcal{X})|_{\mathcal{U}^{k/B}}$ in $\text{CH}(\mathcal{U}^{k/B})$.*

As before, the cycle $\mathcal{Z}(\mathcal{X})$ depends on the morphism $\mathcal{X} \rightarrow B$, even if the notation does not indicate it. The principal source of examples of universally defined cycles on powers X^l is the “natural” cycles on the Hilbert scheme $X^{[k]}$, assuming it is smooth, that is, in the case of dimension $d = 2$ or in any dimension, with $k \leq 3$. Examples of natural cycles on the Hilbert scheme are as before polynomials in the Chern classes of the relative tangent bundle $T_{\mathcal{S}^{[k]}/B}$, but we also have the multiplicity strata and the diagonals or incidence subvarieties in products of Hilbert schemes. For each partition I of $\{1, \dots, k\}$ (or partition of k given by the multiplicities $i_s = |I_s|$), there is an incidence correspondence

$$\Gamma_I \subset X^{l(I)} \times X^{[k]}$$

whose fiber over $(x_1, \dots, x_{l(I)})$ parameterizes the set of subschemes of X with associated cycle $\sum_s i_s x_s$ (this is the correct definition if the x_i 's are $l(I)$ distinct points of X , and it is extended to $X^{l(I)}$ by taking the Zariski closure). The collection of these correspondences induces the de Cataldo-Migliorini decomposition [2]. Being proper over $X^{l(I)}$, the correspondence Γ_I induces a morphism

$$\Gamma_I^* : \text{CH}(X^{[k]}) \rightarrow \text{CH}(X^{l(I)}).$$

The construction of the Hilbert scheme can be done in the relative setting of a smooth family of surfaces $\mathcal{S} \rightarrow B$. The correspondences above then have a relative version $\Gamma_{I/B}$ and we get universally defined cycles on $l(I)$ -th powers of surfaces starting from any “natural cycle” on $\mathcal{S}^{[k]/B}$. One can also introduce products of correspondences $\Gamma_{I,I'} = \Gamma_I \times \Gamma_{I'} \subset \mathcal{S}^{l(I)/B} \times_B \mathcal{S}^{l(I')/B} \times_B \mathcal{S}^{[k]/B} \times_B \mathcal{S}^{[k]/B}$ and consider $\Gamma_{I,I'}^* \Delta_{\mathcal{S}^{[k]/B}}$. All these cycles are easily seen to be universally defined on powers of surfaces.

Coming back to Definition 3.1, we see that, compared to the previous sections, the novelty is the fact that we have obvious new universally defined cycles appearing on powers, namely the diagonals. Note that we can also use the notion of universally defined cycle $\mathcal{Z}_I$ on products of $i_1, \dots, i_s$ -th powers of d -dimensional varieties, which is the obvious variant of Definition 3.1.

In order to deal with the combinatorics of the diagonals, let us introduce some definitions. A partition $I = \{I_1, \dots, I_l\}$ of $\{1, \dots, k\}$ is a decomposition of $\{1, \dots, k\}$ as a disjoint union

$$\{1, \dots, k\} = I_1 \sqcup \dots \sqcup I_l,$$

where the I_i 's are not empty. We denote the integer l by $l(I)$. To such a partition I is associated a diagonal, respectively a relative diagonal if we work with a morphism $\mathcal{X} \rightarrow B$,

$$\Delta_I \subset X^k, \text{ resp } \Delta_I \subset \mathcal{X}^{k/B},$$

defined as the set $\{(x_1, \dots, x_k), x_i = x_j \text{ if } i, j \in I_s \text{ for some } s\}$. If $I = \{\{1\}, \dots, \{k\}\}$, $\Delta_I = X^k$, and if $I = \{\{1, \dots, k\}\}$, Δ_I is the small diagonal where all points x_i coincide. Note that Δ_I is almost canonically isomorphic to $X^{l(I)}$ (resp. $\mathcal{X}^{l(I)/B}$), where the isomorphism

depends in fact of the ordering (indexing) of the set $\{I_1, \dots, I_{l(I)}\}$. (In practice, we can choose the natural ordering.) We will thus consider Δ_I as a morphism from $X^{l(I)}$ to X^k defined by such an ordering. This morphism maps $(x_1, \dots, x_l)$ to $(x'_1, \dots, x'_k)$, where $x'_i = x_s$ whenever i belongs to I_s .

Thanks to Theorem 1.7 which describes universally defined cycles on products, Conjecture 1.8 is equivalent to the following statement.

Conjecture 3.2. *Let $\mathcal{Z}$ be a universally defined cycle on k -th powers of d -dimensional varieties. Then there exists a unique set of universally defined cycles $\mathcal{T}_I$ indexed by the partitions I of $\{1, \dots, k\}$, where each $\mathcal{T}_I$ is universally defined on products of $l(I)$ smooth varieties of dimension d , such that for any variety X over a field,*

$$\mathcal{Z}(X) = \sum_I \Delta_{I*}(\mathcal{T}_I(X, \dots, X)) \text{ in } \text{CH}(X^k). \quad (43)$$

Here we recall that $\mathcal{T}_I$ provides a cycle in $\text{CH}(X_1 \times \dots \times X_l)$ for any smooth d -dimensional varieties $X_1, \dots, X_l$, and the meaning of $\mathcal{T}_I(X, \dots, X)$ is that we apply $\mathcal{T}_I$ to the case where $X_1 = \dots = X_l = X$.

Indeed, Theorem 1.7 applied to each $\mathcal{T}_I$ combined with Conjecture 3.2 provides the desired formula

$$\mathcal{Z}(X) = \sum_I \Delta_{I,*} P_I(\text{pr}_1^* c_1(X), \dots, \text{pr}_1^* c_d(X), \dots, \text{pr}_{l(I)}^* c_1(X), \dots, \text{pr}_{l(I)}^* c_d(X)) \text{ in } \text{CH}(X^k), \quad (44)$$

where P_I is a polynomial in the variables $c_{1,1}, \dots, c_{d,1}, \dots, c_{1,l(I)}, \dots, c_{d,l(I)}$, which is of weighted degree $\leq d$ in each set of variables $c_{1,s}, \dots, c_{d,s}$.

A cycle of the form (44) will be said standard.

3.1 An auxiliary construction

We start with a construction which provides for each universally defined cycle $\mathcal{Z}$ on k -th powers of varieties of dimension d and each partition I of $\{1, \dots, k\}$ a universally defined cycle $\mathcal{Z}_I$ on products of $i_1, \dots, i_{l(I)}$ -th powers of $l(I)$ varieties of dimension d . Let

$$\mathcal{X}_1 \rightarrow B_1, \dots, \mathcal{X}_{l(I)} \rightarrow B_{l(I)}$$

be smooth morphisms of relative dimension d . Consider the family

$$\begin{aligned} \mathcal{X}_\bullet &:= \mathcal{X}_1 \times B_2 \times \dots \times B_{l(I)} \sqcup B_1 \times \mathcal{X}_2 \times \dots \times B_{l(I)} \sqcup \dots \sqcup B_1 \times \dots \times \mathcal{X}_{l(I)} \\ &\rightarrow B_1 \times \dots \times B_{l(I)}. \end{aligned} \quad (45)$$

Over a point $(b_1, \dots, b_{l(I)}) \in B_1 \times \dots \times B_{l(I)}$, its fiber is the disjoint union $\mathcal{X}_{b_1} \sqcup \dots \sqcup \mathcal{X}_{b_{l(I)}}$. In the sequel, if $X_1, \dots, X_l$ are varieties, we will denote similarly $X_\bullet = \sqcup_i X_i$ (this is the case where the basis B_i are points). Let $I = \{I_1, \dots, I_{l(I)}\}$ and $i_s := |I_s|$. There is a natural inclusion map over $B_1 \times \dots \times B_{l(I)}$

$$\mathcal{X}_1^{i_1/B_1} \times \dots \times \mathcal{X}_{l(I)}^{i_{l(I)}/B_{l(I)}} \hookrightarrow \mathcal{X}_\bullet^{k/(B_1 \times \dots \times B_{l(I)})} \quad (46)$$

whose image is an union of connected components of $\mathcal{X}_\bullet$ determined by I , and which we will denote by

$$\mathcal{X}_1^{I_1/B_1} \times \dots \times \mathcal{X}_{l(I)}^{I_{l(I)}/B_{l(I)}}. \quad (47)$$

Fiberwise, if $I_1 = \{j_{1,1} < \dots < j_{1,i_1}\} \subset \{1, \dots, k\}$, ..., $I_{l(I)} = \{j_{l(I),1} < \dots < j_{l(I),i_{l(I)}}\} \subset \{1, \dots, k\}$, this map sends

$$((x_{1,1}, \dots, x_{1,i_1}), (x_{2,1}, \dots, x_{2,i_2}), \dots, (x_{l(I),1}, \dots, x_{l(I),i_{l(I)}})) \in X_1^{i_1} \times \dots \times X_{l(I)}^{i_{l(I)}} \quad (48)$$

to the element

$$(x_j) \in X_{\bullet}^k,$$

where, if $j = i_{s,t} \in I_s$, we set $x_j = x_{s,t}$.

Given a universally defined cycle $\mathcal{Z}$ on k -th powers of varieties of dimension d and a partition I of $\{1, \dots, k\}$, we define, for each families $\mathcal{X}_1 \rightarrow B_1, \dots, \mathcal{X}_{l(I)} \rightarrow B_{l(I)}$ of smooth varieties of dimension d , the cycle

$$\mathcal{Z}_I(\mathcal{X}_1, \dots, \mathcal{X}_{l(I)}) \in \text{CH}(\mathcal{X}_1^{i_1/B_1} \times \dots \times \mathcal{X}_{l(I)}^{i_{l(I)}/B_{l(I)}}) \quad (49)$$

as the restriction of $\mathcal{Z}(\mathcal{X}_{\bullet})$ to the subvariety $\mathcal{X}_1^{I_1/B_1} \times \dots \times \mathcal{X}_{l(I)}^{I_{l(I)}/B_{l(I)}}$ of $\mathcal{X}_{\bullet}^{k/(B_1 \times \dots \times B_{l(I)})}$ determined by I that we constructed above. It is clear that the cycle $\mathcal{Z}_I$ is universally defined on products $X_1^{i_1} \times \dots \times X_{l(I)}^{i_{l(I)}}$ of powers of $l(I)$ smooth varieties of dimension d with the exponents $i_1, \dots, i_{l(I)}$ determined by I .

The next construction goes in the other direction: Given a universally defined cycle $\mathcal{T}$ on products of $i_1, \dots, i_{l(I)}$ -th powers of d -dimensional varieties with $\sum_s i_s = k$, for any smooth morphism $\mathcal{X} \rightarrow B$ of relative dimension d , we get by setting $\mathcal{X}_1 = \mathcal{X}, \dots, \mathcal{X}_k = \mathcal{X}$, a cycle $\mathcal{T}(\mathcal{X}, \dots, \mathcal{X}) \in \text{CH}(\mathcal{X}^{i_1/B} \times \dots \times \mathcal{X}^{i_{l(I)}/B})$. We will denote by

$$\mathcal{T}^{\delta}(\mathcal{X}) \in \text{CH}(\mathcal{X}^{k/B}) \quad (50)$$

the restriction of this cycle to

$$\mathcal{X}^{k/B} \subset \mathcal{X}^{i_1/B} \times \dots \times \mathcal{X}^{i_{l(I)}/B},$$

where $\mathcal{X}^{k/B}$ is naturally identified with the inverse image of the small diagonal of B in B^k under the morphism $\mathcal{X}^{i_1/B} \times \dots \times \mathcal{X}^{i_{l(I)}/B} \rightarrow B^{l(I)}$. $\mathcal{T}^{\delta}$ is universally defined on k th powers of d -dimensional varieties.

Starting from a universally defined cycle $\mathcal{Z}$ on k -th powers of d -dimensional varieties, the composition of these two operations produces for each I a cycle $\mathcal{Z}_I^{\delta}$ which is another universally defined cycle on k -th powers of d -dimensional varieties. Note that for $I = \{\{1, \dots, k\}\}$, $\mathcal{Z}_I^{\delta} = \mathcal{Z}$ using the axioms of Definition 3.1.

Assume that a universally defined cycle $\mathcal{Z}$ on k -th powers of d -dimensional varieties is standard, hence given by a formula of the form (44). Then given a partition I of $\{1, \dots, k\}$, and varieties $X_1, \dots, X_{l(I)}$, we apply formula (44) to the variety $X_{\bullet}$ of (45) and restrict to the component (47). Then we observe that

$$\Delta_J(X_{\bullet}) \cap (X_1^{I_1} \times \dots \times X_{l(I)}^{I_{l(I)}}) = \emptyset$$

if Δ_I is not contained in Δ_J , which is equivalent to the fact that some diagonal identity $x_i = x_j$ satisfied in Δ_J is not satisfied in Δ_I , or equivalently, at least one J_s is not contained in any I_t .

If, to the contrary, Δ_I is contained in Δ_J , (that is, the partition Δ_J is a refinement of the partition of Δ_I , meaning that each J_s is contained in a I_t), then

$$\Delta_J(X_{\bullet}) \cap (X_1^{I_1} \times \dots \times X_{l(I)}^{I_{l(I)}}) = \Delta_{J,I} \subset X_1^{I_1} \times \dots \times X_{l(I)}^{I_{l(I)}}. \quad (51)$$

In the last formula, recalling that the partition J refines the partition I , it provides a partition of each I_s , hence a diagonal for each $X_s^{I_s}$ and taking their product, this defines the desired diagonal $\Delta_{J,I} \subset X_1^{I_1} \times \dots \times X_{l(I)}^{I_{l(I)}}$. We will view it as a morphism from $X_1^{i_1^J} \times \dots \times X_{l(I)}^{i_{l(I)}^J}$ to $X_1^{I_1} \times \dots \times X_{l(I)}^{I_{l(I)}}$. Here i_s^J is the number of elements of the partition J contained in I_s . It follows from these two observations and from (44) that, for each partition I of $\{1, \dots, k\}$

$$\mathcal{Z}_I(X_1, \dots, X_{l(I)}) = \quad (52)$$

$$\sum_{\Delta_I \subset \Delta_J} \Delta_{J,I*}(P_J(\text{pr}_1^* c_1(X_{j(1)}), \dots, \text{pr}_1^* c_d(X_{j(1)}), \dots, \text{pr}_{l(J)}^* c_1(X_{j(l(J))}), \dots, \text{pr}_{l(J)}^* c_d(X_{l(J)})))$$

in $\text{CH}(X_1^{I_1} \times \dots \times X_{l(I)}^{I_{l(I)}})$, where the map $j : \{1, \dots, l(J)\} \rightarrow \{1, \dots, l(I)\}$ maps s to t whenever $J_s \subset I_t$.

We now restrict to the case where $X_i = X$, and observe that, when $X_i = X$ for all i , $\Delta_{J,I} = \Delta_J$ for those J refining I . Then (52) provides for any I the following formula

$$\mathcal{Z}_I^\delta(X) = \sum_{\Delta_I \subset \Delta_J} \Delta_{J*}(P_J(\text{pr}_1^*c_1(X), \dots, \text{pr}_1^*c_d(X), \dots, \text{pr}_{l(I)}^*c_1(X), \dots, \text{pr}_{l(I)}^*c_d(X))) \quad (53)$$

in $\text{CH}(X^k)$.

3.2 Proof of Theorem 1.9

Let us use the construction developed in the previous section to prove the following

Theorem 3.3. (Cf. Theorem 1.9) *Let $\mathcal{Z}$ be a standard universally defined cycle on k -th powers of d -dimensional smooth varieties. If $\mathcal{Z}(X)$ is cohomologous to 0 on X^k for any smooth d -dimensional variety X defined over $\mathbb{C}$, $\mathcal{Z}(X) = 0$ in $\text{CH}(X^k)$ for any such X .*

Proof. Recall that the universally defined cycles on products $\mathcal{Z}_I$ associated with $\mathcal{Z}$ are obtained by computing $\mathcal{Z}$ on $X_1^{i_1} \times \dots \times X_{l(I)}^{i_{l(I)}}$ seen as a connected component of $X_\bullet^k$ where $X_\bullet = \sqcup X_i$. It follows from this definition that, if $\mathcal{Z}$ is cohomologous to 0 on any smooth d -dimensional variety over $\mathbb{C}$, then each $\mathcal{Z}_I$ is cohomologous to 0 on products of powers $X_1^{i_1} \times \dots \times X_{l(I)}^{i_{l(I)}}$. It remains to show that this implies that the polynomials P_I appearing in (44) are 0. We do this by decreasing induction on $l(I)$, using the formula (52). When $l(I) = k$, so $I = \{\{1\}, \dots, \{k\}\}$, we get in particular

$$\mathcal{Z}_{\{\{1\}, \dots, \{k\}\}}(X_1, \dots, X_k) =$$

$$P_{\{\{1\}, \dots, \{k\}\}}(\text{pr}_1^*c_1(X_1), \dots, \text{pr}_1^*c_d(X_1), \dots, \text{pr}_k^*c_1(X_k), \dots, \text{pr}_k^*c_d(X_k)) \text{ in } \text{CH}(X_1 \times \dots \times X_k),$$

and this cycle has to be cohomologous to 0 on $X_1 \times \dots \times X_k$ for any choice of X_i ; we already saw (see Proposition 2.7) that this implies $P_I = 0$. For the general case, we assume that we already proved that $P_J = 0$ for $l(J) > l$. Let I be a partition of $\{1, \dots, k\}$ with $l(I) = l$. Formula (52) taken in cohomology then gives, when the varieties X_i are projective so that the Gysin morphism Δ_{I*} is well defined on cohomology

$$[\mathcal{Z}_I(X_1, \dots, X_{l(I)})] = \Delta_{I*}P_I(\text{pr}_1^*c_1(X_1), \dots, \text{pr}_1^*c_d(X_1), \dots, \text{pr}_{l(I)}^*c_1(X_{l(I)}), \dots, \text{pr}_{l(I)}^*c_d(X_{l(I)}))$$

in $H^*(X_1 \times \dots \times X_k, \mathbb{Q})$. Here the map Δ_I is the diagonal inclusion of $X_1 \times \dots \times X_{l(I)}$ in $X_1^{I_1} \times \dots \times X_{l(I)}^{I_{l(I)}}$ determined by I . The vanishing of $[\mathcal{Z}_I(X_1, \dots, X_{l(I)})]$ and the injectivity of the Gysin morphism Δ_{I*} for projective varieties X_i then implies that

$$P_I(\text{pr}_1^*c_1(X_1), \dots, \text{pr}_1^*c_d(X_1), \dots, \text{pr}_{l(I)}^*c_1(X_{l(I)}), \dots, \text{pr}_{l(I)}^*c_d(X_{l(I)}))$$

is cohomologous to 0 on $X_1 \times \dots \times X_{l(I)}$ for any smooth d -dimensional projective varieties $X_1, \dots, X_{l(I)}$, so that $P_I = 0$, by Proposition 2.7 again. $\square$

3.3 Application to uniqueness

Using the formalism above, we prove the following uniqueness result.

Theorem 3.4. *Let d and k be fixed and let Y be a smooth variety over $\mathbb{C}$. Consider a collection $P_\bullet = (P_I)$ of polynomials with coefficients in $\text{CH}(Y)$, where I runs through the set of partitions of $\{1, \dots, k\}$, and each P_I is a polynomial in the variables*

$$x_{1,1}, \dots, x_{d,1}, \dots, x_{1,l(I)}, \dots, x_{d,l(I)},$$

of weighted degree $\leq d$ in each set of variables $x_{1,j}, \dots, x_{d,j}$ (where $\deg x_{i,j} = i$). Then the Y -universal cycle $\mathcal{Z}_{P_\bullet}$ on k -th powers of d -dimensional varieties defined by

$$\mathcal{Z}(X) = \sum_I \Delta_{I*} P_I(\mathrm{pr}_1^* c_1(X), \dots, \mathrm{pr}_1^* c_d(X), \dots, \mathrm{pr}_{l(I)}^* c_1(X), \dots, \mathrm{pr}_{l(I)}^* c_d(X)) \in \mathrm{CH}(Y \times X^k) \quad (54)$$

vanishes for any smooth d -dimensional variety X over $\mathbb{C}$ if and only if $P_\bullet = 0$.

Remark 3.5. Even in the case where Y is a point, Theorem 3.4 may seem a bit surprising, especially when combined with Theorem 3.3. Indeed, consider for example the case of standard universal 0-cycles on X^k . If X is smooth projective and connected, then the cohomology class of a 0-cycle on X is encoded by its degree. It follows that all 0-cycles indexed by partitions and Chern polynomials appearing in formula (44) have proportional cohomology classes in X^k . So for given connected X , the statement is wrong. There are two reasons behind Theorem 3.4: first of all X is not connected and this is crucial since then the group $H_0(X^k, \mathbb{Z})$ of cycle classes of 0-cycles encodes the connected components of X^k and in particular the diagonals. The second point is general complex cobordism, which says (for $k = 1$) that there are no *universal* polynomial relations between Chern numbers, that is relations satisfied by all X 's.

Proof of Theorem 3.4. Assume $P_\bullet$ satisfies the property that the associated cycle $\mathcal{Z}_{P_\bullet}(X^k)$ given by formula (54) vanishes for smooth d -dimensional varieties X over $\mathbb{C}$. As in the previous proof, this implies as well by definition that each $\mathcal{Z}_{P_\bullet, I}(X_1, \dots, X_{l(I)})$ vanishes for any smooth d -dimensional varieties $X_1, \dots, X_{l(I)}$. We use now formula (52). Next, as in the previous proof, this implies by decreasing induction on $l = l(I)$ that, denoting by $\Delta_{I_\bullet}$ the inclusion of $X_1 \times \dots \times X_{l(I)}$ in $X_1^{I_1} \times \dots \times X_{l(I)}^{I_{l(I)}}$ given by the small diagonal inclusion on each summand, each cycle

$$\Delta_{I_\bullet*} P_I(\mathrm{pr}_1^* c_1(X_1), \dots, \mathrm{pr}_1^* c_d(X_1), \dots, \mathrm{pr}_{l(I)}^* c_1(X_{l(I)}), \dots, \mathrm{pr}_{l(I)}^* c_d(X_{l(I)}))$$

vanishes in $\mathrm{CH}(Y \times X_1^{I_1} \times \dots \times X_{l(I)}^{I_{l(I)}})$ for each set of smooth d -dimensional varieties $X_1, \dots, X_{l(I)}$. Arguing inductively on $l(I)$ and replacing Y by $Y' := Y \times X_1^{i_1} \times \dots \times X_{l(I)-1}^{i_{l(I)-1}}$, we conclude that it suffices to consider the case where $l(I) = 1$, and P is a polynomial in $c_1, \dots, c_d$ of weighted degree $\leq d$ with coefficients in $\mathrm{CH}(Y')$. We then have to prove that if $P \neq 0$, then for any $k > 0$, there exists a smooth d -dimensional variety X such that $\Delta_* P(c_1(X), \dots, c_d(X)) \neq 0$ in $\mathrm{CH}(Y' \times X^k)$. As Δ_* is injective when X is projective, we are finally reduced to the statement for $k = 1$, which is proved in [11], [9]. $\square$

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