Appendix: On linear subspaces contained in the secant varieties of a projective curve

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1. Introduction.

If $C \subset \mathbb{P}^N$ is a curve imbedded in projective space, one can consider the secant variety $\Sigma_d = \bigcup_{Z \in C^{(d)}} \langle Z \rangle$ swept out by the linear spans of d-uples of points of C. This Σ_d contains the $\mathbb{P}^{d-1}$'s parametrized by $Z \in C^{(d)}$ (here we are assuming that d is not large with respect to N). More precisely, Σ_d is birational to a projective bundle of rank $d-1$ over $C^{(d)}$. On the other hand, if d is large enough, $C^{(d)}$ also contains positive dimensional projective spaces, corresponding to linear systems on C. Deciding whether or not Σ_d contains linear subspaces other than those contained in some of the $\mathbb{P}^{d-1}_Z$'s is thus a non trivial problem.

Some time ago, C. Soulé obtained estimates for the maximal dimension of a linear subspace contained in Σ_d, and asked me whether an ad hoc geometric argument would lead to other results.

One answer in this direction is as follows:

We assume that C is smooth of genus $g > 0$ and that the embedding $C \subset \mathbb{P}^N$ is given by the sections of a line bundle $L \otimes \omega_C$, with $\deg(L) = m$. We then show:

Theorem. *If $m \geq 2d+3$, and $\delta \geq d-1$, any $\mathbb{P}^\delta$ contained in Σ_d is one of the $\mathbb{P}^{d-1} = \langle Z \rangle$, $Z \in C^{(d)}$. In particular, Σ_d contains no projective space $\mathbb{P}^\delta$, for $\delta \geq d$.*

Thanks. I wish to thank Christophe Soulé for interesting discussions and for providing the motivation to write this Note.

2. Proof of the theorem.

We first recall a few basic facts about secant varieties of curves (see [1]). First of all, since $m \geq 2d+1$, for any effective divisor Z of degree $k \leq 2d$ on C, we have $H^1(L \otimes \omega_C(-Z)) = 0$, hence the linear span of Z is of dimension $k-1$. Let now $E \to C^{(d)}$ be the vector bundle with fiber $H^0(L \otimes \omega_{C|Z})$ at $Z \in C^{(d)}$. Since the restriction map $H^0(L \otimes \omega_C) \to H^0(L \otimes \omega_{C|Z})$ is surjective for any $Z \in C^{(d)}$, there is a well defined morphism $\alpha : \mathbb{P}(E^*) \to \mathbb{P}^N$, whose image is exactly the secant variety Σ_d. Since sections of $L \otimes \omega_C$ separates any $2d$ points on C, it follows that α is one to one over $\Sigma_d - \Sigma_{d-1}$.

An easy computation shows that for any $Z \in C^{(d)}$, and for any x in the linear span of Z, but not in the linear span of any $Z' \not\subseteq Z$, the differential of α is of maximal rank, so that $\Sigma_d \backslash \Sigma_{d-1}$ is smooth of dimension $2d-1$. The projectivized tangent space to Σ_d at $\alpha(x)$ is easy to describe, at least when Z is a reduced divisor $\sum_1^d z_i$: indeed this is a $\mathbb{P}^{2d-1}$ which contains $\langle Z \rangle$ and also each projective line tangent to C at some point $z_i \in Z$, as one sees by deforming Z fixing z_j, $j \neq i$. It follows that it must be equal to the linear span of the divisor $2Z$. By continuity, this description of the projectivized tangent space to Σ_d remains true at any point of $\Sigma_d - \Sigma_{d-1}$.

We now start the proof of the theorem. We suppose that $\delta \geq d-1$, and assume that some projective space $\mathbb{P}^\delta$ is contained in Σ_d. Assuming $\mathbb{P}^\delta$ is not contained in one of the $\mathbb{P}^{d-1}_Z$'s we shall derive a contradiction.

Note that by induction on d, we may assume that $\mathbb{P}^\delta$ is not contained in Σ_{d-1}. Let $\widetilde{\mathbb{P}}^\delta$ be the closure of $\alpha^{-1}(\mathbb{P}^\delta \backslash \mathbb{P}^\delta \cap \Sigma_{d-1})$ in $\mathbb{P}(E^*)$. Denote by $\pi : \widetilde{\mathbb{P}}^\delta \to C^{(d)}$ the restriction to $\widetilde{\mathbb{P}}^\delta$ of the structural projection $\mathbb{P}(E^*) \to C^{(d)}$. Let $W := \pi(\widetilde{\mathbb{P}}^\delta)$ and $w := \dim W$. Our assumption is that $w > 0$. We shall denote by P_v the fiber $\pi^{-1}(v)$. It is a projective space $\mathbb{P}^\delta \cap \langle Z_v \rangle$, which is generically of dimension $s = \delta - w$.

We start with the following observation:

Lemma 1. *Under our assumption* $\dim W > 0$ *we have the inequality*

$$w > \delta - w \,. \tag{1}$$

Proof. Indeed, we may assume that for v, v' two generic distinct points of W, the supports of the associated divisors Z_v, $Z_{v'}$ of C are disjoint. Otherwise, Z_v would contain a fixed point $x \in C$, for any $v \in W$. But projecting C from x, we then get a curve $C' \subset \mathbb{P}^{N-1}$, such that Σ'_{d-1} contains a $\mathbb{P}^{\delta-1}$ which is not a $\mathbb{P}^{d-2}_Z$; since we may assume the theorem proven for $(m-1, d-1)$, this is impossible.

Now choose v, v' as above. The projective spaces $\langle Z_v \rangle$ and $\langle Z_{v'} \rangle$ do not meet, hence the projective spaces $P_v = \langle Z_v \rangle \cap \mathbb{P}^\delta$, $P_{v'} = \langle Z_{v'} \rangle \cap \mathbb{P}^\delta$ do not meet. Since they are of dimension s in a $\mathbb{P}^\delta$, it follows that $2s < \delta$, or $w > \delta - w$. ∎

Next we observe that, at each point $\alpha(x, Z)$ of $\mathbb{P}^\delta - (\mathbb{P}^\delta \cap \Sigma_{d-1})$, $\mathbb{P}^\delta$ is contained in the projectivized tangent space of Σ_d at $\alpha(x, Z)$, that is in $\langle 2Z \rangle$. Hence for any $v \in W$, the corresponding divisor $Z_v \in C^{(d)}$ satisfies

$$\mathbb{P}^\delta \subset \langle 2\, Z_v \rangle \,.$$

We next study the infinitesimal variation of $\langle 2\,Z_v\rangle \subset \mathbb{P}^N$. Let $H := \mathcal{O}_{\mathbb{P}^N}(1)$. Then we have the identification

$$H^0(\mathbb{P}^N, H) \simeq H^0(C, L\otimes\omega_C)\,, \tag{2}$$

which by definition of the linear span, induces an identification

$$H^0(\mathbb{P}^N, H\otimes I_{\langle 2Z_v\rangle}) \simeq H^0(C, L\otimes\omega_C(-2\,Z_v))\,. \tag{3}$$

If $h \in T_{W,v}$, the infinitesimal deformation of $\langle 2\,Z_v\rangle$ in the direction h is described by an homomorphism:

$$\varphi_h : H^0(\mathbb{P}^N, H\otimes I_{\langle 2Z_v\rangle}) \to H^0(\langle 2\,Z_v\rangle, H_{|\langle 2Z_v\rangle})\,.$$

We have now an isomorphism induced by (2) and (3):

$$H^0(\langle 2\,Z_v\rangle, H_{|\langle 2Z_v\rangle}) \simeq H^0(L\otimes\omega_{C|2Z_v})\,. \tag{4}$$

We have the following

Lemma 2. *Under the isomorphisms* (3) *and* (4)*, if we identify h to an element $u_h \in H^0(\mathcal{O}_C(Z_v)_{|Z_v})$, φ_h identifies to the multiplication*

$$u_h : H^0(C, L\otimes\omega_C(-2\,Z_v)) \to H^0(Z_v, L\otimes\omega_C(-Z_v)_{|Z_v})$$

followed by the inclusion

$$H^0(Z_v, L\otimes\omega_C(-Z_v)_{|Z_v}) \hookrightarrow H^0(2\,Z_v, L\otimes\omega_{C|2Z_v})\,.$$

The proof is straightforward once we recall the construction of φ_h by differentiating under the parameters the equations vanishing on $\langle 2\,Z_v\rangle$. ∎

We know that the spaces $\langle 2\,Z_v\rangle$, for $v\in W$, contain $\mathbb{P}^\delta$. Infinitesimally, this translates into the fact that for any $h\in T_{W,v}$, the image of φ_h vanishes on $\mathbb{P}^\delta$, that is, is contained in

$$\mathrm{Ker}(H^0(\langle 2\,Z_v\rangle, H_{|\langle 2Z_v\rangle}) \to H^0(\mathbb{P}^\delta, H_{|\mathbb{P}^\delta}))\,.$$

¿From the description of φ_h given in Lemma 2, we see that Im φ_h is contained in

$$K := \mathrm{Ker}(H^0(\langle 2\,Z_v\rangle, H_{|\langle 2Z_v\rangle}) \to H^0(\langle Z_v\rangle, H_{|\langle Z_v\rangle}))\,.$$

Indeed, via the isomorphism (4), K identifies to

$$\mathrm{Ker}(H^0(L\otimes\omega_{C|2Z_v}) \to H^0(L\otimes\omega_{C|Z_v})) = \mathrm{Im}\ H^0(L\otimes\omega_C(-Z_v)_{|Z_v}) \to H^0(L\otimes\omega_{C|2Z_v})\,.$$

Finally, note that the restriction map $K \to H^0(\mathbb{P}^\delta, H_{|\mathbb{P}^\delta})$ has rank equal to the dimension of

$$\mathrm{Ker}(H^0(\mathbb{P}^\delta, H_{|\mathbb{P}^\delta}) \to H^0(\mathbb{P}^\delta \cap \langle Z_v\rangle, H_{|\mathbb{P}^\delta \cap \langle Z_v\rangle})),$$

which is equal to $\delta - s$, since $\mathbb{P}^\delta \cap \langle Z_v\rangle = P_v$ is of dimension s.

Denote now by $V \subset H^0(\mathcal{O}_C(Z_v)_{|Z_v})$ the tangent space to W at v. Lemma 2 and the estimate above give us the following conclusion:

Lemma 3. *Under our assumptions, the multiplication map*

$$\mu : V \otimes H^0(C, L \otimes \omega_C(-2\, Z_v)) \to H^0(L \otimes \omega_C(-Z_v)_{|Z_v})$$

has its image contained in a subspace of codimension at least w. □

We now derive a contradiction. We observe first that since $\widetilde{\mathbb{P}^\delta}$ is a rational variety dominating W, W is contained in a linear system $|D| \subset C^{(d)}$. Hence $\mathcal{O}_C(D) = \mathcal{O}_C(Z_v)$ for all $v \in W$, and the fact that $W \subset |D|$ translates infinitesimally into the fact that $V = T_{W,v}$ is contained in the image of the restriction map:

$$H^0(\mathcal{O}_C(Z_v)) \to H^0(\mathcal{O}_C(Z_v)_{|Z_v}).$$

Let now $\widetilde{V}$ be the inverse image of V under this restriction map. Then rk $\widetilde{V} = w + 1$, and Lemma 3 shows that the multiplication map

$$\widetilde{\mu} : \widetilde{V} \otimes H^0(C, L \otimes \omega_C(-2\, Z_v)) \to H^0(C, L \otimes \omega_C(-Z_v))$$

has its image contained in a space of codimension at least w.

Now we have the equality:

$$\mathrm{rk}\ H^0(C, L \otimes \omega_C(-Z_v)) = d + \mathrm{rk}\ H^0(C, L \otimes \omega_C(-2\, Z_v)),$$

since $H^1(C, L \otimes \omega_C(-2\, Z_v)) = 0$. So we conclude that

$$\mathrm{rk}\ \widetilde{\mu} \le h^0(C, L \otimes \omega_C(-2\, Z_v)) + d - w. \tag{5}$$

On the other hand, we can apply Hopf lemma to $\widetilde{\mu}$, and the inequality in Hopf lemma must be strict here, since the line bundle $L \otimes \omega_C(-2\, Z_v)$ is very ample, being of degree at least $2g + 1$, and C is not rational. This gives us:

$$\mathrm{rk}\ \widetilde{\mu} > w + 1 + h^0(C, L \otimes \omega_C(-2\, Z_v)) - 1. \tag{6}$$

Combining (5) and (6), we get:

$$d - w > w. \tag{7}$$

But this contradicts inequality (1), since $\delta \ge d - 1$. □

References.

[1] A. Bertram : Moduli of rank 2 vector bundles, theta divisors, and the geometry of curves in projective space, *J. Diff. Geom.* 35, 1992, 429-469.