TRANSCENDENTAL METHODS IN THE STUDY OF ALGEBRAIC CYCLES

Claire VOISIN

Université d'Orsay et IHES

In these lectures, in contrast with the orientation of M. Green's lectures, we put the accent on the global arguments used when working with families of algebraic varieties (monodromy arguments), and also on the relations between non representability of Chow groups and the "transcendental part of Hodge structures", which explains the title, excepted for the fact that there are very few things that cannot be done within algebraic geometry.

Each lecture is introduced, and they are independent, excepted for lecture 5 - lecture 6, and lecture 7 - lecture 8. The lectures should be read in parallel with those of M. Green and J.P. Murre which they are supposed to complete. They are organized as follows :

1) Divisors
2) Topology and Hodge theory
3) Noether-Lefschetz locus
4) Monodromy
5) 0-cycles I
6) 0-cycles II
7) Griffiths groups
8) Application of the Noether-Lefschetz locus to threefolds.

I am very grateful to the C.I.M.E. foundation and to Fabio Bardelli for giving me the opportunity to deliver these lectures and to collaborate with M. Green and J.P. Murre. I also thank the IHES for its hospitality during the academic year of 1992/93.

M. Green et al.: LNM 1594, A. Albano and F. Bardelli (Eds.), pp. 153–222, 1994.

Lecture one : Divisors

0. This lecture is devoted to the classical and well understood subject of divisors on an algebraic variety. What we want to do is to review the main features of the theory of divisors, the codimension one cycles, from the point of view of algebraic geometry and Hodge theory, which fit very well in this case. In particular, we want to insist on the finiteness statements which have been shown by Clemens and Mumford to be specific of the codimension one cycles.

In contrast, the remaining lectures will illustrate the fact that when the objects constructed from the Hodge theory become more transcendental, the corresponding theory of algebraic cycles is less harmonious.

1. Recall that on a smooth algebraic variety X, one can identify the Weil divisors, which are formal sums $\Sigma n_i D_i$ of codimension one irreducible subvarieties affected with integral coefficients $n_i \in \mathbf{Z}$, and Cartier divisors, which are sections of the Zariski sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$ that is are described in a covering (U_i) of X by Zariski open sets by a collection of rational functions $\psi_i \in \mathcal{K}_X^*$ such that ψ_i/ψ_j is invertible in $U_i \cap U_j$. To the Cartier divisor ψ_i corresponds its local factorization into product of prime, locally principal, ideals. To such a Cartier divisor is also associated an algebraic line bundle, the rank one $\mathcal{O}_X$-submodule of the constant sheaf $\mathcal{K}_X$ generated by ψ_i^{-1} on U_i. The isomorphism class of this line bundle is determined by the class in $H^1_{\text{zar}}(\mathcal{O}^*_{X,\text{alg}})$ of the Čech cocycle $g_{ij} = \psi_i/\psi_j \in \mathcal{O}^*_{U_{ij}}$.

The line bundles associated to two Cartier divisors (ψ_i), (ψ_i') are isomorphic if and only if there is a global rational function ψ on X such that $\psi_i' = \psi\psi_i$ modulo an invertible function on U_i, and then the corresponding Weil divisors D and D' differ by the principal divisor $\operatorname{div}\psi$. One obtains this way a bijection between the group $CH^1(X) :=$ {Weil divisors modulo principal Weil divisors} and the group Pic X of algebraic line bundles modulo linear equivalence (i.e. isomorphisms). The abelian group structure of Pic X is given by the tensor product of line bundles.

The study of Pic X splits into two parts : One introduces the notion of algebraic equivalence between line bundles. A line bundle L on X is algebraically equivalent to zero if there is an irreducible curve C and a line bundle $\mathcal{L}$ on $C \times X$, two points c and $c' \in C$ such that $\mathcal{L}_{|c\times X} \simeq \mathcal{O}_X$ and $\mathcal{L}_{|c'\times X} \simeq L$. These line bundles form a subgroup $\operatorname{Pic}^0 X \subset \operatorname{Pic} X$.

In [5] one can find, for a complete variety X, a purely algebraic construction of $\operatorname{Pic}^0 X$ as an abelian variety, that is a complete commutative algebraic group. One of the main tools is the identification of the space of the first order deformations of a line bundle to the space $H^1(\mathcal{O}_X)$ via the correspondence: {multiplicative one cocycle $g_{ij} + \epsilon f_{ij}$ on $X \times \operatorname{Spec} k[\epsilon]/\epsilon^2 \to$ additive one cocycle $f_{ij}/g_{ij} \in \mathcal{O}_{v_{ij}}$}.

The quotient $\operatorname{Pic} X/\operatorname{Pic}^0 X$ can also be studied by purely algebraic methods. Using vanishing theorems one can prove that if $S \subset X$ is a smooth surface, complete intersection of ample divisors, one has an isomorphism $\operatorname{Pic}^0 X \simeq \operatorname{Pic}^0 S$ and an injection $\operatorname{Pic} X \subset \operatorname{Pic} S$. So $\operatorname{Pic} X/\operatorname{Pic}^0 X \hookrightarrow \operatorname{Pic} S/Pic^0 S$. Now on the surface S, the Grothendieck-Hirzebruch-Riemann-Roch formula is a powerful tool to study line bundles. The results that one can obtain this way are the following :

a) A line bundle is said to be numerically equivalent to zero if it is of degree zero on any curve $C \subset X$. This determines a subgroup $\operatorname{Num} X \subset \operatorname{Pic} X/\operatorname{Pic}^0 X$ and one has : $\operatorname{Num} X$ is a finite group.

b) The quotient $NS(X) = \operatorname{Pic} X/\operatorname{Num} X$ is a finitely generated group of rank $\rho(X)$, called the Néron-Severi group.

c) Hodge index theorem : If X is a surface, the intersection theory puts an intersection form on $NS(X)$, [1], which by definition of numerical equivalence is non degenerate. Then its index is $\rho(X) - 1$.

2. Now we assume that X is a complex projective algebraic variety and we turn to the Hodge theoretic description of line bundles. The starting point is the theorem of Serre [6], also called "GAGA principle" :

2.1. Theorem : The functor $E \to E^{an}$ which to a coherent sheaf of $\mathcal{O}_X$-modules associates the corresponding coherent sheaf of $\mathcal{O}_X^{an}$-modules, for the usual topology, via the continuous map of schemes $X^{an} \to X^{zar}$ is an equivalence of category.

It follows that for algebraic X we can identify $\operatorname{Pic} X$ with the group of analytic line bundles, which is itself isomorphic to $H^1(X, \mathcal{O}_X^{an*})$. (As before to a line bundle corresponds the class of the Čech cocycle of its transition functions, constructed from a set of trivializations). The cohomology is now understood in the usual topology. In the sequel, we will use $\mathcal{O}_X, \mathcal{O}_X^*$ for $\mathcal{O}_X^{an}, \mathcal{O}_X^{an*}$ and work with the usual topology. For the cohomology of coherent sheaves, the confusion is allowed by Serre's theorem.

The second main tool is the exponential exact sequence.

2.2. $0 \to \mathbb{Z} \overset{2i\pi}{\to} \mathcal{O}_X \overset{\exp}{\to} \mathcal{O}_X^* \to 0$, which shows that $\mathcal{O}_X^*$ is quasi isomorphic up to a shift to the complex: $\mathbb{Z}(1) : 0 \to \mathbb{Z} \overset{2i\pi}{\to} \mathcal{O}_X \to 0$, where $\mathcal{O}_X$ is put in degree one. This complex is the first Deligne complex and its cohomology $H^2_D(X, \mathbb{Z}(1)) \simeq H^1(\mathcal{O}^*_X)$ is the first Deligne cohomology group. The generalization of it will be explained by Jacob Murre. There is an obvious map of complexes : $\mathbb{Z}(1) \to \mathbb{Z}$, which has for kernel the sheaf $\mathcal{O}_X$ concentrated in degree one. So one can associate to a line bundle L its Deligne-Chern class $C_1^D(L) \in H^1(\mathcal{O}^*_X) \simeq H^2_D(X, \mathbb{Z}(1))$ and its topological Chern class $c_1(L) \in H^2(X, \mathbb{Z})$. Using the exponential exact sequence, $c_1(L)$ can also be represented as a Čech cocycle $\alpha_{ijk} = \frac{1}{2i\pi}(\log g_{ij} + \log g_{jk} + \log g_{ki}) \in \mathbb{Z}$, and it follows that its image in $H^2(X, \mathbb{C})$ identifies, via the De Rham isomorphisms, to the class of the closed two form :

2.3. $\omega_L = \frac{1}{2i\pi} \partial\overline{\partial} \log h(\sigma)$, where h is a hermitian metric on L and σ is a local nowhere zero holomorphic section of L on which ω_L actually does not depend. This is the Chern-Weil construction of the first Chern class.

From the formula 2.3 it is not difficult to show that $c_1(L)$ is also Poincaré dual of the homology class $\Sigma n_i[D_i] \in H_{2n-2}(X, \mathbb{Z})$ of the divisor $\Sigma n_i D_i$ of any meromorphic section of L (Lelong formula).

If one considers the long exact sequence associated to 2.2, one finds that the following is exact at the middle :

2.4. $\quad H^1(\mathcal{O}^*_X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$, which gives :

2.5. The Lefschetz theorem on $(1,1)$ classes :

An integral class $\alpha \in H^2(X, \mathbb{Z})$ is the first Chern class of a Line bundle (or the Poincaré dual of the homology class of a Weil divisor) if and only if its $(0,2)$ component vanishes.

Here we have identified the map $H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$ to the composite map $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C}) \to H^2(X, \mathcal{O}_X)$, and the last map is described in the DeRham/Dolbeault cohomology as the map $\{d\text{-closed 2-form } \omega\} \to \{(0,2)$ component of ω, which is $\overline{\partial}$-closed$\}$. Using the Hodge decomposition $H^2(X, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$, this map is also the projection on the last factor, and its kernel is $F^1H^2(X, \mathbb{C}) := H^{2,0}(X) \oplus H^{1,1}(X)$. Finally using the fact that under complex conjugation $\overline{H^{2,0}(X)} = H^{0,2}(X)$ one sees that a class $\alpha \in H^2(X, \mathbb{Z}) \cap F^1H^2(X)$ is in fact in $H^{1,1}_{\mathbb{R}}(X) = \{$ classes of real closed 2-forms of type $(1,1)\}$, as is expected from formula 2.3.

Returning to the beginning of the long exact sequence associated to 2.2, one sees that the kernel of the map c_1 is the quotient $H^1(\mathcal{O}_X)/H^1(X, \mathbb{Z})$. What one learns then from the Hodge theory is the following :

One has a decomposition $H^1(X,\mathbf{C}) = H^{1,0} \oplus H^{0,1}$, where $H^{0,1} \simeq H^1(\mathcal{O}_X) \simeq \overline{H^{1,0}(X)} = \overline{H^0(\Omega_X)}$. It follows that the composite $H^1(X,\mathbf{R}) \to H^1(X,\mathbf{C}) \to H^{0,1}(X)$ is an isomorphism of real vector spaces, and that $H^1(X,\mathbf{Z})$ projects to a lattice in $H^1(\mathcal{O}_X)$. The quotient $H^1(\mathcal{O}_X)/H^1(X,\mathbf{Z})$ is then a complex torus, with the complex structure given by the complex structure on its tangent space $H^1(\mathcal{O}_X)$. It is in fact an algebraic torus (or an abelian variety) : As explained in [4], a complex torus $T = \mathbf{C}^m/\Gamma$ is algebraic if one has a skew symmetric form $\omega_{\mathbf{Z}}$ on Γ, whose extension $\omega_{\mathbf{R}}$ to $\mathbf{C}^m = \Gamma \otimes \mathbf{R}$ is of type $(1,1)$ for the complex structure on $\mathbf{C}^m$, and whose associated sesquilinear form h is positive definite on $\mathbf{C}^n$, where $h(X,X) = -\omega(X,iX)$.

Equivalently, one can consider the complexification $\omega_{\mathbf{C}}$ of ω to a form on $\Gamma \otimes \mathbf{C}$, use the splitting $\Gamma \otimes \mathbf{C} \simeq \Gamma^{1,0} \oplus \Gamma^{0,1}$ given by the complex structure on T and define h on $\Gamma^{1,0}$ by

2.6. $$h_\omega(X,X) = \tfrac{i}{2}\, \omega_{\mathbf{C}}(X,\overline{X}).$$

By 2.5, $\omega \in \Lambda^2 H_1(T,\mathbf{Z})^* \simeq H^2(T,\mathbf{Z})$ will then be the class of a line bundle, which is ample by the positivity of h.

To construct such a polarization on $T = H^1(\mathcal{O}_X)/H^1(X,\mathbf{Z})$, one fixes $\alpha \in H^2(X,\mathbf{Z})$ the Chern class of an ample line bundle on X, and one defines :

2.7. $$\omega_{\mathbf{Z}}(\varphi,\psi) = -\int_X \alpha^{n-1}\varphi \wedge \psi$$

Using the previous notations, we have $\Gamma \otimes \mathbf{C} = H^1(X,\mathbf{C})$ with $\Gamma^{1,0} \simeq H^1(\mathcal{O}_X)$.

The fact that $\omega_{\mathbf{R}}$ is of type $(1,1)$ on T is equivalent to the vanishing of $\omega_{\mathbf{C}|H^1(\mathcal{O}_X)}$, which is clear by $\omega_{\mathbf{C}}(\varphi^{0,1},\psi^{0,1}) = -\int_X \alpha^{n-1} \wedge \varphi^{0,1} \wedge \psi^{0,1}$ and $\alpha^{n-1} \wedge \varphi^{0,1} \wedge \psi^{0,1} = 0$ on X.

The positivity condition reduces then to the statement that $h_{\mathbf{C}}(\varphi^{0,1},\varphi^{0,1}) = -\frac{i}{2}\int_X \alpha^{n-1} \wedge \varphi^{0,1} \wedge \overline{\varphi^{0,1}} = \frac{i}{2}\int_X \alpha^{n-1} \wedge \varphi^{1,0} \wedge \overline{\varphi^{1,0}}$ is strictly positive when $\varphi^{0,1} = \overline{\varphi^{1,0}}$ is a non-zero element in $H^1(\mathcal{O}_X)$. This is a particular case of the "Hodge-Riemann bilinear relations" which will be explained in the next lecture. In fact the positivity of $h_\omega(\varphi^{0,1},\varphi^{0,1})$ can be checked directly by a reduction to the case of a curve : α being the class of an ample line bundle on X, there is a smooth curve $C \subset X$, such that $N\alpha^{n-1}$ is the Poincaré dual of the homology class of C, for some integer $N > 0$. Then :

2.8. $\frac{i}{2}\int_X \alpha^{n-1}\overline{\varphi^{0,1}} \wedge \varphi^{0,1} = \frac{1}{N}\frac{i}{2}\int_C \overline{\varphi^{0,1}} \wedge \varphi^{0,1}$ and it is easy to check that $i\overline{\varphi^{0,1}} \wedge \varphi^{0,1}$ is a positive 2-form on C (with respect to the orientation given by its complex structure).

3. The fact that T is an abelian variety will now imply that it is isomorphic to $\mathrm{Pic}^0 X$. It is clear that algebraically equivalent line bundles are topologically equivalent, hence have the same c_1. So points of $\mathrm{Pic}^0 X$ correspond to points in T.

On the other hand one can construct a line bundle on $T \times X$ called the Poincaré line bundle, using the Lefschetz theorem on $(1,1)$ classes :

Consider $H^1(T,\mathbf{Z}) \otimes H^1(X,\mathbf{Z}) \subset H^2(T \times X,\mathbf{Z})$ in the Künneth decomposition. By definition of T, $H^1(T,\mathbf{Z})$ is canonically the dual of $H^1(X,\mathbf{Z})$ so we have a natural class $e \in H^2(T \times X,\mathbf{Z})$ corresponding to $\mathrm{Id} \in H^1(X,\mathbf{Z})^* \otimes H^1(X,\mathbf{Z})$.

Next we show that e is of type $(1,1)$ in the Hodge decomposition of $H^2(T \times X)$. That is we have to check that $e \in H^{1,0}(T) \otimes H^{0,1}(X) \oplus H^{0,1}(T) \otimes H^{1,0}(X)$. But from $T = H^1(\mathcal{O}_X)/H^1(X,\mathbf{Z})$, we deduce that $(1,0)$-forms on T identify to linear forms on $H^1(X,\mathbf{C})$ which vanish on $H^{1,0}(X)$ and $(0,1)$-forms on T identify to linear forms on $H^1(X,\mathbf{C})$ which vanish on $H^{0,1}(X)$. If (ω_i) is a basis for $H^{1,0}(X)$, $(\overline{\omega}_i)$ the conjugate basis of $H^{0,1}(X)$, and $(\omega_i^*, \overline{\omega}_i^*)$ the dual basis of $H^1(X,\mathbf{C})^* = H^1(T,\mathbf{C})$ one finds that $e = \Sigma\, \omega_i^* \otimes \omega_i + \Sigma\, \overline{\omega}_i^* \otimes \overline{\omega}_i$, where the ω_i^*'s vanish on $H^{0,1}(X)$ hence are in $H^{0,1}(T)$ and the $\overline{\omega}_i^*$'s vanish on $H^{1,0}(X)$ hence are in $H^{1,0}(T)$. So e is of type $(1,1)$.

It follows from 2.5 that $e = c_1(\mathcal{L})$ for some line bundle $\mathcal{L}$ on $T \times X$, which is uniquely defined up to line bundles with vanishing Chern classes, coming from T and X. $\mathcal{L}$ is uniquely defined if we impose $\mathcal{L}_{|0 \times X} = \mathcal{O}_X$ and $\mathcal{L}_{|T \times x} = \mathcal{O}_T$ for some fixed point $x \in X$.

T is now a smooth connected algebraic variety which parametrizes line bundles on X via $t \mapsto \mathcal{L}_t := \mathcal{L}_{|T \times X}$. By definition of algebraic equivalence $\mathcal{L}_t$ is then algebraically equivalent to $\mathcal{O}_X = \mathcal{L}_0$. To conclude that $T = \mathrm{Pic}^0 X$ it suffices now to check:

3.1. The map $E : T \to T$ defined by $E(t) = \mathcal{L}_t \in \mathrm{Ker}(c_1 : \mathrm{Pic}\, X \to H^2(X,\mathbf{Z})) \simeq T$ is the identity.

3.1. is a very general fact concerning the Abel-Jacobi map and has a generalization given by the theorem on normal functions. So it may be useful to see the topological meaning of this statement.

First of all, let us describe in more geometric terms the map $\mathrm{Ker}\, c_1 \to H^1(\mathcal{O}_X)/H^1(X,\mathbf{Z})$: Coming back to Weil divisors, an element of $\mathrm{Ker}\, c_1$ is represented by $D = \Sigma n_i D_i$ such that $\Sigma n_i D_i$ is homologous to zero on X. Then if Γ is a $2n-1$ real chain such that $\partial\Gamma = \Sigma n_i D_i$, $\int_\Gamma$ acts on $H^{n,n-1}(X)$ and is well defined up to periods $\int_T$ for $T \in H_{2n-1}(X,\mathbf{Z})$. So one obtains a point $\Phi(D)$ in $H^{n,n-1}(X)^*/H_{2n-1}(X,\mathbf{Z})$. ($\Phi$ is the

Abel-Jacobi map for divisors; its careful definition, in particular the fact that $\int_\Gamma$ is well defined, will be given by M. Green in a more general context).

Now by the Poincaré duality $H^{n,n-1}(X)^*/H_{2n-1}(X,\mathbf{Z}) \simeq H^1(\mathcal{O}_X)/H^1(X,\mathbf{Z}) = T$. One version of Abel's theorem is the statement that $\Phi(D)$ is equal to $c_1^{\mathcal{D}}(D) \in T$, via the last isomorphism. The proof of 3.1 is then immediate. A meromorphic section of $\mathcal{L}$ gives $\mathcal{D} \subset T \times X$ such that the homology class of $\mathcal{D}$ is the Poincaré dual of $c_1(\mathcal{L})$. For $t \in T$, we choose a path γ_t from 0 to t in T. Then $\bigcup_{s\in\gamma_t} D_s$ gives a $2n-1$ chain Γ_t in X with boundary $D_t - D_0$, and we have the identification $E(t) = \int_{\Gamma_t} \in H^{n,n-1}(X)^*/H_{2n-1}(X,\mathbf{Z}) \simeq T$.

It follows that the map $E_* : H_1(T,\mathbf{Z}) \to H_1(T,\mathbf{Z}) \simeq H_{2n-1}(X,\mathbf{Z})$ induced in homology by E is described geometrically as $\gamma \mapsto \mathcal{D}_\gamma = \bigcup_{t\in\gamma} D_t$, for γ a loop in T. In other words one has $E_*(\gamma) = p_{2*}(p_1^*\gamma \cap [\mathcal{D}])$ where p_1, p_2 are the projections from $T \times X$ to T and X, and $[\mathcal{D}]$ is the homology class of $\mathcal{D}$. Now $[\mathcal{D}]$ is Poincaré dual of $c_1(\mathcal{L})$, and the identification of $c_1(\mathcal{L})$ to $\mathrm{Id} \in H^1(T,\mathbf{Z}) \otimes H^1(X,\mathbf{Z})$ gives immediately $E_* = \mathrm{Id}_{H_1(T,\mathbf{Z})}$. Hence $E = \mathrm{Id}_T$. (It is not difficult to show that E is additive).

It follows now that the map c_1 embeds $\mathrm{Pic}\, X/\mathrm{Pic}^0 X$ in the finitely generated group $H^2(X,\mathbf{Z})$. The group $\mathrm{Num}\, X$ identifies to the torsion of $H^2(X,\mathbf{Z})$ for the following reason: by 2.5, torsion classes are first Chern classes of line bundles, and if $\alpha = c_1(L)$ one has for $C \subset X, d^0 L_{|C} = \int_c \alpha = 0$ for $\alpha \in \mathrm{Tors}\, H^2(X,\mathbf{Z})$. So we have the inclusion $\mathrm{Tors}\, H^2(X,\mathbf{Z}) \subset \mathrm{Num}\, X$. The converse uses the weak Lefschetz theorem and the Hodge index theorem, which will be explained in the next lecture : the first one will imply that if $\alpha = c_1(L)$ is a non-torsion class in $H^2(X,\mathbf{Z})$, for a surface $S \subset X$, a complete intersection of ample divisors, $\alpha_{|S} = c_1(L_{|S})$ is a non-torsion class in $H^2(S,\mathbf{Z})$. The second one will imply that the intersection form of $H^2(S,\mathbf{Z})/\mathrm{Torsion}$ restricted to the set of divisor classes is non degenerate. Hence there is a curve $C \subset S$ such that $d^0 L_{|C} \neq 0$.

References :

[1] A. Beauville, Surfaces algébriques complexes, chapitre 1, Astérisque n° 54, Société Mathématique de France (1978).

[2] P. Griffiths, J. Harris, Principles of algebraic Geometry, pp 128-148, Pure and Applied Mathematics, John Wiley and Sons (1978).

[3] A. Grothendieck, Techniques de construction et théories d'existence en géométrie algébrique IV : Les schémas de Hilbert, Séminaire Bourbaki 1960/61, exposé n° 221.

[4] D. Mumford, Abelian varieties, chapter one, Tata Institute for fundamental Research, studies in Mathematics, Oxford University Press 1970.

[5] C.S. Seshadri, La variété de Picard d'une variété complète, Séminaire Chevalley 1958/59, exposé n°8.

[6] J.-P. Serre, Géométrie algébrique et géométrie analytique, Ann. de l'Inst. Fourier 6 (1956) 1-42.

[7] A. Weil, Variétés kählériennes, chapitres 5 et 6, Actualités Scientifiques et Industrielles 1267, Hermann, seconde édition 1971.

Lecture two : Topology and Hodge theory

0. In this lecture, we describe the consequences of the Hodge theory on the topology of algebraic varieties. The most important are the Lefschetz decomposition and the Hodge index theorem, or more generally, the Hodge Riemann bilinear relations, which lead to the notion of polarized Hodge structure.

From the point of view of topology, the weak Lefschetz theorem shows that the cohomology of a complex projective variety X of dimension n differs from the cohomology of its hyperplane section Y only in degrees n and $n-1$.

This suggests that working inductively on the dimension, one has only to study the primitive middle dimensional cohomology of X and Y to understand its relationship with the theory of algebraic cycles. However, excepted for the case of line bundles, the weak Lefschetz theorem has no direct analogue at the level of algebraic cycles.

The weak Lefschetz theorem has a more precise version given by the Lefschetz decomposition, which is proved by the Hodge theory, despite of its algebraic flavour. Interpretation of it in terms of motives is the contents of one of the standard conjectures [3]. We begin with the Morse theoretic proof of the weak Lefschetz theorem [4], and explain how the hard Lefschetz theorem follows then from the Hodge-Riemann bilinear relations. For the Lefschetz decomposition and polarizations on Hodge structures we follow [5].

1. Let $U \subset \mathbb{C}^N$ be a smooth complex analytic subvariety of dimension n. Let $<,>$ be the standard hermitian inner product on $\mathbb{C}^N$. Then for a general point $0 \in \mathbb{C}^N$, the function $f_0 : U \to \mathbf{R}$ defined by $f_0(X) = <\vec{0x}, \vec{0x}>$ is a Morse function on U, that is an exhaustion function with only one non degenerate critical point for each critical value. The Morse theory says that for each critical point x_0 of such a function f, such that $\mathrm{Hess}_{x_0} f$ has index k, the set $U_{f(x_0)+\epsilon} := \{x \in U / f(x) \leq f(x_0) + \epsilon\}$, for small ϵ, is obtained from $U_{f(x_0)-\epsilon}$ by glueing a k-disk on a $k-1$ sphere contained in $U_{f(x_0)-\epsilon}$.

Now we have the following (cf. [4]) :

1.1. Lemma : If $f = f_0$ is the squared distance function with respect to a hermitian metric on $\mathbb{C}^N$, the index k of any non degenerate critical point of f_0 satisfies :

$$k \leq n = \dim_{\mathbb{C}} U.$$

From this one deduces :

1.2. Theorem : An affine variety of complex dimension n has the homotopy type of a CW-complex of real dimension $\leq n$.

As an immediate consequence, one obtains the following vanishing theorems :

1.3. Corollary : Let U be an affine variety of dimension n. Then $H_k(U, \mathbf{Z}) = 0 = H^k(U, \mathbf{Z})$ for $k > n$.

Now, if $X \subset \mathbf{P}^N$ is a smooth projective variety of dimension n and $Y = \mathbf{P}^{N-1} \cap X$ is a smooth hyperplane section, $U := X \backslash Y$ is an affine variety of dimension n, $U \subset \mathbf{P}^N \backslash \mathbf{P}^{N-1} = \mathbf{C}^N$, and the Poincaré duality says that $H_k(U, \mathbf{Z}) = H^{2n-k}(X, Y, \mathbf{Z})$. The long exact sequence of relative cohomology for the pair (X, Y) and the corollary 1.3 now imply :

1.4. The weak Lefschetz theorem : The inclusion $j : Y \hookrightarrow X$ induces isomorphisms $j^* : H^k(X, \mathbf{Z}) \to H^k(Y, \mathbf{Z})$ for $k < n-1$ and an injection $j^* : H^{n-1}(X, \mathbf{Z}) \to H^{n-1}(Y, \mathbf{Z})$.

Let $\omega \in H^2(X, \mathbf{Z})$ be the Poincaré dual of the homology class of $Y \subset X$. Using Poincaré duality on X and Y, one obtains $j_* : H^k(Y) \to H^{k+2}(X)$, dual of $j^* : H^{2n-k-2}(X) \to H^{2n-k-2}(Y)$.

By the definition of ω it is then immediate that $L = L_X := j_* j^* : H^k(X) \to H^{k+2}(X)$ is equal to the cup-product with ω.

Identifying a neighbourhood of Y in X with a neighbourhood of the zero section of its normal bundle N in X, ω is also identified with the Thom class of N, and it is then a standard fact that $L_Y := j^* j_* : H^k(Y) \to H^{k+2}(Y)$ is equal to the cup-product with $\omega_{|Y}$. Now we have :

1.5. The hard Lefschetz theorem : For $k \leq n$, $L^{n-k} : H^k(X, \mathbf{Q}) \to H^{2n-k}(X, \mathbf{Q})$ is an isomorphism.

By the weak Lefschetz theorem and induction on the dimension it suffices to check it for $k = n-1$, as shown by the following diagram where both vertical maps are isomorphisms for $k \leq n-2$:

$$\begin{array}{ccc} H^k(X,\mathbb{Q}) & \xrightarrow{L^{n-k}} & H^{2n-k}(X,\mathbb{Q}) \\ j^* \downarrow & & \uparrow j_* \\ H^k(Y,\mathbb{Q}) & \xrightarrow[L^{n-k-1}]{} & H^{2n-k-2}(Y,\mathbb{Q}) \end{array}$$

For $k = n-1$, we still have the commutative diagram :

$$\begin{array}{ccc} H^{n-1}(X,\mathbb{Q}) & \xrightarrow{L} & H^{n+1}(X,\mathbb{Q}) \\ j^* \downarrow & & \uparrow j_* \\ H^{n-1}(Y,\mathbb{Q}) & \simeq & H^{n-1}(Y,\mathbb{Q}) \end{array}$$

where j^* is injective and its dual j_* is surjective. So via j^*, the kernel of L identifies with the kernel of the intersection form $<>_Y$ on $H^{n-1}(Y)$, restricted to $\operatorname{Im} j^* = (\operatorname{Ker} j_*)^\perp$. We have $\operatorname{Ker} j_* \subset \operatorname{Ker}(L_Y : H^{n-1}(Y,\mathbb{Q}) \to H^{n+1}(Y,\mathbb{Q}))$ and $\operatorname{Ker} j_*$ is a sub-Hodge structure of $\operatorname{Ker} L_Y$. The Hodge-Riemann bilinear relations to be explained below imply :

1.6. On $\operatorname{Ker} L_Y \cap H^{p,q}(Y)$, $p+q = n-1$, the hermitian form $< \varphi, \psi >_H = i^{p-q} \int_Y \varphi \wedge \overline{\psi}$ is non degenerate of a definite sign.

In particular it remains non degenerate of a definite sign on $\operatorname{Ker} j_* \cap H^{p,q}(Y)$, and it follows that $<>_Y$ is non degenerate on $\operatorname{Ker} j_*$, hence on $(\operatorname{Ker} j_*)^\perp$. So $L : H^{n-1}(X,\mathbb{Q}) \to H^{n+1}(X,\mathbb{Q})$ is injective, hence an isomorphism.

If one defines the primitive cohomology $H^k(X)^0$, for $k \leq n$ as the kernel of $L^{n-k+1} : H^k(X,\mathbb{Q}) \to H^{2n-k+2}(X,\mathbb{Q})$ the hard Lefschetz theorem implies now :

1.7. Lefschetz decomposition : The natural map $\varphi := \oplus L^k : \bigoplus_{m-k \leq n} H^{m-2k}(X)^0 \to H^m(X)$ is an isomorphism.

Notice that $L^k : H^{m-2k}(X)^0 \to H^m(X)$ is a morphism of Hodge structure, so the Lefschetz decomposition is in fact a decomposition of $H^m(X,\mathbb{Q})$ into a direct sum of primitive sub-Hodge structures.

Sketch of proof of 1.7. By the hard Lefschetz theorem, one may assume that $m \leq n$. Now one works by induction on m. If 1.7 is known for $m-2$, one considers $L^{n-m+1}: H^m(X) \to H^{2n-m+2}(X)$. Its kernel is by definition $H^m(X)^0$. Now $L^{n-m+2}: H^{m-2}(X) \to H^{2n-m+2}(X)$ is an isomorphism, which implies that L^{n-m+1} is surjective and that $H^m(X) \simeq H^m(X)^0 \oplus LH^{m-2}(X)$, so 1.7 is also true for m, by injectivity of $L: H^{m-2}(X) \to H^m(X)$.

The projection from $H^m(X)$ to $H^{m-2}(X)$ given by the isomorphism $H^m(X) \simeq H^m(X)^0 \oplus LH^{m-2}(X)$ is induced up to coefficient by the operator Λ of Hodge theory, which acts on forms (see §2). It is not known if it can be represented by an algebraic cycle of codimension $n-1$ in $X \times X$ (acting on $H^\bullet(X)$ via Poincaré duality and Künneth decomposition), although this is obviously the case for the operator L.

2. We turn now to the Hodge theoretic approach to the Lefschetz theorems. The weak and hard Lefschetz theorems will be obtained as a consequence of the Lefschetz decomposition, which in Kähler geometry exists at the level of forms.

Let X be a Kähler manifold of dimension n. Let ω be the Kähler form, and let $L: A^k(X) \to A^{k+2}(X)$ be the pointwise operator of multiplication by ω on complex k-forms on X. One defines the Hodge operator $*: A^k(X) \to A^{2n-k}(X)$ by $(\varphi, \psi)\,\mathrm{Vol} = \varphi \wedge *\psi$, where $\mathrm{Vol} = \frac{\omega^n}{n!}$ and $(,)$ is the induced pointwise hermitian metric on $A^k(X)$.

Let $\Lambda = *^{-1}L*$. Then obviously Λ is the formal adjoint of L for the hermitian metric $<,>$ defined on $A^k(X)$ by $<\varphi,\psi> = \int_M (\varphi,\psi)\,\mathrm{Vol}$.

One says that a form $\varphi \in A^k(X)$ is primitive if it satisfies $\Lambda\varphi = 0$.

One has then the following result of hermitian geometry, which comes from the computation of the commutator $[L,\Lambda]$ as a number operator ($[L,\Lambda]$ acts as multiplication by $p-n$ on p-forms).

2.1. Proposition : Any k-form $\varphi \in A^k(X)$ can be written uniquely as $\varphi = \sum_{r \geq \mathrm{Sup}(k-n,0)} L^r \varphi_{k-2r}$ where φ_{k-2r} is primitive of degree $k-2r$. There exists non commutative polynomials $P_{k,r}(L,\Lambda)$ with rational coefficients such that $\varphi_{k-2r} = P_{k,r}(L,\Lambda)(\varphi)$.

The existence of the Lefschetz decomposition 1.7 follows now from the following facts:

2.2. i) Let $\varphi \in A^k(X)$, $k \leq n$; then $\Lambda\varphi = 0$ is equivalent to $L^{n-k+1}\varphi = 0$.

ii) The operators L and Λ commute with the Laplacian operator Δ.

ii) implies that if φ is harmonic its primitive components φ_{k-2r} are also harmonic, so 2.1 and 2.2 i) imply the surjectivity in 1.7. The injectivity follows also from 2.1 : Suppose $0 = \Sigma L^r \varphi_{k-2r}$ in $H^k(X)$, with $L^{n-k+2r+1}\varphi_{k-2r} = 0$ in $H^{2n-k+2r+2}(X)$. One may assume that φ_{k-2r} is harmonic so $L^r\varphi_{k-2r}$ is also harmonic, and the equalities $0 = \Sigma L^r\varphi_{k-2r}$, $0 = L^{n-k+2r+1}\varphi_{k-2r}$ hold at the level of forms. By 2.2 i) and unicity in 2.1, one deduces $\varphi_{k-2r} = 0$.

2.3. On the forms $\varphi \in A^k(X)$, we have defined the hermitian inner product $<\varphi,\psi> = \int_X (\varphi,\psi)\,\mathrm{Vol} = \int_X \varphi \wedge *\psi$. For a non zero form φ, one has $<\varphi,\varphi> > 0$. One deduces from this :

2.4. Hodge-Riemann bilinear relations : Let $Q_k(\varphi,\psi) = (-1)^{k(k-1/2)} \int_X \varphi\wedge\psi\wedge\omega^{n-k}$, for $\varphi,\psi \in H^k(X)$, $k \leq n$. Then :

i) $Q_k(\varphi,\psi) = 0$ for $\varphi \in H^{p,q}(X), \psi \in H^{p',q'}(X)$ and $(p',q') \neq (q,p)$.

ii) $i^{(p-q)}Q_k(\varphi,\overline{\varphi}) = (-1)^{k(k-1)/2} i^{(p-q)} \int_X \varphi \wedge \overline{\varphi} \wedge \omega^{n-k} > 0$, for $\varphi \in H^{p,q}(X)$, $\varphi \neq 0$, and φ primitive.

This follows from the following fact of hermitian geometry :

2.5. If $\varphi \in A^{p,q}(X)$ is primitive $(p+q = k \leq n)$, one has the equality $*\varphi = (-1)^{k(k-1)/2}$ $i^{(p-q)}\frac{1}{(n-k)!}L^{n-k}\overline{\varphi}$.

Of course, to deduce 2.4 of 2.5, one uses the fact that if φ is a primitive cohomology class, its harmonic representative is a primitive form, by 2.2 ii).

The consequences of 2.4 depend on the parity of k :

A) If k is even, 2.4 describes completely the indices of the real symmetric intersection forms Q_k on $H^k(X)$. In particular if n is even one can deduce from 2.4 and 1.7 (cf. [5], p. 78) the following :

2.6. Hodge index theorem : The topological intersection form $Q(\varphi,\psi) = \int_X \varphi \wedge \psi$ on $H^n(X)$ for n even has for index $i(X) = \sum_{(a,b)} (-1)^a h^{a,b}$.

Let us also mention the following consequence of 2.4, which is very important for the consistency of the Hodge conjecture, when applied to the monodromy groups.

2.7. Let X be an algebraic variety and ω the class of an ample line bundle on X. For each integer p consider $Hdg^{2p}(X) := H^{2p}(X,\mathbf{Z}) \cap H^{p,p}(X)$. Then the group of automorphisms of $H^{2\cdot}(X,\mathbb{Z})$ preserving the forms Q_k, the operator L and the subspace $Hdg^{2\cdot}(X) \subset H^{2\cdot}(X,\mathbb{Z})$ acts as a finite group on $Hdg^{2\cdot}(X)$.

B) When k is odd, as explained in the first lecture, one can interpret 2.4 as the positivity condition necessary to polarize certain complex tori constructed from the Hodge decomposition on $H^k(X)^0$. These tori are defined as the quotients $F/H^k(X,\mathbb{Z})^0$ where $F = \sum_{\substack{p+q=k \\ p=q+1(4)}} H^{p,q}(X)^0$ is considered as a quotient of $H^k(X,\mathbb{C})^0$. They are the Weil intermediate jacobians, which unfortunately do not in general vary holomorphically with X ([2]).

In contrast, the Griffiths intermediate jacobians (see M. Green's lectures) vary holomorphically with X, but are only complex tori, having a line bundle with non degenerate indefinite curvature, the signs of which are determined by 2.4. These two tori contain the "algebraic part of the intermediate jacobian" which is the complex torus $L^{\ell,\ell+1}/L_{\mathbb{Z}}^{2\ell+1}$, where $L \subset H^{2\ell+1}(X,\mathbb{Z})^0$ is maximal to satisfy : $L_{\mathbb{C}} = L_{\mathbb{C}} \cap F^{l+1}H^{2\ell+1}(X,\mathbb{C})^0 \oplus \overline{L_{\mathbb{C}} \cap F^{l+1}H^{2\ell+1}(X,\mathbb{C})^0}$. This torus is an abelian variety polarized by $Q_{2\ell+1}$ according to 2.4 and (Lecture one, 2.6 - 2.7).

3. To conclude this lecture we mention the following applications of the Lefschetz decomposition and the Hodge-Riemann bilinear relations :

The first one is due to Deligne [1] :

3.1. Theorem : Let $f : \mathcal{X} \to B$ be a smooth and projective morphism; then the Leray spectral sequence

$$H^p(R^q f_*\mathbb{C}) \Rightarrow H^{p+q}(X,\mathbb{C}) \text{ degenerates at } E_2.$$

The point is that a relatively ample line bundle will give a flat Lefschetz decomposition on the flat bundle $R^q f_*\mathbb{C}$. The operators $L^k : R^q f_*\mathbb{C} \to R^{q+2k} f_*\mathbb{C}$ are compatible with d_2, and it suffices to check the vanishing of d_2 on the primitive cohomology : but if n is the dimension of the fibers, one has the following diagram :

$$\begin{array}{ccc} H^p((R^q f_*\mathbb{C})^0) & \xrightarrow{L^{n-q+1}} & H^p(R^{2n-q+2} f_*\mathbb{C}) \\ \downarrow d_2 & & \downarrow d_2 \\ H^{p+2}(R^{q-1} f_*\mathbb{C}) & \xrightarrow{L^{n-q+1}} & H^{p+2}(R^{2n-q+1} f_*\mathbb{C}) \end{array}$$

Now the first L^{n-q+1} is zero and the second one is an isomorphism, which implies that the first d_2 is zero. The vanishing of the other d_r's is proved in the same way.

For the second application, let us say that a Hodge structure over $\mathbf{Q}(H^k_{\mathbf{Q}}, H^k_{\mathbf{C}} = \bigoplus_{p+q=k} H^{p,q})$ is polarized if there is a bilinear form Q on $H^k_{\mathbf{Q}}$ symmetric for k even, skew for k odd, defined over $\mathbf{Q}$ and satisfying the conditions 2.4.

Then one has :

3.2. Let H be a polarized Hodge structure and let $L \subset H$ be a sub-Hodge structure. Then L is a direct factor in H, i.e. there exists a sub-Hodge structure $L' \subset H$ such that $L \oplus L' = H$.

It suffices to note that Q is non degenerate on $L_{\mathbf{Q}}$ because of the definiteness of $i^{(p-q)}Q(\varphi, \overline{\varphi})$ on $H^{p,q}$ and that $L^{\perp} := L'$ is also a sub-Hodge structure of H by the first condition in 2.4.

References :

[1] P. Deligne, Théorèmes de Lefschetz et critères de dégénérescence de suites spectrales, Publ. Math. IHES, 35 (1968) 107-126.

[2] P. Griffiths, Periods of integrals on algebraic manifolds I, Amer. J. Math. 90 (1968), 568-626.

[3] S.L. Kleiman, Algebric cycles and the Weil conjectures. Dans : Dix exposés sur la cohomologie des schémas, North-Holland, Amsterdam (1963) 359-386.

[4] J. Milnor, Morse theory, Part I, Princeton University Press (1963), Annals of Mathematics Studies, number 51.

[5] A. Weil, Variétés kählériennes, chapitres I et II, Actualités scientifiques et industrielles 1267, Hermann, seconde édition 1971.

Lecture 3 : Noether-Lefschetz loci

0. This lecture is devoted to the Noether-Lefschetz loci associated with a variation of Hodge structure, that is the loci where an integral class is a Hodge class. The name is borrowed from the case of a variation of Hodge structure given by the H^2 of a family of varieties, where by the Lefschetz theorem on $(1,1)$ classes these loci can also be defined as the sets of points in the family corresponding to varieties having a line bundle with prescribed first Chern class. It also refers to the celebrated Noether-Lefschetz theorem, proved rigorously by Lefschetz, which says that a general surface in $\mathbf{P}^3$ of degree at least four carries no line bundle other than the multiples of $\mathcal{O}(1)$.

For Hodge structures of higher degree, such interpretation depends on the Hodge conjecture, and conversely one could hope that the compared study of the deformation of subschemes and the deformation of the associated Hodge classes would lead to a proof of the "variational Hodge conjecture" [7]. In fact, excepted for particular situations (special cycles in hypersurfaces, Lagrangian subvarieties in symplectic manifolds) this study is very difficult to carry out, and the only general criterion, the notion of semi-regularity due to Bloch [1], subsequently refined by Ran [6], cannot be checked in general. (That is, general principles do not give the existence of semi-regular representative of a Hodge class, even modulo an ample class, excepted in the case of divisors). We start with the Hodge theoretic local description of the Noether-Lefschetz loci, and continue with the geometric description of the deformation theory of the Hodge class associated to a subscheme [1]. We also introduce some infinitesimal invariants of a Hodge class and explain its relations with geometry, following [2]. Finally, we turn to the case of divisors, explain shortly some recent results in Noether-Lefschetz theory for surfaces in $\mathbf{P}^3$, and conclude with the proof of the infinitesimal criterion of M. Green ([3], [4]) which will be used as an existence criterion for one cycles in threefolds in lecture 8.

1. Let U be a connected and simply connected complex space and let $H^{2k}_{\mathbf{Z}}$, $\mathcal{H}^{2k}$, $F^{\ell}\mathcal{H}^{2k}$ be a variation of Hodge structure on U. So we have $\mathcal{H}^{2k} = H^{2k}_{\mathbf{Z}} \otimes \mathcal{O}_U$, a flat bundle with the Gauss-Manin connection $\nabla : \mathcal{H}^{2k} \to \mathcal{H}^{2k} \otimes \Omega_U$, such that $\nabla H^{2k}_{\mathbf{Z}} = 0$, and $F^{\ell}\mathcal{H}^{2k}$ is a decreasing filtration by holomorphic subbundles satisfying $\nabla F^{\ell}\mathcal{H}^{2k} \subset F^{\ell-1}\mathcal{H}^{2k} \otimes \Omega_U$. Now the local system $H^{2k}_{\mathbf{Z}}$ is trivial on U, so one can identify it with the group of its global sections and with its fiber at any point (notation $\lambda \to \lambda_t,\ t \in U$).

One says that $\lambda_t \in H^{2k}_{\mathbf{Z}(t)}$ is a Hodge class if it belongs to $F^k H^{2k}_{(t)}$.

For any $\lambda \in H^{2k}_{\mathbf{C}} := H^{2k}_{\mathbf{Z}} \otimes \mathbf{C}$, one defines $U_\lambda \subset U$ by $U_\lambda = \{t \in U / \lambda_t \in F^k\mathcal{H}^{2k}_{(t)}\}$. One has then :

1.1. Lemma : U_λ is analytic and can be defined locally by $h^{k-1,k+1}$ holomorphic equations, where $h^{k-1,k+1} = \operatorname{rank} \mathcal{H}^{k-1,k+1}$, $\mathcal{H}^{k-1,k+1} = F^{k-1}/F^k\mathcal{H}^{2k}$.

The fact that U_λ is analytic is clear, because U_λ is defined by the vanishing of the projection of the flat, hence holomorphic section $\lambda \in \mathcal{H}^{2k}$ in the quotient $\mathcal{H}^{2k}/F^k\mathcal{H}^{2k}$.

To prove the second statement, one uses transversality : One chooses $\mathcal{G} \subset \mathcal{H}^{2k}$ such that $\mathcal{G} \oplus F^{k-1}\mathcal{H}^{2k} = \mathcal{H}^{2k}$. One has then a holomorphic projection $\pi : \mathcal{H}^{2k} \to F^{k-1}\mathcal{H}^{2k}$. Let $0 \in U_\lambda$ and let U'_λ be defined in a neighbourhood of 0 by the vanishing of $\pi(\lambda)$ in the quotient $F^{k-1}/F^k\mathcal{H}^{2k}$. In U'_λ, λ belongs to $\mathcal{G} \oplus F^k\mathcal{H}^{2k}$, so $\lambda = \lambda_1 + \lambda_2$, $\lambda_1 \in F^k\mathcal{H}^{2k}, \lambda_2 \in \mathcal{G}$.

Now $\mathcal{G}$ has an induced connection $\nabla_{\mathcal{G}} = \pi_{\mathcal{G}} \circ \nabla$, and by transversality, one finds on U'_λ :

$$\nabla \lambda_1 \in F^{k-1}\mathcal{H}^{2k} \otimes \Omega_{U'_\lambda}, \ \nabla\lambda = 0 \Rightarrow \nabla_{\mathcal{G}}\lambda_2 = 0.$$

Because $\lambda_2 = 0$ at 0, it follows that $\lambda_2 = 0$ on a neighbourhood of 0 in U'_λ, hence that $U'_\lambda = U_\lambda$ in this neighbourhood .

Now, if one wants to extend these definitions to the case of a non simply connected U and preserve the fact that the U_λ are analytic, one has to work with polarized Hodge structures and restrict the definition to the case of an integral class λ. Otherwise, it could happen that the monodromy on λ along U_λ produces infinitely many new classes λ' in such a way that the analytic space U_λ has in fact infinitely many branches U'_λ at one point. This situation is excluded for λ integral, and for polarized Hodge structures, by the finiteness of the monodromy action on the set of Hodge classes, mentioned in lecture 2.

1.2. Now we want to describe at finite orders near 0 the subscheme U_λ. Suppose we have a map $S_m \underset{j_m}{\mapsto} U_\lambda$, where $S_m = \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^m$, such that $j_m(0) = 0$, and let $S_m \subset S_{m+1} \underset{j_{m+1}}{\mapsto} U$ be an extension of j_m to a map with value in U. Then $j_{m+1}(S_{m+1}) \subset U_\lambda$ if and only if the flat class λ remains in $F^k\mathcal{H}^{2k}$ on S^{m+1}. Now let $\tilde{\lambda}$ be a holomorphic section of $j^*_{m+1}F^k\mathcal{H}^{2k}$ extending λ (by assumption, on S_m, λ is a section of $F^k\mathcal{H}^{2k}$). Then because λ is flat on S_m, $\nabla\tilde{\lambda} = \epsilon^{m-1}d\epsilon\,\varphi$ where φ is in $F^{k-1}\mathcal{H}^{2k}_{(0)}$ by transversality. $\tilde{\lambda}$ can be modified by an arbitrary section of the form $\epsilon^m\psi$, hence $\nabla\tilde{\lambda}$ can be modified by an arbitrary element of the form $\epsilon^{m-1}d\epsilon\,\psi$, where $\psi \in F^k\mathcal{H}^{2k}_{(0)}$. So the image $0_{m+1}(\lambda)$ of φ in $F^{k-1}\mathcal{H}^{2k}_{(0)}/F^k\mathcal{H}^{2k}_{(0)} = \mathcal{H}^{k-1,k+1}_0$ does not depend on $\tilde{\lambda}$, and one has : $j_{m+1}(S_{m+1}) \subset U_\lambda \iff$ there exists an extension $\tilde{\lambda} \in j^*_{m+1}F^k\mathcal{H}^{2k}$ which is flat $\iff 0_{m+1}(\lambda) = 0$ in $\mathcal{H}^{k-1,k+1}_{(0)}$.

In particular, for $m = 1$, we have described the Zariski tangent space of U_λ at 0 :

1.3. Lemma : $TU_{\lambda(0)} = \operatorname{Ker} \overline{\nabla}(\lambda^{k,k})$, where $\overline{\nabla} : \mathcal{H}^{k,k}_{(0)} \to \operatorname{Hom}(TU_{(0)}, \mathcal{H}^{k-1,k+1}_{(0)})$ is the linear map constructed by transversality from the Gauss-Manin connection :

$$\begin{array}{cccc}
\nabla : & F^{k+1}\mathcal{H}^{2k} & \to & F^k\mathcal{H}^{2k} \otimes \Omega_U \\
 & \downarrow & & \downarrow \\
\nabla : & F^{k}\mathcal{H}^{2k} & \to & F^{k-1}\mathcal{H}^{2k} \otimes \Omega_U \\
 & \downarrow & & \downarrow \\
\overline{\nabla} : & \mathcal{H}^{k,k} & \to & \mathcal{H}^{k-1,k+1} \otimes \Omega_U
\end{array}$$

and $\lambda^{k,k} \in \mathcal{H}^{k,k}_{(0)}$ is the projection of the class λ, which is by assumption in $F^k\mathcal{H}^{2k}$ at 0.

The lemma follows from the equality $\overline{\nabla}(\lambda)(j_{2*}\partial/\partial\epsilon) = 0_2(\lambda)$, which is clear by the description of $0_2(\lambda)$ given above. Finally, from lemmas 1.1 and 1.3, we deduce :

1.4. Lemma : If $\overline{\nabla}(\lambda^{k,k})$ is surjective at $0 \in U_\lambda$, U_λ is smooth of codimension $h^{k-1,k+1}$ at 0.

2. The case where λ is the Hodge class of subvariety :

2.1. We assume now that we have a family of smooth complex varieties $\mathcal{X} \underset{\pi}{\to} U$, and we consider the associated variation of Hodge structures : $H^{2k}_{\mathbf{Z}} = R^{2k}\pi_*\mathbf{Z}$, $F^\ell/F^{\ell+1}\mathcal{H}^{2k} = R^{2k-\ell}\pi_*(\Omega^\ell_{\mathcal{X}/U})$. Let $Z_0 \subset X_0$ be a smooth subvariety (Bloch works with $\ell.c.i$ subschemes). Let $\lambda \in H^{2k}(X_0, \mathbf{Z}) \cap H^{k,k}(X_0, \mathbf{Z})$ be the associated Hodge class. We want to compare the deformation theory of the pair (Z_0, X_0) and the subscheme U_λ defined in 1). Let us assume that for $S_m \underset{j_m}{\to} U$, as in 1.2, we could extend Z_0 to a subvariety

$$\begin{array}{ccc}
Z_m & \hookrightarrow & \mathcal{X}_m \\
\pi_2 \searrow & & \swarrow \pi \\
 & S_m &
\end{array}$$

then this implies that $j_m(S_m) \subset U_\lambda$; to see this we have to show that the flat extension λ_m of λ is in $F^k\mathcal{H}^{2k}{}_{|S_m}$. But if n is the dimension of the fibers X_t, $\mathcal{H}^{2k}$ is dual of $\mathcal{H}^{2n-2k}$ and it is easy to see that the flat extension λ_m of λ, as an element of $\mathcal{H}^{2n-2k*}$ is given by the restriction map $\mathcal{R}^{2n-2k}\pi_*(\Omega^\bullet_{\mathcal{X}_m/S_m}) \to \mathcal{R}^{2n-2k}\pi_*(\Omega^\bullet_{Z_m/S_m}) = \mathcal{R}^{n-k}\pi_*(K_{Z_m/S_m}) = \mathcal{O}_{S_m}$, which implies clearly that it annihilates $F^{n-k+1}\mathcal{H}^{2n-2k}{}_{|S_m} = \mathcal{R}^{2n-2k}\pi_*F^{n-k+1}(\Omega^\bullet_{\mathcal{X}_m/S_m})$.

2.2. If we have an extension $j_{m+1} : S^{m+1} \to U$ of j_m as in 1) we have now the obstruction $0_{m+1}(\lambda) \in \mathcal{H}_{(0)}^{k-1,k+1}$ constructed in 1.2, and the obstruction $0'_{m+1}(Z_m) \in H^1(N_{Z_0})$ to extend Z_m to a subvariety $Z_{m+1} \subset \mathcal{X}_{m+1}$, where N_{Z_0} is the normal bundle of Z_0 in X_0. The obstruction $0'_{m+1}(Z_m)$ can be defined as the extension class of the exact sequence :

$$0 \to \mathcal{O}_{Z_0} \to J_m^{\mathcal{X}_{m+1}} \otimes \mathcal{O}_{Z_0} \to \underbrace{J_m^{\mathcal{X}_m} \otimes \mathcal{O}_{Z_0}}_{\| \; N_Z^*} \to 0$$

where $J_m^{\mathcal{X}_{m+1}}, J_m^{\mathcal{X}_m}$ are ideal sheaves of Z_m in $\mathcal{X}_{m+1}$ and $\mathcal{X}_m$ respectively. More analytically the deformation $\mathcal{X}_{m+1}$ can be represented by a form $\alpha = \sum_0^m \epsilon^i \alpha_i$ where $\alpha_i \in A^{0,1}(T_{X_0}^{1,0})$, (that is α_i is a $(0,1)$ form on X_0 with value in the bundle $T_{X_0}^{1,0}$ of $(1,0)$ vector fields), and α has to satisfy up to order m, the integrability condition : $(*)$ $\overline{\partial}\alpha - [\alpha, \alpha] = 0$. Because $Z_m \subset \mathcal{X}_m$ is analytic, we may assume that up to order $m-1, \alpha_{|Z_0}$ is in $A^{0,1}(Z_0)(T_{Z_0}^{1,0})$. Then $(*)$ together with the fact that $\alpha_0 = 0$ shows that the image of $\alpha_{m|Z_0}$ in $A^{0,1}(Z_0)(N_{Z_0})$ is $\overline{\partial}$ closed, hence has a class in $H^1(N_{Z_0})$, which gives the obstruction $0'_{m+1}(Z_m)$.

To compare $0'_{m+1}(Z_m)$ and $0_{m+1}(\lambda)$, Bloch introduces the semi-regularity map γ : $H^1(N_{Z_0}) \to H^{k+1}(\Omega_{X_0}^{k-1}) = \mathcal{H}_{(0)}^{k-1,k+1}$ which is defined as the dual of the composite :

2.2.1. $\quad H^{n-k-1}(\Omega_{X_0}^{n-k+1}) \to H^{n-k-1}(Z_0, \Omega_{X_0}^{n-k+1}{}_{|Z_0}) \to H^{n-k-1}(Z_0, K_{Z_0} \otimes N_{Z_0}^*)$

where the last map is induced by the exact sequence : $0 \to N_{Z_0}^* \to \Omega_{X_0|Z_0} \to \Omega_{Z_0} \to 0$, which gives to the $n-k+1^{th}$ power :

$$\Omega_{X_0}^{n-k+1} \to K_{Z_0} \otimes N_{Z_0}^*.$$

Bloch shows : [1].

2.3. Theorem : One has the relation $0_{m+1}(\lambda) = \gamma(0'_{m+1}(Z_m))$ in $H^{k-1,k+1}(X_0)$.

The proof of Bloch is purely algebraic : an analytic proof for smooth Z_0 can be sketched as follows : as an element of $H^{n-k+1,n-k-1}(X_0)^*$ the obstruction $0_{m+1}(\lambda)$ has the following description : Let $\varphi \in F^{n-k+1}H^{2n-2k}(X_0)$ and let $\tilde{\varphi} = \sum_0^m \epsilon^i \varphi_i$ represents a holomorphic section of $j_{m+1}^*(F^{n-k+1}\mathcal{H}^{2n-2k})$ extending φ, where φ_k are $2n-2k$ closed forms on X_0, such that $\Sigma\epsilon^i\varphi_i$ is in $F^{n-k+1}A^{2n-2k}$ up to order m for the complex structure defined by α (cf. 2.2); then

2.3.1. $\quad O_{m+1}(\lambda)(\varphi) = \int_{Z_0} \varphi_m.$

Now using the fact that $\alpha = \sum_0^m \epsilon^i \alpha_i$ with $\alpha_{i|z_0} \in A^{0,1}(Z_0)(T^{1,0}_{Z_0})$ for $i < m$, one finds that $\varphi_{i|z_0} = 0$ for $i < m$ and that :

2.3.2. $\varphi_{m|z_0} = \int \alpha_m(\varphi_0)_{|z_0}$.

It is then clear by 2.3.1, 2.3.2 , 2.2.1 and the description of $0'_{m+1}(Z_m)$ as the projection of $\alpha_{m|z'_0}$ that $(0_{m+1}(\lambda)(\varphi) = \int_{Z_0} \varphi_m = \int_{Z_0} \operatorname{Int} \alpha_m(\varphi_m) =< \gamma^*(\varphi),\ 0'_{m+1}(Z_m) >$, which is the contents of the theorem.

As a consequence of theorem 2.3 one has now :

2.4. Theorem : If γ is injective, the locus U_λ coincide schematically with the image in U of the space of deformations of the pair (Z_0, X_0).

One says that Z is semi-regular in this case, because this is the generalization of the notion of regularity $(H^1(N_{Z_0}) = 0)$ which implies the stability of Z_0 under small deformations of X_0 [5].

2.5. We want to explain now the construction of invariants of Hodge classes [2], which are an attempt to recover information on the ideal of a subscheme from the infinitesimal behaviour of its Hodge class.

Consider $Z_0 \subset X_0$, a subvariety of codimension k. Associated to a (maybe infinitesimal) deformation $\mathcal{X} \to U$ of X_0, we have the map $\varphi = \oplus\varphi_p : TU_{(0)} \to \oplus \operatorname{Hom}(H^{p,q}(X_0), H^{p-1,q+1}(X_0))$ describing the infinitesimal variation of Hodge structure of X_0. These maps can be iterated and by the flatness of the Gauss-Manin connection, we obtain that the ℓ^{th} iterations $\varphi_p(U_1) \circ \ldots \circ \varphi_{p+\ell-1}(U_\ell) : H^{p+\ell-1,2n-2k-p-\ell+1}(X_0) \to H^{p-1,2n-2k-p+1}(X_0)$ is symmetric in $U_1, \ldots, U_\ell$. So we obtain in particular a map :

$$\psi^\ell : H^{n-k+\ell,n-k-\ell}(X_0) \to \operatorname{Hom}(S^\ell TU_{(0)}, H^{n-k,n-k}(X_0)).$$

The invariants that Griffiths and Harris associate to $\lambda = [Z] \in H^{k,k}(X_0)$ is then :

2.5.1. $H^{n-k+\ell,n-k-\ell}(-\lambda) = \{\omega \in H^{n-k+\ell,n-k-\ell}(X_0)/\forall V \in S^\ell TU_{(0)}, \psi^\ell(\omega)(V).\lambda = 0\}$.

Now the relation with the geometry is given by :

2.6. Lemma : The image of $H^{n-k-\ell}(\Omega^{n-k+\ell}_{X_0} \otimes I_{Z_0})$ in $H^{n-k-\ell}(\Omega^{n-k+\ell}_{X_0})$ is contained in $H^{n-k+\ell,n-k-\ell}(-\lambda)$.

The lemma follows from the description of $\psi^\ell(\omega)(V)$ as the interior product of ω by the image of $V = U_1 \otimes \ldots \otimes U_\ell$ in $H^\ell(\Lambda^\ell T_{X_0})$, where we use the Kodaira-Spencer map

$TU_{(0)} \to H^1(T_{X_0})$: It is clear then that for $\omega \in H^{n-k-\ell}(\Omega_{X_0}^{n-k+\ell} \otimes I_{Z_0}), \psi_\ell(\omega)(V) \in H^{n-k}(\Omega_{X_0}^{n-k} \otimes I_{Z_0})$ which integrates to zero on Z_0.

One can find examples of cycles in hypersurfaces for which equality holds in 2.6.

In the case of a class $\lambda \in H^2(S,\mathbf{Z}) \cap H^{1,1}(S)$ on a surface S, the space $H^{2,0}(-\lambda) \subset H^0(K_S)$ is made of sections of the canonical bundle and it was believed for a time that for surfaces in $\mathbf{P}^3$, if one has a reduced component $\mathcal{S}_\lambda$ of the Noether-Lefschetz locus, along which the space $H^{2,0}(-\lambda)$ is non zero, the class λ is supported on the divisor of a form ω in $H^{2,0}(-\lambda)$. This was proved in [8] for degree less than seven, and disproved in general in [9].

3. We finish this lecture with the proof of a very important lemma, due to M. Green, which gives an infinitesimal and purely algebraic criterion for the existence and density of the Noether-Lefschetz locus for divisors. We will work in the geometric setting although the proof works for any VHS of weight two.

So let $\mathcal{S} \xrightarrow{\pi} U$ be a family of smooth projective varieties parametrized by a smooth and connected basis U, and consider the associated variation of Hodge structure : $H^2_{\mathbf{Z}} = R^2\pi_*\mathbf{Z}, F^2\mathcal{H}^2 = \mathcal{H}^{2,0} \subset F^1\mathcal{H}^2 \subset \mathcal{H}^2 = H^2_{\mathbf{Z}} \otimes \mathcal{O}_U$, with $\mathcal{H}^{1,1} = F^1\mathcal{H}^2/F^2\mathcal{H}^2, \mathcal{H}^{0,2} = \mathcal{H}^2/F^1\mathcal{H}^2$, and the corresponding infinitesimal variation of Hodge structure :

3.1.

$$\begin{array}{lccc}
\nabla : & F^2\mathcal{H}^2 & \to & F^1\mathcal{H}^2 \otimes \Omega_U \\
& \downarrow & & \downarrow \\
\nabla : & F^1\mathcal{H}^2 & \to & \mathcal{H}^2 \otimes \Omega_U \\
& \downarrow & & \downarrow \\
\overline{\nabla} : & \mathcal{H}^{1,1} & \to & \mathcal{H}^{0,2} \otimes \Omega_U \\
& \downarrow & & \downarrow \\
& 0 & & 0
\end{array}$$

For $\lambda \in \mathcal{H}^{1,1}_{(0)} = H^1(\Omega_{S_0})$, we have the map :

$$\overline{\nabla}(\lambda) : TU_{(0)} \to \mathcal{H}^{0,2}_{(0)} = H^2(\mathcal{O}_{S_0}).$$

We define the $\mathcal{C}^\infty$ bundle $\mathcal{H}^{1,1}_{\mathbf{R}}$ as the real part of the bundle $\mathcal{H}^{1,1}$; more precisely, let $\mathcal{H}^2_{\mathbf{R}} = H^2_{\mathbf{R}} \otimes \mathcal{C}^\infty_{\mathbf{R}}(U)$ and $\mathcal{H}^{1,1}_{\mathbf{R}} = \mathcal{C}^\infty F^1\mathcal{H}^2 \cap \mathcal{H}^2_{\mathbf{R}}$. Then $\mathcal{H}^{1,1}_{\mathbf{R}}$ is the sheaf of sections of the real vector bundle on U with fiber $H^{1,1}_{\mathbf{R}}(S_t) = H^{1,1}(S_t) \cap H^2(S_t, \mathbf{R})$. We shall use the notation $H^{1,1}_{\mathbf{R}}$ for the total space of this vector bundle.

We have then :

3.2. Lemma (M. Green) : Assume that there exists $\lambda \in H^1(\Omega_{S_0})$ such that $\overline{\nabla}(\lambda) : TU_{(0)} \to H^2(\mathcal{O}_{S_0})$ is surjective. Then the set $\{(t,\lambda)/\lambda \in H^2(S_t, \mathbf{Q}) \cap H^{1,1}(S_t)\}$ is dense in $H^{1,1}_{\mathbf{R}}$.

Proof : The assumption is a Zariski open property on the algebraic vector bundle $\mathcal{H}^{1,1}$ on U so if it satisfied at 0, it will also be satisfied in a Zariski open set of U; furthermore if it is satisfied by $\lambda \in H^1(\Omega_{S_0})$ it is also satisfied by λ in a Zariski open dense subset of $H^{1,1}_{\mathbf{R}}(S_0)$, because $H^{1,1}(S_0) = H^{1,1}_{\mathbf{R}}(S_0) \otimes \mathbf{C}$. So we have only to prove :

3.2.1. Let $\lambda \in H^{1,1}_{\mathbf{R}}(S_0)$, satisfying the condition : $\overline{\nabla}(\lambda) : TU_{(0)} \to H^2(\mathcal{O}_{S_0})$ is surjective; then there exists a sequence (t_n, λ_n) with $\lambda_n \in H^2(S_n, \mathbf{Q}) \cap H^{1,1}(S_{t_n})$, and (t_n, λ_n) converges to $(0, \lambda)$ in $H^{1,1}_{\mathbf{R}}$.

In a neighbourhood of 0, the local system $H^2_{\mathbf{Z}}$ is trivial, hence we have a flat trivialization of the bundle $\mathcal{H}^2$. this gives a diagram of holomorphic maps :

$$\begin{array}{ccc} F^1H^2 & \overset{P_1}{\to} & H^2(S_0, \mathbf{C}) \\ \searrow & & \nearrow P \\ & H^2 & \end{array} ;$$

where F^1H^2 is the total space of $F^1\mathcal{H}^2$, and H^2 is the total space of $\mathcal{H}^2$. P_t gives an isomorphism $H^2(S_t, \mathbf{C}) \simeq H^2(S_0, \mathbf{C})$ which preserves the rational structure, and by definition of a flat trivialization the Gauss-Manin connection is described by $P(\nabla\sigma) = dP(\sigma)$ for a section $\sigma : U \to H^2$ of the bundle $\mathcal{H}^2$.

It follows from this and the definition of $\overline{\nabla}$, that the condition "$\overline{\nabla}(\lambda)$ surjective " is equivalent to : The map P_1 is a submersion at $\lambda \in F^1H^2_{(0)} = F^1H^2(S_0)$. Now λ being real, $P_1(\lambda) \in H^2(S_0, \mathbf{R})$, so it can be approximated by $\lambda_n \in H^2(S_0, \mathbf{Q})$. P_1 being submersive at λ, one can find t_n, such that $\lim_{n\infty} t_n = 0$, and $\widetilde{\lambda}_n \in F^1H^2_{(t_n)}, \lim_{n\infty} \widetilde{\lambda}_n = \lambda$, such that $P_1(\widetilde{\lambda}_n) = \lambda_n$ Then $\widetilde{\lambda}_n \in F^1H^2(S_{t_n}) \cap H^2(S_{t_n}, \mathbf{Q})$, because the flat trivialization preserves the rational structure. So the lemma is proved.

3.3. One should notice that this criterion does not work for higher weight variation of Hodge structures $H^{2k}_{\mathbf{Z}}$, with $F^{k+2}\mathcal{H}^{2k} \neq 0$. The point is that one can analogously construct the holomorphic map $P_k : F^k H^{2k} \to H^{2n}(X_0, \mathbf{C})$ but by transversality it will not be submersive at any point when $F^{k+2}\mathcal{H}^{2k} \neq 0$.

This suggests that the Noether-Lefschetz locus is not dense in general, and may be not dense, for example, for four dimensional hypersurfaces of $\mathbf{P}^5$ of degree at least six ($h^{4,0} \neq 0$), but I don't know how to prove this.

References :

[1] S. Bloch : Semi-regularity and De Rham cohomology, Inventiones Math. 17, S1. 66 (1972).

[2] P. Griffiths-J. Harris : Infinitesimal variations of Hodge structure II : An infinitesimal invariant of Hodge classes, Compositio Math. Vol 50 (1985) 207-265.

[3] C. Giliberto, J. Harris, R. Miranda : General components of the Noether-Lefschetz locus and their density in the space of all surfaces, Math. Ann. 282 (1988) 666-680.

[4] S.0. Kim : Noether-Lefschetz locus for surfaces, Trans. Amer. Math. Soc. 1 (1991).

[5] K. Kodaira : On stability of compact submanifolds of complex manifolds, Amer. J. Math. Vol. 85 (1963) 79-94.

[6] Z. Ran : Hodge theory and the Hilbert scheme, J. of Differential Geom. 37 (1993) 191-198.

[7] J. Steenbrink : Some remarks about the Hodge conjecture, in *Hodge theory* proceedings Sant Cugat, Lecture Notes in Math. n° 1246, 165-175, Springer Verlag.

[8] C. Voisin : Sur le lieu de Noether-Lefschetz en degré 6 et 7, Compositio Mathematica 75 (1990) 47-68.

[9] C. Voisin : Contrexemple à une conjecture de J. Harris, C.R. Acad; Sci. Paris t. 313, Série I, 685-687 (1991).

Lecture 4. Monodromy

In this lecture we describe the theory of Lefschetz degenerations from the point of view of topology and Hodge theory. This theory is fully understood and modern developments of (mixed) Hodge theory have produced more general results concerning the degenerations of algebraic varieties. In fact the degenerations that have been considered to give a unified treatment of monodromy and asymptotic Hodge theory are the degenerations to a normal crossing variety, which are, by the semi-stable reduction theorem of Mumford, the most general ones, up to base change. We refer to [10] for an efficient survey of the general results.

However, for the applications we have in mind, the Lefschetz theory has the advantage of giving an explicit description of the middle homology of a variety and of the monodromy action on the middle homology of its hyperplane sections. Furthermore, normal crossing varieties which have been used to give supplementary results in the theory of algebraic cycles ([4], [5], [12]) don't have triple intersections, so behave locally as a Lefschetz degeneration with parameters. We refer to [4] for a full treatment of the generalized Picard-Lefschetz formula and of the degeneration of Hodge structure in this case.

We begin with the description of the vanishing cycle associated to a node, and sketch the proof of the Picard-Lefschetz formula. We apply this to the Lefschetz description of the vanishing homology of X and Y, where $Y \subset X$ is a hyperplane section. We give then several applications of the Picard-Lefschetz formula. We conclude with a concrete (but non rigorous) description of the behaviour of the Hodge filtration near a Lefschetz degeneration, in the spirit of [7], [9], and its relations with algebraic cycles : The limit Hodge class of the vanishing cycle in even dimension, the generalized intermediate jacobian and its extension class in odd dimension. We refer to [2], [4] for a rigorous treatment, and for applications of this last point.

1. Let $V \subset \mathbf{C}^n$ be a neighbourhood of 0 and let $f : U \to \mathbf{C}$ be a holomorphic map, such that $df(x) \neq 0$ for $x \neq 0$, and $df(0) = 0$, but $\mathrm{Hess}_0 f := \Sigma \frac{\partial^2 f}{\partial z_i \partial z_j} dz_i dz_j$ is a non degenerate symmetric bilinear form. Then the holomorphic Morse lemma says that shrinking U if necessary, there exists holomorphic coordinates u_i centered at zero such that :

1.1.1. $\quad f = \Sigma u_i^2.$

Now we assume that U is a closed ball of radius 1 and that f is given by (1.1.1). Consider for $|t| < 1$ and for a choice of $\sqrt{t}$ the sphere $S^{n-1}_{\sqrt{t}} = \{(u_1, \ldots, u_n) \ / \ u_i = \sqrt{t}\ v_i, v_i \text{ real}, \Sigma v_i^2 = 1\} \subset U_t = \{u = (u_1, \ldots, u_n)/f(u) = t\}$.

For $t \neq 0$, $u \in U_t$, if one writes $\frac{1}{\sqrt{t}} u_i = x_i + iy_i$, one has :

1.1.2
$$u \in U_t \Longleftrightarrow \left| \begin{array}{l} \Sigma\, x_i^2 - y_i^2 = 1,\ \Sigma\, x_i y_i = 0 \\ \Sigma\, x_i^2 - y_i^2 \leq \frac{1}{|t|} \end{array} \right.$$

and this represents U_t as a disk bundle in the tangent bundle of $S^{n-1}_{\sqrt{t}}$.

1.2. The boundaries of the U_t's over the disk $\Delta_{\frac{1}{2}}$ of radius $\frac{1}{2}$ form a compact C^∞ fibration, and the U_t's themselves form a fibration into varieties with boundary over the circle $S_{\frac{1}{2}} = \partial\Delta_{\frac{1}{2}}$. One can construct a trivialization of the pull-back $U'_{[02\pi]}$ of this last fibration to $[02\pi] \xrightarrow{\frac{1}{2}\exp i\theta} S_{\frac{1}{2}}$, such that the induced trivialization of the boundaries descend to $\partial U_{S_{\frac{1}{2}}}$ and extends over the disk $\Delta_{\frac{1}{2}}$. The resulting homeomorphism

$$\begin{array}{cccc} \Phi : & U'_0 & \simeq & U'_{2\pi} \\ & \| & & \| \\ & U_{\frac{1}{2}} & \rightrightarrows & U_{\frac{1}{2}} \end{array}$$

satisfies then : $\Phi_{|\partial U_{\frac{1}{2}}=\mathrm{Id}}$.

The monodromy along the circle $S_{\frac{1}{2}}$ is the action of the map Φ on the pair $(U_{\frac{1}{2}}, \partial U_{\frac{1}{2}})$. It is in fact described by the maps :$\alpha_k : H_k(U_{\frac{1}{2}}, \partial U_{\frac{1}{2}}) \to H_k(U^0_{\frac{1}{2}})$ where $U^0_{\frac{1}{2}}$ is the interior set of $U_{\frac{1}{2}}$, and α_k is defined by : $\alpha_k(\gamma) = \Phi(\gamma) - \gamma$, for γ a k-chain with boundary in $\partial U_{\frac{1}{2}}$.

Now the action of Φ being trivial on H_0, the only non trivial α_k is α_{n-1} which necessarily sends the generator of $H_{n-1}(U_{\frac{1}{2}}, \partial U_{\frac{1}{2}})$ (called the transverse cycle and represented by a fiber of the disk bundle $U_{\frac{1}{2}} \to S^{n-1}_{\sqrt{\frac{1}{2}}}$) to a multiple of the generator δ of $H_{n-1}(U^0_{\frac{1}{2}})$ (called the vanishing cycle and represented by the sphere $S^{n-1}_{\sqrt{\frac{1}{2}}}$). One can check by explicit computation that the missing coefficient is 1, if one gives compatible orientations of these generators, using the natural real orientation of the tangent disk bundle $U_{\frac{1}{2}}$. In other words, we have the local Picard-Lefschetz formula :

1.2.1. $\alpha_{n-1}(\gamma) = (\gamma, \delta)\delta$ where the product $(,)$ is the intersection between $H_{n-1}(U_{\frac{1}{2}}, \partial U_{\frac{1}{2}})$ and $H_{n-1}(U^0_{\frac{1}{2}})$ given by the orientation above.

1.3. Now if we have a family of compact complex varieties $\mathcal{X} \underset{\varphi}{\to} \Delta$ such that φ is smooth over Δ^*, and $X_0 = \varphi^{-1}(0)$ has only one node at x_0, that is φ behaves as in 1.1 at x_0, one chooses a neighbourhood U of x_0 as in 1.1, one assumes that Δ is small (say $\Delta = \Delta_{\frac{1}{2}}$ as in 1.2) so that $\varphi_{|\bigcup_{t\in\Delta_{\frac{1}{2}}} \partial U_t}$ is a fibration, and one extends the trivialization (1.2) of $\varphi_{|\bigcup_{t\in\Delta_{\frac{1}{2}}} \partial U_t}$

to $\mathcal{X}\setminus \underset{t}{U}\, U_t^0$. Restricting this to $S_{\frac{1}{2}}$ and taking the pull-back to $[02\pi]$, we glue this last trivialization with the trivialization of $U'_{[02\pi]}$, and obtain a map :

$$\begin{array}{rccc} \Phi: & X'_0 & \simeq & X'_{2\pi} \\ & \| & & \| \\ & X_{\frac{1}{2}} & \xrightarrow{\sim} & X_{\frac{1}{2}} \end{array}$$

satisfying : $\Phi = \mathrm{Id}$ outside $U_{\frac{1}{2}}$.

The action ρ of Φ on $H.(X_{\frac{1}{2}})$ is called the monodromy action (of the positive generator of $\pi_1(\Delta^*)$) and because $\Phi = \mathrm{Id}$ outside $U_{\frac{1}{2}}, \Phi_* - \mathrm{Id}$ clearly factors as :

$$H_k(X_{\frac{1}{2}}) \underset{\text{excision}}{\rightarrow} H_k(U_{\frac{1}{2}}, \partial U_{\frac{1}{2}}) \underset{\alpha_k}{\rightarrow} H_k(U_{\frac{1}{2}}^0) \underset{\text{inclusion}}{\rightarrow} H_k(X_{\frac{1}{2}}).$$

Finally we have to note that the orientation of $U_{\frac{1}{2}}$ used in 1.2 differs from the complex orientation by a factor $(-1)^{n(\frac{n+1}{2})}$ coming from the rearrangement of variables in 1.1.2 $(x_i, y_i)_{i=1\dots n} \rightarrow (y_1, \dots, y_n, x_1, \dots, x_n)$. So if we use intersection on $H_{n-1}(X_{\frac{1}{2}})$ instead of the product (,) of 1.2, the Picard-Lefschetz formula now reads :

1.3.1. $\rho(\gamma) = \gamma + (-1)^{n(\frac{n+1}{2})} < \gamma, \delta > \delta$, for $\gamma \in H_{n-1}(X_{\frac{1}{2}})$, the monodromy being trivial on the other homology groups.

2.

2.1. Returning to the local situation 1.1, 1.2, there is a retraction of $\underset{t\in\Delta_{\frac{1}{2}}}{U} U_t$ on the union of $U_{\frac{1}{2}}$ and the "cone over the vanishing cycle" $\Gamma_\delta := \underset{t\in[0\frac{1}{2}]}{U} S^{n-1}_{\sqrt{t}}$, which is equal on $\underset{t\in\Delta_{\frac{1}{2}}}{U} \partial U_t$ to the retraction given by the trivialization of the boundaries in 1.2. The boundary of Γ_δ is $S^{n-1}_{\sqrt{\frac{1}{2}}}$ because S_0^{n-1} is shrinked to a point. In the global situation $\mathcal{X} \rightarrow \Delta$, we can glue this retraction with the one given by the trivialization of $\mathcal{X}\setminus U$ and we obtain : $\mathcal{X}$ retracts on $X_{\frac{1}{2}} \cup \Gamma_\delta$, where Γ_δ is an n-disk glued on the sphere $S^{n-1}_{\sqrt{\frac{1}{2}}} \subset X_{\frac{1}{2}}$.

2.2. Suppose now that X is a smooth projective variety of dimension n and $(X_t)_{t\in\mathbf{P}^1}$ is a Lefschetz pencil of hypersurfaces, that is X_t has at most one node, which is not on the base locus. Assume X_∞ is smooth, and let $\widetilde{X}$ be the blow-up of the base locus $X_0 \cap X_\infty$. Then $\mathcal{X} = \widetilde{X}\setminus X_\infty$ admits a map φ to $\mathbf{C} = \mathbf{P}^1\setminus\infty$ which satisfies locally the assumptions in 1.3. One fixes a regular value 0 of φ and for each critical value t_i of φ, one chooses a path γ_i from 0 to t_i, in such a way that the γ_i's meet only at 0. The plane $\mathbf{C}$ retracts on the union of the γ_i's so by smoothness of φ outside $U\Delta_i$ (Δ_i a small disk around t_i), $\mathcal{X}$

retracts first on $\varphi^{-1}(U\Delta_i \cup \gamma_i)$. Each $\mathcal{X}_{\Delta_i}$ retracts then on $X_{t'_i} \bigcup_{S^{n-1}_{t'_i}} \Gamma_{\delta_i}$, where t'_i is the intersection of γ_i with the $\partial\Delta_i$, and finally $\varphi^{-1}[t'_i 0]$ is naturally isomorphic to $X_0 \times [t'_i, 0]$. So $\mathcal{X}$ has the homotopy type of X_{t_0} with disks $\Gamma_{t'_i}$ glued over spheres $S'^{n-1}_{t'_i} \subset X_0$, the images of the spheres $S^{n-1}_{t'_i}$ in X_0, via the induced isomorphism $X_{t'_i} \simeq X_0$.

2.3. It follows that the relative homology $H_n(\mathcal{X}, X_0)$ is generated by the disks $\Gamma_{t'_i}$, and that the kernel of the map $H_{n-1}(X_0) \to H_{n-1}(\mathcal{X})$ is generated by the classes δ_i of the spheres $S'^{n-1}_{t'_i}$. In fact one can deduce from the hard Lefschetz theorem that this result remains true for X instead of $\mathcal{X}$, [1] (here one needs the assumption that X_∞ is smooth).

2.4. As a corollary of this, and the Picard-Lefschetz formula 1.3, one finds a particular case of the global invariant cycle theorem proved in general by Deligne [6] using the degeneration of Leray spectral sequence (lecture 2) and the theory of mixed Hodge structures :

Theorem. In the situation of 2.2, if $\gamma \in H_{n-1}(X_0, \mathbb{Q}) \simeq H^{n-1}(X_0, \mathbb{Q})$ is invariant under the monodromy action $\rho : \pi_1(\mathbb{P}^1 \setminus \{t_1, \ldots, t_N\}, 0) \to \operatorname{Aut} H^{n-1}(X_0, \mathbb{Q})$, then γ is in the image of the restriction map $j^* : H^{n-1}(X, \mathbb{Q}) \to H^{n-1}(X_0, \mathbb{Q})$.

This comes from $\operatorname{Im} j^* = (\operatorname{Ker} j_*)^\perp = < \delta_i >^\perp$, and the following consequence of 1.3: $\gamma \perp \delta_i \Leftrightarrow \gamma$ is invariant under the monodromy action, generated by the Picard-Lefschetz reflections $\gamma \mapsto \gamma \pm < \gamma, \delta_i > \delta_i$.

We explain now two applications of the Picard-Lefschetz theory : The first one is :

2.5. The Noether-Lefschetz theorem : Let S be a general surface in $\mathbb{P}^3$ of degree $d \geq 4$. Then $\operatorname{Pic} S = \mathbb{Z}$, generated by $\mathcal{O}_S(1)$.

One notes first the following : If S is general, the Néron-Severi group of S is globally invariant under the monodromy action. This holds because a line bundle L on a general S is defined on the universal surface $\mathcal{S} \to U$, where $U \to V$ is a finite Galois cover of the moduli space of S. So the monodromy action on $c_1(L)$ just exchanges $c_1(L)$ with $c_1(L_\gamma)$ where L_γ is obtained from L by action of the Galois group of $U \to V$. Now if $c_1(L)$ is primitive and non zero, for a Lefschetz pencil of surfaces $(S_t)_{t \in \mathbb{P}^1}$ with $S_0 = S$, there is a vanishing cycle δ_i such that $c_1(L).\ \delta_i \neq 0$ because the intersection form $\langle , \rangle$ on S is non degenerate on $H^2(S)^0$, generated by the δ_i's. So if $c_1(L) \neq 0$, $NS(S) \otimes \mathbb{Q}$ contains one δ_i by the Picard-Lefschetz formula. Finally we have :

Sublemma. The monodromy acts transitively on the set of vanishing cycles.

Admitting this, we see that the hypotheses would imply that $NS(S) \otimes \mathbf{Q}$ contains all the δ_i's, that is contains $H^2(S, \mathbf{Q})^0$. This is absurd if $H^2(\mathcal{O}_S) \neq 0$ (see lecture one), that is when $d \geq 4$.

For the proof of the sublemma, one uses the inclusion of $\mathbf{P}^1$ into the space $\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^3}(d))) = \mathbf{P}^k$. One knows that $\pi_1(\mathbf{P}^1 \backslash \{t_1, \ldots, t_N\}, 0\} \twoheadrightarrow \pi_1(\mathbf{P}^k \backslash \mathcal{D})$ where $\mathcal{D}$ is the discriminant hypersurface. $\mathcal{D}$ being irreducible it is easy to see that all the $\gamma_i^{-1}.\delta\Delta_i.\gamma_i$'s are conjugate in $\pi_1(\mathbf{P}^k \backslash \mathcal{D})$ hence in $\pi_1(\mathbf{P}^1 \backslash \{t_1, \ldots, t_N\}, 0\})$ which proves the lemma, using the Picard-Lefschetz formula.

2.6. The second application can be found in [9] : Let us consider the family $\mathcal{X} \underset{\pi}{\rightarrow} U$ of smooth automorphism free hypersurfaces of degree d in $\mathbf{P}^{2m}$, modulo $\mathbf{P}G\ell$ action. There is the associated family of intermediate Jacobians $J \rightarrow U$, with sheaf of holomorphic sections given by $\mathcal{J} : \mathcal{H}^{2m-1}/F^m\mathcal{H}^{2m-1} \oplus H_{\mathbf{Z}}^{2m-1}$, where $H_{\mathbf{Z}}^{2m-1} = R^{2m-1}\pi_*\mathbf{Z}$, $\mathcal{H}^{2m-1} = H_{\mathbf{Z}}^{2m-1} \otimes \mathcal{O}_U$ and F^kH^{2m-1} is the Hodge filtration. A normal function ν is a section of $\mathcal{J}$. (In principle, one should impose to it infinitesimal conditions and growth conditions at the boundary but we don't need it). We will say that ν is flat if it is locally the projection in $\mathcal{J}$ of a flat section $\tilde{\nu}$ of $\mathcal{H}^{2m-1}$. By infinitesimal considerations one can show that such $\tilde{\varphi}$ is then unic up to a section of $H_{\mathbf{Z}}^{2m-1}$, and this remain true on an tale cover of U.

Now we have :

2.6. Proposition : Let $V \underset{r}{\rightarrow} U$ be an tale cover of U and let ν be a flat normal function on V. Then ν is a torsion section of $\mathcal{J}$.

Proof : Fix $0 \in V$. Then $r_* : \pi_1(V, 0) \rightarrow \pi_1(U, 0)$ has image of finite index, say N. Let $\tilde{\varphi}$ be a flat lifting of ν near 0 and let $\tilde{\varphi}(0) \in H^{2m-1}(X_0, \mathbf{C})$ be its value at 0. We have to check that $\tilde{\varphi}(0) \in H^{2m-1}(X_0, \mathbf{Q})$. If $\gamma : [01] \rightarrow V$ is a loop based at 0, ν being locally flat we can follow $\tilde{\varphi}$ along γ and get near $\gamma(1)$ a new lifting $\tilde{\varphi}'$ of ν which is flat. By the unicity statement, we have $\tilde{\varphi}'(0) - \tilde{\varphi}(0) \in H^{2m-1}(X_0, \mathbf{Z})$. But by definition of the monodromy on the local system $H_{\mathbf{C}}^{2m-1}, \tilde{\varphi}'(0) - \tilde{\varphi}(0) = \rho(\gamma)(\tilde{\varphi}'(0)) - \tilde{\varphi}'(0)$. It remains to prove :

2.6.1. If $\eta \in H^{2m-1}(X_0, \mathbf{C})$ satisfies : $\forall \gamma \in \pi_1(V, 0), \rho(\gamma)(\eta) - \eta \in H^{2m-1}(X_0, \mathbf{Z})$, then $\eta \in H^{2m-1}(X_0, \mathbf{Q})$. To see this, one chooses a Lefschetz pencil in U, with loops γ_i acting by Picard-Lefschetz reflections associated to the δ_i's, the vanishing cycles, which generate $H^{2m-1}(X_0, \mathbf{Z})$ by 2.3. Then $\gamma_i^N \in r_*(\pi_1(V, 0))$ and acts by the transformation $\eta \rightarrow \eta \pm N < \eta, \delta_i > \delta_i$. So the assumption implies : $\forall i, < \eta, \delta_i > \in \mathbf{Q}$, hence $\eta \in H^{2m-1}(X_0, \mathbf{Q})$, because $\langle , \rangle$ is non degenerate and defined over $\mathbf{Q}$.

3. We want now to explain the Hodge theory on the central fiber of a Lefschetz degeneration. The results depend on the parity of n. This follows from the Picard-Lefschetz formula which shows that the monodromy is of order two for $n-1$ even, and of infinite order for $n-1$ odd (or trivial).

However, we will use in both cases Griffiths' arguments which are not completely general but give quickly a concrete description of the limit Hodge structure, and are well adapted to the Lefschetz degenerations. For the general case one should work with normal crossing model of X_0 and introduce the logarithmic log complex [10] [11].

We will work with families $X_t \subset X$ where X_t is a hypersurface of X ample enough for its $(n-1)^{th}$ primitive cohomology to be realized by residues of meromorphic n-forms on X, the Hodge filtration corresponding to the pole order filtration (see the M. Green lecture on hypersurfaces). So $F^kH^{n-1}(X_t)$ will be generated by residues of n-forms with poles of order at most $n-k$ along X_t.

A) $n-1$ even : To kill the monodromy, which is of order two, one makes a base change $t = u^2$. The spheres $S^{n-1}_{\sqrt{t}} = S^{n-1}_u$ give then a univalued locally constant section δ_u of $H^{n-1}_{\mathbf{Z}} = H_{n-1,\mathbf{Z}}$ on the punctured disk with coordinate u. One verifies using the description of the U_t's in 1.1, that $\delta_u^2 = \pm 2$ (in particular $\delta_u \neq 0$) so over $\mathbf{Q}$ one has a splitting $H^{n-1}_{\mathbf{Q}} = <\delta_u> \oplus <\delta_u>^{\perp}$ and $<\delta_u>^{\perp}$ is by the Picard-Lefschetz formula the invariant part of $H^{n-1}_{\mathbf{Z}}$ on Δ_t^*. Cycles γ_t in $<\delta_u>^{\perp} \subset H_{n-1}(X_t, \mathbf{Z})$ can be represented by chains in $X_t \backslash U_t$ (see 2.1), hence have a limit γ_0 in $X_0 \backslash U_0$.

Clearly if γ_t is such a cycle, and ω_t/f_t^k is a holomorphically varying family of n-forms on X with pole of order $\leq k$ along X_t one has $\lim_{t \to 0} \int_{\gamma_t} \mathrm{Res}_{X_t}(\omega_t/f_t^k) = \int_{\gamma_0} \mathrm{Res}_{X_0}(\omega_0/f_0^k)$ where the last term makes sense because γ_0 is supported away from $\mathrm{Sing}\, X_0$.

It remains to study the behaviour of $\int_{\delta_u} \mathrm{Res}_{X_t}(\omega_u/f_t^k)$, which is a local problem, since the δ_u are supported near x_0. The result is then :

3.1. Proposition : Assume $\frac{df_t}{dt}\big|_{t=0}$ does not vanish at x_0 (Lefschetz assumption). Then for $2(n-k) > n-1$, $\lim_{t \to 0} \int_{\delta_u} \mathrm{Res}_{X_t}(\omega_u/f_t^k) = 0$ and for $2(n-k) = n-1$, $\lim_{u \to 0} \int_{\delta_u} \mathrm{Res}_{X_t}(\omega_u/f_t^k)$ exists and vanishes if and only if ω_0 vanishes at x_0.

This shows concretely how to extend the Hodge filtration $F^mH^{n-1}(X_0)$ over 0, for $2m \leq n-1$: Using $H^{n-1}(X_{u^2}) \simeq H_{n-1}(X_{u^2})^*$ one defines $F^mH^{n-1}(X_{u^2})$ as the space generated by the limits $\lim_{u \to 0} \int_{\gamma_u} \mathrm{Res}_{X_{u^2}}(\omega_u/f_t^k)$, for $k = n-m$, where γ_u is any locally constant section of $H_{n-1,\mathbf{Z}}$ over Δ_u. One needs supplementary assumptions to check that this filtration has correct rank (satisfied for X_t ample enough), and one extends it to a Hodge filtration on $H_0^{n-1} = \lim_{t\to 0} H^{n-1}(X_n, \mathbb{Z}) \otimes \mathbb{C}$ using complex conjugaison. The fact that this limit filtration really puts a Hodge structure on H_0^{n-1} remains to be proved, and one has more generally the following theorem, which is proved by the decreasing distance property of the period map :

3.2. Theorem : [8] For a polarized variation of Hodge structures $(H^k_{\mathbb{Z}}, F^\ell \mathcal{H}^k)$ on Δ^* without monodromy, the period map extends at 0 and defines a pure Hodge structure on $H^k_{\mathbb{Z}(0)}$.

We note finally that by proposition 3.1, the limit δ_0 of the vanishing cycle annihilates $F^{m+1}H_0^{n-1}$, where $2m = n-1$, hence is a Hodge class. A geometric way of interpreting it as an algebraic cycle is to construct the normal crossing model of X_0 (after base change $t = u^2$) as the union of the minimal desingularization $\tilde{X}_0$ of X_0 and a $(n-1)$ quadric Q which intersect $\tilde{X}_0$ along the exceptional divisor, which is identified to a hyperplane section Q' of Q. Then one can check that the spheres S_u converge to the generator of $H_{n-1}(Q \backslash Q')$, that is the difference of the two rullings of Q.

B) $n-1$ odd, $n = 2m$. In this case the vanishing cycle δ_u may have trivial homology class in X_0. Then there is no monodromy and by theorem 3.2 there is a pure Hodge structure on the fiber H_0^{n-1}. More generally, one can consider a central fiber with several nodes, and define the defect of X_0 as the number of relations between the homology classes of the associated vanishing cycles. Under some vanishing assumptions on X, one can identify this defect to the corank of the restriction map $H^0(K_X(mX_0)) \to H^0(K_X(mX_0)_{|Z})$, where Z is the singular locus of X_0 [5].

This is strongly related to the following analog of Prop. 3.1 :

3.3. Proposition : 1) for $k \leq m$, $\lim_{t\to 0} \int_{\delta_t} \mathrm{Res}_{X_t}(\omega_t/f_t^k)$ exists and is equal to zero for $k < m$, and is a non zero multiple of $\omega_0(x_0)$ for $k = m$.

(Notice that now the δ_t's give an invariant section of $H_{n-1,\mathbb{Z}}$ over Δ^*).

2) For γ a multivalued section of $H_{n-1,\mathbb{Z}}$ over Δ^*, and $k \leq m$, $\int_{\delta_t} \mathrm{Res}_{X_t}(\omega_t/f_t^k)$ has a logarithmic growth near zero and its monodromy is described by 1) and the Picard-Lefschetz formula.

So for $<\gamma_t, \delta_t>= 0$, that is γ_t has a limit γ_0 which is supported in $X_0 \backslash \{x_0\}$, one can define the limit periods $\int_{\gamma_0} \mathrm{Res}_{X_0}(\omega_0/f_0^k)$, $k \leq m$, where under some vanishing assumptions on (X, X_t), the $\mathrm{Res}_{X_0}(\omega_0/f_0^k) = \lim_{t\to 0} \mathrm{Res}_{X_t}(\omega_t/f_t^k)$ generate the fiber at 0 of the extended Hodge bundle $F^m\mathcal{H}^{2m-1}$, which is characterized by the growth condition in 3.3.2).

The intermediate jacobian $J(X_t)$ of X_t is the compact complex torus $F^mH^{2m-1}(X_t)^*/H_{2m-1}(X_t, \mathbf{Z})$ and the generalized intermediate jacobian $J(X_0)$ of X_0 is defined as the partial torus $(F^mH_0^{2m-1})^*/$ periods.

This torus has for quotient the intermediate jacobian of $\tilde{X}_0$, which under the same assumptions is realized by projecting $(F^mH_0^{2m-1})^*$ to $((F^mH_0^{2m-1})^0)^*$, where $(F^mH_0^{2m-1})^0$ is the hyperplane generated by the residues $\mathrm{Res}(\omega_0/f_0^m)$, with $\omega_0(x_0) = 0$. (We now assume that δ is non zero). By 3.3 i) $\int_\delta$ project to zero in $((F^mH_0^{2m-1})^0)^*$ so $J(X_0) \to J(\tilde{X}_0)$ represents $J(X_0)$ as an extension of $J(\tilde{X}_0)$ by $\mathbf{C}^*$. Such an extension is classified by $J(\tilde{X}_0)^\vee/\pm 1 = J(\tilde{X}_0)/\pm 1$ and one has the following :

3.4. Theorem : [2], [4], [5] When the vanishing cycle has non zero homology class in X_t, the two rullings of the exceptional divisor of $\tilde{X}_0$ are homologous and the image of their difference in $J(\tilde{X}_0)$ by the Abel-Jacobi map describes the extension :

$$0 \to \mathbf{C}^* \to J(X_0) \to J(\tilde{X}_0) \to 0.$$

References

[1] A. Andreotti, T. Frenkel, The second Lefschetz theorem on hyperplane sections, Global analysis, A symposium in honor of K. Kodaira, Princeton University Press, Princeton (1969) 1-20.

[2] F. Bardelli, Polarized mixed Hodge structures, On irrationality of threefolds via degenerations, Ann. Mat. Pura Appl. (4) 137 (1984) 287-369.

[3] J. Carlson, The Geometry of the extension class of a mixed Hodge structure, Proceedings of Symposia in Pure Mathematics, Vol. 46 (1987).

[4] H. Clemens, Degeneration technics in the study of threefolds in *Algebraic threefolds*, Lecture Notes in Math. n° 947, Springer Verlag (1981).

[5] H. Clemens, double solids, Advances in Mathematics 47, 107-230 (1983).

[6] P. Deligne, Théorie de Hodge II, Publ. Math. IHES (40) (1971) 5-57.

[7] P. Griffiths, On the periods of certain rational integrals I and II, Annals of Math. 90 (1969) 460-541.

[8] P. Griffiths, Periods of integrals on algebraic manifolds III, Publ. Math. IHES 38 (1970) 125-180.

[9] P. Griffiths, A theorem concerning the differential equations satisfied by normal functions associated to algebraic cycles, Amer. J. Math. 101 (1979) 94-131.

[10] D. Morrison, The Clemens-Schmid exact sequence and applications, in *Topics in transcendental Algebraic Geometry*, ed. by P. Griffiths, Ann. of Math. Studies, Study 106, Princeton 1984.

[11] J. Steenbrink, Limits of Hodge structures, Inv. Math. 31 (1976) 229-257.

[12] P. Griffiths - J. Harris, On the Noether-Lefschetz theorem and some remarks on codimension two cycles, Math. Ann. 271, 31-51 (1985).

Lecture 5. 0-cycles I.

Lectures 5 and 6 are devoted to 0-cycles modulo rational equivalence, especially for surfaces. This subject does not seem a priori much related to transcendental aspects of Hodge theory, and in fact all that we will explain belongs to algebraic geometry, even if for simplicity we use at some places the usual topology and the Betti cohomology of complex varieties. However, the relation with our main topic is the discovery by Mumford that the non-representability of Chow groups (here we will be concerned with CH_0), is related to the transcendental character of the corresponding Hodge theory. In the first lecture, we have shown that the Hodge theoretic objects related to divisors are the Picard torus and the set of Hodge classes in H^2. The first one is an abelian variety, and the second one generalizes to the "Tate-Hodge structures", which are made only of Hodge classes. These two kinds of objects are the only algebro-geometric objects that can be extracted from a Hodge structure, and this gives a more transcendental character to the remaining part of the Hodge theory of a variety. In this lecture, we explain the following now classical results : Mumford theorem on infinite dimensionality of the CH_0 group of a surface having non zero holomorphic two-form, subsequent generalizations of it by Roitman, and we present Bloch's conjecture on correspondences between surfaces. We sketch also the argument of Bloch-Kas-Lieberman, which gives for surfaces not of general type the following consequence of Bloch's conjecture : $H^0(K_S) = 0 \Rightarrow CH_0^0(S) = \mathrm{Alb}\, S$. We also sketch Roitman's proof of his two fundamental theorems : CH_0 finite dimensional $\Rightarrow CH_0^0 = \mathrm{Alb}$, and $\mathrm{tors}(CH_0) \simeq \mathrm{tors}(\mathrm{Alb})$. (However we follow largely [2] for the proof of the later statement).

1.

1.1. Let S be a smooth projective surface, and let G be a smooth projective variety. Let

$$\begin{array}{ccc} Z & \xrightarrow{q} & S \\ p\downarrow & & \\ G & & \end{array}$$

be a zero correspondence between G and S, that is $Z \subset G \times S$ is a reduced algebraic subset and p is finite; then for $\omega \in H^0(K_S)$, one can construct a holomorphic two form $Z[\omega]$ on G, either by defining (carefully) the trace $p_*(q^*\omega)$, either by the construction of the associated map $\varphi_Z : G \to S^{(N)}$, such that $\varphi_Z(g) = q(p^{-1}(g))$, where $N = d^0p$, and $S^{(N)}$ is the symmetric product (unfortunately singular) of S, and by showing that the symmetric two-form $\sum_{i=1}^N pr_i^*\omega$ on $S^{(N)} \setminus \operatorname{sing} S^{(N)}$ has a non singular pull-back to G.

For a non reduced Z, $Z = nZ'$, one defines $Z[\omega] = nZ'[\omega]$. The main theorem is then:

1.2. Theorem : ([4]). If Z_1 and Z_2 are two correspondences between G and S, satisfying : $\forall g \in G$, $\varphi_{Z_1}(g)$ and $\varphi_{Z_2}(g)$ are rationally equivalent zero-cycles on S, then $\forall \omega \in H^0(K_S)$, $Z_1[\omega] = Z_2[\omega]$ in $H^0(\Omega^2_G)$.

The general idea is the following : by countability of the Hilbert scheme of rational curves in $S^{(k)}$, $k \in \mathbf{N}$, and by definition of rational equivalence, one may assume that there is an etale map $G' \underset{r}{\rightarrow} G$, a map $\Psi : G' \rightarrow S^{(k)}$ and a map $\Phi : \mathbf{P}^1 \times G' \rightarrow S^{(N+k)}$ such that: for $g' \in G'$, $\Phi(0, g') = \Phi_{Z_1}(g') + \Psi(g')$, and $\Phi(\infty, g') = \varphi_{Z_2}(g') + \Psi(g')$, where we use "+" for the obvious map $S^{(N)} \times S^{(k)} \rightarrow S^{(N+k)}$. Now, for $\omega \in H^0(K_S)$, there is as before a pull-back $\Phi^*(\omega) \in H^0(\Omega^2_{\mathbf{P}^1 \times G'})$ which necessarily is of the form $\pi^*\alpha$, where $\pi : \mathbf{P}^1 \times G' \rightarrow G'$ is the projection to G', and $\alpha \in H^0(\Omega^2_{G'})$. By the obvious additivity with respect to Z of the traces $Z[\omega]$, and by restriction of the above equalities to $\{0\} \times G'$ and $\{\infty\} \times G'$ we find $r^*Z_1[\omega] + \Psi^*[\omega] = \pi^*\alpha_{|\{0\}\times G'} = \alpha = r^*Z_2[\omega] + \Psi^*[\omega]$, that is $Z_1[\omega] = Z_2[\omega]$.

Now $CH_0(S)$ contains for each N the quotient of $S^{(N)}$ by the relation of rational equivalence, which is described in $S^{(N)} \times S^{(N)}$ by a countable union of algebraic subsets. For a cycle $Z \in S^{(N)}$, one defines the dimension d_Z of its orbit 0_Z under rational equivalence as the maximal dimension of an algebraic component of $\{Z' \in S^{(N)} / Z' \underset{rat}{\equiv} Z\}$, and clearly d_Z is a constant for a general cycle Z. 0_Z being roughly the fiber through Z of the map $R^N : S^{(N)} \rightarrow CH_0(S)$, one defines $\operatorname{Im} R^N := 2N - d_Z$, for a general cycle Z.

Mumford applies then 1.2 to show :

1.3. Theorem : [4]. If $H^0(K_S) \neq \{0\}$, $\lim_{N\rightarrow\infty} \dim \operatorname{Im} R^N = \infty$, that is $CH_0(S)$ is not finite dimensional.

The point is the following : Let Z be general in $S^{(N)}$ and choose a component G_Z of 0_Z of maximal dimension d_Z; then one may assume that $Z \in G_Z$, G_Z is smooth at Z, and that Z is made of distinct points $\{z_1, \ldots, z_N\}$. If $\omega \in H^0(K_S)$ is non zero, one may also assume that $\omega(z_i) \neq 0$; by theorem 1.2, the form $\Omega = \Sigma\ pr_i^*\omega$ on the smooth variety $S^{(N)} \backslash \operatorname{sing} S^{(N)} \ni Z$ has to vanish on G_Z near Z, and $\omega(z_i) \neq 0$ implies that Ω is non degenerate (as a two form) at Z; it follows that $\dim_Z G_Z \leq \frac{1}{2} \dim S^{(N)}$. Hence $\dim \operatorname{Im} R^N \geq N$, which implies 1.3.

1.4. It follows a posteriori [5] that in fact $\dim \operatorname{Im} R^N = 2N$, when $H^0(K_S) \neq \{0\}$, that is, a general cycle Z has a zero dimensional orbit 0_Z. A proof of this can be checked as follows : Assume a general cycle moves in its orbit O_Z and fix an ample curve $C \subset S$. Then 0_Z will meet $C + S^{(N_1)} \subset S^{(N)}$. It follows by induction on k that for a general cycle Z in $S^{(N+k)}$ its orbit will meet $C^{(k+1)} + S^{(N-1)}$, and because the image of $C^{(k)}$ in $CH_0(S)$ has dimension bounded by g = genus of C, this would show that $\forall k$, $\dim R^{(N+k)} \leq g+2(N-1)$ in contradiction with 1.3.

We refer to [5] for generalizations and refinements of this kind of results for higher dimensional varieties.

1.5. Let us consider now smooth surfaces Σ, S, and let $Z \subset \Sigma \times S$ be a codimension two algebraic cycle, say $Z = \Sigma n_i Z_i$ with Z_i irreducible and generically finite over Σ. Then as in 1.1, Z gives a map $Z : \omega \to Z[\omega] = \Sigma n_i Z_i[\omega], H^0(K_S) \to H^0(K_\Sigma)$. (We don't need that $Z_i \underset{p_1}{\to} \Sigma$ be finite because it has positive dimensional fibers only over a codimension two subset of Σ, so we can use Hartog's theorem).

Now it is easily seen from the definition that the map Z_i is the $(2,0)$ Hodge component of the map $Z_i : H^2(S) \to H^2(\Sigma)$ which is a morphism of Hodge structures and can be constructed alternatively as :

i) consider the Hodge class $[Z_i] \in H^4(\Sigma \times S, \mathbb{Z})$; then its Künneth $(2,2)$ component lies in $H^2(\Sigma) \otimes H^2(S) \simeq \mathrm{Hom}(H^2(S), H^2(\Sigma))$, which gives our Z_i.

ii) Choose a desingularisation $\widetilde{Z}_i \to Z_i$ of Z_i : then one has $\widetilde{p}_1 : \widetilde{Z}_i \to \Sigma, \widetilde{p}_2 : \widetilde{Z}_i \to S$ and one can define Z_i as the composite :

$$H^2(S) \underset{\widetilde{p}_2^*}{\to} H^2(\widetilde{Z}_i) \simeq H_2(\widetilde{Z}_i) \underset{\widetilde{p}_{1*}}{\to} H_2(\Sigma) \simeq H^2(\Sigma)$$

, where $\simeq$ means Poincaré duality isomorphism. Consider the splitting (over $\mathbb{Q}$) of $H^2(S, \mathbb{Q})$ into $NS(S) \otimes \mathbb{Q}$ and $TH^2(S) = NS(S)^\perp$. Let $\varphi : H^2(S) \to H^2(\Sigma)$ be a morphism of Hodge structures which vanishes on $H^{2,0}(S)$; then $\mathrm{Ker}\,\varphi \cap TH^2(S)$ is a sub-Hodge structure of $TH^2(S)$, and contains $H^{2,0}(S)$. Its orthogonal for $<,>_{|_{TH^2(S)}}$ is defined over $\mathbb{Q}$ and perpendicular to $H^{2,0}(S)$ hence is contained in $TH^2(S) \cap NS(S) \otimes \mathbb{Q} = \{0\}$. So φ vanishes in fact on $TH^2(S)$. From this and theorem 1.2 follows :

1.6. Proposition : Let $Z \subset \Sigma \times S$ be a codimension two cycle; then the induced map of Hodge structures $TH^2(S) \to H^2(\Sigma)$ vanishes if the map $p_{1*}^*(p_2^*().Z) : CH_0(\Sigma) \to CH_0(S)$ induced by Z is zero.

One can refine 1.6 as follows : coming back to the Mumford argument, one sees easily that if the map $p_{1*}(p_2^*().Z) : CH_0(\Sigma) \to CH_0(S)$ is zero on the set of cycles of degree zero and in the kernel of the Albanese map of Σ, then $Z : H^{2,0}(S) \to H^{2,0}(\Sigma)$ vanishes, hence $Z : TH^2(S) \to H^2(\Sigma)$ also vanishes. Bloch [1] has conjectured the converse of the last statement :

1.7. Conjecture : Let $Z \subset \Sigma \times S$ be a codimension two cycle whose (2,2) Künneth component lies in $NS(\Sigma) \otimes NS(S)$ (equivalently Z vanishes on $TH^2(S)$). Then $p_{2*}(p_1^*().Z)$: $CH_0^0(\Sigma) \to CH_0(S)$ factors through the Albanese variety of Σ.

Here $CH_0^0(\Sigma)$ is the set of degree 0 cycles on Σ. We refer to J.P. Murre's lectures for the construction of the degree and Albanese map on CH_0).

A particular case of Bloch's conjecture is given by the diagonal $\Delta_S \subset S \times S$ of a surface with $h^0(K_S) = 0$. Bloch's conjecture implies in this case :

1.8. Subconjecture : If $h^0(K_S) = 0$, $CH_0^0(S) = \operatorname{Alb} S$.

We will explain in the next lecture a proof of this for Godeaux type surfaces.

Using classification of surfaces, Bloch-Kas-Lieberman have proved 1.8 for surfaces with p_g $(= h^0(K_S)) = 0$ and not of general type : In fact classification will imply with some work that for minimal surfaces there are essentially three cases to consider :

i) $q = \dim \operatorname{Alb} S = 1$, and alb: $S \to E$ is a smooth fibration, so $S = E' \times F/G$, G acting on E' by a finite group of translations. Let $\pi : E' \times F \to E' \times F/G$ be the quotient map.

ii) $q = \dim \operatorname{Alb} S = 1$, and alb: $S \to E$ has elliptic fibres. Then one shows that the associated jacobian fibration, which has isomorphic CH_0 group, fails into i).

iii) $q = 0$ and S has an elliptic pencil $S \to \mathbf{P}^1$. Then they show that the associated jacobian fibration $S' \to \mathbf{P}^1$ is a rational surface, hence has $CH_0 = \mathbf{Z}$, using the Castelnuovo criterion : $q(S') = h^0(K_{S'}^{\otimes 2}) = 0 \Rightarrow S'$ is rational. Now it is easy to see that $CH_0(S) = CH_0(S')$.

For case i), one uses the Roitman theorem (2.2) which implies that $\operatorname{Ker}(\text{alb}) \subset CH_0^0(S)$ has no torsion. So it suffices to check that $\pi^*(\operatorname{Ker}\text{alb}) = 0$ in $CH_0^0(E' \times F)$ mod torsion. But if $Z \in \operatorname{Ker}(\text{alb})$, $\pi^*Z = \sum_{i,g} n_i(\tilde{e}_i + g, g.x_i)$, with $\Sigma n_i e_i = 0$ in $\operatorname{Alb} S = E$. Now $h^0(K_S) = 0 \Leftrightarrow F$ has no non zero 1-form invariant under $G \Leftrightarrow F/G = \mathbf{P}^1$. So $\forall_i, \sum_g g x_i = h =$ const. On the other hand, up to torsion, that we don't consider, $\sum_{i,g} n_i(\tilde{e}_i + g, g.x_i) = \sum_{i,g} n_i(\tilde{e}_i, g.x_i) = \sum_i n_i(\tilde{e}_i * (\sum_g g.x_i))$ in $CH_0(E' \times F)$. So $\pi^*Z = (\sum_i n_i\tilde{e}_i) * h$ up to torsion in $CH_0(E' \times F)$. Since $\sum_i n_i\tilde{e}_i =$ torsion point in $CH_0^0(E')$, $\pi^*Z = 0$ up to torsion, and we are done. (The product $*$ that we used between $CH_0(E')$ and $CH_0(F)$ is such that $Z * Z' = \Sigma n_i m_j(z_i, z_j')$ for $Z = \Sigma n_i z_i$, $Z' = \Sigma m_j z_j'$).

2. We turn now to the proof of the following fundamental theorems of Roitman :

2.1. Theorem : [5]. Let X be a projective variety, such that $CH_0(X)$ is finite dimensional: then $CH_0^0(X) \xrightarrow{\text{alb}} \text{Alb}\, X$ is an isomorphism.

2.2. Theorem : [2], [6]. Let X be a projective variety; then the Albanese map induces an isomorphism alb : $\text{Tors}\, CH_0(X) \simeq \text{Tors}(\text{Alb}\, X)$.

For the definition of finite dimensionality in 2.1, we have (over the uncountable field $\mathbb{C}$) different equivalent characterizations. (We use the notations of 1.1 - 1.3 with S replaced by X).

i) There is an integer $d(X)$ such that $\forall N \in \mathbb{N},\ \dim \text{Im}\, R^N \leq d(X)$.

ii) There exists an integer N such that the difference map : $X^{(N)} \times X^{(N)} \to CH_0^0(X)$ is surjective.

The equivalence follows from the argument sketched in 1.4. If i) holds, one chooses N such that $N \dim X - d(X) \geq \dim X$. Then if $C \subset X$ is an ample curve and $Z \in X^{(N)}$ is a general cycle, 0_Z meets $C \times X^{(N-1)}$, so any cycle in $X^{(N)}$ is rationally equivalent to a cycle in $C + X^{(N-1)}$. As in 1.4 it follows that any cycle in $X^{(N+k)}$ is rationally equivalent to a cycle in $C^{(k+1)} + X^{(N-1)}$, and since $C^{(g)} \times C^{(g)} \underset{\text{difference map}}{\longrightarrow} CH_0^0(C)$ is surjective, ii) follows. ii) $\Rightarrow$ i) comes also from the fact that rational equivalence between cycles in $S^{(K)}$ is described by a countable union of algebraic sets in $S^{(K)} \times S^{(K)}$.

2.3. Now the argument for theorem 2.1 goes as follows :

Step 1 : If $CH_0(X)$ is finite dimensional, there exists an abelian variety A, and a family of cycles of degree zero $Z \subset A \times X$ inducing a surjective map of groups $A \underset{f}{\to} CH_0^0(X), f(a) = Z(a)$.

Proof : For any abelian variety A, and for any family of zero-cycles of degree 0 $Z \subset A \times X$ inducing a map of groups $f : A \to CH_0^0(X), f(a) = Z(a)$, the kernel is a countable union of algebraic subsets of A, and it is a group. So A', its connected component through 0, is an abelian variety and we define $\dim f(A) = \dim A/A'$. Let $d(X)$ be as in i). Then one checks $\dim f(A) \leq d(X)$. So there exists (A, Z) as above such that $\dim f(A)$ is maximal. Adding to A the jacobian of any curve $C \subset X$, it is then easy to show that f is surjective.

Step 2 : One may assume that $A \underset{f}{\to} CH_0^0(X)$ has a countable kernel.

Proof : The kernel of f is a countable union of algebraic subsets and it is a subgroup. So an algebraic component of it passing through 0 is an abelian subvariety $B \subset A$, and we can work with an abelian subvariety $A' \subset A$, isogenous to A/B, and with the restricted cycle $Z_{|_{A'}}$.

Step 3 : In $X \times A$, the set $\{(x,a)/x - x_0 \equiv Z(a) = f(a) \text{ in } CH_0^0(X)\}$ is a countable union of algebraic subsets, which projects onto X. So one of its components, say R, projects onto X. Such R is finite over X, because $f : A \to CH_0^0(X)$ has countable kernel. So R gives rise to a zero correspondence between X and A, and we have the following commutative diagram :

2.3.1.
$$\begin{array}{ccc} X & \xrightarrow{g} & A \\ & k \searrow \quad \swarrow f & \\ & CH_0^0(X) & \end{array}$$

where $k(x) = n(x - x_0)$, for $n = \deg(R/X)$, and $g(x) = \text{alb}(A) \circ R(x)$.

$\text{Im}\, g$ generates A as a group, otherwise $\dim(\text{Im}\, g) < \text{Im}\, A$ and $\dim A = \dim CH_0(X)$ by the fact that f has a countable kernel, would contredict the fact that $CH_0^0(X)$ is generated by $k(X)$. (Here we define $\dim CH_0(X) = \dim \text{Im}\, R^N$ (cf. 1.2, 1.3)). By the universal property of the Albanese map, we find now a surjective map $g'' : \text{Alb}\, X \to A$, such that the following diagram commutes :

2.3.2.
$$\begin{array}{ccccc} X & \xrightarrow{g} & A & \xrightarrow{f} & CH_0^0(X) \\ \text{alb} \searrow & & \nearrow g'' & & \downarrow \text{alb} \\ & \text{Alb}\, X & & \xrightarrow[n \times \text{Id}]{} & \text{Alb}\, X \end{array}$$

It follows that the kernel of f is in fact finite, so $CH_0^0(X)$ is an abelian variety A', as a finite quotient of A. Also the map alb : $A' \to \text{Alb}\, X$ is an algebraic map of abelian varieties, and induces by Theorem 2.2 an isomorphism on torsion points, so it is an isomorphism.

Proof of Theorem 2.2. : We follow partially [2], because there is a point which is not clear in Roitman's paper [6]. Notice that the surjectivity is clear because there is a curve C in X such that $JC \to \text{Alb}\, X$ is surjective with connected fibers, by the weak Lefschetz theorem. So we find that k-torsion $(CH_0^0(C)) = k - \text{torsion}(JC) \twoheadrightarrow k - \text{torsion}(\text{Alb}\, X)$. For the injectivity, Bloch does the following :

Step 1 : Let $Z = \Sigma n_i Z_i$ be a k-torsion cycle in X. By definition of rational equivalence, one can find curves C_i in X and rational functions φ_i on C_i such that $\Sigma \,\text{div}\, \varphi_i = kZ$. By birational invariance of CH_0, one can blow up X to $\widetilde{X}$, and replace UC_i by a stable curve C [(1)]. One can choose a smooth surface S containing C such that $\text{Alb}\, S \to \text{Alb}\, \widetilde{X} = \text{Alb}\, X$

[(1)] An important point here is the fact that one can connect the local components of the proper transform of UC_i by rational curves.

is an isomorphism. (By the weak Lefschetz theorem, it suffices to choose for S a complete intersection of ample divisors, and C being stable, smoothness of S is possible). Now, assuming $\mathrm{alb}_X(Z) = 0$, one has $\mathrm{alb}_S(Z) = 0$, and it suffices to show that Z is rationally equivalent to zero in S.

Finally we may add components to C, so that the new curve C' is a very ample divisor in S.

Now the technical point in Bloch's proof (this is curve theory) is the following

Step 2 : We may move Z on C' up to rational equivalence in such a way that it is supported on the smooth part of C' and determines a k-torsion line bundle L_k on C', or a k-torsion point in the generalized jacobian of C'.

Admitting this, the end of the proof goes as follows :

Step 3 : We can deform C' to a smooth curve C'' in the same linear system on S, and the line bundle L_k has accordingly a deformation L''_k as a k-torsion line bundle on C'.

Now one can use here the Roitman argument : Consider the pencil $\mathbf{P}^1$ on S determined by C' and C'' and let $D \to \mathbf{P}^1$ be the covering parametrizing k-torsion line bundles on fibers $C_t)_{t\in\mathbf{P}^1}$. Then the map $D \to CH_0^0(S)$ given by $d \in D \to L_d$ on $C_t \to j_{t*}(L_d) \in CH_0^0(S)$ is constant on connected components of D.

This is because for a component D' of D, $CH_0^0(D')$ is divisible and the image of the induced map $CH_0^0(D') \to CH_0^0(S)$ is contained in the k-torsion of $CH_0^0(S)$, so is 0.

Step 4 : ([2], [6]). Now we have a smooth curve C very ample on S, and may assume that it belongs to a Lefschetz pencil on S. We have a k-torsion line bundle on it, which sends to 0 in $\mathrm{Alb}\, S$, via $j_* : JC \to \mathrm{Alb}\, S$, and we want to show that it goes to zero in $CH_0^0(S)$, via $j_* : JC \to CH_0^0(S)$. The kernel of $j_* : JC \to \mathrm{Alb}\, S$ is a finite quotient of the connected abelian variety $(JC)^0 = (H^0(\Omega_C)^*)^0/H_1(C,\mathbf{Z})^0$ where $(H^0(\Omega_C)^*)^0 = \mathrm{Ker}(j_* : H^0(\Omega_C)^* \to (H^0(\Omega_S))^*)$, and $H_1(C,\mathbf{Z})^0 = \mathrm{Ker}(j_* : H_1(C,\mathbf{Z}) \to H_1(S,\mathbf{Z}))$. The connectedness follows from the surjectivity of this last j_* (Lefschetz). k-torsion points in $\mathrm{Ker}\, j_*$ lift to k'-torsion points in $(JC)^0$ for some k'.

One knows by Lefschetz theory that $H_1(C,\mathbf{Z})^0$ is generated by the vanishing cycles of the pencil (see Lecture 4) and that if the discriminant hypersurface for the linear system associated to C is irreducible, which is true for very ample C, the vanishing cycles are all conjugate under the monodromy action.

The k'-torsion of $(JC)^0$ is generated by the $\frac{1}{k'}\,\delta_i$ modulo $H_1(C,\mathbf{Z})^0$ and by the Roitman argument in step 3 they all have the same image in $CH_0^0(S)$. More precisely, it follows from this argument that the map $j_* : k' - \text{torsion}((JC)^0) \to CH_0^0(S)$ is invariant under monodromy.

So for $v \in \frac{1}{k'}H_1(C,\mathbf{Z})^0$ and δ_i a vanishing cycle, the Picard-Lefschetz formula gives $j_*(\bar{v}) = j_*(\overline{v + (\delta_i|v).\delta_i})$ where $\overline{}$ means reduction modulo $H_1(C,\mathbf{Z})^0$. Hence :

2.4.1. $(k'v|\delta_i) \cdot j_*(\overline{\frac{1}{k'}\delta_i}) = 0.$

Finally, the last trick in Bloch's proof is :

2.4.2. Given δ a vanishing cycle, the integers $(\delta|\delta_i)$ have no common multiple, where δ_i runs through the set of vanishing cycles.

Otherwise by the Picard-Lefschetz formula the cycles δ' obtained from δ by monodromy action would satisfy $\delta' = \delta$ modulo $dH_1(C,\mathbf{Z})^0$, which is absurd because $H_1(C,\mathbf{Z})^0$ is generated by those δ''s and, if it is non zero (which one may assume !), it has rank ≥ 2.

Using 2.4.1, with $v = \frac{1}{k'}\,\delta$, one deduces from 2.4.2 that there exists integers m_i, with $\Sigma m_i = 1$, such that ${}_i m_i j_*(\overline{\frac{1}{k'}\delta_i}) = 0$, and because all $j_*(\overline{\frac{1}{k'}\delta_i}) = 0$ are equal, we find that j_* vanishes on the k'-torsion of $(JC)^0$.

References;

[1] S. Bloch : Lectures on algebraic cycles, Lecture one, Duke University Mathematics series IV, Durham 1980.

[2] S. Bloch : Torsion algebraic cycles and a theorem of Roitman, Compo. Math. 39 (1979) 107-127.

[3] S. Bloch, A. Kas, D. Lieberman : Zero-cycles on surfaces with $p_g = 0$, Compo. Math. 33 (1976) 135-145.

[4] D. Mumford : Rational equivalence of zero-cycles on surfaces, J. Math. Kyoto Univ. 9 (1969) 195-204.

[5] A.A. Roitman : Rational equivalence of zero-cycles, Math. USSR, Sbornik 18 (1972) 571-588.

[6] A.A. Roitman : The torsion of the group of 0-cycles modulo rational equivalence, Annals of Mathematics 111 (1980) 553-569.

Lecture 6 : Zero-cycles II

0. We continue on zero-cycles and turn now to more recent contributions. We first explain the ideas of Bloch-Srinivas [2], which are based on an elementary lemma, but which shed a new light on Mumford's theorem. In Mumford's approach the accent was put on the (local) relation zero-cycles $\leftrightarrow$ holomorphic forms. In the Bloch-Srinivas approach correspondences are considered as global objects which are shown to be essentially controlled by their action on the CH_0 group. This approach leads more immediately to global results on the effect of a correspondence on the Hodge theory of a variety (more precisely on its "motive"), and we shall describe a few applications of it. The rest of the lecture is devoted to a proof [6] of Bloch's "subconjecture" (see lecture 5) for Godeaux surfaces, which are surfaces of the general type with $q = p_g = 0$ obtained as quotients of complete intersection surfaces by a finite group, and to a generalization of the Mumford criterion, in the case of families of surfaces [7]. There we consider a family of 0-cycles $(Z_b)_{b\in B}$ in a family of surfaces $(S_b)_{b\in B}$, and we give a Hodge theoretic criterion for Z_b to be rationally equivalent to zero in S_b, $\forall\ b \in B$. We explain the two following applications: (Here the restriction to surfaces in $\mathbf{P}^3$ is not essential and simply motivated by the fact that the algebra related to infinitesimal variations of Hodge structures is well understood in this case (see M. Green lectures)):

1) If $C \subset S$ is a general plane section of a general surface of degree $d \geq 5$ in $\mathbf{P}^3$, $\operatorname{Ker} j_* : JC \to CH_0^0(S)$ is equal to the torsion of JC.

2) If $S \subset \mathbf{P}^3$ is a general surface of degree $d \geq 7$, two distinct points of S are not rationally equivalent.

It should be noticed that for the second statement, we come back to the local approach of Mumford, in the sense that we study restrictions of holomorphic forms to families of rationally equivalent cycles.

1) 1.1. Let X, Y be smooth algebraic varieties over $\mathbb{C}$, and $V \subset Y$ an algebraic subset. Let $Z \subset X \times Y$ be a zero correspondence between X and Y, so $Z = \Sigma n_i Z_i$ with$Z_i \underset{pr_1}{\longrightarrow} X$ generically finite. There is a common algebraically closed field k of definition of $X, Y, V Z_i$, and we may assume that k has finite transcendance degree over $\mathbb{Q}$. Then the function field $k(X)$ and any algebraic extension of it admits embeddings to $\mathbb{C}$, extending a given embedding $k \subset \mathbb{C}$. The Z_i can be considered as points of Y defined over $k(Z_i)$, and we can find a Galois extension L of $k(X)$ containing all $k(Z_i)$. For each inclusion γ_ℓ of $k(Z_i)$ in L over $k(X)$, $\ell = 1, \ldots, d^0 k(Z_i)/k(X)$, one has the L point Z_i^ℓ of Y obtained from

Z_i by extension of scalars $k(Z_i) \underset{\gamma_\ell}{\longrightarrow} L$. One has then a zero-cycle of Y, defined over $L : Z_L = \sum_{i,\ell} n_i Z_i^\ell$.

Let us choose compatible embeddings

$$\begin{array}{ccc} k(X) & \hookrightarrow & \mathbf{C} \\ \searrow & & \nearrow \\ & L & \end{array} .$$

Then Z_L gives a zero-cycle $Z_{\mathbf{C}} \in Y_{\mathbf{C}}$ by extension of scalars $L \hookrightarrow \mathbf{C}$, and the diagonal $\Delta \subset X \times X$, seen as a point of X defined over $k(X)$, gives a point $\Delta_{\mathbf{C}} \in X_{\mathbf{C}}$. Clearly $Z_{\mathbf{C}} = Z(\Delta_{\mathbf{C}}) = pr_{2*}(pr_1^*(\Delta_{\mathbf{C}}).Z)$.

Now assume that X, Y, V satisfy the following property:

1.1.1. $\forall\, x \in \Delta_{\mathbf{C}},\ Z(x)$ is rationally equivalent to zero in $Y_{\mathbf{C}} \backslash V_{\mathbf{C}}$.

It follows that $Z_{\mathbf{C}}$ is rationally equivalent to zero in $Y_{\mathbf{C}} \backslash V_{\mathbf{C}}$. Then Z_L is rationally equivalent to zero in $Y \backslash V$, over a finite extension of L, which means that there exists a Zariski open set $V \subset X$, and a finite, flat and proper morphism $\varphi : U' \to U$ such that $(\varphi \times \mathrm{Id})^*(Z_{|U \times Y \backslash V})$ is rationally equivalent to zero. It follows that $(\varphi \times \mathrm{Id}_{Y \backslash V})_*(\varphi \times \mathrm{Id})^*(Z_{|U \times Y \backslash V}) = d^0\varphi(Z_{|U \times Y \backslash V})$ is rationally equivalent to zero.

Let $D = X \backslash U$. Then the exact sequence:

$$CH(D \times Y \cup X \times V) \longrightarrow CH(X \times Y) \longrightarrow CH(U \times Y \backslash V) \longrightarrow 0$$

shows that $d^0\varphi.Z$ is rationally equivalent to a cycle supported on $D \times Y \cup X \times V$. So we have

1.2. Proposition: Let X, Y, V, Z be as before, and assume that 1.1.1 holds: $\forall\, x \in X_{\mathbf{C}},\ Z(x) \equiv 0$ in $Y_{\mathbf{C}} \backslash V_{\mathbf{C}}$. Then there exists a divisor D of X such that a multiple of Z is supported on $D \times Y \cup X \times V$, modulo rational equivalence.

(Note that in 1.2, one can obviously allow Z to have components of codimension $\dim Y$ which are not finite over X).

Let us give an easy but useful corollary:

1.2.1. Corollary: Let X, Y be two varieties of the same dimension n and $Z \subset X \times Y$ be a codimension n cycle. Then if Z satisfies property 1.1.1, for some proper algebraic subset V of Y, ${}^tZ \subset Y \times X$ also satisfies this property. In particular if X and Y are surfaces $Z : CH_0^0(X) \to CH_0^0(Y)$ factors through $\mathrm{Alb}\, X$, if and only if ${}^tZ : CH_0^0(Y) \to CH_0^0(X)$ factors through $\mathrm{Alb}\, Y$.

For the last statement, one notes the existence of a curve $C \subset X$ such that $JC \twoheadrightarrow \mathrm{Alb}\, X$. Obviously there is a curve $C' \subset Y$ such that $Z(CH_0^0(C))$ is supported on C'; then one takes for V the curve C', and there exists a curve $D \subset X$ and an integer N such that NZ is supported up to rational equivalence in $D \times Y \cup X \times C'$. It follows that N^tZ factors through a map (given by an algebraic correspondence ($\Gamma : CH_0^0(Y) \to CH_0^0(D)$, hence by the universal property of the Albanese map, $N^tZ_{|CH_0^0(Y)}$ factors through $\mathrm{Alb}\, Y$. The same is true for ${}^tZ$ by divisibility of $\mathrm{Ker}(CH_0^0(Y) \xrightarrow{\mathrm{alb}} \mathrm{Alb}\, 4)$. Notice that divisibility of CH_0^0 is also used in the proof of the first statement.

For the applications in Bloch-Srinivas one considers the diagonal cycle $\Delta \subset X \times X$. Proposition 1.2 then gives:

1.3. Proposition: Assume that $\exists\ V \subset X$ such that $CH_0(X_{\mathbb{C}} \backslash V_{\mathbb{C}}) = 0$ then for some integer N, $N\Delta$ is up to rational equivalence supported on $D \times X \cup X \times V$, for some divisor D of X. We will write $N\Delta \equiv \Gamma_1 + \Gamma_2$, with $\Gamma_1 \subset D \times Y$ and $\Gamma_2 \subset X \times V$.

Remark: It is a very interesting problem to decide whether the integer N can be set equal to 1. This is the case if X is a rational variety, and the minimal such N is a birational invariant.

1.3.1. The correspondence Δ acts on all Chow groups as the identity. The action is given by $\gamma \mapsto pr_{2*}(pr_1^*\gamma.\Delta)$ or by $\gamma \mapsto pr_{1*}(pr_2^*\gamma.\Delta)$. Replacing $N\Delta$ by $\Gamma_1 + \Gamma_2$, one finds that this decomposition, which is obtained only by the consideration of the action of Δ on CH_0, has many implications on the other Chow groups. Let us first recover the Mumford-Roitman theorem:

1.4. Proposition: With the notations of proposition 1.3, one has

$$H^{k,0}(X) = 0, \text{ for } k > \dim V.$$

Proof: Let $\widetilde{D} \xrightarrow[\widetilde{j}]{} X$ be a desingularization of $D \underset{j}{\hookrightarrow} X$. There exists a cycle $\widetilde{\Gamma}_1 \subset \widetilde{D} \times Y$ such that $N\Delta = \widetilde{j}_*\widetilde{\Gamma}_1 + \Gamma_2$ with $\Gamma_2 \subset X \times V$. The action of ${}^t\Gamma_2$ on $H^k(X)$, ${}^t\Gamma_2 = pr_{1*}(pr_2^*(\ ).\Gamma_2)$, factors through the restriction to V, hence annihilates $H^{k,0}(X)$, for $k >$

$\dim V$, so we find: $\forall\, \omega \in H^{k,0}(X)$, $N\omega = j_*({}^t\widetilde{\Gamma}_1(\omega))$. But j_* is a morphism of Hodge structures of bidegree $(1,1)$, so its image does not contain a non zero element of type $(k,0)$, and we find:$N\omega = 0$.

Now Bloch and Srinivas give the analogous consequences on Chow groups. As an example they assume $\dim V \leq 3$ and give the following consequences of 1.3 on $CH^2(X)$:

1.5. Theorem: i) If $\dim V \leq 3$, the Hodge conjecture for rational $(2,2)$ classes on X holds.

ii) If $\dim V \leq 2$, homological equivalence and algebraic equivalence coincide on $CH^2(X)$.

iii) If $\dim V \leq 1$, $CH^2(X)_{\mathrm{hom}}$ is isomorphic to $J^2(X)$ via the Abel-Jacobi map Φ_X.

We will show ii) and iii) only up to torsion, the argument being then very easy. For the analysis of the torsion in the Chow groups we refer to J. Murre's lectures.

Proof of 1.5. i) Let α be a $(2,2)$ integral class in $H^4(X)$. We want to show that a multiple of α is algebraic, and by 1.3.1 we need only to show that ${}^t\Gamma_1(\alpha)$, ${}^t\Gamma_2(\alpha)$ are algebraic. Using desingularizations of D and V, we have ${}^t\Gamma_1(\alpha) = \tilde{j}_*(\beta)$, where β is a $(1,1)$ integral class on $\widetilde{D}$, and ${}^t\Gamma_2(\alpha) = {}^t\widetilde{\Gamma}_2(\alpha_{|\widetilde{V}})$, where $\widetilde{\Gamma}_2$ is the desingularized correspondence Γ_2, between X and $\widetilde{V}$. By the Lefschetz theorem on $(1,1)$ classes, β is algebraic. From $\dim V \leq 3$, we conclude that a multiple of $\alpha_{|\widetilde{V}}$ is also algebraic, because the Hodge conjecture is true in degree 4, for varieties of dimension less than 3. (By the hard Lefschetz theorem, the Lefschetz theorem on $(1,1)$ classes implies the Hodge conjecture in degree $2\dim_{\mathbb{C}}(\;) - 2$). So ${}^t\Gamma_1(\alpha)$ and ${}^t\Gamma_2(\alpha)$ are algebraic, and i) is proved.

ii) Let Z be a codimension two cycle homologous to zero. Then $\dim V \leq 2 \Rightarrow Z.\widetilde{V}$ is algebraically equivalent to zero on $\widetilde{V}$. Also ${}^t\widetilde{\Gamma}_1(Z) \subset \widetilde{D}$ is a divisor in $\widetilde{D}$ homologous, hence algebraically equivalent, to zero. So $NZ = j_*({}^t\widetilde{\Gamma}_1(Z)) + {}^t\widetilde{\Gamma}_2(Z.\widetilde{V})$ is algebraically equivalent to zero.

iii) $\dim V \leq 1 \Rightarrow {}^t\widetilde{\Gamma}_2$ vanishes on $CH^2(X)$. So we have $N\,\mathrm{Id} = j_* \circ {}^t\widetilde{\Gamma}_1$ on $CH^2(X)$, and on $CH^2(X)_{\mathrm{hom}}$, we have the following diagram:

1.5.1.
$$\begin{array}{ccccc} CH^2(X)_{\mathrm{hom}} & \xrightarrow{{}^t\widetilde{\Gamma}_1} & CH^1(\widetilde{D})_{\mathrm{hom}} & \xrightarrow[j_*]{} & CH^2(X)_{\mathrm{hom}} \\ \Phi_X \downarrow & & \Phi_{\widetilde{D}} \downarrow & & \downarrow \Phi_X \\ J^3(X) & \xrightarrow[{}^t\widetilde{\Gamma}_1]{} & J^1(\widetilde{D}) & \xrightarrow[j_*]{} & J^3(X). \end{array}$$

We refer to [6] for the commutativity of 1.5.1. On the last line ${}^t\widetilde{\Gamma}_1$ and j_* are the morphisms between abelian varieties corresponding to the morphisms of Hodge structures ${}^t\widetilde{\Gamma}_2$ and j_* ($J^3(X)$ is an abelian variety by 1.4 and (Lecture 2)). Now ${}^t\widetilde{\Gamma}_2$ annihilates $H^3(X)$, as $\dim V \leq 1$, so on the last line, $j_* \circ {}^t\widetilde{\Gamma}_1 = N \times \mathrm{Id}$. $\Phi_{\widetilde{D}}$ being an isomorphism it follows immediately from 1.5.1 that Φ_X is surjective with kernel contained in the N-torsion of $CH^2(X)$.

Remark: 1.5. i) generalizes [3], and 1.5. ii) generalizes [1] and other works on Fano threefolds.

2) 2.1. We explain now the method of [7] to prove the conjecture 1.8 of lecture 5 for the following type of surfaces:

1) Consider $G = \mathbb{Z}/5\mathbb{Z}$ acting on $\mathbb{P}^3$ by $g_\zeta^*(X_0, \ldots, X_3) = (\zeta X_0, \ldots, \zeta^4 X_3)$, for ζ a primitive 5^{th} root of unity. For a generic $F \in H^0(\mathcal{O}_{\mathbb{P}^3}(5))$, satisfying $g_\zeta^*(F) = F$, $S = V(F)$ is smooth, G acts freely on it, $\Sigma := S/G$ is of general type, and $H^0(K_\Sigma) = H^0(K_S)^{\text{inv}} = 0$, where "inv" means "invariant part under G".

2) Consider $G = \mathbb{Z}/8\mathbb{Z}$ acting on $\mathbb{P}^6$ by $g_\zeta^*(X_0, \ldots, X_6) = (\zeta X_0, \ldots, \zeta^7 X_6)$. Let $Q_i \in H^0(\mathcal{O}_{\mathbb{P}^6}(2))$ be general quadrics satisfying $g_\zeta^* Q_i = \zeta^{2i} Q_i$, $i = 1, \ldots, 4$. Then $S = \cap Q_i$ satisfies the same conclusion as in 1. We shall prove:

2.2. Theorem: [7] for $\Sigma = S/G$ as in 1) or 2), $CH_0^0(\Sigma) = 0$.

Let us first give the argument for case 1):

Step 1: The linear system H made of G-invariant quintic polynomials on $\mathbb{P}^3$ has no base point. So S is covered by smooth curves $C = S \cap S'$, with $S' = V(F')$, $F' \in H$. If $x, y \in S$ are generic there is such a C containing x and y. Let $\varphi : S \to \Sigma$ be the quotient map. By the Roitman theorem, and $\mathrm{Alb}(\Sigma) = 0$, $CH_0^0(\Sigma)$ has no torsion and it suffices to prove $\varphi^* CH_0^0(\Sigma) = 0$. $\varphi^* CH_0^0(\Sigma)$ is generated by cycles $Z = \sum_{g \in G} g.x - g.y$, for generic $x, y \in S$. Let $C \underset{j}{\hookrightarrow} S$ be a curve as above containing x and y. Then $\sum_{g \in G} g.x - g.y$ is a G-invariant 0-cycle Z' of degree zero on C and $Z = j_* Z'$. So it suffices to prove:

2.2.1. For C as above, the map $j_* : (JC)^{\text{inv}} \underset{j_*}{\longrightarrow} CH_0(S)$ is 0, where $(JC)^{\text{inv}}$ is the invariant part of JC under G.

Step 2: Let us consider the pencil $(S_t)_{t \in \mathbb{P}^1}$ determined by S and S'. Each S_t is defined by a G-invariant quintic polynomial so has no invariant holomorphic two form.

By the Lefschetz theorem on $(1,1)$ classes, it follows that $H^2(S_t,\mathbf{Z})^{\mathrm{inv}}$ is generated by classes of G-invariant line bundles on S_t, and because $H^2(S_t,\mathbf{Z})^{\mathrm{inv}}$ is finitely generated, we conclude:

2.2.2. There exists a smooth ramified cover $D \xrightarrow[r]{} \mathbf{P}^1$, such that D parametrizes G-invariant line bundles on fibers S_t (i.e. to $d \in D$ corresponds a line bundle L_d on S_t, $t = r(d)$), and such that, over the open set U of $\mathbf{P}^1$ parametrizing smooth S_t's, the map $\alpha : r_*\mathbf{Z}_{|U} \to (R^2\pi_*\mathbf{Z})^{\mathrm{inv}}{}_{|U}$, which sends 1_d to $c_1(L_d)$, is surjective. Here $\pi : \mathcal{S} \to \mathbf{P}^1$ is the family of surfaces $(S_t)_{t\in\mathbf{P}^1}$.

We have a natural map $\beta : D \to (\mathrm{Pic}\, C)^{\mathrm{inv}}$, which associates $j_t^*(L_d)$ to $d \in D$, where j_t is the inclusion of C in S_t. We also write β for the induced map $JD \to (\mathrm{Pic}^0\, C)^{\mathrm{inv}} = (JC)^{\mathrm{inv}}$. Now we show that:

2.2.3. $(JC)^{\mathrm{inv}}$ is generated by $\mathrm{Im}\,\beta$ and by the various $j_{t_i}^*(\mathrm{Pic}^0(\widetilde{S}_{t_i})^{\mathrm{inv}})$ where the S_{t_i}'s are the singular surfaces of the pencil and $\widetilde{S}_{t_i} \to S_{t_i}$ is a G-invariant desingularization.

To prove 2.2.3, we blow up C in $\mathbf{P}^3$, so that $\mathcal{S} = \widetilde{\mathbf{P}}^3$ and G acts on $\widetilde{\mathbf{P}}^3$. It is well known that there is a natural isomorphism

2.2.4. $$JC \simeq J^3(\widetilde{\mathbf{P}}^3), \quad (JC)^{\mathrm{inv}} \simeq J^3(\widetilde{\mathbf{P}}^3)^{\mathrm{inv}}.$$

The surjective map $\alpha : r_*\mathbf{Z}_{|V} \to (R^2\pi_*\mathbf{Z})^{\mathrm{inv}}{}_{|U}$ induces a surjective map $\alpha : H^1(r_*\mathbf{Z}_{|V}) \to H^1\left((R^2\pi_*\mathbf{Z})^{\mathrm{inv}}{}_{|U}\right)$. By the Leray spectral sequence for $\pi : V := \pi^{-1}(U) \to U$ one has: $H^3(V,\mathbf{Q})^{\mathrm{inv}} = H^1\left((R^2\pi_*\mathbf{Q})^{\mathrm{inv}}{}_{|V}\right)$. So we have:

$$\alpha : H^1\left(r^{-1}(U),\mathbf{Q}\right) \twoheadrightarrow H^3(V,\mathbf{Q})^{\mathrm{inv}}.$$

The map $\beta : JD \to (JC)^{\mathrm{inv}} = J^3(\widetilde{\mathbf{P}}^3)^{\mathrm{inv}}$ induces a map $\beta_* : H^1(D,\mathbf{Z}) \to H^3(\widetilde{\mathbf{P}}^3,\mathbf{Z})^{\mathrm{inv}}$ and it is not difficult to check that the following diagram commutes:

2.2.5. $$\begin{array}{cccc} \alpha & : & H^1\left(r^{-1}(U),\mathbf{Q}\right) & \longrightarrow & H^3(V,\mathbf{Q})^{\mathrm{inv}} \\ & & \uparrow \text{restriction} & & \uparrow \text{restriction} \\ \beta_* & : & H^1(D,\mathbf{Q}) & \longrightarrow & H^3(\widetilde{\mathbf{P}}^3,\mathbf{Q})^{\mathrm{inv}}. \end{array}$$

α and β_* are morphisms of mixed Hodge structures [4], and it follows from the strictness of such maps for the W-filtration that the surjectivity of α implies that of:

2.2.6. $$\beta_* : H^1(D,\mathbf{Q}) \longrightarrow H^3(\widetilde{\mathbf{P}}^3,\mathbf{Q})^{\mathrm{inv}} / \mathrm{Ker}(\text{restriction}).$$

On the other hand, the kernel of the restriction map $H^3(\widetilde{\mathbf{P}}^3, \mathbf{Q})^{\text{inv}} \to H^3(V, \mathbf{Q})^{\text{inv}}$ is equal to:

2.2.7. $$\sum_i \widetilde{k}_{t_i *}\, H^1(\widetilde{S}_{t_i}, \mathbf{Q})^{\text{inv}},$$

where $\widetilde{S}_{t_i} \underset{\widetilde{k}_{t_i}}{\longrightarrow} \widetilde{\mathbf{P}}^3$ is composed of the desingularization map and of the inclusion $k_{t_i} : S_{t_i} \to \widetilde{\mathbf{P}}^3$. $\widetilde{k}_{t_i}$ is induced on rational homology by the corresponding maps:

2.2.8. $$\widetilde{k}_{t_i *} : \mathrm{Pic}^0(\widetilde{S}_{t_i})^{\text{inv}} \longrightarrow CH_1(\widetilde{\mathbf{P}}^3)^{\text{inv}}_{\text{hom}} \underset{\text{Abel-Jacobi}}{\longrightarrow} J^3(\widetilde{\mathbf{P}}^3)^{\text{inv}},$$

which can also be identified to $j_{t_i}^* : \mathrm{Pic}^0(\widetilde{S}_{t_i})^{\text{inv}} \to (JC)^{\text{inv}}$. From 2.2.5, 2.2.7 and 2.2.8 we conclude that the map of abelian varieties $\beta \oplus \sum_i j_{t_i}^* : JD \oplus \bigoplus_i \mathrm{Pic}^0(\widetilde{S}_{t_i})^{\text{inv}} \to JC$ induces a surjective map on rational homology, hence is surjective.

Step 3. We have shown that $(JC)^{\text{inv}}$ is generated by line bundles of degree 0 in $\bigoplus_{t\in\mathbf{P}^1} j_t^* : \bigoplus_{t\in\mathbf{P}^1} \mathrm{Pic}(\widetilde{S}_t)^{\text{inv}} \to \mathrm{Pic}\, C$, (where $\widetilde{S}_t = S_t$ if S_t is non singular).

To conclude that $j_* : (JC)^{\text{inv}} \to CH_0^0(S)^{\text{inv}}$ is 0 it suffices to note the commutativity of the following diagram: (for S_t singular or not)

2.2.9. $$\begin{array}{ccc} \mathrm{Pic}\,\widetilde{S}_t & \xrightarrow{k_t *} & CH_1(\mathbf{P}^3) \\ \downarrow j_t^* & & \downarrow k^* \\ \mathrm{Pic}\, C & \underset{j_*}{\longrightarrow} & CH_0(S)\ , \end{array}$$

where $k_t : \widetilde{S}_t \to \mathbf{P}^3$ is the inclusion, eventually composed with desingularization. If $\Sigma n_t\, j_t^*(L_t)$, $L_t \in \mathrm{Pic}\,\widetilde{S}_t$, has degree 0 on C, $\Sigma n_t k_{t*}(L_t)$ is homologous to zero in $\mathbf{P}^3$, so is rationally equivalent to 0, hence $j_*(\Sigma n_t j_t^*(L_t)) = k^*(\Sigma n_t k_{t*}(L_t)) = 0$ in $CH_0^0(S)$, and 2.2.1 is proved.

2.3. The second case is treated similarly, replacing $\mathbf{P}^3$ by $X = Q_1 \cap Q_2 \cap Q_3$ which is a Fano threefold with a representable CH_1^{hom} group , and H by the linear system $\{Q_4\}$. Going thru the proof one concludes by the analog of the diagram 2.2.9 that the map $k^* : CH_1(X)_{\text{hom}} \to CH_0^0(S)^{\text{inv}}$ is surjective. So $CH_0^0(\Sigma)$ is finite dimensional, hence is zero by $\mathrm{Alb}\,\Sigma = 0$, and by Roitman's theorem (lecture 5).

3) 3.1. We explain now a generalization of Mumford's criterion for 0-cycles in a family of surfaces: [8]

We consider a family of smooth regular projective surfaces over a smooth quasi-projective basis $B : \mathcal{S} \xrightarrow[\pi]{} B$. Let $Z \subset \mathcal{S}$ be a codimension two cycle flat over B such that $\forall\, b \in B$, Z_b has degree zero on S_b. The cycle Z has a class in $H^2(\Omega^2_{\mathcal{S}})$ (see [5]). Hence there is an induced section δ_Z of $\operatorname{Ker} H^0(R^2\pi_*\Omega^2_{\mathcal{S}}) \to H^0(R^2\pi_*\Omega^2_{\mathcal{S}/B})$. We write now the exact sequence: $0 \to \pi^*\Omega_B \to \Omega_{\mathcal{S}} \to \Omega_{\mathcal{S}/B} \to 0$, which gives:

3.1.1. a) $0 \to K \to \Omega^2_{\mathcal{S}} \to \Omega^2_{\mathcal{S}/B} \to 0$ defining K,

b) $0 \to \pi^*\Omega^2_B \to K \to \pi^*\Omega_B \otimes \Omega_{\mathcal{S}/B} \to 0$.

It follows that δ_Z identifies to a section of $R^2\pi_*K$, and then to a section of $\Omega^2_B \otimes R^2\pi_*\mathcal{O}_{\mathcal{S}}/\Psi(\Omega_B \otimes R^1\pi_*\Omega_{\mathcal{S}/B})$ where Ψ is obtained by the long exact sequence associated to 3.1.1. b). In terms of variations of Hodge structure, we have on B the "VHS" $H^2_{\mathbf{Z}} = R^2\pi_*\mathbf{Z}$, $\mathcal{H}^2 = H^2_{\mathbf{Z}} \otimes \mathcal{O}_B$, with Hodge filtration $F^i\mathcal{H}^2$, such that $\mathcal{H}^{0,2} = \mathcal{H}^2/F^1\mathcal{H}^2 = R^2\pi_*\mathcal{O}_{\mathcal{S}}$, $\mathcal{H}^{1,1} = F^1\mathcal{H}^2/F^2\mathcal{H}^2 = R^1\pi_*(\Omega_{\mathcal{S}/B})$. The infinitesimal variation of Hodge structure gives $\overline{\nabla} : \mathcal{H}^{1,1} \to \Omega_B \otimes \mathcal{H}^{0,2}$, and then $\overline{\nabla}_2 : \mathcal{H}^{1,1} \otimes \Omega_B \to \Omega^2_B \otimes \mathcal{H}^{0,2}(\overline{\nabla}_2(\alpha \otimes \omega) = \omega \wedge \overline{\nabla}\alpha)$. Griffith's description of $\overline{\nabla}$ gives that $\overline{\nabla}_2 = \Psi$.

So we have associated to Z the infinitesimal invariant $\delta_Z \in \mathcal{H}^{0,2} \otimes \Omega^2_B / \operatorname{Im} \overline{\nabla}_2$.

3.2. Suppose now that Z satisfies:

3.2.1. $\forall\, b \in B$, Z_b is rationally equivalent to zero in S_b: arguing as in Bloch-Srinivas, we see that up to torsion, modulo rational equivalence, Z is supported over a proper algebraic subset of B. By [5] it follows then that the class of Z vanishes over the complementary of this subset and we conclude:

3.3. Proposition. Assume $\mathcal{H}^{0,2} \otimes \Lambda^2\Omega_B / \operatorname{Im} \overline{\nabla}_2$ has constant rank. Then, if Z satisfies 3.2.1, δ_Z vanishes on B.

Notice that 3.3 is exactly Mumford's theorem in the case where $\mathcal{S}$ does not vary, i.e. $\mathcal{S} = S \times B$, because δ_Z identifies to the trace map $Z : H^0(K_S) \to H^0(\Omega^2_B)$ in this case.

For the applications, we give two descriptions of δ_Z:

3.4. Assume now that Z is a divisor of relative degree 0 in $\mathcal{C}$, where $\mathcal{C} \to B$ is a smooth family of curves over B, and that $\mathcal{C} \underset{j}{\hookrightarrow} \mathcal{S}$ is an inclusion over B.

Then Z has an infinitesimal invariant $\delta\nu_Z \in \mathcal{H}^{0,1}_C \otimes \Omega_B/\operatorname{Im}\overline{\nabla}_C$ (see M. Green's lectures), and if the maps $\overline{\nabla}_C : \mathcal{H}^{1,0}_C \to \mathcal{H}^{0,1} \otimes \Omega_B$, $\overline{\nabla}_{2,C} : \mathcal{H}^{1,0}_C \otimes \Omega_B \to \mathcal{H}^{0,1}_C \otimes \Lambda^2\Omega_B$ (infinitesimal variation of Hodge structure of the family $\mathcal{C}$) are injective one has:

3.4.1. $\delta\nu_Z = 0 \Rightarrow \exists$ locally a flat section of $\mathcal{H}^1_C$, projecting onto ν_Z (the normal function associated to Z), unique up to a section of $H^1_{\mathbb{Z}}$. Now we have:

3.5. Proposition: There exists a natural map $j_* : \mathcal{H}^{0,1}_C \otimes \Omega_B/\operatorname{Im}\overline{\nabla}_C \longrightarrow \mathcal{H}^{0,2} \otimes \Omega^2_B/\operatorname{Im}\overline{\nabla}_2$ such that $j_*(\delta\nu_Z) = \delta_Z$, where δ_Z is the invariant of 3.1 for the cycle $j(Z)$.

3.6. Assume finally that $Z \subset \mathcal{S}$ is given by $\sum_i n_i\sigma_i(B)$, with $\sigma_i : B \to \mathcal{S}$, sections of π, and $\Sigma n_i = 0$.

Then if $N = \dim B$ one has an isomorphism at $0 \in B$:

3.6.1. $$\mathcal{H}^{0,2}_{(0)} \otimes \Omega^2_{B(0)}/\operatorname{Im}\overline{\nabla}_{2(0)} = \left[H^0(\Omega^N_{\mathcal{S}} \otimes \pi^*K^{-1}_{B|S_0})/H^0\left(\pi^*(\Omega^N_B \otimes K_B^{-1})_{|S_0}\right)\right]^*.$$

Then one proves:

3.7. Proposition: $\delta Z_{(0)}$, as an element of the dual of $H^0(\Omega^N_{\mathcal{S}} \otimes \pi^*K^{-1}_{B|S_0})/H^0(\pi^*\Omega^N_B \otimes K_B^{-1}{}_{|S_0})$, is equal to

$$\Sigma n_i\sigma_i^* : H^0(\Omega^N_{\mathcal{S}} \otimes \pi^*K_B^{-1}{}_{|S_0}) \longrightarrow H^0(\Omega^N_{B(0)} \otimes K^{-1}_{B(0)}) \simeq \mathbb{C}.$$

(Notice that $\Sigma n_i = 0 \Rightarrow \Sigma n_i\sigma_i^*$ vanishes on $H^0(\pi^*\Omega^N_B \otimes K_B^{-1}{}_{|S_0})$). Now we explain two applications of 3.3, 3.5, 3.7:

3.8. Theorem: [8] Let $S \subset \mathbb{P}^3$ be a general surface of degree $d \geq 5$, and $C \subset S$ a general plane section. Then $\operatorname{Ker} j_* : JC \to CH^0_0(S)$ is equal to the torsion of JC.

Sketch of proof: Let B be the moduli space of the pair (C, S) (or its smooth part). We have to show: If $U \to B$ is étale and

$$\begin{array}{ccccc} Z_U & \subset & \mathcal{C} & \underset{j}{\subset} & \mathcal{S} \\ & \searrow & \downarrow & \swarrow & \\ & & U & & \end{array}$$

is as in 3.4 and satisfies $\forall\ u \in U,\ j_*(Z_u)$ is rationally equivalent to zero in S_u, then Z_u is of torsion in JC_u. Now we use the following (this uses the nice properties of the jacobian rings describing the variation of Hodge structure of the pair (C, S)):

3.8.1. For $d \geq 5$, the map j_* of 3.5 is injective, for

$$\begin{array}{ccc} \mathcal{C} & \underset{j}{\subset} & \mathcal{S} \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

as above, when B is the universal deformation of (C, S). So if we have such Z_U, we find that $\delta\nu_{Z_u} = 0$, by 3.3 and 3.5, and then from $\overline{\nabla}^C_U$, $\overline{\nabla}^C_{2,U}$ injective, which is also a consequence of the general properties of jacobian rings, we conclude that ν_{Z_U} is a locally flat normal function. We use then the monodromy argument of lecture 4 to conclude that ν_{Z_U} is a torsion normal function, that is: $Z_u \in \mathrm{Tors}(JC_u)$ for $u \in U$.

3.9. Theorem: [8] Let S be a surface in $\mathbf{P}^3$ general of degree $d \geq 7$. Then if $p \neq q$ are points of S, they are not rationally equivalent in S.

Sketch of proof: Again by standard arguments it suffices to show: Let $U \to B$ be an étale cover of the moduli space of S. Let $\sigma_1,\ \sigma_2 : U \to \mathcal{S}_U$ be two distinct sections of the universal surface $\mathcal{S}_U$. Then for u general in U, $\sigma_1(u)$ and $\sigma_2(u)$ are not rationally equivalent in S_u.

If this is not the case, we apply 3.3 and 3.7 and this gives: For any $0 \in U$, the maps $\sigma_1^* : H^0(\Omega^N_{\mathcal{S}} \otimes \pi^* K_U^{-1}{}_{|S_0} \to \mathbb{C}$ and $\sigma_2^* : H^0(\Omega^N_{\mathcal{S}} \otimes \pi^* K_U^{-1}{}_{|S_0}) \to \mathbb{C}$ are equal ($N = \dim U$). For $\sigma_1(0) \neq \sigma_2(0)$, this contradicts the following:

3.10. Proposition: [8] For $d \geq 7$, $\mathcal{S} \underset{\pi}{\longrightarrow} B$ the local universal deformation of S_0, $N = \dim B$, the vector bundle $\Omega^N_{\mathcal{S}|S_0}$ is very ample on S_0.

From 3.8, one deduces another proof of a theorem of Xu: for $d \geq 5$, a general surface S in $\mathbf{P}^3$ of degree d contains no rational curve.

From 3.10, one has also the following geometric corollary: let $d \geq 7$, and C be a fixed curve ; then for a general surface S of degree d, there is no non constant map from C to S.

References:

[1] S. Bloch - J.P. Murre: On the Chow groups of certain types of Fano threefolds. Comp. Math. Vol. 39 (1979), 47–105.

[2] S. Bloch - V. Srinivas: Remarks on correspondences and algebraic cycles, Amer. J. Math. 105 (1983), 1235–1253.

[3] A. Conte - J.P. Murre: The Hodge conjecture for fourfolds admitting a covering by rational curves, Math. Ann. 238 (1978), 79–88.

[4] P. Deligne: Théorie de Hodge II, Publ. Math. IHES (40) (1971), 5–57.

[5] F. Elzein: Complexe dualisant et applications à la classe fondamentale d'un cycle, Bull. Soc. Math. France, Mémoire 58 (1978), 4–66.

[6] F. Elzein, S. Zucker: Extendability of normal functions associated to algebraic cycles, in *Topics in transcendental algebraic geometry*, ed. by P. Griffith, Annals of mathematics studies, Study 106, Princeton (1984), 269–288.

[7] C. Voisin: Sur les zéro-cycles de certaines hypersurfaces munies d'un automorphisme, Annali della Scuola Norm. Sup. di Pisa, Vol. 29 (1993), p. 473-492.

[8] C. Voisin: Variations de structure de Hodge et zéro-cycles des surfaces générales, preprint IHES (1993).

Lecture 7 : Griffiths group.

This lecture is devoted to another phenomenon which holds only for cycles of codimension ≥ 2 : The non triviality of the Griffiths group {cycles homologous to zero modulo cycles algebraically equivalent to zero}, [6], and more spectacularly in contrast with the divisor case, its non finite generation [3].

This last fact was discovered only recently, and this is related to the difficulty of constructing interesting cycles of codimension at least two, (excepted for the zero-cycles, for which the above-mentioned phenomena do not hold). We will devote the next lecture to this problem.

For codimension two cycles, it is at the moment conjectured that the Griffiths group can be detected using the Abel-Jacobi map, more precisely its projection in the transcendental part of the intermediate jacobian J^3 ([6], [8]). However, in the paper [8], it is shown that for higher codimension cycles, the Griffiths group can be non trivial modulo torsion, even if the corresponding intermediate jacobian is trivial.

The lecture is organized as follows : We first describe Griffiths' argument [6] for the non triviality of the Griffiths group, and continue with a sketch of Clemens' method to get infinite generation of the Griffiths group. We follow then [8], and explain the application of "Hodge theoretic connectivity" to non triviality in the Griffiths group of cycles restricted to general complete intersection subvarieties. We finally describe the ideas of Bloch and Ogus, and present, in a non rigorous way, the Bloch-Ogus resolution and the Bloch-Ogus formula for the group of cycles modulo algebraic equivalence [2].

1.

1.1. Griffiths [6] worked with quintic hypersurfaces in $\mathbb{P}^4$. This is the smallest degree for which these hypersurfaces have $h^{3,0} \neq 0$. These varieties also satisfy the following property, which suggests that they have an interesting CH^2 group, by analogy of what was known previously for cubics and quartics :

1.1.1. Fact : A generic quintic in $\mathbb{P}^4$ has a finite number $N > 1$ of (rigid) lines.

Now Griffiths proved, using 1.1., that the group $\mathrm{Hom}^2 / \mathrm{Alg}^2$ of cycles of codimension two homologous to zero modulo algebraic equivalence is generally a non torsion group :

1.2. Theorem : Let $X \subset \mathbb{P}^4$ be a general quintic 3-fold, $\ell_1 \neq \ell_2$ two distinct lines of X. Then $\ell_1 - \ell_2$ is a non-torsion element of $\mathrm{Hom}^2 / \mathrm{Alg}^2(X)$.

Step I : One has $H_2(X,\mathbb{Z}) = H^2(X,\mathbb{Z})^* = H^2(\mathbb{P}^4,\mathbb{Z})^*$ by weak Lefschetz theorem, and ℓ_1 and ℓ_2 have the same degree, so they are homologous. So one can use Φ_X, the Abel-Jacobi map of X, which gives

$$\Phi_X(\ell_1 - \ell_2) \in J^3(X) = H^3(X,\mathbb{C})/F^2H^3(X) \oplus H^3(X,\mathbb{Z}).$$

As explained by M. Green, the image of Φ_X on cycles algebraically equivalent to zero is contained in $J(X)^{\text{alg}} :=$ the maximal abelian subvariety of $J^3(X)$, with tangent space contained in $H^{1,2}(X) \subset H^3(X,\mathbb{C})/F^2H^3(X,\mathbb{C})$.

From the injectivity of $\overline{\nabla}$: $H^{1,2}(X) \to \text{Hom}(H^1(T_X), H^3(\mathcal{O}_X))$ (see M. Green, Lecture 4), one deduces now :

1.2.1. If X is general [(1)], no non zero integral class $\alpha \in H^3(X,\mathbb{Z})$ is contained in $F^1H^3(X)$.

It follows that $J(X)^{\text{alg}} = 0$, for general X. So we conclude that Theorem 1.2 follows from :

1.2.2. Proposition : If X is general, $\Phi_X(\ell_1 - \ell_2)$ is a non torsion point of $J^3(X)$, for $\ell_1 \neq \ell_2$ two lines in X.

Step II : Let $X \subset Y$, with Y a smooth quintic in $\mathbb{P}^5$ containing two planes P_1 and P_2 such that $P_1 \cap X = \ell_1$, $P_2 \cap X = \ell_2$. Let $(X_t)_{t\in\mathbb{P}^1}$ be a Lefschetz pencil of hyperplane sections of Y, such that $X_0 = X$.

On the open set $U \subset \mathbb{P}^2$ parametrizing smooth X_t's, there is a holomorphic section ν of the fibration $J \to U$ of intermediate jacobians with fiber $J_t = J^3(X_t)$, given by :

1.2.3. $\nu(t) = \Phi_{X_t}(\ell_1^t - \ell_2^t)$, where $\ell_i^t = P_i \cap X_t$ is a line in X_t, $i = 1, 2$.

1.2.2 is then equivalent to :

1.2.4. Proposition : ν is not a torsion section of J.

Step III : The sheaf of holomorphic sections of J over U is given by : $\mathcal{J} = \mathcal{H}^3/F^2\mathcal{H}^3 \oplus H^3_{\mathbb{Z}}$, where if $\mathcal{X}_U \underset{\pi}{\to} U$ is our family of threefolds, as usual $H^3_{\mathbb{Z}} = R^3\pi_*\mathbb{Z}, \mathcal{H}^3 = H^3_{\mathbb{Z}} \otimes \mathcal{O}_U$, and $F^2\mathcal{H}^3_{(t)} = F^2H^3(X_t) \subset H^3(X_t,\mathbb{C})$.

---

(1) Here "general" has not the usual sense : it means : locally outside a countable union of analytic subsets, instead of : outside a countable union of algebraic subsets.

Using the exact sequence :

1.2.5. $0 \to H^3_{\mathbf{Z}} \to \mathcal{H}^3/F^2\mathcal{H}^3 \to \mathcal{J} \to 0$. $\nu \in H^0(\mathcal{J})$ gives a natural element $[\nu] \in H^1(U, H^3_{\mathbf{Z}}) = \mathrm{Ker}(H^4(\mathcal{X}_U, \mathbf{Z}) \to H^4(X_t, \mathbf{Z}))$.

Then an important fact is :

1.2.6. Proposition : Let $\widetilde{Y}$ be the blow-up of Y along the base-locus of the pencil; let $\pi : \widetilde{Y} \to \mathbf{P}^1$ and $\tau : \widetilde{Y} \to Y$ be the natural maps. Then $\mathcal{X}_U = \pi^{-1}(U) \hookrightarrow \widetilde{Y}$, and we have: $[\nu] =$ restriction to $\mathcal{X}_U$ of $\tau^*([P_1 - P_2]) \in \mathrm{Ker}\, H^4(\widetilde{Y}) \to H^4(X_t)$.

Now 1.2.4 follows from 1.2.6, because $[P_1 - P_2] \in H^4(Y, \mathbf{Z})$ is not of torsion (this follows from the computation of the intersection $(P_1 - P_2)^2$), and more precisely no multiple of it is supported on some fiber of the pencil, by the Lefschetz assumption. This implies easily that the restriction of $\tau^*[P_1 - P_2]$ to $\mathcal{X}_U$ is not of torsion, so ν is not of torsion.

Finally, 1.2.6 is proved as follows : Consider the restrictions π_1, π_2 of π to the proper transforms $\widetilde{P}_1, \widetilde{P}_2$ of P_1, P_2 in $\widetilde{Y}$. There is a finite number of points $s_1, \ldots, s_k$ of U, over which $\pi_1 \cup \pi_2$ fails to be a fibration. Let $V = U \setminus \{s_1, \ldots, s_k\}$. The restriction $H^1(U, R^3\pi_*\mathbf{Z}) \to H^1(V, R^3\pi_*\mathbf{Z})$ being injective it suffices to prove 1.2.6 on V. Let $0 \in V$, and $G = \pi_1(V, 0)$. Let $L_0 = H^3(X_0, \mathbf{Z})$. Then G acts on L_0 by the monodromy representation, and $H^1(V, R^3\pi_*\mathbf{Z}) = H^1(G, L_0)$ is represented by cocycles $g \to \alpha_g \in L_0$, well defined up to coboundary $\alpha_g = {}^g\beta - \beta$, $\beta \in L_0$. To find a representative of $\tau^*[P_1 - P_2]_{|X_V}$ in $H^1(G, L_0)$, one notes that on the universal cover $\widetilde{V}$ of V, one can trivialize the pulled-back family of pairs $((\widetilde{P}_1 \cup \widetilde{P}_2)_{\widetilde{V}}, X_{\widetilde{V}})$. Because $(P_1 - P_2).X_0$ is homologous to zero in X_0, one can then find on $\widetilde{V}$ a continuously varying family of real 3-chains $(v \to \gamma_v \subset X_v)_{v \in \widetilde{V}}$, such that $\forall v \in \widetilde{V}, \partial\gamma_v = P_1 \cap X_v - P_2 \cap X_v = \ell_1^v - \ell_2^v$. If $\widetilde{0} \in \widetilde{V}$ is a point of $\widetilde{V}$ over 0, for any $g \in G$, there is a canonical isomorphism $g : X_{g^{-1}(\widetilde{0})} \simeq X_{\widetilde{0}}$, such that $g(\ell_i^{g^{-1}\widetilde{0}}) = \ell_i^{\widetilde{0}}$, for $i = 1, 2$, and we can construct the cocycle $g \mapsto g.\gamma_{g^{-1}(\widetilde{0})} - \gamma_{\widetilde{0}} \in H_3(X_{\widetilde{0}}, \mathbf{Z}) \equiv H^3(X_{\widetilde{0}}, \mathbf{Z}) = L_0$. This gives our representative. On the other hand, by definition of the Abel-Jacobi map, the γ_v's give a global lifting $\widetilde{\nu}$ of ν on $\widetilde{V}$ to $F^2\mathcal{H}^3_{\widetilde{V}}$, if one defines $\widetilde{\nu}(v) = \int_{\gamma_v} \in F^2H^3(X_v)^*$ for $v \in \widetilde{V}$. Then clearly, by the definition of $[\nu]$ in 1.2.5, a representative of $[\nu]$ in $H^1(G, L_0)$ is given by the cocycle : $g \mapsto g.\widetilde{\nu}(g^{-1}\widetilde{0}) - \widetilde{\nu}(\widetilde{0}) \in H_3(X_{\widetilde{0}}, \mathbf{Z}) \subset F^2H^3(X_{\widetilde{0}})^*$, that is by : $g \mapsto \int_{g.\gamma_{g^{-1}(\widetilde{0})} - \gamma_{\widetilde{0}}}$, so 1.2.6 is proved.

2. Griffiths' discovery left open the possibility that the group $\mathrm{Hom}^2 / \mathrm{Alg}^2$ (which is in any case a countable group, because there are only countably many components of Chow varieties parametrizing effective cycles of codimension two in a given variety) is a finitely generated group. However, Clemens, working also with the quintic hypersurfaces, has shown that this is not the case, even modulo torsion ([3]) : (See also [7] for generalization to other K-trivial complete intersections.

2.1. Theorem : Let X be a general quintic threefold in $\mathbf{P}^4$; then $\mathrm{Hom}^2 / \mathrm{Alg}^2(X)$ has infinite rank over $\mathbf{Q}$. More precisely its image in JX (see 1.2, Step I), tensorized by $\mathbf{Q}$ has infinite rank over $\mathbf{Q}$.

The proof is somewhat delicate, and we will only sketch the main ideas. First of all, there is an interesting statement, of independent interest from the point of view of the geometry of Calabi-Yau threefolds :

2.2. Theorem : (Clemens) If X is general as above, X contains infinitely many rigid rational curves.

This step is done as follows : One construct to begin with a surface $S \subset \mathbf{P}^3$ smooth of degree 4, having infinitely many smooth rational curves L_n. Then if X is a generic quintic containing S, one shows that singularities of X are nodes which are not on the L_n's, and that the normal bundle of L_n in X is $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$ ($L_n \simeq \mathbf{P}^1$). Such curves then deform with X, by the Kodaira stability theorem.

The second geometric point is to show, again working with the surface S, that one can specialize X to X_0 having a node on any given L_{n_0}, and no node on the other L_n's.

Finally a careful study of the generic normal bundle of L_{n_0} in the desingularization $\widetilde{X}_0$ of X_0 shows that under deformations of X_0 smoothing the node on L_{n_0}, L_{n_0} deforms with X_0, only when one takes a double cover B_2 of the basis B ramified along the discriminant locus (the locus where the node is preserved).

Consider now

$$\begin{array}{ccc} \mathcal{X} & \rightarrow & B \\ \uparrow & & \uparrow \\ \mathcal{X}_2 & \rightarrow & B_2 \end{array},$$

and let i be the involution of B_2 over B. Then on B_2 one has the cycle $L_{n_0}(b) - L_{n_0}(ib) \subset X_b$. Now, the most difficult part of Clemens' argument is :

2.3. The family of intermediate jacobians $J^3(X_b)$ for smooth X_b, $b \in B_2$, extends over the discriminant locus; the central fiber $J^3(X_0)$ has two components, and the normal function $\nu_{L_{n_0}}(b) = \Phi_{X_b}(L_{n_0}(b) - L_{n_0}(ib))$ extends over 0 in the component which does not pass through 0. (This uses essentially Theorem 3.4 of Lecture 5).

Admitting this, the rest of the argument goes as follows : Let C be a plane section of X. For each L_n, of degree d_n, $5L_n - d_nC$ is homologous to zero in X, so we have $\Phi_X(5L_n - d_nC) \in J^3(X)$. Assume there is for general X a relation :

$(*)$ $\Sigma_n c_n \Phi_X(5L_n - d_nC) = 0$, $c_n \in \mathbb{Z}$. Then going to X_0 having a node on L_{n_0}, and letting the monodromy act on $(*)$ by $L_{n_0}(b) \to L_{n_0}(ib)$, $L_n(b) \to L_n(b)$, for $n \neq n_0$, one concludes that c_{n_0} must be even. So all c_n must be even, and it remains only to prove :

2.4. Let $G \subset J^3(X)$ be a countable subgroup such that $G \otimes \mathbb{Z}/2\mathbb{Z}$ has infinite rank. Then $G \otimes \mathbb{Q}$ has infinite rank.

We will give in the next lecture a somewhat different approach to the infinite generation of $\Phi_X(CH^2(X)\,\text{hom})$.

3. Now we turn to the results of Nori, which are essentially new for codimension ≥ 3 cycles. One of the most striking consequences of [8] is :

3.1 Theorem : [8]. For $d \geq 3$, $n > d$. there exist varieties X of dimension n with a cycle $Z \subset X$ of codimension d. homologous to zero and Abel-Jacobi equivalent to zero, such that no multiple of Z is algebraically equivalent to zero.

This results in fact of the following more precise statement :

3.2 Theorem : [8] Let X be a variety and Z be a cycle of codimension d, satisfying $[Z] \neq 0$ in $H^{2d}(X, \mathbb{Q})$. Then for $n > d$ and $Y \subset X$ a general complete intersection of sufficiently ample divisors in X, such that $\dim Y = n > d$, $Z \cap Y$ is not algebraically equivalent to zero in Y.

3.2 implies 2.1 because we can take in 3.2 a variety X of dimension $2d$, having no odd dimensional cohomology, with an algebraic cycle Z of codimension d, primitive with respect to an ample divisor L, and consider a complete intersection $Y \subset X$ of dimension n satisfying $d < n \leq 2d - 2$ (here one needs $d \geq 3$), of divisors in (L^{n_i}), satisfying 3.9. Then $J^{2d-1}(Y) = 0$, and $Z_{|_Y}$ satisfies 3.1. This way we get all (n, d) satisfying $d \geq 3$, $2d - 2 \geq n > d$. We can take products with a projective space to obtain the other (n, d)'s.

The theorem 3.2 is a consequence of the following Theorem 3.3, the proof of which will be given by M. Green.

Let X be a projective variety of dimension $n + k$, L_i $i = 1, \ldots, k$ be ample divisors on X. Fix positive integers $n_1, \ldots, n_k$ and let $S = \bigoplus_1^k H^0(X, L^{n_i})$.

Let $Y_S \subset S \times X$ be the universal complete intersection. Then :

3.3 Theorem : For $n_i \gg 0$, $m \leq 2n$, and any smooth base change $T \to S$, one has $H^m(T \times X, Y_T, \mathbb{Q}) = 0$.

3.4. Now we show how Theorem 3.3 implies Theorem 3.2. So, under the assumptions of Theorem 3.1, and Y being a general complete intersection of divisors in $|L^{n_i}|$, n_i as in 3.3, assume that $Z \cap Y$ is algebraically equivalent to zero.

Then there exists an etale morphism $S' \to S$, a smooth family of curves $\mathcal{C} \underset{\varphi}{\to} S'$, a divisor D on $\mathcal{C}$, homologous to zero on the fibers of φ, a cycle Γ of codimension d in $\mathcal{C} \times_{S'} Y_{S'}$, such that, writing j for the natural map $Y_{S'} \to X$, one has :

3.4.1. $j^*Z = p_{2*}(\Gamma . p_1^* D)$, in $CH^d(Y_{S'})$ (this is the definition of algebraic equivalence, put in family). Here we work with the diagram :

$$(*) \qquad \begin{array}{ccc} & \mathcal{C} \times_{S'} Y_{S'} & \\ p_1 \swarrow & & \searrow p_2 \\ \mathcal{C} & & Y_{S'} \underset{j}{\to} X \end{array}$$

Now let $T = \mathcal{C}$. So $\mathcal{C} \times_{S'} Y'_S = Y_T$. The cycle Γ has a class γ in $H^{2d}(Y_T, \mathbb{Q})$, and by Theorem 3.3 and $2d < 2n$, one finds that this class extends to $\tilde{\gamma} \in H^{2d}(T \times X, \mathbb{Q})$. Consider then :

3.4.2. $\beta = \varphi_*(p_1^*[D].\tilde{\gamma}) \in H^{2d}(S' \times X, \mathbb{Q})$, where we consider now the following diagram :

$$(**) \qquad \begin{array}{ccc} & T \times X & \\ p_1 \swarrow & & \searrow \varphi \\ T & & S' \times X \end{array},$$

which contains $(*)$. Then by 3.4.1, $\beta_{|Y_{S'}} = j^*[Z] \in H^{2d}(Y_{S'}, \mathbb{Q})$, where j is as before the natural map $Y_{S'} \to X$. But by Theorem 3.3 applied to S', the restriction map $H^{2d}(S' \times X, \mathbb{Q}) \to H^{2d}(Y_{S'}, \mathbb{Q})$ is injective, so we find :

3.4.3. $\beta = p_2^*[Z]$, where p_2 is the second projection $S' \times X \to X$.

But if one chooses $s \in S'$, one finds that in $H^{2d}(X, \mathbb{Q})$, $\beta_{|s \times X} = [Z] = \varphi_{s*}(p_1^*[D_s].\tilde{\gamma}_{|C_s \times X})$, where one works with the diagram :

$$(***) \qquad \begin{array}{ccc} & C_s \times X & \\ p_1 \swarrow & & \searrow \varphi_s = \varphi_{|C_s} \\ C_s & & s \times X \end{array}.$$

So $\beta|_{s\times X} = 0$, because $[D_s] = 0$ and $[Z] = 0$ in $H^{2d}(X, \mathbb{Q})$, hence Theorem 3.2 is proved.

4. To conclude this lecture we want to describe the Bloch-Ogus resolution and Bloch-Ogus formula for the group of cycles modulo algebraic equivalence [2].

Recall the following classical result :

4.1. Lemma : Let X be a smooth projective variety and $D = \Sigma n_i D_i$ be a divisor on X. Then a multiple of D is homologous to zero if and only if there exists a (necessarily closed) holomorphic one form ω with logarithmic poles along the support $|D|$ of D, and such that $\mathrm{Res}_{|D|}\,\omega = \Sigma n_i 1_{D_i}$.

The notion of logarithmic pole is well defined when $|D|$ is a divisor with normal crossings, which we can achieve by blowing up X. Then locally $|D|$ admits for equation $z_1, \ldots, z_k$, in some coordinate systems $(z_1, \ldots, z_n)$ on X, and ω has logarithmic pole if it is of the form : $\sum_1^k f_i \dfrac{dz_i}{dz_i} + \sum_{i>k} g_i dz_i$, for holomorphic functions f_i, g_i.

The residues of ω are then the fonctions $\frac{1}{2i\pi} f_i$ on D_i. They are constant if ω is closed.

The lemma follows from the exact sequence :

4.1.1. $\quad 0 \to \Omega_X \to \Omega_X(\log|D|) \xrightarrow{\mathrm{Res}} \bigoplus_i \mathcal{O}_{D_i} \to 0$. See [9].

It is easy to show that the induced map : $\bigoplus_i H^0(\mathcal{O}_{D_i}) \to H^1(\Omega_X)$ sends $\Sigma n_i 1_{D_i}$ to the De Rham class of D in $H^1(\Omega_X)$. So if $C_1^{\mathbb{R}}(\Sigma n_i D_i) = 0$, $\Sigma_i n_i 1_{D_i} = \mathrm{Res}\,\omega$, as we wanted. Such a form ω is closed by the degeneration at E_1 of the Hodge-DeRham spectral sequence for the logarithmic complex $\Omega_X^{\cdot}(\log|D|)$. See [9]).

4.2. More generally, if X is smooth not necessarily compact and $Y \subset X$ is a smooth hypersurface, one has a residue $\mathrm{Res}_Y : H^k(X\backslash Y) \to H^{k-1}(Y)$ see [6], and if one has $Y_1, Y_2 \subset X$ intersecting transversally, one has :

4.2.1

$$H^k(X\backslash Y_1 \cup Y_2) \xrightarrow{\mathrm{Res}_{Y_2}} H^{k-1}(Y_2\backslash Y_1 \cap Y_2) \xrightarrow{\mathrm{Res}_{Y_1\cap Y_2}} H^{k-2}(Y_1 \cap Y_2)$$

$$H^k(X\backslash Y_1 \cup Y_2) \xrightarrow[\mathrm{Res}_{Y_1}]{} H^{k-1}(Y_1\backslash Y_2 \cap Y_1) \xrightarrow[\mathrm{Res}_{Y_2\cap Y_1}]{} H^{k-2}(Y_1 \cap Y_2)$$

Now the very important point is the following fact :

4.2.2. i) $\mathrm{Res}_{Y_1\cap Y_2} \circ \mathrm{Res}_{Y_2} = -\,\mathrm{Res}_{Y_2\cap Y_1} \circ \mathrm{Res}_{Y_1}$

ii) $\mathrm{Res}_Y\, H^k(X) = 0$.

Furthermore Lemma 4.1 has now the following integral version :

4.2.3. $\Sigma n_i D_i$ is homologous to zero $\Longleftrightarrow \Sigma n_i 1_{Y_i} = \Sigma\, \mathrm{Res}_{Y_i}(\eta)$, for some $\eta \in H^1(X \backslash \cup Y_i, \mathbb{Z})$.

4.2.2 is the essential point for the construction of the Bloch-Ogus resolution, although there are in fact big difficulties coming from the singularities of the subvarieties we consider.

4.3. Define the Zariski sheaf $\mathcal{H}^k$ on X as associated to the presheaf $U \to H^k(U, A)$ (where A may be integral, rational or complex coefficients), and define $H^k(\mathbb{C}(X)) = \varinjlim_{V \subset X} H^k(X)$. Then $\mathcal{H}^k$ is also $R^k j_*(A)$ where $j : X^{\mathrm{an}} \to X^{zar}$ is the continuous map from the usual topology to the Zariski topology.

4.2.2 (with a lot of work) gives a complex :

$$0 \to \mathcal{H}^k \to H^k(\mathbb{C}(X)) \to \bigoplus_{\substack{\mathrm{cod}\, D=1 \\ D \mathrm{irred}}} H^{k-1}(\mathbb{C}(D)) \to \bigoplus_{\substack{\mathrm{cod}\, Z=2 \\ Z \mathrm{irred}}} H^{k-2}(\mathbb{C}(Z)) \to$$

4.3.1 $$\ldots \to \bigoplus_{\substack{\mathrm{cod}\, Z=k \\ Z \mathrm{irred}}} A_Z \to 0.$$

Here one sees $H^\ell(\mathbb{C}(Z))$ as a constant sheaf supported on the irreducible subvariety Z.

The main result of Bloch-Ogus is then :

4.4. Theorem : [2]. 4.3.1 is exact.

So we have a resolution of $\mathcal{H}^k$ by acyclic sheaves, for the Zariski topology, which has length $\leq k$.

If one considers the end of this resolution, for $A = \mathbb{Z}$, one finds :

4.5. $$H^k_{\mathrm{zar}}(\mathcal{H}^k_{\mathbb{Z}}) = \bigoplus_{\substack{\mathrm{cod}\, Z=k \\ Z \mathrm{irred}}} (\mathbb{Z}_Z) / \mathrm{Res}\Big(\bigoplus_{\substack{\mathrm{cod}\, D=k-1 \\ D \mathrm{irred}}} H^1(\mathbb{C}(D), \mathbb{Z})\Big).$$

But now by 4.2.3, one sees that $\Sigma n_i 1_{Z_i}$. codim $Z_i = k$ defines a zero element in $H^k_{\mathrm{zar}}(\mathcal{H}^k_{\mathbb{Z}})$ if and only if it is a combinaison of cycles homologous to zero in varieties of codimension $k-1$, that is iff $\Sigma n_i Z_i$ is algebraically equivalent to zero. In other words, one has the Bloch-Ogus formula :

4.6. $$CH^k(X)/\mathrm{alg.eq} \simeq H^k_{\mathrm{zar}}(\mathcal{H}^k).$$

4.7. This fascinating formula did not prove very useful for the understanding of the groups $CH^k/\text{alg.eq.}$, but the Bloch-Ogus theory has very interesting consequences on the spectral sequence associated to the map j of 4.3. An important consequence of Theorem 4.4 is :

4.7.1. $H^p_{\text{zar}}(\mathcal{H}^q) = 0$ for $p > q$.

For $p+q = 3, 4$ it follows that the spectral sequence $H^p_{\text{zar}}(\mathcal{H}^q) \Rightarrow H^{p+q}(X)$ degenerates quickly, and gives rise to the following exact sequence.

4.7.2. $H^3(X) \to H^0_{\text{zar}}(\mathcal{H}^3) \xrightarrow{d_2} H^2_{\text{zar}}(\mathcal{H}^2) \to H^4$, where the first map is given by the restrictions $H^3(X) \to H^3(U, \mathbb{Z}), U \subset X$, and is in many cases injective, and the last one identifies to the cycle class defined on $CH^2(X)/\text{alg.eq.}$

For complex coefficients a section of $\mathcal{H}^3$ can be understood as a meromorphic form which is locally the sum of a holomorphic form and of an exact form. The obstruction for such a form to define a global cohomology class lies in the group $(\text{Hom}^2 / \text{Alg}^2(X)) \otimes \mathbb{C}$, which is generally non trivial by the Griffiths theorem 1.2 (see [1], [2]).

References :

[1] M.F. Atiyah, W.V.D. Hodge : Integrals of the second kind on an algebraic variety, Annals of Maths. Vol. 62, n°1 (1955) 56-91.

[2] S. Bloch, A. Ogus, Ann. Scient. Ec. Norm. Sup. 4ème Série, t.7 (1974) 181-202.

[3] H. Clemens : Homological equivalence, modulo algebraic equivalence, is not finitely generated, Publ. Math. IHES 58 (1983) 19-38.

[4] H. Clemens : Neron models for families of intermediate jacobians, Publ. Math. IHES 58 (1983) 5-18.

[5] N. Coleff, M. Herrera, D. Lieberman : Algebraic cycles as residues of meromorphic forms, Math. Ann. (1980) 73-87.

[6] P.A. Griffiths : On the periods of certain rational integrals I, II, Ann. of Math. 90 (1969) 460-541.

[7] K.H. Paranjape : Curves on threefolds with trivial canonical bundle, Proc. Indian Acad. Sci. Math. 101 (1991) n°3, 199-213.

[8] M.V. Nori : Algebraic cycles and Hodge theoretic connectivity, Inv. Math. 111(2) 349-373 (1993).

[9] S. Zucker : Degeneration of Hodge bundles (after Steenbrink) in "Topics in transcendental algebraic geometry" ed. by P. Griffiths, Annals of Mathematics Studies, Study 106, Princeton (1984) 121-141.

Lecture 8 : Application of the Noether-Lefschetz locus to threefolds.

This last lecture describes the applications of the criterion of M. Green (see Lecture 3) for the existence and density of the Noether-Lefschetz locus, to the study of one-cycles on threefolds, which were worked out in [15], [16], [17].

The simplest application is a purely infinitesimal (hence algebraic) method to solve the Hodge conjecture for the algebraic part of the intermediate jacobian of certain threefolds [15]. Of course to apply the method, one needs information which depend on the particular case we consider, but the method applies well in some cases where the geometry is so intricated that the existence of interesting codimension two cycles is not obvious.

We also sketch an infinitesimal proof of Clemens' theorem [17]. Here we don't use the full power of M. Green's criterion, because we specialize to quintics containing the Fermat surface S of degree five : A crucial point to get the "infinite generation" is the fact that $\mathrm{rank}(\mathrm{Pic}\, S)^{\mathrm{prim}} \geq 2$, and the existence of such surface is not predicted by this criterion. (Note that one can always construct surfaces S with $\mathrm{rank}(\mathrm{Pic}\, S)$ large in a given threefold, but they will not correspond to components of the Noether-Lefschetz locus of the right codimension, a condition which is important for all computations).

The most convincing application is the generalization of the Griffiths theorem ([6], and Th. 1.2 of Lecture 7) to any non-rigid Calabi-Yau threefold, that is a smooth projective threefold X satisfying $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = \{0\}$ and K_X trivial) : we prove :

Theorem [16] : Let X be a Calabi-Yau threefold satisfying $H^1(T_X) \neq 0$. Then for a general deformation X' of X, one has : $\mathrm{Hom}^2 / \mathrm{Alg}^2(X')$ is not a torsion group. More precisely, the Abel-Jacobi map $\Phi_{X'}$ of X' is not of torsion.

1.

1.1. For complete intersection threefolds X which have a negative canonical bundle, it is known how to parametrize the intermediate jacobian $J^3(X)$ using the Abel-Jacobi map, as predicted by the Hodge conjecture (see J.P. Murre's lecture on the Hodge conjecture): These varieties contain a positive dimensional family D of lines, and the Abel-Jacobi map $\Phi_X : \mathrm{Alb}\, D \to J^3(X)$ is surjective. Bardelli [11] has constructed an example of a complete intersection X with effective canonical bundle and with an involution i without fixed point such that $(JX)^-$ is contained in the algebraic part of JX, that is $H^{3,0}(X)^- = 0$, and has constructed geometrically interesting codimension two cycles in X, which give a parametrization of $(JX)^-$ by algebraic cycles.

1.2. We sketch now an argument which works for other varieties X of dimension 3 with a group of automorphisms G, such that the invariant part of $J^3(X)$ under G, or its orthogonal is contained in $J^3(X)^{\text{alg}}$ [15]. Assuming that one of these two possibilities occurs, we will refer to them respectively as case i) and case ii).

Case i) is then equivalent to $H^{3,0}(X)^{\text{inv}} = 0$, case ii) is equivalent to G acts trivially on $H^{3,0}(X)$.

1.3. Now assume that X carries a G-invariant line bundle with enough G-invariant sections, and such that :

Case i) for a smooth $S = V(\sigma)$, $\sigma \in H^0(L)^{\text{inv}}$, the number $h^{2,0}(S)^{\text{inv}} := \dim H^{2,0}(S)^{\text{inv}}$ satisfies : $h^{2,0}(S)^{\text{inv}} < h^0(N_S X)^{\text{inv}}$.

Case ii) for a smooth $S \in V(\sigma)$, $\sigma \in H^0(L)^{\text{inv}}$, the number $h^{2,0}(S)^{\sharp} := \dim(h^{2,0}(S)^{\sharp} := \dim(H^{0,2}(S)^{\text{inv}\,\perp})$ satisfies : $h^{2,0}(S)^{\sharp} < h^0(N_S X)^{\text{inv}}$.

These assumptions are made plausible by the adjunction formula, which gives :

$$H^0(K_S)^{\text{inv}} = H^0(K_X(L)_{|S})^{\text{inv}} \hookleftarrow H^0(K_X)^{\text{inv}} \otimes H^0(N_S X)^{\text{inv}} \ (= 0 \text{ in case i}))$$

$$H^0(K_S)^{\sharp} = H^0(K_X(S)_{|S})^{\sharp} \hookleftarrow H^0(K_X)^{\sharp} \otimes H^0(N_S X)^{\text{inv}} \ (= 0 \text{ in case ii}))$$

Now as explained in Lecture 3, $(h^{2,0})^{\text{inv}}$ (resp. $h^{2,0}(S)^{\sharp}$ is the expected codimension of the components of the Noether-Lefschetz loci corresponding to invariant (resp. skew) classes λ, on the space of G-invariant surfaces. So assumption 1.3 means that in both cases, the Noether-Lefschetz locus, if non empty has positive dimensional components. For (S, λ) in a component $\mathcal{S}_\lambda$ of the Noether-Lefschetz locus, with $\lambda \in H^2(S, \mathbf{Z})^{\text{inv}} \cap H^{1,1}(S)$ in case i), $\lambda \in H^2(S, \mathbf{Z})^{\sharp} \cap H^{1,1}(S)$ in case ii), one considers λ as an element of $CH_1(S)^{\text{inv}}$ (resp. $CH_1(S)^{\sharp}$ in case ii), where a one-cycle Z is skew, if $\sum_{g \in G} gZ = 0$). Here one assumes that $H^1(\mathcal{O}_S) = 0$, for simplicity.

If $j : S \hookrightarrow X$ is the inclusion, $j_*\lambda$ gives an element of $CH_1(X)^{\text{inv}}$ (resp. $CH_1(X)^{\sharp}$ in case ii).

Using theAbel-Jacobi map Φ_X, one deduces a map :

1.3.1. $\mathcal{S}_\lambda \underset{\alpha_\lambda}{\rightarrow} (JX)^{\text{inv}}$ (resp. $(JX)^{\sharp}$) in case ii), depending on the choice of a base point, that is well defined up to constant, and $\mathcal{S}_\lambda$ being positive dimensional, one can hope that the image of α_λ generates $(JX)^{\text{inv}}$ (resp. $(JX)^{\sharp}$).

1.4. Now we explain how one can make this work with only algebraic assumptions on the infinitesimal variation of Hodge structure of $S \subset X$.

Let $U \subset \mathbf{P}(H^0(L)^{\text{inv}})$ parametrize the smooth invariant surfaces $S \subset X$, so $TU_{(S)} = H^0(N_S X)^{\text{inv}} = H^0(L_{|_S})^{\text{inv}}$. On U, we consider in case i) the invariant variation of Hodge structure $H^{2\,\text{inv}}_{\mathbf{Z}}, \mathcal{H}^{2\,\text{inv}}, F^k\mathcal{H}^{2\,\text{inv}}$, with infinitesimal variation described at $(S) \in U$ by =

$$\begin{array}{cccc} \overline{\nabla}: & H^1(\Omega_S)^{\text{inv}} & \to & \text{Hom}(TU_{(S)}, H^2(\mathcal{O}_S)^{\text{inv}}) \\ & \| & & \\ & \mathcal{H}^{1,1\,\text{inv}}_{(S)} & \to & \text{Hom}(TU_{(S)}, \mathcal{H}^{0,2\,\text{inv}}_{(S)}) \end{array}$$

and we write $\lambda \in H^1(\Omega_S)^{\text{inv}} \to \overline{\nabla}(\lambda) \in \text{Hom}(TU_{(S)}, H^2(\mathcal{O}_S)^{\text{inv}})$. In case ii) we work with $H^{2\sharp}_{\mathbf{Z}}$, and the associated VHS. Now to apply M. Green's Lemma, we refine assumption 1.3 as follows :

1.4.1. For S generic in $U, \exists \lambda \in H^1(\Omega_S)^{\text{inv}}$ (resp. $H^1(\Omega_S)^\sharp$ in case ii)), such that $\overline{\nabla}(\lambda) : TU_{(S)} \to H^2(\mathcal{O}_S)^{\text{inv}}$ is surjective (resp. $\overline{\nabla}(\lambda) : TU_{(S)} \twoheadrightarrow H^2(\mathcal{O}_S)^\sharp$) in case ii)).

Then M. Green's Lemma says that for generic S and generic λ in $H^1(\Omega_S)^{\text{inv}}_{\mathbf{R}}$ (resp. $H^1(\Omega_S)^\sharp_{\mathbf{R}}$) we can approximate it by (S_n, λ_n) such that : $\lambda_n \in H^2(S_n, \mathbb{Q})^{\text{inv}} \cap H^{1,1}(S_n)$, (resp. $\lambda_n \in H^2(S_n, \mathbb{Q})^\sharp \cap H^{1,1}(S_n)$).

Also for large enough n, (λ_n, S_n) also satisfies assumption 1.4.1 so that $\mathcal{S}_{\lambda_n}$ is smooth of maximal codimension at S_n. (See Lecture 3). This is the existence step for our construction.

Next we explain which infinitesimal condition is needed for the Abel-Jacobi map α_{λ_n} of 1.3.1 to be non trivial.

For $\lambda \in H^1(\Omega_S)^{\text{inv}}$ (resp. $H^1(\Omega_S)^\sharp$ in case ii)) one defines : $TU_\lambda := \text{Ker}\,\overline{\nabla}(\lambda) : TU_{(S)} \to H^2(\mathcal{O}_S)^{\text{inv}}$ (resp. $H^2(\mathcal{O}_S)^\sharp$). Then let $\alpha_{\lambda_*} : TU_\lambda \to H^2(\Omega_X)$ be defined as follows : One has $TU_{(S)} \subset H^0(L_{|_S})^{\text{inv}}$, hence a map :

$$\mu_\lambda : TU_{(S)} \to H^1(\Omega_S(L))^{\text{inv}} \text{ (resp. } H^1(\Omega_S(L))^\sharp).$$

Consider the exact sequence : $0 \to \mathcal{O}_S(-L) \to \Omega_{X|S} \to \Omega_S \to 0$, which gives $\delta : H^1(\Omega_S(L))^{\text{inv}} \to H^2(\mathcal{O}_S)^{\text{inv}}$ (resp. $\delta : H^1(\Omega_S(L))^\sharp \to H^2(\mathcal{O}_S)^\sharp$). Then one has

(See M. Green's lecture on hypersurfaces) :

$$\delta \circ \mu_\lambda = \overline{\nabla}(\lambda)$$

One concludes that $\mu_\lambda(TU_\lambda)) \subset H^1(\Omega_X(L)_{|S})^{\text{inv}}$ (resp. $H^1(\Omega_X(L))_{|S})^\sharp$), using $H^1(\mathcal{O}_S) = 0$, and this goes naturally to $H^2(\Omega_X)^{\text{inv}}$ (resp. $H^2(\Omega_X)^\sharp$), using the exact sequence $0 \to \Omega_X \to \Omega_X(L) \to \Omega_X(L)_{|S} \to 0$.

This defines our α_{λ_*}, and one can check :

1.4.2. Lemma : If $\lambda \in H^1(\Omega_S) \cap H^2(S, \mathbf{Z})$, α_{λ_*} is the differential of the map α_λ of 1.3.1, where one identifies $TU_{(\lambda)}$ to $TS_{\lambda(S)}$. (See Lecture 3).

If λ is only rational, $N\lambda \in H^2(S, \mathbf{Z}) \cap H^1(\Omega_S)$, for some $N \in \mathbf{N}$, and one has obviously: $\alpha_{\lambda_*} = \frac{1}{N}$ (differential of $\alpha_{N\lambda}$).

Consider now the following assumption (a formal assumption on $IVHS$).

1.4.3. For generic $S \in U$, $\lambda \in H^1(\Omega_S)^{\text{inv}}$ (resp. $H^1(\Omega_S)^\sharp$), the map $\alpha_{\lambda *} : TU_{(\lambda)} \to H^2(\Omega_X)$ is non zero.

If 1.4.3 is true for (S, λ), satisfying assumption 1.4.1, it is also true for some (S, λ), with λ real satisfying 1.4.1. Then we approximate (S, λ) by (S_n, λ_n) as in 1.4.1. For n large enough, $TU_{(\lambda_n)}$ has the minimal dimension, so 1.4.3 will also hold at (S_n, λ_n). According to 1.4.2, this means that $(\alpha_{\lambda_n})*$ is non zero at (S_n, λ_n) and $\mathcal{S}_{\lambda_n}$ being smooth at S_n, it follows that α_{λ_n} is non zero.

One can generally conclude that α_{λ_n} is in fact surjective, by checking that $(JX)^{\text{inv}}$ or $(JX)^\sharp$ is a simple abelian variety. Hence we have shown under the formal assumptions 1.4.1 and 1.4.3 that the Hodge conjecture holds for $(JX)^{\text{inv}}$ or $(JX)^\sharp$.

1.5. Examples and variant i) ([15]). Consider a generic quintic polynomial in $\mathbf{P}^4$ invariant under the involution i acting by $i^*(X_0, \ldots, X_4) = (-X_0, -X_1, X_2, X_3, X_4)$ then $h^{3,0}(X)^- = 0$. One considers the quintic fourfolds $Y \subset \mathbf{P}^5$ invariant under the extended involution i acting on $\mathbf{P}^5$: $i^*(X_0, \ldots, X_5) = (-X_0, -X_1, X_2, \ldots, X_5)$, and containing X. These quintics have $h^{4,0} = 0$, and by Bloch-Srinivas the Hodge conjecture is true for them in degree 4. So the method of the Noether-Lefschetz locus can be applied and it is verified in [15] that the analogs of 1.4.1, 1.4.3 hold true.

ii) More generally, the method works for all K-trivial complete intersections with an involution, excepted the one constructed by Bardelli.

2. We explain now how one can prove Clemens' theorem by infinitesimal methods [17].

2.1 Theorem : Let X be a general quintic threefold in $\mathbf{P}^4$. Then the one-cycles on X homologous to zero, and supported on a hyperplane section of X generate by the Abel-Jacobi map a countable subgroup of JX of infinite rank over $\mathbf{Q}$.

Let $S \subset X$ be a hyperplane section, and let $\lambda \in H^1(\Omega_S)^{\text{prim}} \cap H^2(S, \mathbf{Z})$ be an algebraic class.

Note that by K_X trivial, and the adjunction formula, one has $\dim H^0(N_S X) = \dim H^0(K_S) = \dim H^2(\mathcal{O}_S)$.

Working with the same notations as in 1, with $U \subset \mathbf{P}(H^0(\mathcal{O}_X(1))$ the Zariski open set parametrizing smooth surfaces, we suppose now :

2.2. Assumption : The map $\overline{\nabla}(\lambda) : H^0(N_S X) \to H^2(\mathcal{O}_S)$ is an isomorphism.

Then consider $V \subset \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^4}(5)))$ the open set parametrizing smooth quintics, and let $\mathcal{U} \underset{\pi}{\to} V$ be the fibration with fiber $\mathcal{U}_t$ = open set in $\mathbf{P}(H^0(\mathcal{O}_{X_t}(1)))$ parametrizing smooth surfaces in X_t. (S, X) is a point of $\mathcal{U}$, and one can consider the component $\mathcal{S}_\lambda \subset \mathcal{U}$ of the Noether-Lefschetz locus passing through (S, X). By the surjectivity of the map $\overline{\nabla}(\lambda)$ in 2.2 $\mathcal{S}_\lambda$ is smooth of codimension $h^{2,0}$ at (S, X), and the kernel of $\pi_* : T\mathcal{S}_{\lambda(S,X)} \to TV_{(X)}$ is clearly equal to $\text{Ker}(\overline{\nabla}(\lambda) : H^0(N_S X) \to H^2(\mathcal{O}_S))$. Also π_* is surjective by the surjectivity of $\overline{\nabla}(\lambda)$. So we conclude :

2.3 Lemma : Assumption 2.2 implies that $\mathcal{S}_\lambda \xrightarrow{\pi} V$ is etale in a neighbourhood of (S, X).

2.4. Consider the Fermat quintic surface $S : \sum_0^3 X_i^5 = 0$. It has a primitive Picard group of rank $\rho_{\text{prim}} \geq 2$. Let us denote by $NS^{1,1}_{\text{prim}} \subset H^1(\Omega_S)^{\text{prim}}$ the subspace of $H^1(\Omega_S)^{\text{prim}}$ generated over $\mathbf{C}$ by algebraic classes.

Using an easy algebraic characterization of $NS^{1,1}_{\text{prim}}$, and the jacobian description of the $IVHS$ of S (cf. M. Green's Lecture), one checks now :

2.5 Lemma. Let X be a generic quintic threefold containing S, and let λ be a generic element of $NS^{1,1}_{\text{prim}}$. Then the map $\overline{\nabla}(\lambda) : H^0(N_S X) \to H^2(\mathcal{O}_S)$ is an isomorphism.

Because $NS^{1,1}_{\text{prim}}$ is a $\mathbf{C}$-vector space with a $\mathbf{Q}$-structure given by $NS_{\text{prim}} \otimes \mathbf{Q}$, and because the asumption on λ is Zariski open, it follows that the conclusion of 2.5 holds for infinitely many elements of $\mathbf{P}(NS_{\text{prim}} \otimes \mathbf{Q})$.

In other words, assumption 2.2 holds for infinitely many non proportional algebraic classes in S, for some fixed X containing S.

2.6. By 2.3 and 2.5, we have now in a neighbourhood of a generic $X \in V$ containing S, countably many components $\mathcal{S}_{\lambda_n}$, tales over V and to each of them we associate a holomorphic family of one-cycles homologous to zero in the fibers $(X_v)_{v\in V}$: to $v \in V$ we associate $j_{v*}(\lambda_{n,v})$, where $v \to S_v \underset{j_v}{\hookrightarrow} X_v$ is a local section of $\mathcal{S}_{\lambda_n} \underset{\pi}{\to} V$, and $\lambda_{n,v} \in H^2(S_v, \mathbf{Z})_{\text{prim}} \cap H^{1,1}(S_v) \simeq CH_1(S_v)_{\text{prim}}$, because $S_v \in \mathcal{S}_{\lambda_n}$.

We consider the associated germs of normal functions $\nu_n \in \mathcal{J}(V)$, defined as $\nu_n(v) = \Phi_{X_v}\ (j_{v*}(\lambda_{n,v}))$.

To each such germ, we associate its infinitesimal invariant at X (See M. Green's Lecture on normal functions), that is :

2.6.1 : $\quad \delta\nu_{n(X)} \in H^{1,2}(X) \otimes \Omega_{V(X)}/\overline{\nabla}(H^{2,1}(X)).$

Now the technical point is the following :

2.6.2 Lemma : Let S be the Fermat quintic. Then there exists two non proportional algebraic classes $\lambda_1, \lambda_2 \in NS(S)_{\text{prim}}$, satisfying : For X general containing S, there exists an infinite number of non proportional combinations $\alpha\lambda_1 + \beta\lambda_2,\ \alpha, \beta \in \mathbf{Z}$, such that :

i) $\alpha\lambda_1 + \beta\lambda_2$ satisfies 2.2, for the inclusion $S \subset X$.

ii) the associated germs of normal functions $\nu_{\alpha\lambda_1+\beta\lambda_2}$ on V near X have infinitesimal invariants $\delta\nu_{\alpha\lambda_1+\beta\lambda_2}(X)$ which are independent over $\mathbf{Q}$, in $H^{1,2}(X)\otimes\Omega_{V(X)}/\overline{\nabla}(H^{2,1}(X))$.

Now the infinitesimal proof of Theorem 2.1 is almost finished. Assume there is for a general point near X a relation with a finite number of integral coefficients between the normal functions $\nu_{\alpha\lambda_1+\beta\lambda_2}$, defined in a common open set of V containing X. By a countability argument one may assume that a fixed relation holds in this open set. But this relation will then also hold between the associated $\delta\nu_{\alpha\lambda_1+\beta\lambda_2}$ at X which contredicts their independence over $\mathbf{Q}$.

2.7 Remark : This method has been successfully applied in [12], where the authors use to begin with the Noether-Lefschetz locus and conclude with Clemens argument, and in [10] where theorem 2.1 is proved for cycles of dimension 3 in cubic hypersurfaces of dimension 7.

3. We conclude our lecture with the following theorem, which makes a full use of M. Green's criterion :

3.1 Theorem : [16] Let X be a non-rigid Calabi-Yau threefold. Then for a general deformation X' of X, the Abel-Jacobi map $\Phi_{X'}$ of X' is not of torsion.

As in Griffiths' theorem, this implies that $\mathrm{Hom}^2 / \mathrm{Alg}^2(X')$ is not of torsion, because for a Calabi-Yau threefold, the map $\overline{\nabla} : H^{1,2} \to \mathrm{Hom}(H^1(T_X), H^3(\mathcal{O}_X))$ is an isomorphism (which identifies, via K_X trivial $\Rightarrow T_X \simeq \Omega^2_X$, to the Serre duality) and its injectivity implies as in Lecture 7 that JX' has no algebraic part, for general X'.

The first step is done in [18], where one checks M. Green's criterion for sufficiently ample surfaces in X :

3.2 Theorem : Let X be a Calabi-Yau threefold, and $L \to X$ an ample line bundle. Then for n large enough, there is a smooth $S \in |L^n|$, and a $\lambda \in H^1(\Omega_S)^0 := \mathrm{Ker}(H^1(\Omega_S) \to H^2(\Omega^2_X))$ satisfying :

3.2.1. $\overline{\nabla}(\lambda) : H^0(N_S X) \to H^2(\mathcal{O}_S)$ is an isomorphism.

Notice, and this point was also crucial for the proof of 2.1 that "K_X trivial" $\Rightarrow$ $\dim H^0(K_S) = \dim H^0(N_S X)$, and this fact suggested the use of the Noether-Lefschetz locus instead of the rigid rational curves of Clemens, because one expected from it that the Noether-Lefschetz locus for surfaces in X is essentially 0-dimensional.

In fact, applying M. Green's criterion and using 3.2, we find now :

3.3 Corollary : For $n \gg 0$, the Noether-Lefschetz locus in $U \subset \mathbb{P}(H^0(X, L^n))$ (the open set parametrizing smooth surfaces) has countably many reduced 0-dimensional components, which are dense in U.

Note also the following very important refinement of 3.3, which is part of M. Green's Lemma :

The couples (S_n, λ_n) where λ_n is rational algebraic on S_n and $\{S_n\}$ is a zero-dimensional reduced component of $\mathcal{S}_{\lambda_n}$ are dense in the total space of $\mathcal{H}^{1,1}_{\mathbf{R},\mathrm{prim}}$ on U. (See Lecture 3 for notations).

3.4. Now we do the same construction as in §2.

3.4.1. One knows that the local universal deformation V of X is smooth. For n large enough, let $\mathcal{U} \overset{\pi}{\to} V$ parametrizing smooth surfaces $S_v \in |L^n_v|$ on X_v, $v \in V$. Let $S \subset X$, $\lambda \in H^2(S, \mathbf{Z})^0 \cap H^{1,1}(S)$ be an algebraic class satisfying 3.2.1. Then as in 2.3 it follows that $\mathcal{S}_\lambda \subset \mathcal{U}$ is smooth of codimension $h^{2,0}(S) = \dim$ (fibers of $\pi : \mathcal{U} \to V$), and that $\pi_{|\mathcal{S}_\lambda} : \mathcal{S}_\lambda \to V$ is etale over a neighbourhood of (X) in V.

3.4.2. As in 2.6, we conclude that we have now countably many germs of families $(Z_\lambda(v))$ on V near (X), of 1-cycles homologous to zero in the fibers X_v, and countably many corresponding germs of normal functions $\nu_\lambda(v)$ on V near (X), which are in bijection with the countably many 0-dimensional reduced components of the Noether-Lefschetz locus of $U = \mathcal{U}_{(X)}$.

The Theorem 3.1 then follows from :

3.5 Theorem : There is at least one ν_λ which is not of torsion.

We want to sketch the proof of this. The invariant used here is a refinement of the infinitesimal invariant of a normal function, and we want to describe it.

3.5.1. Let $S \underset{j}{\hookrightarrow} X$ be smooth, and $\lambda \in H^2(S,\mathbb{Z})^0 \cap H^{1,1}(S)$. Then $\lambda \in CH_1(S)^0$ $(H^1(\mathcal{O}_S) = 0)$, so we have $j_*(\lambda) \in CH_1(X)_{\mathrm{hom}}$, and we have also $\Phi_X(j_*\lambda) \in J^3(X)$.

Now using the Deligne description of the Abel-Jacobi map [14], $\Phi_X(j_*\lambda)$ can be described as follows :

Let $Y = X \backslash S$. Then there is an exact sequence :

3.5.2. $0 \to H^3(X,\mathbb{Z}) \to H^3(Y,\mathbb{Z}) \underset{\mathrm{Res}}{\longrightarrow} H^2(S,\mathbb{Z})^0$, and $H^3(Y,\mathbb{C})$ has a Hodge filtration $F^iH^3(Y)$, [13], such that :

3.5.3. $\mathrm{Res}\, F^iH^3(Y) = F^{i-1}H^2(S)^0$, $F^iH^3(Y) \cap H^3(X) = F^iH^3(X)$. By 3.5.2, λ admits a lifting $\lambda_{\mathbb{Z}} \in H^3(Y,\mathbb{Z})$. By 3.5.3, λ admits a lifting λ_F in $F^2H^3(Y)$. Then $\lambda_{\mathbb{Z}} - \lambda_F \in H^3(X,\mathbb{C})$ and is well defined modulo $H^3(X,\mathbb{Z})$ and $F^2H^3(X)$. This gives our point $\Phi_X(j_*\lambda) \in J^3(X) = H^3(X,\mathbb{C})/F^2H^3(X) \oplus H^3(X,\mathbb{Z})$.

As a corollary, we have :

3.5.2. $\Phi_X(j_*\lambda)$ is a torsion point of $J^3(X) \Leftrightarrow \lambda$ admits a lifting in $H^3(Y,\mathbb{Q}) \cap F^2H^3(Y)$.

Consider $W \subset H^3(Y,\mathbb{R})$ defined as $W = F^2H^3(Y) \cap H^3(Y,\mathbb{R})$. Then by the residue, W is isomorphic to $H^{1,1}_{\mathbb{R}}(S)^0 \ni X$ and Corollary 3.5.4 rewrites as :

3.5.5. $\Phi_X(j_*\lambda)$ is of torsion $\Leftrightarrow$ the lifting $\tilde{\lambda}$ of λ in W lies in $H^3(Y,\mathbb{Q}) \subset H^3(Y,\mathbb{R})$.

We put this in family : consider the normal function $\nu_\lambda(v) = \Phi_{X_v}(j_{v*}(\lambda_v)) \in J^3(X_v)$. Let $Y_v = X_v \backslash S_v$, where $S_v \underset{j_v}{\hookrightarrow} X_v$ is the point of $\mathcal{S}_\lambda \underset{\text{local isom.}}{\overset{\pi}{\longrightarrow}} V$ over v near S.

Then we have a $\mathcal{C}^\infty$ real bundle $\mathcal{W}$ on $\mathcal{S}_\lambda$, with fiber $W_{(v)} = H^3(Y_v), \mathbf{R}) \cap F^2H^3(Y_v)$, and $\mathcal{W}$ is isomorphic to $(\mathcal{H}^{1,1}_{\mathbf{R}})^0$. On $\mathcal{S}_\lambda$, λ gives a section of $(\mathcal{H}^{1,1}_{\mathbf{R}})^0$ and has a unic lifting $\tilde{\lambda}$ in $\mathcal{W} \subset \mathcal{H}^3_{\mathbf{R},Y}$ (the flat real $\mathcal{C}^\infty$-bundle with fiber $H^3(Y_v, \mathbf{R})$). 3.5.5 gives now :

3.5.6. If ν_λ is of torsion, $\tilde{\lambda}_v$ is in fact in $H^3(Y_v, \mathbf{Q})$ for any $v \in V$, so in particular, via the inclusion $\mathcal{W} \subset \mathcal{H}^3_{\mathbf{R},Y}$, $\tilde{\lambda}$ is a flat section of $\mathcal{H}^3_{\mathbf{R},Y}$.

3.5.7. The infinitesimal variation of the Hodge filtration of the open sets Y_v over $\mathcal{S}_\lambda$, and more generally of the sets $Y_t = X_v \backslash S_t$, for $v \in V$, $t \in \mathcal{U}$, $\pi(t) = v$, gives the following diagrams :

3.5.8.

$$\begin{array}{cccc} \nabla^Y & : & F^2\mathcal{H}^3_Y & \to & F^1\mathcal{H}^3_Y \otimes \Omega_{\mathcal{U}} \\ & & \downarrow & & \downarrow \\ \overline{\nabla}^Y & : & F^2/F^3\mathcal{H}^3_Y & \to & F^1/F^2\mathcal{H}^3_Y \otimes \Omega_{\mathcal{U}} \end{array},$$

which restricts on $\mathcal{S}_\lambda$ to :

$$\begin{array}{cccc} \nabla^Y & : & F^2\mathcal{H}^3_Y & \to & F^1\mathcal{H}^3_Y \otimes \Omega_{\mathcal{S}_\lambda} \\ & & \downarrow & & \downarrow \\ \overline{\nabla}^Y & : & F^2/F^3\mathcal{H}^3_Y & \to & F^1/F^2\mathcal{H}^3_Y \otimes \Omega_{\mathcal{S}_\lambda} \end{array},$$

$\tilde{\lambda}$ is a $\mathcal{C}^\infty$ section of $F^2\mathcal{H}^3_Y$ on $\mathcal{S}_\lambda$, which is flat. One deduces then from 3.5.8 :

3.5.9. $\forall v \in \mathcal{S}_\lambda$, $\overline{\nabla}^Y(\overline{\lambda}_v)$ vanishes in $F^1/F^2\mathcal{H}^3_Y \otimes \Omega_{\mathcal{S}_\lambda}(v)$, where $\overline{\lambda}_v$ is the value at v of the projection in $F^2/F^3\mathcal{H}^3_Y$ of $\tilde{\lambda}$. Note that $\overline{\lambda}_v$ belongs to the image $\overline{W}_v$ of $W \otimes \mathbf{C}$ in $F^2/F^3H^3(Y_v)$, and that via Res, $\overline{W}_v$ is isomorphic to $H^1(\Omega_{S_v})^0$.

We conclude now the proof of 3.5.

3.5.10. If all the ν_λ's of 3.4.2 were of torsion, one would have, using 3.5.9 and the density statement in 3.3. : For generic $S \subset X$, $S \in |L^n|$, for generic $\lambda \in H^1(\Omega_S)^0$, the natural lifting $\overline{\lambda}$ of λ in $\overline{W}(Y_S)$ ($Y_S = X \backslash S$) satisfies :

3.5.11. $\overline{\nabla}^Y(\overline{\lambda}) \in \Omega_{\mathcal{U}} \otimes F^1/F^2H^3(Y_S)$ vanishes in $(\operatorname{Ker} \overline{\nabla}(\lambda))^* \otimes F^1/F^2H^3(Y_S)$.

It is shown in [16], using the results of [18], that 3.5.11 is not true. Note that 3.5.11 is an algebraic statement except for the data of the space $\overline{W} \subset F^2/F^3H^3(Y_S)$ which does not vary holomorphically with S. We prove that 3.5.11 is false for any subspace $\overline{W}$ of $F^2/F^3H^3(Y_S)$ isomorphic to $H^1(\Omega_S)^0$ via Res, (for S generic). So 3.5 is proved.

References.

We use the same references as for Lecture 7, and add the following :

[10] A. Albano-A. Collino : On the Griffith group of the cubic sevenfold, Math. Ann., Vol. 299, No. 4 (1994) 715-726

[11] F. Bardelli : On Grothendieck generalized conjecture for a family of threefolds with geometric genus one. Proc. of the Conf. Alg. Geometry, Berlin 1985, 14-23, Teubner texte Math 92, Leipzich (1986).

[12] F. Bardelli - S. Müller-Stach : Algebraic cycles on certain Calabi-Yau threefolds, Math. Z., 215, No. 4 (1994) 569-582

[13] P. Deligne : Théorie de Hodge II, Publ. Math. IHES 40 (1972) pp 5-57.

[14] F. El Zein - S. Zucker : Extendability of normal functions associated to algebraic cycles. In *Topics in transcendental Algebraic Geometry*, ed. by P. Griffiths, Annals of Mathematics Studies, Study 106, Princeton (1984) 269-288.

[15] C. Voisin : Sur les zero cycles de certaines hypersurfaces munies d'un automorphisme, Annali della Scuola Norm. Sup di Pisa, Vol. 29 (1993) 473-492.

[16] C. Voisin : Sur l'application d'Abel-Jacobi des variétés de Calabi-Yau de dimension 3, to appear in Ann. de l'ENS Paris.

[17] C. Voisin : Une approche infinitésimale du théorème de H. Clemens sur les cycles d'une quintique générale de $\mathbf{P}^4$, J. Algebraic Geometry 1 (1992) 157-174.

[18] C. Voisin : Densité du lieu de Noether-Lefschetz pour les sections hyperplanes des variétés de Calabi-Yau, International J. of Math., Vol. 3, n° 5 (1992) 699-715.