Footnotes to papers of O'Grady and Markman

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Pour Olivier, avec amitié

Abstract

In this paper, we first generalize to any hyper-Kähler manifold X with $b_3(X) \neq 0$ results proved by O'Grady for hyper-Kähler manifolds of generalized Kummer type. In the second part, we restrict to hyper-Kähler manifolds of generalized Kummer type and prove, using results of Markman, that their Kuga-Satake correspondence is algebraic.

0 Introduction

This paper provides complements to the recent papers [15] by O'Grady and [12] by Markman. Hyper-Kähler manifolds X of generalized Kummer type are obtained by deforming the generalized Kummer varieties $K_n(A)$ constructed by Beauville [1] starting from an abelian surface A. The manifold $K_n(A)$ is defined as the subset of the punctual Hilbert scheme $A^{[n+1]}$ consisting of 0-dimensional subschemes with trivial Albanese class. For $n \geq 2$, one has $b_2(X) = 7$, $b_3(X) = 8$. Both papers are concerned with the intermediate Jacobians $J^3(X)$ for X as above. Recall that $J^3(X)$ is the complex torus built from the Hodge structure on $H^3(X, \mathbb{Z})$, which in this case is of level 1 since $H^{3,0}(X) = 0$, and is thus an abelian variety when X is projective. As $b_3(X) = 8$, $J^3(X)$ is an abelian fourfold. O'Grady proves the following results.

Theorem 0.1. *(O'Grady [15]) (1) $J^3(X)$ is a Weil abelian fourfold.*

(2) For a very general projective deformation of X, the Kuga-Satake abelian variety $\mathrm{KS}(X)$ *of* $(H^2(X, \mathbb{Q})_{\mathrm{tr}}, (\ ,\ ))$ *is isogenous to a power of* $J^3(X)$.

Let us explain both statements. A Weil abelian fourfold is an abelian fourfold that admits an endomorphism $\phi : A \to A$ satisfying a quadratic equation $\phi^2 = -d\mathrm{Id}$, with $d > 0$, with the following extra condition: consider the action $\phi_{\mathbb{C}}$ of ϕ on $H^1(X, \mathbb{C})$ by pullback. Then $\phi_{\mathbb{C}}$ preserves $H^{1,0}(A)$ and $H^{0,1}(A)$ since it is a morphism of Hodge structures and thus it has eigenvalues either $i\sqrt{d}$ or $-i\sqrt{d}$ on these 4-dimensional spaces. The Weil condition is that $\phi_{\mathbb{C}}$ has both eigenvalues $i\sqrt{d}$ and $-i\sqrt{d}$ with multiplicity 2 on $H^{1,0}(A)$ (hence also on $H^{0,1}(A)$). It guarantees that A has a 2-dimensional space of Weil Hodge classes of degree 4. More precisely, denoting K the number field $\mathbb{Q}[\sqrt{-d}]$, $H^1(A, \mathbb{Q})$ is a 4-dimensional K-vector space and the condition above guarantees that the 2-dimensional subspace

$$\bigwedge_K^4 H^1(A, \mathbb{Q}) \subset \bigwedge^4 H^1(A, \mathbb{Q})$$

consists of classes of Hodge type $(2, 2)$, hence of Hodge classes.

Concerning the point (2), let us define a Hodge structure of hyper-Kähler type as the data of a weight 2 (effective, rational or integral) Hodge structure $(H^2, F^i H^2_{\mathbb{C}})$ with $h^{2,0} = 1$, equipped with a nondegenerate quadratic form satisfying the first Hodge-Riemann relations, namely

$$(H^{2,0}, F^1 H^2_{\mathbb{C}}) = 0$$

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and the condition that the restriction of $(\ ,\ )$ to the real 2-plane $(H^{2,0} \oplus H^{0,2}) \cap H^2_{\mathbb{R}}$ is positive definite. This is the structure one gets on the degree 2 cohomology of a hyper-Kähler manifold, the intersection pairing being given by the Beauville-Bogomolov quadratic form. We will say that we have a polarized Hodge structure of hyper-Kähler type if furthermore $(\ ,\ )$ is negative definite on the space $H^{1,1}_{\mathbb{R}} := H^{1,1} \cap H^2_{\mathbb{R}}$. We get such a structure by considering the transcendental degree 2 cohomology of a projective hyper-Kähler manifold. The Kuga-Satake variety $\mathrm{KS}(H^2, (\ ,\ ))$ associated to a Hodge structure $(H^2, (\ ,\ ))$ of hyper-Kähler type is a complex torus, or weight 1 Hodge structure (which is defined up to isogeny if we work with rational Hodge structures) built by a formal process (see [10], [5] and Section 2). If the considered Hodge structure is a polarized Hodge structure of hyper-Kähler type, $\mathrm{KS}(H^2, (\ ,\ ))$ is an abelian variety. In the case of a very general polarized weight 2 Hodge structure of generalized Kummer type, we have $b_{2,tr} = 6$ and the corresponding Kuga-Satake variety is isogenous to a power of a Weil abelian fourfold. The point (1) thus follows from (2) and (2) itself is a consequence of a certain universality property of the Kuga-Satake construction proved in [3], (and [7] in a slightly different setting, see Section 2), and of the following result.

Theorem 0.2. *(O'Grady [15]) Let X be a hyper-Kähler $2n$-fold of generalized Kummer deformation type with $n \geq 2$. Then the composite map map*

$$\bigwedge^2 H^3(X, \mathbb{Q}) \to H^6(X, \mathbb{Q}) \stackrel{Q_X^{n-2}}{\to} H^{4n-2}(X, \mathbb{Q})$$

is surjective.

Here $Q_X \in H^4(X, \mathbb{Q})$ is a cohomology class which is constructed using the Beauville-Bogomolov form (see Section 1.1). Our first result is the following generalization of Theorem 0.2.

Theorem 0.3. *Let X be a hyper-Kähler $2n$-fold such that $b_3(X) \neq 0$. Then*

(1) The composite map

$$\bigwedge^2 H^3(X, \mathbb{Q}) \to H^6(X, \mathbb{Q}) \stackrel{Q_X^{n-2}}{\to} H^{4n-2}(X, \mathbb{Q})$$

is surjective.

(2) Assuming X is projective, the intermediate Jacobian $J^3(X)$ contains a simple component of the Kuga-Satake abelian variety of $H^2(X, \mathbb{Q})_{\mathrm{tr}}$.

(3) One has $b_3(X) \geq 2^k$, where $k = \frac{b_2(X)-2}{2}$ if $b_2(X)$ is even, $k = \frac{b_2(X)-1}{2}$ if $b_2(X)$ is odd.

We will also prove similar results, and in particular the bound

$$b_{2n-1}(X) \geq 2^k, \text{ if } H^{2n-1}(X, \mathbb{Q}) \neq 0,\ H^{2n-3}(X, \mathbb{Q}) = 0. \quad (1)$$

where k is as in (3). In particular, we get the following corollary in dimension 6.

Corollary 0.4. *Let X be a hyper-Kähler 6-fold such that $b_{\mathrm{odd}}(X) \neq 0$. Then $b_{\mathrm{odd}}(X) \geq 2^k$, where $k = \frac{b_2(X)-2}{2}$ if $b_2(X)$ is even, $k = \frac{b_2(X)-1}{2}$ if $b_2(X)$ is odd.*

The Betti numbers of hyper-Kähler manifolds have been studied in [17], [8] which establishes very precise bounds in dimension 4 and in [18] which claims similar bounds in dimension 6 (but the proof seems to be incomplete). The paper [9] gives very precise conjectural bounds (for example bounds on b_2 depending only on the dimension), depending on a conjecture on the Looijenga-Lunts representation [11], [19]. The subject remains however wide open.

A key point in both cases is the fact that the weight 3 or weight $2n-1$ Hodge structure one considers is of Hodge level 1, that is, they satisfy the property $h^{p,q} = 0$ for $|p-q| > 1$.

As we already mentioned, the points (2) and (3) follow, using this observation, from the point (1), and from a universality property for the Kuga-Satake weight 1 Hodge structure, proved by Charles in [3], even in the unpolarized case.

The second part of this paper provides a complement to Markman's paper [12]. In this paper, Markman proves the Hodge conjecture for the Weil Hodge classes on the Weil abelian fourfolds appearing in Theorem 2.2. He also proves that the Abel-Jacobi map

$$\Phi_X : \mathrm{CH}^2(X)_{\mathrm{alg}} \to J^3(X),$$

defined on the group of codipmension 2 cycles on X algebraically equivalent to 0, is surjective for a projective hyper-Kähler manifold X of generalized Kummer deformation type. This statement was expected as a consequence of the generalized Hodge conjecture because $H^{3,0}(X) = 0$ (see [22]).

Our second result is the following

Theorem 0.5. *For X a projective hyper-Kähler manifold of generalized Kummer deformation type with $n \geq 2$, the Kuga-Satake correspondence between X and its Kuga-Satake variety* $\mathrm{KS}(X)$ *is algebraic.*

To explain this statement, the Kuga-Satake construction in the polarized case produces an abelian variety $\mathrm{KS}(X)$ associated to the polarized Hodge structure $(H^2(X,\mathbb{Q})_{\mathrm{tr}}, (\ ,\ ))$ which has the property that $H^2(X,\mathbb{Q})_{\mathrm{tr}}$ is a Hodge substructure of $H^2(\mathrm{KS}(X),\mathbb{Q})$. The Hodge conjecture predicts the existence of a correspondence between X and $\mathrm{KS}(X)$, that is an algebraic cycle Γ of codimension $2n$ with $\mathbb{Q}$-coefficients in $X \times \mathrm{KS}(X)$, such that Γ_* induces the given embedding $H^2(X,\mathbb{Q})_{\mathrm{tr}} \hookrightarrow H^2(\mathrm{KS}(X),\mathbb{Q})$. The meaning of the "algebraicity of the Kuga-Satake correspondence" is the existence of such cycle Γ (see [6] for a general discussion).

The algebraicity of the Kuga-Satake correspondence is known for projective $K3$ surfaces with Picard number at least 17 [14]. It is also known by work of Paranjape [16] for $K3$ surfaces with Picard number 16 obtained as desingularizations of double covers of $\mathbb{P}^2$ ramified along 6 lines. Some hyper-Kähler examples involving cubic fourfolds have been exhibited in [21].

1 Applications of the hard Lefschetz theorem

1.1 Degree 3 cohomology: complement to a paper of O'Grady

Let X be a hyper-Kähler manifold of dimension $2n$ with $n \geq 2$. The Beauville-Bogomolov quadratic form q_X is a nondegenerate quadratic form on $H^2(X,\mathbb{Q})$, whose inverse gives an element of $\mathrm{Sym}^2 H^2(X,\mathbb{Q})$. By Verbitsky [2], the later space imbeds by cup-product in $H^4(X,\mathbb{Q})$, hence we get a class

$$Q_X \in H^4(X,\mathbb{Q}). \tag{2}$$

The O'Grady map $\phi : \bigwedge^2 H^3(X,\mathbb{Q}) \to H^{4n-2}(X,\mathbb{Q})$ is defined by

$$\phi(\alpha \wedge \beta) = Q_X^{n-2} \cup \alpha \cup \beta. \tag{3}$$

The following result was first proved by O'Grady [15] in the case of a hyper-Kähler manifold of generalized Kummer deformation type.

Theorem 1.1. *Let X be a hyper-Kähler manifold of dimension $2n$. Assume $H^3(X,\mathbb{Q}) \neq 0$. Then the O'Grady map map $\phi : \bigwedge^2 H^3(X,\mathbb{Q}) \to H^{4n-2}(X,\mathbb{Q})$ is surjective.*

Proof. We can choose the complex structure on X to be general, so that $\rho(X) = 0$, and this implies that the Hodge structure on $H^2(X,\mathbb{Q})$ (or equivalently $H^{4n-2}(X,\mathbb{Q})$ as they are isomorphic by combining Poincaré duality and the self-duality given by the Beauville-Bogomolov form) is simple. As the morphism ϕ is a morphism of Hodge structures, its image is a Hodge substructure of $H^{4n-2}(X,\mathbb{Q})$, hence either ϕ is surjective, or it is 0. Theorem 1.1 thus follows fropm the next proposition. $\square$

Proposition 1.2. *The map ϕ is not identically* 0*.*

Proof. Let $\omega \in H^2(X,\mathbb{R})$ be a Kähler class. Then we know that the ω-Lefschetz intersection pairing $\langle\, ,\, \rangle_\omega$ on $H^3(X,\mathbb{R})$, defined by

$$\langle \alpha, \beta\rangle_\omega := \int_X \omega^{2n-3} \cup \alpha \cup \beta$$

is nondegenerate. This implies that the cup-product map

$$\psi : \bigwedge^2 H^3(X,\mathbb{Q}) \to H^6(X,\mathbb{Q})$$

has the property that $\operatorname{Im}\psi$ pairs nontrivially with the image of the map

$$\operatorname{Sym}^{2n-3} H^2(X,\mathbb{Q}) \to H^{4n-6}(X,\mathbb{Q})$$

given by cup-product. Note that the Hodge structure on $H^3(X,\mathbb{Q})$ has Hodge level 1, so that the Hodge structure on the image of $\operatorname{Im}\psi$ in $\operatorname{Sym}^{2n-3} H^2(X,\mathbb{Q})^*$ is a Hodge structure of level at most 2. We now argue as in [20]. We choose X very general so that the Mumford-Tate group of the Hodge structure on $H^2(X,\mathbb{Q})$ is the orthogonal group $O(q_X)$. Any Hodge substructure of $\operatorname{Sym}^{2n-3} H^2(X,\mathbb{Q})^* \cong \operatorname{Sym}^{2n-3} H^2(X,\mathbb{Q})$ is thus a direct sum of $O(q_X)$-subrepresentations of $\operatorname{Sym}^{2n-3} H^2(X,\mathbb{Q})$. Elementary representation theory of $O(q_X)$ then shows that the irreducible $O(q_X)$-subrepresentations of $\operatorname{Sym}^{2n-3} H^2(X,\mathbb{Q})$ are the subspaces

$$Q_X^l \operatorname{Sym}^{2n-3-2l} H^2(X,\mathbb{Q})^0,$$

where we see here Q_X as an element of $\operatorname{Sym}^2 H^2(X,\mathbb{Q})$, and

$$\operatorname{Sym}^k H^2(X,\mathbb{Q})^0 \subset \operatorname{Sym}^k H^2(X,\mathbb{Q})$$

can be defined after passing to $\mathbb{C}$-coefficients as the subspace generated by α^k with $q_X(\alpha) = 0$ (this definition is correct with $\mathbb{Q}$-coefficients only if the quadratic form q_X has a zero).

The irreducible Hodge structure on $Q_X^l \operatorname{Sym}^{2n-3-2l} H^2(X,\mathbb{Q})^0$ has Hodge level > 2 when $2n-3-2l > 1$ since it contains the class $Q_X^l \sigma_X^{2n-3-2l}$ which is of type $(4n-6-4l, 2l)$, where σ_X generates $H^{2,0}(X)$. It follows that $\operatorname{Im}\psi$ can pair nontrivially only with $Q_X^{n-2} H^2(X,\mathbb{Q})$, hence the map $Q_X^{n-2}\psi$, which is the O'Grady map, is nonzero, which concludes the proof. $\square$

1.2 Cohomology of degree $2n-1$

For other odd degree $2k-1 \leq 2n-1$, one may wonder what the hard Lefschetz theorem gives. The proof of Proposition 1.2 will give as well:

Proposition 1.3. *The composition*

$$\psi' : \bigwedge^2 H^{2k-1}(X,\mathbb{Q}) \to H^{4k-2}(X,\mathbb{Q}) \to \operatorname{Sym}^{2n-2k+1} H^2(X,\mathbb{Q})^*,$$

where the first map is the cup-product and the second one is Poincaré dual to the cup-product map $\operatorname{Sym}^{2n-2k+1} H^2(X,\mathbb{Q}) \to H^{4n-4k+2}(X,\mathbb{Q})$*, is nontrivial (and even, nondegenerate).*

However, we do not know a priori the Hodge level of $H^{2k-1}(X,\mathbb{Q})$ so we do not know to which irreducible component of the $O(q)$-representation of $\operatorname{Sym}^{2n-2k+1} H^2(X,\mathbb{Q})^*$ the image $\operatorname{Im}\psi'$ can map nontrivially. In the case of degree $2k-1 = 2n-1$, we have only one piece, namely $H^2(X,\mathbb{Q})^*$, hence we get:

Corollary 1.4. *If X is a hyper-Kähler manifold of dimension $2n$ with $H^{2n-1}(X,\mathbb{Q}) \neq 0$, the cup-product map*

$$\bigwedge^2 H^{2n-1}(X,\mathbb{Q}) \to H^2(X,\mathbb{Q})^*.$$

is surjective.

Proof. Indeed, the map ψ is nonzero and a morphism of Hodge structures, the right hand side being a simple Hodge structure for a very general complex structure on X. □

We will also use in next section the following observation.

Lemma 1.5. *Let X be a hyper-Kähler $2n$-fold. Then the Hodge structure on the quotient*

$$H^{2n-1}(X,\mathbb{Q})^0 := H^{2n-1}(X,\mathbb{Q})/H^2 \cup H^{2n-3}(X,\mathbb{Q}) \tag{4}$$

has Hodge level 1. In particular, if $H^{2n-3}(X,\mathbb{Q}) = 0$, the Hodge structure on $H^{2n-1}(X,\mathbb{Q})$ has Hodge level 1.

Proof. The statement is that the (p,q)-components of $H^{2n-1}(X,\mathbb{C})^0$ vanish unless $(p,q) = (n,n-1)$ or $(p,q) = (n-1,n)$. We thus have to show that any class in $H^{p,q}(X)$ with $p > n$ or $q > n$ belongs to $H^2(X,\mathbb{C}) \cup H^{2n-3}(X,\mathbb{C})$. This follows from the fact that, as σ_X is a symplectic holomorphic form, the cup-product map by σ_X^l induces a vector bundle isomorphism

$$\sigma_X^l \wedge : \Omega_X^{n-l} \to \Omega_X^{n+l},$$

hence an isomorphism $\sigma_X^l \cup : H^{n-l,q}(X) \cong H^{n+l,q}(X)$. This proves the statement for $p > n$ and the other statement follows by Hodge symmetry. □

2 Universality of the Kuga-Satake correspondence and applications

We start with an effective rational Hodge structure $(H^2, F^i H^2_{\mathbb{C}})$ of weight 2 with $h^{2,0} = 1$ equipped with a symmetric nondegenerate intersection pairing $(\ ,\ )$ satisfying the conditions

$$(\sigma, F^1H^2) = 0,\ (\sigma, \overline{\sigma}) > 0,$$

where σ generates $H^{2,0}$. Note that $(\ ,\ )$ satisfies only part of the Hodge-Riemann relations so that the Hodge structure is not in general polarized. We will call such data a Hodge structure of hyper-Kähler type (although it also encodes the quadratic form) because this is the structure that we have on the degree 2 cohomology $H^2(X,\mathbb{Q})$ of a hyper-Kähler manifold equipped with the Beauville-Bogomolov form q_X. The Kuga-Satake correspondence first constructed in [10] associates to a Hodge structure H^2 of hyper-Kähler type a weight 1 Hodge structure H^1_{KS}, which has the property that there is an injective morphism of Hodge structures

$$H^2 \to \mathrm{End}\, H^1_{\mathrm{KS}},$$

of bidegree $(-1,-1)$. Note that the Hodge structure on both sides has Hodge level 2. When the Hodge structure of hyper-Kähler type on H^2 is polarized by $(\ ,\ )$, which means that our data have the extra property that the pairing $(\ ,\ )$ restricted to $H^{1,1}_{\mathbb{R}}$ is negative definite, the Hodge structure on H^1_{KS} is polarized, hence is the Hodge structure on the H^1 of an abelian variety.

The construction of H^1_{KS} can be summarized as follows: The $\mathbb{Q}$-vector space H^1_{KS} is defined as $\mathrm{Cliff}\,(H^2, (\ ,\ ))$, that is, it is the quotient of the tensor algebra $\otimes H^2$ by the ideal generated by the relations $x^2 = (x,x)1$, $x \in H^2$. The weight 1 Hodge structure on H^1_{KS} is given by a complex structure on the real vector space $H^1_{\mathrm{KS},\mathbb{R}}$. It is constructed as follows.

Consider the subspace $(H^{2,0} \oplus H^{0,2})_{\mathbb{R}} \subset H^2_{\mathbb{R}}$. It is of dimension 2, naturally oriented, and the restriction of the form $(\,,\,)$ to this real plane is positive definite. Choose a positively oriented orthonormal basis (e_1, e_2) of $(H^{2,0} \oplus H^{0,2})_{\mathbb{R}}$. Then $e := e_1 e_2 \in \mathrm{Cliff}\,(H^2_{\mathbb{R}}, (\,,\,))$ does not depend on the choice of basis and satisfies $e^2 = -1$. Left Clifford multiplication by e thus defines the desired complex structure on $\mathrm{Cliff}\,(H^2_{\mathbb{R}}, (\,,\,)) = H^1_{\mathrm{KS},\mathbb{R}}$.

Clifford multiplication on the left induces a morphism

$$H^2 \to \mathrm{End}\, H^1_{\mathrm{KS}}$$

which is a morphism of Hodge structures of bidegree $(-1,-1)$. This is equivalent to saying that Clifford multiplication on the left by $H^{1,1}$ preserves the Hodge decomposition of $H^1_{\mathrm{KS},\mathbb{C}}$ and that Clifford multiplication on the left by $H^{2,0}$ shifts the Hodge decomposition of $H^1_{\mathrm{KS},\mathbb{C}}$ by $(-1,-1)$. The first fact follows because $H^{1,1}$ is orthogonal to $H^{2,0}$ for $(\,,\,)$, so multiplication by elements of $H^{1,1}$ anticommutes with Clifford multiplication by elements of $H^{2,0}$ or $H^{0,2}$, hence commutes with Clifford multiplication by $e_1 e_2$. The second fact is an easy computation.

The weight 1 Hodge structure H^1_{KS} is not simple. In fact it has a big algebra of endomorphisms given by right Clifford multiplication on the Clifford algebra. These endomorphisms obviously commute with left Clifford multiplication by e, hence provide automorphisms of Hodge structure of H^1_{KS}. To start with, we can restrict the construction to the even Clifford algebra $C^+(H^2, (\,,\,))$ generated by the tensor products $v_1 \otimes \ldots \otimes v_k$, $v_i \in H^2$, with k even, which clearly provides a Hodge substructure of $H^1_{\mathrm{KS},\mathbb{Q}}$ since multiplication on the left by e preserves $C^+(H^2_{\mathbb{R}}, (\,,\,))$. We can do similarly with the odd part $C^-(H^2, (\,,\,))$ of the Clifford algebra, which provides another Hodge substructure. Multiplication on the right by a given element $v_0 \in H^2$ with $(v, v) \neq 0$ provides an isomorphism

$$C^+(H^2, (\,,\,)) \cong C^-(H^2, (\,,\,)),$$

so that, denoting $H^1_{\mathrm{KS}+}$, $H^1_{\mathrm{KS}-}$ the weight 1 Hodge structures so obtained, we have an isomorphism

$$H^1_{\mathrm{KS}+} \to H^1_{\mathrm{KS}-}$$

given by right Clifford multiplication by v_0, and we get an injective (but not canonical) morphism of Hodge structures

$$H^2 \to \mathrm{End}\, H^1_{\mathrm{KS}+}$$

given by

$$v \mapsto (\alpha \mapsto v\alpha v_0).$$

When the Mumford-Tate group of the Hodge structure on H^2 is the orthogonal group $O((\,,\,))$, using representation theory of the orthogonal group, one can describe up to isogeny the complex tori appearing as subquotient of the Kuga-Satake complex torus (see [3], [6]). Note that in the geometric case, it follows from the local surjectivity of the period map that the Mumford-Tate group is the orthogonal group $O((\,,\,))$. When the dimension h of H^2 is odd, the Kuga-Satake complex torus is a power of a simple torus of dimension $2^{\frac{h-3}{2}}$ or $2^{\frac{h-1}{2}}$. When h is even, the situation is much more delicate, as the classification of the subquotients depends on the discriminant of the quadratic form $(\,,\,)$. In this case, the Kuga-Satake complex torus is a sum of powers of one or two simple complex tori which can be of dimension $2^{h/2}$, $2^{h/2-1}$ or $2^{h/2-2}$ (see [3]). The numbers above are obtained starting from the fact that the even Clifford algebra $C^+(H^2)$ has dimension 2^{h-1} and that the action on it by right multiplication by elements of $C^+(H^2)$ (which are morphisms of Hodge structures since they commute with the left multiplication by e) splits it as a direct sum of weight 1 Hodge structures. We now consider the polarized case. The Kuga-Satake Hodge structure $H^1_{\mathrm{KS}+}$ is then polarized and thus is the weight 1 Hodge structure on the degree 1 rational cohomology of an abelian variety, that we will denote $\mathrm{KS}(H^2, (\,,\,))$, and is defined up to isogeny. The two dual complex tori appearing above are then isomorphic. In the case where

h is even, the simple abelian variety one gets has a quadratic endomorphism which makes it a Weil abelian variety. In the geometric case, where we start from the degree 2 cohomology of a hyper-Kähler manifold X, equipped with the Beauville-Bogomolov form q_X, we assume X is polarized by an ample class $l \in \mathrm{NS}(X)$ and put

$$(H^2, (\ ,\ )) = (H^2(X, \mathbb{Q})^{\perp_{q_X} l}, q_X)$$

$$\mathrm{KS}(X) = \mathrm{KS}(H^2, (\ ,\ )).$$

In fact, we can also define $(H^2(X, \mathbb{Q})^{\perp_{q_X} l}, q_X)$ without using the Beauville-Bogomolov form, since $H^2(X, \mathbb{Q})^{\perp_{q_X} l}$ is the group of l-primitive classes, and, up to a rational coefficient, the restricted form q_X is proportional to the Lefschetz intersection pairing defined by l.

The following universality property is proved in [3] (see also [7] for a slightly different statement, proved only in the polarized case).

Theorem 2.1. *Let $(H^2, (\ ,\ ))$ be a Hodge structure of hyper-Kähler type. Assume the Mumford-Tate group of H^2 is $SO(H^2, (\ ,\ ))$. Let H be a simple effective weight 1 Hodge structure, such that for nonnegative integers a, b of the same parity, there exists an injective morphism of Hodge structures of bidegree $(\frac{a-b}{2} - 1, \frac{a-b}{2} - 1)$*

$$H^2 \hookrightarrow H^{\otimes a} \otimes (H^\vee)^{\otimes b}.$$

Then H is a subquotient of the Kuga-Satake Hodge structure $H^1_{\mathrm{KS}+}$. In particular

$$\dim H \geq 2^k, \text{ where } k = \frac{h-1}{2} \text{ if } h \text{ is odd, } k = \frac{h-2}{2} \text{ if } h \text{ is even.}$$

If h is divisible by 4 and the signature of $(\ ,\ )$ is $(3, h-3)$, the last inequality can be improved to $\dim H \geq 2^{\frac{h}{2}}$.

A first application of this universality property (or rather a variant of it) was given in [7] where we proved the Matsushita conjecture on the moduli map for Lagrangian fibration of projective hyper-Kähler manifolds, at least in the case where $b_2(X) \geq 5$, assuming the Mumford-Tate group is maximal. A second application (also in the projective case, with $a = b = 1$) was given by O'Grady in [15]. Let X be a projective hyper-Kähler manifold of generalized Kummer deformation type and dimension ≥ 4. One has $b_2(X) = 7$, hence for a very general projective such hyper-Kähler manifold, $b_2(X)_{tr} = 6$, so that $\mathrm{KS}(X)$ is isogenous to a sum of copies of a simple abelian fourfold of Weil type. Using Theorem 1.1 (that he had proved by an explicit computation in that case), the fact that $b_3(X) = 8$, and the universality property of Theorem 2.1, O'Grady proved the following result.

Theorem 2.2. *The intermediate Jacobian $J^3(X)$ of a projective hyper-Kähler manifold of generalized Kummer deformation type with $\rho(X) = 1$ is a Weil abelian fourfold. The Kuga-Satake variety of $(H^2(X, \mathbb{Q})_{tr}, q_X)$ is isogenous to a sum of two copies of $J^3(X)$.*

2.1 Applications to Betti numbers

In this section, we are going to apply the previous results to get inequalities involving the Betti numbers of hyper-Kähler manifolds.

Theorem 2.3. *Let X be a hyper-Kähler manifold. Assume that $b_3(X) \neq 0$. Then*

$$b_3(X) \geq 2^k, \tag{5}$$

where $k = \frac{b_2(X)-1}{2}$ if $b_2(X)$ is odd, $k = \frac{b_2(X)-2}{2}$ if $b_2(X)$ is even.

If $b_2(X)$ is divisible by 4, the last inequality can be improved to $b_3(X) \geq 2^{b_2(X)/2}$.

Proof. By Theorem 1.1, we have a surjective morphism of Hodge structures

$$\bigwedge^2 H^3(X,\mathbb{Q}) \to H^{4n-2}(X,\mathbb{Q}) \cong H^2(X,\mathbb{Q})^*,$$

which gives as well an injective morphism of Hodge structures

$$H^2(X,\mathbb{Q}) \hookrightarrow \bigwedge^2 H^3(X,\mathbb{Q})^* \hookrightarrow H^3(X,\mathbb{Q})^* \otimes H^3(X,\mathbb{Q})^*.$$

Choosing the complex structure on X very general so that the Mumford-Tate group of the Hodge structure on $H^2(X,\mathbb{Q})$ is the orthogonal group of $(\ ,\ )$, we can thus apply Theorem 2.1, which gives (5). □

We now turn to the Betti number b_{2n-1}. We prove the following

Theorem 2.4. *Let X be a hyper-Kähler manifold such that $H^{2n-3}(X) = 0$ and $H^{2n-1}(X) \neq 0$. Then*

$$b_{2n-1}(X) \geq 2^k, \tag{6}$$

where $k = \frac{b_2(X)-1}{2}$ if $b_2(X)$ is odd, $k = \frac{b_2(X)-2}{2}$ if $b_2(X)$ is even.

If $b_2(X)$ is divisible by 4, the last inequality can be improved to $b_{2n-1}(X) \geq 2^{b_2(X)/2}$.

Proof. By Corollary 1.4, the cup-product map

$$\bigwedge^2 H^{2n-1}(X,\mathbb{Q}) \to H^{4n-2}(X,\mathbb{Q})$$

is surjective. As we assumed $H^{2n-3}(X,\mathbb{Q}) = 0$, the Hodge structure on $H^{2n-1}(X,\mathbb{Q})$ has Hodge level 1 by Lemma 1.5. We are thus exactly as in the situation of Theorem 2.3 and the same arguments give inequality (6). □

In the case of a hyper-Kähler manifold X of dimension $2n = 6$, we get

Corollary 2.5. *Let X be a hyper-Kähler 6-fold such that $H^{\mathrm{odd}}(X) \neq 0$. Then*

$$b_{odd}(X) \geq 2^k, \tag{7}$$

where $k = \frac{b_2-2}{2}$ if b_2 is even, $k = \frac{b_2-1}{2}$ if b_2 is odd.

Proof. We observe that, in dimension 6, if $H^{\mathrm{odd}}(X) \neq 0$, then either $H^3(X,\mathbb{Q}) \neq 0$ or, $H^3(X,\mathbb{Q}) = 0$ and $H^5(X,\mathbb{Q}) \neq 0$. In the first case we apply Theorem 2.3 and in the second case we apply Theorem 2.4. □

3 Algebraicity of the Kuga-Satake correspondence

Let X be a projective complex manifold. Assume that $h^{2,0}(X) = 1$, so that the Hodge structure on $H^2(X,\mathbb{Q})$ is of hyper-Kähler type (choosing a polarization l on X, the Lefschetz intersection pairing $(\ ,\ )_{\mathrm{lef}}$ defined by

$$(\alpha,\beta)_{\mathrm{lef}} = \int_X l^{n-2}\alpha \cup \beta$$

gives the desired intersection form). In the case of a hyper-Kähler manifold of dimension $2n$, the Beauville-Bogomolov intersection pairing on $H^2(X,\mathbb{Q})$ is independent of the choice of a polarization, but when we restrict it to the l-primitive cohomology $H^2(X,\mathbb{Q})_{\mathrm{prim}} = H^2(X,\mathbb{Q})^{\perp l^{2n-1}}$, the two pairings coincide up to a scalar coefficient. Let $\mathrm{KS}(X)$ be the Kuga-Satake abelian variety (defined up to isogeny) associated to the polarized Hodge structure

$(H^2(X,\mathbb{Q})_{\rm prim}, (\ ,\ )_{\rm lef})$. Using an adequate polarization on $\mathrm{KS}(X)$, the injective morphism of Hodge structures

$$H^2(X,\mathbb{Q})_{\rm prim} \hookrightarrow \mathrm{End}\,(H^1_{\mathrm{KS}+}(H^2(X,\mathbb{Q})_{\rm prim}, (\ ,\ ))) = \mathrm{End}\, H^1(\mathrm{KS}(X),\mathbb{Q})$$

gives an injective morphism of Hodge structures

$$H^2(X,\mathbb{Q})_{\rm prim} \hookrightarrow H^1(\mathrm{KS}(X),\mathbb{Q})^{\otimes 2}. \tag{8}$$

whose image is contained in $\bigwedge^2 H^1(\mathrm{KS}(X),\mathbb{Q}) = H^2(\mathrm{KS}(X),\mathbb{Q})$.

A morphism of Hodge structures $\beta : H^2(X,\mathbb{Q})_{\rm prim} \to H^2(\mathrm{KS}(X),\mathbb{Q})$ as in (8) provides a Hodge class (see [22])

$$\alpha \in H^{4n-2}(X,\mathbb{Q}) \otimes H^2(\mathrm{KS}(X),\mathbb{Q}) \subset H^{4n}(X\times \mathrm{KS}(X),\mathbb{Q}). \tag{9}$$

The Hodge conjecture thus predicts that there is a cycle $\Gamma \in \mathrm{CH}^{2n}(X\times \mathrm{KS}(X))_{\mathbb{Q}}$ such that $[\Gamma] = \alpha$, hence in particular

$$[\Gamma]_* = \beta : H^2(X,\mathbb{Q})_{\rm prim} \to H^2(\mathrm{KS}(X),\mathbb{Q}).$$

When this holds, we will say that the Kuga-Satake correspondence is algebraic.

In the case where X is an abelian surface, so $b_2(X)_{tr} \leq 5$, or more generally any projective $K3$ surface with $\rho \geq 17$, the algebraicity of the class α above is proved by Morrison [14]. In that case, the Kuga-Satake variety is isogenous to a sum of copies of the abelian surface itself.

In the next case, where $b_{2,tr} = 6$, we already mentioned that the Kuga-Satake variety is isogenous to a sum of copies of a 4-dimensional abelian variety which is of Weil type (assuming the maximality of the Mumford-Tate group). This case appears geometrically with $K3$ surfaces with Picard number 16 and the first family of such $K3$ surfaces for which the Kuga-Satake correspondence was known to be algebraic was found by Paranjape [16]. The Paranjape $K3$ surfaces are obtained by desingularizing double covers of $\mathbb{P}^2$ ramified along the union of six lines.

The geometric situation we consider is the same as in [15], [12]. X is a projective hyper-Kähler manifold of generalized Kummer type. In particular, we know by O'Grady theorem (Theorem 2.2) that $J^3(X)$ is isogenous to a component of the Kuga-Satake variety $\mathrm{KS}(X)$. We prove now the following result.

Theorem 3.1. *Let X be a projective hyper-Kähler manifold of generalized Kummer type. Then the Kuga-Satake correspondence of X is algebraic.*

This theorem should be actually considered as an addendum to Markman's paper [12]. The result will indeed follow from the following result (Theorem 3.2) of Markman. As we already mentioned, for X as above, the Hodge structure on $H^3(X,\mathbb{Q})$ is of Hodge level 1, that is, of type $(2,1)+(1,2)$. The generalized Hodge conjecture thus predicts that the degree 3 cohomology of X is supported on a (singular) divisor of X, and this is equivalent to the fact that the Griffiths Abel-Jacobi map

$$\Phi_X : \mathrm{CH}^2(X)_{\rm alg} \to J^3(X) \tag{10}$$

is surjective (see [22]).

Theorem 3.2. *(Markman [12]) For a projective hyper-Kähler manifold of Kummer deformation type, the Abel-Jacobi map (10) is surjective.*

Proof of Theorem 3.1. An equivalent version of Theorem 3.2 says that there exists a codimension 2 cycle $\mathcal{Z} \in \mathrm{CH}^2(J^3(X)\times X)_{\mathbb{Q}}$ such that the map $[\mathcal{Z}]_* : H_1(J^3(X),\mathbb{Q}) \to H^3(X,\mathbb{Q})$ is the natural identification $H_1(J^3(X),\mathbb{Q}) \cong H^3(X,\mathbb{Q})$. We recall here that $J^3(X)$ is the complex torus $H^3(X,\mathbb{C})/(F^2H^3(X,\mathbb{C}) \oplus H^3(X,\mathbb{Z}))$ built from the Hodge structure on $H^3(X,\mathbb{Z})$ so that $H_1(J^3(X),\mathbb{Z}) = H^3(X,\mathbb{Z})$ canonically. Note that we can assume that the cohomology class $[\mathcal{Z}] \in H^4(J^3(X) \times X,\mathbb{Q})$ belongs to the Künneth component $H^1(J^3(X),\mathbb{Q}) \otimes H^3(X,\mathbb{Q})$. Indeed, using the action of the maps of multiplication by k on $J^3(X)$, the Künneth components of $[\mathcal{Z}]$ are all algebraic, and the Künneth components not in $H^1(J^3(X),\mathbb{Q}) \otimes H^3(X,\mathbb{Q})$ induce the zero map $H_1(J^3(X),\mathbb{Q}) \to H^3(X,\mathbb{Q})$.

Next, by another result of Markman [13], the class $Q_X \in H^4(X,\mathbb{Q})$ introduced in (2) is algebraic on hyper-Kähler manifolds of generalized Kummer type. It is thus the class of a cycle $\mathcal{Q}_X \in \mathrm{CH}^2(X)_{\mathbb{Q}}$. On $J^3(X) \times X$, we consider the following cycle

$$\Gamma := \mathcal{Z}^2 \cdot \mathrm{pr}_X^* \mathcal{Q}_X^{2n-2}, \tag{11}$$

where $\mathrm{pr}_X : J^3(X) \times X \to X$ denotes the second projection. We prove the following

Claim 3.3. *The map $[\Gamma]_* : H_2(J^3(X),\mathbb{Q}) \to H^{4n-2}(X,\mathbb{Q})$ identifies with the O'Grady map $\phi : \bigwedge^2 H^3(X,\mathbb{Q}) \to H^6(X,\mathbb{Q}) \stackrel{Q_X}{\to} H^{4n-2}(X,\mathbb{Q})$ of (3).*

Proof. Recall that we assumed that $[\mathcal{Z}] \in H^1(J^3(X),\mathbb{Q}) \otimes H^3(X,\mathbb{Q})$. Taking a basis e_i of $H^3(X,\mathbb{Q})$, which provides a basis f_i of $H_1(J^3(X),\mathbb{Q})$ and the dual basis f_i^* of $H^1(J^3(X),\mathbb{Q})$, we can thus write

$$[\mathcal{Z}] = \sum_i \mathrm{pr}_{J^3(X)}^* f_i^* \cup \mathrm{pr}_X^* e_i, \tag{12}$$

since $[\mathcal{Z}]_*(f_i) = e_i$. We now deduce from (12)

$$[\Gamma] = -\sum_{i,j} \mathrm{pr}_{J^3(X)}^*(f_i^* \cup f_j^*) \cup \mathrm{pr}_X^* e_i \cup \mathrm{pr}_X^* e_j \cup \mathrm{pr}_X^* Q_X,$$

which immediately implies the claim. □

The claim implies the theorem since we already identified the intermediate Jacobian with a component of the Kuga-Satake variety, in such a way that the transpose of the map (8) is the O'Grady map. Thus the map (8) and its transpose are induced by an algebraic cycle. □

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