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CONIVEAU 2 COMPLETE INTERSECTIONS AND EFFECTIVE CONES

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Abstract. Griffiths computation of the Hodge filtration on the cohomology of a smooth hypersurface X of degree d in $\mathbb{P}^n$ shows that it has coniveau $\geq c$ once $n \geq dc$. The generalized Hodge conjecture (GHC) predicts that the cohomology of X is then supported on a closed algebraic subset of codimension at least c. This is essentially unknown for $c \geq 2$. In the case where $c = 2$, we exhibit a geometric phenomenon in the variety of lines of X explaining the estimate for the coniveau, and show that (GHC) would be implied in this case by the following conjecture on effective cones of cycles of intermediate dimension: Very moving subvarieties have their class in the interior of the effective cone.

0 Introduction

The goal of this paper is first of all to propose a strategy to attack the generalized Hodge conjecture for coniveau 2 complete intersections, and secondly to state a conjecture concerning the cones of effective cycle classes in intermediate dimensions. Our main results show that the generalized Hodge conjecture for coniveau 2 complete intersections would follow from a particular case of this effectiveness conjecture.

A rational Hodge structure of weight k is given by a $\mathbb{Q}$-vector space L together with a Hodge decomposition

$$L_{\mathbb{C}} = \bigoplus_{p+q=k} L^{p,q}$$

satisfying Hodge symmetry

$$\overline{L^{p,q}} = L^{q,p}.$$

The coniveau of such a Hodge structure is the smallest integer c such that $L^{k-c,c} \neq 0$.

When the Hodge structure comes from geometry, the notion of coniveau is conjecturally related to codimension by the generalized Grothendieck–Hodge conjecture. Suppose X is a smooth complex projective variety and $L \subset H^k(X, \mathbb{Q})$ is a Hodge substructure of coniveau c.

Conjecture 0.1 (cf. [Gr]). *There exists a closed algebraic subset $Z \subset X$ of codimension c such that L vanishes under the restriction map $H^k(X, \mathbb{Q}) \to H^k(U, \mathbb{Q})$, where $U := X \setminus Z$.*

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Notice that it is a non-trivial fact that the kernel of this restriction map is indeed a Hodge substructure of coniveau $\geq c$. This needs some arguments from mixed Hodge theory (see [D], [Gr] or [V2, II, 4.3.2]).

REMARK 0.2. When a Hodge substructure L has coniveau c, we can consider the generalized Grothendieck–Hodge conjecture for L and for any coniveau $c' \leq c$, that is, ask whether L vanishes on the complementary set of a closed algebraic subset of codimension c'. When this holds, we will say that L satisfies the generalized Grothendieck–Hodge conjecture for coniveau c'.

Consider a complete intersection $X \subset \mathbb{P}^n$ of r hypersurfaces of degree $d_1 \leq \cdots \leq d_r$. By the Lefschetz hyperplane sections theorem, the only interesting Hodge structure on the cohomology of X is the Hodge structure on $H^{n-r}(X,\mathbb{Q})$, and in fact on the primitive part of it (that is the orthogonal of the restriction of $H^*(\mathbb{P}^n,\mathbb{Q})$ with respect to the intersection pairing). We will say that X has coniveau c if the Hodge structure on $H^{n-r}(X,\mathbb{Q})_{prim}$ has coniveau c.

The coniveau of a complete intersection can be computed using Griffiths residues and the comparison of pole order and Hodge filtration (see [G] or [V2, II, 6.1.2]). The result is as follows:

Theorem 0.3. *X has coniveau $\geq c$ if and only if*
$$n \geq \sum_i d_i + (c-1)d_r\,.$$

For $c=1$, the result is obvious, as $coniveau(X) \geq 1$ is equivalent to $H^{n-r,0}(X) = H^0(X,K_X) = 0$, that is X is a Fano complete intersection. In this case, the generalized Hodge–Grothendieck conjecture is known to be true, using the correspondence between X and its Fano variety of lines F. Denoting by
$$\begin{array}{ccc} P & \xrightarrow{q} & X \\ {\scriptstyle p}\big\downarrow & & \\ F & & \end{array}$$
the incidence correspondence, where p is the tautological $\mathbb{P}^1$-bundle on F, one can show (see for example [Sh]) that taking a $n-r-2$-dimensional complete intersection $F_{n-r-2} \subset F$ and restricting P to it, the resulting morphism of Hodge structures
$$q'_* \circ p'^* : H^{n-r-2}(F_{n-r-2},\mathbb{Q}) \to H^{n-r}(X,\mathbb{Q})$$
is surjective, where $P' = p^{-1}(F_{n-r-2})$ and p', q' are the restrictions of p, q to P'. It follows that $H^{n-r}(X,\mathbb{Q})$ vanishes on the complement of the (singular) hypersurface $q(P') \subset X$.

In the case of coniveau 2, the numerical condition given by Theorem 0.3 becomes
$$n \geq \sum_i d_i + d_r\,. \tag{0.1}$$
The geometric meaning of this bound is not so obvious. Furthermore the generalized Grothendieck–Hodge conjecture for coniveau 2 is not known to hold in this range. For a general hypersurface X of degree d in $\mathbb{P}^n$, it is known to hold (for coniveau 2) only when the dimension n becomes much larger than the degree d, so that X becomes covered by a family of planes (see [EVL]). (A slightly weaker condition has

been obtained by Otwinowska [O], see section 3.) The numerical range in which the result is known looks like

$$n \geq \tfrac{d^2}{4} + O(d)$$

which is very different from the bound $n \geq 2d$ of Theorem 0.3. For specific hypersurfaces of coniveau $\geq c$ defined by an equation $F(X_0, \ldots, X_n)$ of the form

$$F = F_1(X_0, \ldots, X_{d+s}) + F_2(X_{d+s+1}, \ldots, X_{2d+s}) + \cdots + F_c(X_{(c-1)d+s+1} \ldots, X_{cd+s})\,,$$

(so $n = cd + s$), the generalized Hodge Conjecture 0.1 is proved in [V1] which proves more generally that they satisfy the corresponding Bloch–Beilinson conjecture (Conjecture 0.4) on Chow groups of hypersurfaces.

CONJECTURE 0.4. *If coniveau$(X) \geq c$, the Chow groups $CH_i(X)_{hom,\mathbb{Q}}$ of cycles homologous to zero modulo rational equivalence are trivial for $i \leq c-1$.*

Notice that the previously mentioned papers [EVL], [O] are also devoted to the study of cycles of low dimension, and that the generalized Grothendieck–Hodge conjecture is deduced from vanishings for these. We refer to [BS], [S] or [V2, II, 10.3.2], for the proof that the Bloch–Beilinson conjecture implies the generalized Hodge–Grothendieck conjecture.

We propose in this paper a strategy to prove the generalized Hodge conjecture for coniveau 2 complete intersections, which does not involve the study of Chow groups and the construction of 2-cycles (replacing the lines used in the coniveau 1 case).

Our method is based on the following result which in section 1 allows us to give a geometric proof of the numerical estimate (0.1) for the coniveau 2 property : First of all, let us make the following definition:

DEFINITION 0.5. *A smooth k-dimensional subvariety $V \subset Y$, where Y is smooth projective, is* very moving *if it has the following property: through a generic point $y \in Y$, and given a generic vector subspace $W \subset T_{Y,y}$ of rank k, there is a deformation $V' \subset Y$ of V in Y which is smooth and passes through y with tangent space equal to W at y.*

Let $X \subset \mathbb{P}^n$ be a generic complete intersection of multidegree $d_1 \leq \cdots \leq d_r$. For a generic section $G \in H^0(X, \mathcal{O}_X(n - \sum_i d_i - 1))$ with zero set $X_G \subset X$, consider the subvariety $F_G \subset F$ of the variety of lines contained in X_G. Then by genericity, F and F_G are smooth of respective dimensions $2n - 2 - \sum_i d_i - r$ and $n - r - 2$. One can show that the deformations of F_G are given by deformations of G.

Theorem 0.6. *When*

$$n \geq \sum_i d_i + d_r$$

the subvariety $F_G \subset F$ is "very moving".

Cones of effective cycles have been very much studied in codimension 1 or in dimension 1 (cf. [BoDPP]), but essentially nothing is known in intermediate (co)dimensions. Let us say that an algebraic cohomology class is big if it belongs to the interior of the effective cone. In [P], it is shown that when $\dim W = 1$, and $W \subset V$ is moving and has ample normal bundle, its class $[W]$ is big. We will give here an example working in any dimension ≥ 4 and in codimension 2, showing that in higher dimensions, a moving variety $W \subset V$ with ample normal bundle may not have big class. Here by *moving*, we mean that a generic deformation of W in V may be imposed to pass through a generic point of V.

We make the following conjecture for "very moving" subvarieties.

CONJECTURE 0.7. *Let V be smooth and projective and let $W \subset V$ be a very moving subvariety. Then the class $[W]$ of W is big.*

This conjecture, in the case of codimension 2 complete intersections in projective space, predicts rather mysterious effectiveness statements concerning projective bundles on Grassmannians. We will show this effectiveness result exactly in the same range as appears in [O], which together with Theorem 0.8 below gives another proof of the fact that the generalized Hodge–Grothendieck for coniveau 2 is satisfied in this numerical range.

We finally prove the following result (Theorem 3.1) in section 3:

Theorem 0.8. *Assume $n \geq \sum_i d_i + d_r$ and the subvariety $F_G \subset F$ introduced above has a big class (that is satisfies Conjecture 0.7). Then the complete intersections of multidegree $d_1 \leq \ldots \leq d_r$ in $\mathbb{P}^n$ satisfy the generalized Hodge conjecture for coniveau 2.*

Our method reproves the known results concerning the generalized Hodge conjecture for coniveau 2, that is proves it in the same range as [O], but the spirit is very different, and the two methods lead in fact to a different statement, which we explain to conclude this introduction. It is generally believed that to solve the generalized Hodge conjecture for coniveau c for $H^k(X,\mathbb{Q})$, $k = \dim X$, one should produce a family of cycles $(Z_b)_{b\in B}$, $\dim B = k-2c$, of dimension c in X, such that the incidence family

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{q} & X \\ {\scriptstyle p}\big\downarrow & & \\ B & & \end{array}$$

induces a surjective map

$$q_*p^* : H^{k-2c}(B,\mathbb{Q}) \to H^k(X,\mathbb{Q})\,.$$

This obviously implies that $H^k(X,\mathbb{Q})$ vanishes away from $q(\mathcal{Z})$ and thus that the generalized Hodge conjecture for coniveau c is satisfied. In general such cycles are provided by the proof that Chow groups of dimension $< c$ are small (cf. [BS] or [V2, II, proof of Th. 10.31]).

However if the generalized Hodge conjecture holds for coniveau c and for $H^k(X,\mathbb{Q})$, it does not imply the existence of such family, unless we also have a Lefschetz type conjecture satisfied. To see this more precisely, suppose the Hodge conjecture holds true for $H^k(X,\mathbb{Q})$ and for coniveau c. Then there exists a closed algebraic subset $Z \subset X$ of codimension c such that $H^k(X,\mathbb{Q})$ vanishes on $X \setminus Z$. Introduce a desingularization $\tau : \widetilde{Z} \to X$ of Z. Then the vanishing of $H^k(X,\mathbb{Q})$ on $X \setminus Z$ implies by strictness of morphisms of mixed Hodge structures (see [D] or [V2, II, 4.3.2]) that

$$\tau_* : H^{k-2c}(\widetilde{Z},\mathbb{Q}) \to H^k(X,\mathbb{Q})$$

is surjective. Observe now that $\dim \widetilde{Z} = k-c$. If the Lefschetz standard conjecture is satisfied by $\widetilde{Z}$, there exists a variety B of dimension $k-2c$ and a cycle $\mathcal{T} \subset B \times \widetilde{Z}$

of codimension $k-2c$ (a family of cycles on $\widetilde{Z}$ of dimension c parameterized by B), such that the map

$$q_*p^* : H^{k-2c}(B,\mathbb{Q}) \to H^{k-2c}(\widetilde{Z},\mathbb{Q})$$

hence also the map

$$\tau_* \circ q_*p^* : H^{k-2c}(B,\mathbb{Q}) \to H^k(X,\mathbb{Q})$$

are surjective, where p and q are the maps from $\mathcal{T}$ to B and $\widetilde{Z}$ respectively.

Hence the parametrization of $H^k(X,\mathbb{Q})$ by algebraic cycles of dimension c does not follow from the generalized Hodge conjecture for coniveau c, but also needs a Lefschetz standard conjecture applied to a certain subvariety of X.

1 A Geometric Interpretation of the Coniveau 2 Condition

In this section, we will give a geometric interpretation of the numerical condition (0.1), relating it to a positivity property of a certain cycle class on the variety of lines of the considered complete intersection. We will also show how to deduce Theorem 0.3 for coniveau 2 from this positivity property.

Let thus X be a generic complete intersection of multidegree $d_1 \leq \cdots \leq d_r$ in $\mathbb{P}^n$. Thus the variety F of lines in X is smooth of dimension $2n-2-\sum_i d_i - r$. For G a generic polynomial of degree $n-\sum_i d_i - 1$, let $X_G \subset X$ be the hypersurface defined by G and $F_G \subset F$ the variety of lines contained in X_G. Thus F_G is smooth of dimension

$$2n-2-\sum_i d_i - r - (n-\sum_i d_i) = n-r-2 = \dim X - 2\,.$$

Recall the incidence diagram

$$\begin{array}{rcl} q: & P & \longrightarrow X \\ & \downarrow & \\ & F & \end{array}$$

which, for $n-\sum_i d_i \geq 0$, induces an injective morphism of Hodge structures,

$$p_*q^* : H^{n-r}(X,\mathbb{Q}) \to H^{n-r-2}(F,\mathbb{Q})\,.$$

LEMMA 1.1. *For a primitive cohomology class $a \in H^{n-r}(X,\mathbb{Q})_{prim}$, the class $\eta := p_*q^*a \in H^{n-r-2}(F,\mathbb{Q})$ satisfies the following two properties (property 1 will be used only in section 3):*

1. *(See [Sh]) η is primitive with respect to the Plücker polarization $l := c_1(\mathcal{L})$ on F.*
2. *η vanishes on subvarieties F_G:*

$$\eta_{|F_G} = 0\,. \tag{1.2}$$

Proof. For the proof of the first statement, recall first that primitive cohomology $H^{n-r-2}(F,\mathbb{Q})_{prim}$ is defined as the kernel of

$$\cup l^{n-\sum_i d_i+1} : H^{n-r-2}(F,\mathbb{Q}) \to H^{3n-2\sum_i d_i - r}(F,\mathbb{Q})$$

because $\dim F = 2n-2-\sum_i d_i - r$. On the basis U parameterizing smooth complete intersections X such that F is smooth of the right dimension, the composed maps

$$\cup l^{n-\sum_i d_i+1} \circ p_*q^* : H^{n-r}(X)_{prim} \to H^{n-r-2}(F)$$

give a morphism of local systems. The point is now the following: suppose X degenerates to a generic X_0 with one ordinary double point x_0. Then the family Z_0 of lines in X_0 passing through x_0 has dimension $n-\sum_i d_i$. It follows that it does not meet the generic intersection K of $n-\sum_i d_i+1$ members of the Plücker linear system $|\mathcal{L}|$. Choose an X_ϵ which is generic and close enough to X_0. Then X_ϵ contains a vanishing sphere S_ϵ which is arbitrarily close to x_0. Thus the cycle $p_\epsilon(q_\epsilon^{-1}(S_\epsilon))\subset F_\epsilon$ does not meet a small perturbation $K_\epsilon\subset F_\epsilon$ of K. It follows that if δ_ϵ is the class of S_ϵ (well defined up to a sign depending on a choice of orientation), $\gamma_\epsilon := p_{\epsilon *}q_\epsilon^*(\delta_\epsilon)\in H^{n-r-2}(F_\epsilon)_{prim}$ is primitive. Hence we conclude that δ_ϵ belongs to the kernel of this morphism of local systems. As the monodromy along U, acting on $H^{n-r}(X,\mathbb{Q})_{prim}$, acts transitively on vanishing cycles (transported from the δ_ϵ's), and the later generate $H^{n-r}(X,\mathbb{Q})_{prim}$ (cf. [V2, II, 3.2.2]), it follows that the morphism $\cup l^{n-\sum_i d_i+1}\circ p_*q^* : H^{n-r}(X)_{prim}\to H^{n-r-2}(F)$ is zero.

The second statement is elementary. Indeed, as $X_G\subset X$ is a smooth member of $\left|\mathcal{O}_X\left(n-\sum_i d_i-1\right)\right|$, by the Lefschetz theorem on hyperplane sections, the restriction map

$$H^{n-r}(X,\mathbb{Q})_{prim}\to H^{n-r}(X_G,\mathbb{Q})$$

is zero. Now we have the following commutative diagram:

$$\begin{array}{ccc} H^{n-r}(X,\mathbb{Q})_{prim} & \longrightarrow & H^{n-r}(X_G,\mathbb{Q}) \\ {\scriptstyle p_*q^*}\big\downarrow & & \big\downarrow{\scriptstyle p_{G*}q_G^*} \\ H^{n-r-2}(F,\mathbb{Q}) & \longrightarrow & H^{n-r-2}(F_G,\mathbb{Q}) \end{array}$$

where

$$\begin{array}{ccc} P_G & \xrightarrow{q_G} & X_G \\ {\scriptstyle p_G}\big\downarrow & & \\ F_G & & \end{array}$$

is the incidence diagram for X_G and the horizontal maps are restriction maps. Thus we have $\eta_{|F_G}=p_{G*}q_G^*(a_{|X_G})=0$. □

The main result of this section is the following:

Theorem 1.2. *If X is as above and $n\ge\sum_i d_i+d_r$, the subvarieties $F_G\subset F$, where $\deg G=n-\sum_i d_i-1$ are very moving (cf. Definition 0.5).*

Before giving the proof, let us use it, combined with Lemma 1.1, to give a geometric proof of the numerical estimate (0.1) of Theorem 0.3 for coniveau 2.

Corollary 1.3 (cf. Theorem 0.3). *If the inequality $n\ge\sum_i d_i+d_r$ is satisfied, X has coniveau ≥ 2.*

Proof. The fact that X has coniveau ≥ 2 is equivalent (as we know already that $H^{n-r,0}(X)=0$) to the vanishing $H^{n-r-1,1}(X)_{prim}=0$. Let $a\in H^{n-r-1,1}(X)_{prim}$ and consider

$$\eta := p_*q^*a\in H^{n-r-2,0}(F)=H^0(F,\Omega_F^{n-r-2})\,.$$

Then $\eta=0$ iff $a=0$. We use now statement 2 in Lemma 1.1. This gives us

$$\eta_{|F_G}=0 \text{ in } H^0(F_G,\Omega_{F_G}^{n-r-2})$$

for generic G. On the other hand, Theorem 1.2 tells us that for generic G, the $n-r-2$-dimensional subvariety F_G passes through a generic point $\Delta\in F$ with a generic tangent space. It follows immediately that $\eta_{|F_G}=0$ implies $\eta=0$. □

Proof of Theorem 1.2. We fix a line $\Delta \in F$. We want to study the differential of the map ϕ which, to a polynomial G vanishing on Δ, associates the tangent space at Δ of F_G, assuming it has the right dimension. What we will prove is the fact that the differential of ϕ is generically surjective when the bound is realized. Let $W := \phi(G)$, that is $W \subset T_{F,\Delta}$ is the tangent space to F_G. Then the differential of ϕ is a linear map

$$d\phi(G) : H^0\Big(X, \mathcal{I}_\Delta\Big(n - \sum_i d_i - 1\Big)\Big) \to \mathrm{Hom}(W, T_{F,\Delta}/W) \tag{1.3}$$

where the right-hand side is the tangent space to the Grassmannian of rank $n-r-2$-dimensional subspaces of $T_{F,\Delta}$. We remark that $d\phi(G)$ factors through

$$H^0\Big(\Delta, \mathcal{I}_{\Delta,X}/\mathcal{I}^2_{\Delta,X}\Big(n - \sum_i d_i - 1\Big)\Big) = H^0(\Delta, N^*_{\Delta/X}\Big(n - \sum_i d_i - 1)\Big).$$

Furthermore, we observe that the natural map

$$H^0\Big(X, \mathcal{I}_{\Delta,X}\Big(n - \sum_i d_i - 1\Big)\Big) \to H^0\Big(\Delta, \mathcal{I}_{\Delta,X}/\mathcal{I}^2_{\Delta,X}\Big(n - \sum_i d_i - 1\Big)\Big)$$

is surjective. Indeed, the restriction map

$$\begin{aligned} H^0\Big(\mathbb{P}^n, \mathcal{I}_{\Delta,\mathbb{P}^n}\Big(n - \sum_i d_i - 1\Big)\Big) &\to H^0\Big(\Delta, \mathcal{I}_{\Delta,\mathbb{P}^n}/\mathcal{I}^2_{\Delta,\mathbb{P}^n}\Big(n - \sum_i d_i - 1\Big)\Big) \\ &= H^0\Big(\Delta, N^*_{\Delta/\mathbb{P}^n}\Big(n - \sum_i d_i - 1\Big)\Big) \end{aligned}$$

is surjective. Furthermore, the conormal exact sequence,

$$0 \to \bigoplus_j \mathcal{O}_\Delta(-d_j) \to N^*_{\Delta/\mathbb{P}^n} \to N^*_{\Delta/X} \to 0\,,$$

shows that the cokernel of the map

$$H^0\Big(\Delta, N^*_{\Delta/\mathbb{P}^n}\Big(n - \sum_i d_i - 1\Big)\Big) \to H^0\Big(\Delta, N^*_{\Delta/X}\Big(n - \sum_i d_i - 1\Big)\Big)$$

is

$$H^1\Big(\Delta, \bigoplus_j \mathcal{O}_\Delta(n - \sum_i d_i - 1 - d_j)\Big). \tag{1.4}$$

As $d_r = \mathrm{Sup}\{d_j\}$, the vanishing of the space (1.4) is clearly implied (and in fact equivalent to) by the inequality $n - \sum_i d_i - 1 - d_r \geq -1$ of (0.1).

We now show the surjectivity of the map induced by $d\phi(G)$,

$$\overline{d\phi(G)} : H^0\Big(\Delta, N^*_{\Delta/X}\Big(n - \sum_i d_i - 1\Big)\Big) \to \mathrm{Hom}(W, T_{F,\Delta}/W)\,.$$

Let $G \in H^0(X, \mathcal{I}_{\Delta/X}(n - \sum_i d_i - 1))$ be generic and $W = T_{F_G,\Delta}$. Then the quotient $T_{F,\Delta}/W$ identifies to $H^0\big(\Delta, \mathcal{O}_\Delta\big(n - \sum_i d_i - 1\big)\big)$. Indeed, F_G is defined as the zero set of a transverse section of $S^{n-\sum_i d_i - 1}\mathcal{E}$, and thus the normal bundle of F_G in F identifies to $S^{n-\sum_i d_i - 1}\mathcal{E}_{|F_G}$, with fiber $H^0\big(\Delta, \mathcal{O}_\Delta\big(n - \sum_i d_i - 1\big)\big)$ at Δ. Furthermore we observe that as Δ is generic and $n - \sum_i d_i > 1$, the normal bundle $N_{\Delta/X}$ is generated by sections. Of course $N^*_{\Delta/X}\big(n - \sum_i d_i - 1\big)$ is generated by global sections too. Finally, we note that by definition, the space W is the space $H^0(\Delta, N_{\Delta/X_G})$. As $\deg G = n - \sum_i d_i - 1$, X_G is a generic Fano complete intersection of index 2, and it thus follows that the normal bundle N_{Δ/X_G} is generically a direct sum of copies of $\mathcal{O}_\Delta$.

We have the following lemma:

LEMMA 1.4. (1) *The space $W = T_{F_G,\Delta} \subset H^0(\Delta, N_{\Delta/X})$ identifies to the kernel of the contraction map with*

$$G \in H^0\Big(\Delta, \mathcal{I}_{\Delta,X}/\mathcal{I}^2_{\Delta,X}\Big(n - \sum_i d_i - 1\Big)\Big) = H^0\Big(\Delta, N^*_{\Delta/X}\Big(n - \sum_i d_i - 1\Big)\Big),$$

with value in $H^0\big(\Delta, N^*_{\Delta/X}(n - \sum_i d_i - 1)\big)$.

(2) *Using the inclusion $W \subset H^0(\Delta, N_{\Delta/X})$, the map $\overline{d\phi(G)}$ is induced by the contraction map between $H^0(\Delta, N_{\Delta/X})$ and $H^0\big(\Delta, N^*_{\Delta/X}(n - \sum_i d_i - 1)\big)$, with value in $H^0\big(\Delta, \mathcal{O}_\Delta(n - \sum_i d_i - 1)\big)$.*

Postponing the proof of this lemma, we now conclude as follows. Let $G \in H^0\big(\Delta, N^*_{\Delta/X}(n - \sum_i d_i - 1)\big)$ be generic and let $W = \operatorname{Ker} d\phi(G)$, where $d\phi(G)$ has been identified in Lemma 1.4 to contraction by G, with value in $H^0\big(\Delta, \mathcal{O}_\Delta(n - \sum_i d_i - 1)\big)$. We have to show that the map given by contraction

$$H^0\Big(\Delta, N^*_{\Delta/X}\Big(n - \sum_i d_i - 1\Big)\Big) \to \operatorname{Hom}\Big(W, H^0\Big(\Delta, \mathcal{O}_\Delta\Big(n - \sum_i d_i - 1\Big)\Big)\Big)$$

is surjective. This problem now concerns vector bundles on $\Delta = \mathbb{P}^1$: we have a vector bundle E of rank $s = n - r - 1$ and degree $k = n - 1 - \sum_i d_i$ on $\Delta = \mathbb{P}^1$, such that $E^*(k)$ is generated by global sections. We choose a generic element G of $H^0(\Delta, E^*(k))$. We know that G gives a surjective map $E \to \mathcal{O}_\Delta(k)$ with kernel K which is a trivial vector bundle. Denote by $W \subset H^0(\Delta, E)$ the kernel of the contraction map with G, with value in $H^0(\mathcal{O}_\Delta(k))$; thus $W = H^0(\Delta, K)$ and we have to show that the contraction map induces a surjective map,

$$H^0\big(\Delta, E^*(k)\big) \to \operatorname{Hom}\big(W, H^0(\Delta, \mathcal{O}_\Delta(k)\big).$$

We now consider the composed map

$$H^0\big(\Delta, E^*(k)\big) \to H^0\big(\Delta, K^*(k)\big) \to \operatorname{Hom}\big(H^0(\Delta, K), H^0(\Delta, \mathcal{O}_\Delta(k))\big)$$

and want to show that it is surjective. The first map is surjective as its cokernel is $H^1(\Delta, \mathcal{O}_\Delta) = 0$. The second map is surjective exactly when $H^0(\Delta, K(-1)) = 0$ which follows from the fact that K is trivial. $\square$

Proof of Lemma 1.4. (1) We have the inclusions

$$\Delta \subset X_G \subset X,$$

which give the normal bundles exact sequence,

$$0 \to N_{\Delta/X_G} \to N_{\Delta/X} \stackrel{dG}{\to} \mathcal{O}_\Delta\Big(n - \sum_i d_i - 1\Big) \to 0.$$

The space W is by definition the space $H^0(\Delta, N_{\Delta/X_G})$ hence by the exact sequence above, it is the kernel of

$$dG : H^0(\Delta, N_{\Delta/X}) \to H^0\Big(\Delta, \mathcal{O}_\Delta\Big(n - \sum_i d_i - 1\Big)\Big).$$

We just have to observe that the map dG above is nothing but contraction with the image of G in $\mathcal{I}_\Delta/\mathcal{I}^2_\Delta\big(n - \sum_i d_i - 1\big) = N^*_{\Delta/X}\big(n - \sum_i d_i - 1\big)$, which follows from the construction of the normal bundles sequence.

(2) It is an immediate consequence of (1) and the following more general statement: Let $\phi : V \to H$ be a surjective map and let $W := \mathrm{Ker}\,\phi$. A small deformation of ϕ, given by an element of $\mathrm{Hom}(V,H)$ gives a deformation of $W \in \mathrm{Grass}(w,V)$. Thus we have a natural rational map

$$a : \mathrm{Hom}(V,H) \to \mathrm{Grass}(w,V)$$

where $w = rk\,W$. Then the differential $h \in \mathrm{Hom}(V,H) \mapsto da(h) \in \mathrm{Hom}(W,V/W) = \mathrm{Hom}(W,H)$ of this map is simply the map

$$h \in \mathrm{Hom}(V,H) \mapsto h_{|W}\,.$$

This last statement follows from the standard construction of the isomorphism $T_{\mathrm{Grass}(w,V),W} \cong \mathrm{Hom}(W,V/W)$. This concludes the proof of (2). $\square$

2 A Conjecture on Cones of Effective Cycles

Let Y be a smooth projective complex variety and let $Alg^{2k}(Y) \subset H^{2k}(Y,\mathbb{R})$ be the vector subspace generated by classes of codimension k algebraic cycles. This vector space contains the effective cone

$$E^{2k}(Y) \subset Alg^{2k}(Y)$$

generated by classes of effective cycles.

Definition 2.1. *A class $\alpha \in Alg^{2k}(Y)$ is said to be big if α is an interior point of $E^{2k}(Y)$.*

If $h = c_1(H)$ where H is an ample line bundle on Y, h^k belongs to the interior of the cone $E^{2k}(Y)$. Indeed, for any effective cycle $Z \subset Y$ of codimension k, the class $Nh^k - [Z]$ is effective for N large enough. Applying this to a basis of $Alg^{2k}(Y)$ consisting of effective cycles, one concludes that for some open set $U \subset Alg^{2k}(Y)$ containing 0, $h^k - U \subset E^{2k}(Y)$.

Thus we get the following:

Lemma 2.2. *A class $\alpha \in Alg^{2k}(Y)$ is big if and only if, for some $\epsilon > 0$, $\alpha - \epsilon h^k \in E^{2k}(Y)$.*

In the divisor case, the effective cone, or rather its closure, the pseudo-effective cone, is now well understood by work of Boucksom, Demailly, Paun and Peternell [BoDPP]. The case of higher codimension is not understood at all. To start with, there is the following elementary result for divisors, which we will show to be wrong in codimension 2 and higher.

Lemma 2.3. *Let D be an effective divisor on Y, and assume that $\mathcal{O}_Y(D)_{|D}$ is ample. Then $[D]$ is big.*

Proof. Indeed, one shows for any divisor E on Y, by vanishing on D, that the natural map

$$H^1\big(Y,\mathcal{O}_Y((n-1)D-E)\big) \to H^1\big(Y,\mathcal{O}_Y(nD-E)\big)$$

is surjective for large n, hence must be an isomorphism for large n. Thus the restriction map

$$H^0\big(Y,\mathcal{O}_Y(nD-E)\big) \to H^0\big(D,\mathcal{O}_D(nD-E)\big)$$

is surjective for large n. The space $H^0(D, \mathcal{O}_D(nD - E))$ is non-zero for n large, by ampleness of $\mathcal{O}_D(D)$, and it thus follows that the space $H^0(Y, \mathcal{O}_Y(nD - E))$ is non-zero too. Taking for E an ample divisor and applying Lemma 2.2 gives the result. □

Note that when D is smooth, $\mathcal{O}_D(D)$ is also the normal bundle $N_{D/Y}$. Let us now construct examples of codimension 2 smooth subvarieties $W \subset Y$ with ample normal bundle, such that $[W]$ is not big.

We start from a generic hypersurface Z of degree d in $\mathbb{P}^{d+2}$ with $d \geq 3$, and denote by Y its variety of lines. Let now $\mathbb{P}^{d+1} \subset \mathbb{P}^{d+2}$ be an hyperplane, and let $W \subset Y$ be the subvarieties of lines contained in $Z' := Z \cap \mathbb{P}^{d+1}$. By genericity, W and Y are smooth, and W is the zero set of a transverse section of the tautological rank-2 bundle $\mathcal{E}$ on F, with fiber $H^0(\Delta, \mathcal{O}_\Delta(1))$ at the point $\Delta \in F$. Thus the normal bundle of W in Y is isomorphic to $\mathcal{E}_{|W}$. This vector bundle is globally generated on W, and it is ample if the natural map

$$\mathbb{P}(\mathcal{E}) \to Z'$$

is finite to one. This is indeed the case for generic Z' by [J]. Thus W has ample normal bundle. Let us now show

PROPOSITION 2.4. *The class $[W] \in Alg^4(Y, \mathbb{R})$ is not big.*

Proof. The $d+1$-dimensional hypersurface Z has $H^{d,1}(Z)_{prim} \neq 0$, and this provides using the incidence diagram (cf. section 1) a non-zero holomorphic form $\eta \in H^{d-1,0}(Y)$ vanishing on W. Note that $\dim W = d - 1$. Assume that $[W]$ is big. By Lemma 2.2, one has

$$[W] = \epsilon h^2 + e\,, \tag{2.5}$$

where $e = \sum_i e_i[E_i]$ is a class of an effective codimension 2 cycle with real coefficients and h is the class of an ample divisor. Let us integrate the form $i^{d-1}(-1)^{(d-1)(d-2)/2}\eta \wedge \overline{\eta}$ on both sides. As $\eta_{|W} = 0$, the left-hand side is zero. On the other hand, we have by the second Hodge–Riemann relations (for holomorphic forms):

$$\int_{E_i} i^{d-1}(-1)^{\frac{(d-1)(d-2)}{2}}\eta \wedge \overline{\eta} \geq 0\,, \quad \int_Y h^2 \cup i^{d-1}(-1)^{\frac{(d-1)(d-2)}{2}}\eta \wedge \overline{\eta} > 0\,.$$

Thus the integral on the right is > 0, which is a contradiction. □

Note that the variety W also has the property that it is moving, that is, its deformations sweep out Y. The proof of Lemma 2.4 shows however that it is not very moving (see section 1). Let us make the following conjecture :

CONJECTURE 2.5. *Let $W \subset Y$ be a smooth variety which is very moving. then $[W]$ is big.*

REMARK 2.6. We could slightly weaken the conjecture above by considering varieties which are very moving and have ample normal bundle. Indeed, the varieties we will consider in the sequel and for which we would need to know that their class is big not only are very moving (cf. Theorem 1.2) but also have ample normal bundle.

The conjecture is true for divisors by Lemma 2.3. We also have

PROPOSITION 2.7. *Conjecture 2.5 is true for curves.*

Proof. Indeed, by Moïshezon–Nakai criterion, the dual cone to the effective cone of curves is the ample cone. A point $c=[C]$ which is in the boundary of the effective cone of curves must thus have zero intersection with a nef class e, that is a point in the boundary of the ample cone. Consider now a one-dimensional family $(C_t)_{t\in B}$ of curves deforming C and passing through a given generic point $y\in Y$. Such a family exists as C is very moving. Let $\tau:\Sigma\to X$ be a desingularization of this family. Then Σ contains two divisor classes c and σ which correspond to the fibers of the map $\Sigma\to B$ and to the section of this map given by the point y. The nef class $\tau^*e\in H^2(\Sigma,\mathbb{R})$ satisfies

$$\langle\tau^*e,c\rangle=0\,,\quad \langle\tau^*e,\tau^*e\rangle\geq 0\,,$$

while the class c satisfies $\langle c,c\rangle=0$, where the $\langle\,,\rangle$ denotes the intersection product on $H^2(\Sigma,\mathbb{R})$. The Hodge index theorem then tells that the classes c and τ^*e must be proportional. But as $\langle c,\sigma\rangle>0$ and $\langle\tau^*e,\sigma\rangle=0$, it follows that in fact $\tau^*e=0$. The contradiction comes from the following: as $e\neq 0$ there is a curve $\Gamma\subset X$, which we may assume to be in general position, such that $\deg_\Gamma e>0$. Now, because C is very moving, there exist for each $\gamma\in\Gamma$ curves C_γ which are deformations of C and pass through y and γ. We can thus choose our one-dimensional family to be parameterized by a cover $r:\Gamma'\to\Gamma$, in such a way that, for any $\gamma'\in\Gamma'$, $r(\gamma')\in C_{\gamma'}$. But then the surface $\tau(\Sigma)$ contains the section $\gamma'\mapsto r(\gamma')\in C_{\gamma'}$ which is sent via τ onto Γ, and as $\deg_\Gamma e>0$, we conclude that $\tau^*e\neq 0$. □

Another example where Conjecture 2.5 holds is the following:

LEMMA 2.8. *Let $V\subset G(1,n)$ be a very moving subvariety. Then $[V]$ is big.*

Proof. The cone of effective cycles on $G(1,n)$ is very simple because (cf. [M]) we have, for each pair of complementary dimensions, dual bases of the cohomology which consist of classes of Schubert cycles, which are effective; and furthermore, as the tangent space of the Grassmannian is globally generated, two effective cycles of complementary dimension on $G(1,n)$ have non-negative intersection (cf. [FL]). If z is an effective class on $G(1,n)$, write $z=\sum_i\alpha_i\sigma_i$ where σ_i are Schubert classes of dimension equal to $\dim z$. Then $\alpha_i=z\cdot\sigma_i^*\geq 0$. Thus the cone of effective cycles is the cone generated by classes of Schubert cycles. It follows from this argument that for the class $[V]$ to be big, it suffices that

$$V\cdot\sigma_i^*>0$$

for every Schubert cycle σ_i^* of complementary dimension. These inequalities are now implied by the fact that V is very moving. Indeed, choose a smooth point $x\in\sigma_i^*$. Then we can choose a deformation V' of V passing through x, smooth at x and meeting transversally σ_i^* at x. Thus there is a non-zero contribution to $V\cdot\sigma_i^*$ coming from the point x. The intersection $V\cap\sigma_i^*$ could be non-proper away from x, but because the tangent space of the Grassmannian is generated by global sections, each component of the intersection $V\cap\sigma_i^*$ has a non-negative contribution to $V\cdot\sigma_i^*$ by [FL]. Thus $V\cdot\sigma_i^*>0$. □

2.1 On Conjecture 2.5 for the varieties F_G. We finally turn to the study of Conjecture 2.5 for the subvarieties $F_G\subset F$. For X a generic complete intersection of multidegree $d_1 \leq \cdots \leq d_r$ in $\mathbb{P}^n$, we have seen that $F_G \subset F$ is very moving, where $\deg G = n - \sum_i d_i - 1$. We now have the following:

Theorem 2.9. *The class $[F_G] \in H^{2n-2\sum_i d_i}(F, \mathbb{Q})$ is big when*

$$3n - 4 - \sum_i \frac{(d_i+1)(d_i+2)}{2} \geq n - r - 2\,. \tag{2.6}$$

Before proving this, let us explain the geometric meaning of this bound. The number $3n - 6 - \sum_i \frac{(d_i+1)(d_i+2)}{2}$ is simply the expected dimension (that is, generically, the actual dimension) of the family of planes contained in X. The number $3n - 4 - \sum_i \frac{(d_i+1)(d_i+2)}{2}$ is thus the expected dimension of the family Z' of lines contained in a plane contained in X. On the other hand, the dimension $n - r - 2$ appearing on the right is the dimension of the varieties F_G.

Proof of Theorem 2.9. First of all, by Proposition 2.15 proved below, it suffices to consider the case where

$$3n - 4 - \sum_i \frac{(d_i+1)(d_i+2)}{2} = n - r - 2\,.$$

What we shall do is to compute the class of the subvariety Z' of F introduced above, which we describe as follows: let $G_2 \to G$ be the partial flag manifold parameterizing pairs (Q, Δ), $\Delta \subset Q$, where Δ is a line in $\mathbb{P}^n$ and Q is a plane in $\mathbb{P}^n$. Let $\pi : F_2 \to F$ be the subvariety of G_2 consisting of such pairs with $\Delta \in F$. Thus $\pi : F_2 \to F$ is a projective bundle of relative dimension $n - 2$. We consider the variety $Z \subset F_2$ consisting of pairs $(\Delta, P) \in F_2$ where P is also contained in X. It has dimension $3n - 4 - \sum_i \frac{(d_i+1)(d_i+2)}{2}$ which by assumption is equal to $n - r - 2$. Thus $Z' = \pi(Z)$ has dimension $n - r - 2$ and its codimension is $n - \sum_i d_i$.

Theorem 2.9 is an immediate consequence of the following two statements (Lemma 2.10 and Proposition 2.11). □

Lemma 2.10. *Assume there exists a class $P \in H^{2n-2\sum_i d_i}(F, \mathbb{Q})$, which has the following properties:*

1. *P is an effective class on F;*
2. *P can be written as*
$$P = -\epsilon l^{n-\sum_i d_i} + c_2 R\,,$$
where R is any algebraic class of F.

Then the class $c_{n-\sum_i d_i}(S^{n-\sum_i d_i - 1}\mathcal{E}) \in H^{2n-2\sum_i d_i}(F, \mathbb{Q})$ is big.

Proposition 2.11. *The class $[Z']$ (which is by definition effective) of the variety Z' defined above is given by a polynomial expression $P = P(l, c_2)$ satisfying property 2 of Lemma 2.10.*

We used above the notation $c_2 = c_2(\mathcal{E}) \in H^4(F, \mathbb{Q})$ and $l = c_1(\mathcal{E}) \in H^2(F, \mathbb{Q})$, $\mathcal{E}$ being as before the (restriction to F of) the tautological rank-2 vector bundle with fiber $H^0(\Delta, \mathcal{O}_\Delta(1))$ at $\Delta \in F$.

Proof of Lemma 2.10. Introduce a formal splitting of $\mathcal{E}$ or equivalently formal roots x, y of its Chern polynomial, so that

$$c_2 = xy\,, \quad l = x + y\,.$$

Then $S^{n-\sum_i d_i-1}\mathcal{E}$ has for formal roots the expressions $kx+(n-\sum_i d_i-1-k)y$, with $0\leq k\leq n-\sum_i d_i-1$. Thus we get

$$c_{n-\sum_i d_i}(S^{n-\sum_i d_i-1}\mathcal{E})=\prod_{k=0}^{n-\sum_i d_i-1} kx+\Big(n-\sum_i d_i-1-k\Big)y$$

which one can rewrite as

$$c_{n-\sum_i d_i}\big(S^{n-\sum_i d_i-1}\mathcal{E}\big)=\Big(n-\sum_i d_i\Big)^2 xy\prod_{k=1}^{n-\sum_i d_i-2} kx+\Big(n-\sum_i d_i-1-k\Big)y\,.$$

We claim that the class $Q:=\prod_{k=1}^{n-\sum_i d_i-2} kx+(n-\sum_i d_i-1-k)y$ is big on the Grassmannian hence a fortiori on F. To see this, we apply the argument of Lemma 2.8. We just have to show that

$$Q(x,y)\cdot\sigma>0$$

for all Schubert cycles of dimension $n-\sum_i d_i-2$ on $G(1,n)$. We now use the fact that

$$\Big(n-\sum_i d_i-1\Big)^2 c_2Q=c_{n-\sum_i d_i}\big(S^{n-\sum_i d_i-1}\mathcal{E}\big)\,.$$

Any Schubert cycle of dimension $n-\sum_i d_i-2$ on $G(1,n)$ is of the form $\sigma=c_2\sigma'$ where σ' is a Schubert cycle of dimension $n-\sum_i d_i$ on $G(1,n)$. One then has

$$\Big(n-\sum_i d_i-1\Big)^2 Q(x,y)\cdot\sigma=\Big(n-\sum_i d_i\Big)^2 c_2Q(x,y)\cdot\sigma'=c_{n-\sum_i d_i}\big(S^{n-\sum_i d_i-1}\mathcal{E}\big)\cdot\sigma'.$$

One shows easily that for any Schubert cycle σ' of dimension $n-\sum_i d_i$, one has

$$c_{n-\sum_i d_i}\big(S^{n-\sum_i d_i-1}\mathcal{E}\big)\cdot\sigma'>0$$

unless $\sigma'=\sigma_{0,n-\sum_i d_i+1}$ is the Schubert cycle of lines passing through a point and contained in a linear space of dimension $n-\sum_i d_i+1$. But in this case, $c_2\sigma'=0$. This proves the claim.

Now we proved that $c_{n-\sum_i d_i}(S^{n-\sum_i d_i-1}\mathcal{E})=c_2Q$, where Q is big on F. Assume that there is a class P of codimension $n-\sum_i d_i$ on G which is effective and of the form $-\epsilon l^{n-\sum_i d_i}+c_2R$, with R algebraic. As the class Q is big on F, it is in the interior of the effective cone of F, and thus for some small ϵ'

$$Q-\epsilon' R$$

is effective on F. Thus $c_2(Q-\epsilon' R)$ is also effective on F and we get

$$c_{n-\sum_i d_i}\big(S^{n-\sum_i d_i-1}\mathcal{E}\big)=c_2Q=\epsilon' c_2R+E\,,$$

with E effective. Replacing c_2R by $P+\epsilon l^{n-\sum_i d_i}$, where P is effective, we get

$$c_{n-\sum_i d_i}\big(S^{n-\sum_i d_i-1}\mathcal{E}\big)=E+\epsilon' P+\epsilon'\epsilon l^{n-\sum_i d_i},$$

where $E+\epsilon' P$ is effective. By Lemma 2.2, $c_{n-\sum_i d_i}\big(S^{n-\sum_i d_i-1}\mathcal{E}\big)$ is then big. □

Proof of Proposition 2.11. For simplicity, we give the proof for $r=1$ and thus denote $d_r=d$ with $n=2d$. Let $\pi:F_2\to F$ be as above. F_2 is a $\mathbb{P}^{n-2}$-bundle over F. Let $P_2\to F_2$ be the universal plane parameterized by F_2. As P_2 is sent naturally to $\mathbb{P}^n$, this $\mathbb{P}^2$-bundle admits a natural polarization $\mathcal{O}(1)$. Then

$$P_2=\mathbb{P}(\mathcal{F})$$

for a certain rank-3 vector bundle on F_2. By definition, for each pair $(\Delta, Q) \in F_2$, one has $\Delta \subset Q$, which gives a natural surjective map of bundles on F_2

$$\mathcal{F} \to \pi^*\mathcal{E} \to 0 .$$

Let $\mathcal{H}$ be its kernel and let $h := c_1(\mathcal{H}) \in H^2(F_2, \mathbb{Q})$. The $\mathbb{P}^{n-2}$-bundle F_2 over F is polarized by the line bundle $\mathcal{L}_2 := \det \mathcal{F}$ coming from the Grassmannian $G(2,n)$ via the natural map $F_2 \to G(2,n)$, $(\Delta, Q) \mapsto Q$. From the exact sequence

$$0 \to \mathcal{H} \to \mathcal{F} \to \pi^*\mathcal{E} \to 0 , \tag{2.7}$$

one deduces the exact sequence

$$0 \to \mathcal{H} \otimes S^{d-1}\mathcal{F} \to S^d\mathcal{F} \to \pi^* S^d\mathcal{E} \to 0 .$$

Observe now that the defining equation f of X induces a section of $S^d\mathcal{F}$ which by definition vanishes in $S^d\mathcal{E}$. Thus we get a section $\tilde{f}$ of $\mathcal{H} \otimes S^{d-1}\mathcal{F}$ such that $Z = V(\tilde{f})$. It follows (together with some transversality arguments implying that $\tilde{f}$ vanishes transversally) that

$$[Z] = c_N(\mathcal{H} \otimes S^{d-1}\mathcal{F}) , \quad N = \operatorname{rank} S^{d-1}\mathcal{F} = \frac{d(d+1)}{2} .$$

It remains to compute

$$[Z'] = \pi_*[Z] .$$

It is clear that $[Z']$ is a polynomial in l and c_2. As we are interested only in showing that the coefficient of l^{n-d} in this polynomial is negative, we can do formally as if $\mathcal{E} = \mathcal{L} \oplus \mathcal{O}_F$. As $\mathcal{H} \otimes S^{d-1}\mathcal{F}$ is filtered with successive quotients isomorphic to

$$\mathcal{H}^{\otimes i} \otimes S^{d-i}\mathcal{E} , \quad i = 1 \ldots , d ,$$

we get

$$[Z] = \prod_{i=1}^{d} c_{d-i+1}(\mathcal{H}^{\otimes i} \otimes S^{d-i}\mathcal{E})$$

and modulo c_2 this is equal to

$$\prod_{i=1}^{d} \prod_{j=0}^{d-i} \big(ih + (d-i-j)l\big) .$$

Observe now that by the exact sequence (2.7), the $\mathbb{P}^{n-2}$-bundle P_2 polarized by $\mathcal{H} = \mathcal{L}_2 \otimes \pi^*\mathcal{L}^{-1}$ is isomorphic to $\mathbb{P}(\mathcal{K}_1)$, where $\mathcal{K}_1$ is the kernel of the evaluation map $V \otimes \mathcal{O}_F \to \mathcal{E}$, where $V := H^0(\mathcal{O}_{\mathbb{P}^n}(1))$, so that we have the exact sequence

$$0 \to \mathcal{K}_1 \to V \otimes \mathcal{O}_F \to \mathcal{E} \to 0 .$$

By definition of Segre classes, we then have

$$\pi_* h^{n-2+i} = s_i(\mathcal{K}_1^*) = c_i(\mathcal{E}^*) .$$

As we compute modulo c_2, we get

$$\pi_* h^{n-2+i} = 0 , \quad i \neq 0, 1 ,$$
$$\pi_* h^{n-2} = 1 , \quad p_* h^{n-1} = -l .$$

Hence it follows that we have the equality

$$[Z'] = \pi_* \Bigg(\prod_{i=1}^{d} \prod_{j=0}^{d-i} \big(ih + (d-i-j)l\big) \Bigg) = (\alpha_{n-2} - \alpha_{n-1}) l^{n-d} \bmod c_2 ,$$

where we write

$$M := \prod_{i=1}^{d}\prod_{j=0}^{d-i}\big(ih+(d-i-j)l\big) = \sum_{i\le N}\alpha_i h^i l^{N-i}.$$

The important point is now the following: Let us factor in M all the terms corresponding to $j=d-i$. Then

$$M = d!h^d \prod_{i\ge1, j\ge1, i+j\le d}(ih+jl)\,.$$

Let $M' := \prod_{i\ge1,j\ge1,i+j\le d}(ih+jl) = \sum_{i\le N-d}\beta_i h^i l^{N-d-i}$. Then we have

$$\alpha_{n-2}-\alpha_{n-1} = d!(\beta_{n-d-2}-\beta_{n-d-1})\,.$$

Thus we have to show that $\beta_{n-d-2}-\beta_{n-d-1}<0$. But now we observe that the degree of the homogeneous polynomial M' is $N-d$ with $N=n-2+n-d$. Hence $\deg M' = 2n-2d-2$. The polynomial M' is symmetric of degree $2n-2d-2$ in l and h, and it suffices to show that its coefficients β_i are strictly increasing in the range $i\le n-d-1=\deg M'/2$. This can be checked by hand or proved by geometry using the fact that

$$(d!)^2(xy)^d M'(x,y) = \prod_{1\le i\le d} c_{i+1}(S^i\mathcal{E})\,,$$

if x,y are the formal roots of $\mathcal{E}$, and that the numbers $(d!)^2(\beta_i-\beta_{i+1})$, $2i\le 2n-2d-2$ can be interpreted as intersection numbers of $\prod_{1\le i\le d} c_{i+1}(S^i\mathcal{E})$ with a Schubert cycle. $\square$

Finally, let us rephrase Conjecture 2.5 in the case of subvarieties $F_G\subset F$, in terms of geometry of the Grassmannian $G:=G(1,n)$ of lines in $\mathbb{P}^n$. For simplicity, let us consider the case of hypersurfaces of degree d in $\mathbb{P}^n$. The bound (0.1) then becomes

$$n\ge 2d\,.$$

Let again $V := H^0(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(1))$ with symmetric power $S^dV = H^0(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(d))$. Recall that we denote by $\mathcal{E}$ the rank-2 vector bundle with fiber $H^0(\Delta,\mathcal{O}_\Delta(1))$ on $G(1,n)$. There is a natural evaluation map

$$S^dV\otimes\mathcal{O}_G\to S^d\mathcal{E} \tag{2.8}$$

with kernel we denote by $\mathcal{K}_d$. The projective bundle

$$\pi:\mathbb{P}(\mathcal{K}_d^*)\to G(1,n)$$

(where we use the Grothendieck notation) thus parameterizes pairs (f,Δ) such that $f_{|\Delta}=0$. The natural projection $\rho:\mathbb{P}(\mathcal{K}_d^*)\to\mathbb{P}(S^dV^*)$, which to (Δ,f) associates f, has for fibre over f the variety F_f of lines contained in the hypersurface X_f defined by f.

Now we consider the subvarieties $F_{f,G}\subset F_f$, where G has degree $n-d-1$. These are defined as zero sets of generic, hence transverse, sections of $S^{n-d-1}\mathcal{E}$. Thus their cohomology class is given by

$$[F_G] = c_{n-d}(S^{n-d-1}\mathcal{E}_{|F})\,.$$

Summarizing and applying Lemma 2.2, Conjecture 2.5 tells that for some $\epsilon>0$, (a multiple of) the class $\pi^*(c_{n-d}(S^{n-d-1}\mathcal{E})-\epsilon l^{n-d})$ is effective on the generic fiber of the map $\rho:\mathbb{P}(\mathcal{K}_d^*)\to\mathbb{P}(S^dV^*)$. An elementary Hilbert scheme argument then shows that we can put this in family, and combining this with the description of the cohomology ring of a projective bundle, this gives us the following reformulation of Conjecture 2.5 in this case: we denote by H the class $c_1(\mathcal{O}(1))$ on the projective bundle $\mathbb{P}(\mathcal{K}_d^*)$.

CONJECTURE 2.12. *Assume $n \geq 2d$. Then for some algebraic class*

$$\alpha' \in H^{2n-2d-2}\big(\mathbb{P}(\mathcal{K}_d^*), \mathbb{Q}\big),$$

and for some small $\epsilon > 0$, the class

$$\pi^*\big(c_{n-d}(S^{n-d-1}\mathcal{E}) - \epsilon l^{n-d}\big) + H\alpha'$$

is effective on $\mathbb{P}(\mathcal{K}_d^*)$. In other words, the class $c_{n-d}(S^{n-d-1}\mathcal{E})$ belongs to the interior of the cone of degree $2n-2d$ classes α on G such that $\pi^*\alpha + H\alpha'$ is effective on $\mathbb{P}(\mathcal{K}_d^*)$ for some algebraic class $\alpha' \in H^{2n-2d-2}(\mathbb{P}(\mathcal{K}_d^*), \mathbb{Q})$.

Theorem 2.9 proves Conjecture 2.12 in the range $3n - 4 - \frac{(d+1)(d+2)}{2} \geq n - 3$, that is

$$n \geq \frac{1}{2}\left(\frac{(d+1)(d+2)}{2} + 1\right).$$

REMARK 2.13. More generally, suppose we have a variety Y and a vector bundle $\mathcal{F} \to Y$, with associated projective bundle $\pi : \mathbb{P}(\mathcal{F}) \to Y$. Let $h = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))$. We can then introduce the convex cone $E^{2k}(Y, \mathcal{F})$ consisting of classes $c \in Alg^{2k}(Y)$ such that $\pi^*c + hc' \in E^{2k}(\mathbb{P}(\mathcal{F}))$, for some $c' \in Alg^{2k-2}(\mathbb{P}(\mathcal{F}))$. There is an obvious inclusion

$$E^{2k}(Y) \subset E^{2k}(Y, \mathcal{F}).$$

When $\mathcal{F}$ is ample, the whole of $Alg^{2k}(Y)$ is contained in $E^{2k}(Y, \mathcal{F})$ and it follows that $E^{2k}(Y)$ is contained in the interior of $E^{2k}(Y, \mathcal{F})$. In contrast, if $\mathcal{F}$ is trivial, it is obvious that

$$E^{2k}(Y) = E^{2k}(Y, \mathcal{F}).$$

In our case, the class $c_{n-d}(S^{n-d-1}\mathcal{E})$ is effective and even numerically effective but not in the interior of the effective cone of $Alg^{2n-2d}(G(1,n))$. The bundle $\mathcal{K}_d^*$ is generated by sections but not ample.

REMARK 2.14. The conjecture cannot be improved by replacing $c_{n-d}(S^{n-d-1}\mathcal{E})$ by $c_{n-d-1}(S^{n-d-2}\mathcal{E})$. Indeed, on a generic Fano variety of lines F in a degree d hypersurface X, the class $[F_G] = c_{n-d-1}(S^{n-d-2}\mathcal{E})$ of the subvariety of lines in X_G, $\deg G = d - n - 2$ is not big. In fact, $\dim F_G = n - 2$ and F_G has 0 intersection with the variety F_x of lines through a generic point $x \in X$, which has dimension $n - d - 1 = \dim F - n + 2$. As F_x is moving, this implies that $[F_G]$ is not big. The same argument works for any class in $H^{2n-2d-2}(F, \mathbb{Q})$ which is divisible by c_2.

The crucial case for Conjecture 2.12 is the case where $n = 2d$. Indeed, we have the following:

PROPOSITION 2.15. *If the Conjecture 2.12 is true for a given pair (n, d), $n \geq 2d$, then it is true for the pair $(n+1, d)$.*

Proof. Let us denote by $\mathcal{K}_d^{n+1}$ the kernel of the evaluation map (2.8) on $G(1, n+1)$, for $V = H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$. We assume that the conjecture is satisfied for n, d with $n \geq 2d$ and show that it is satisfied for $n+1, d$.

Choose a pencil of hyperplanes $(H_t)_{t\in\mathbb{P}^1}$ on $\mathbb{P}^{n+1}$, with base-locus B. Then $G(1,n+1)$ contains the Plücker hyperplane section G_B consisting of lines meeting B. It is singular and admits a natural desingularization

$$G' \to G_B \subset G(1,n+1)$$

where $G' = \sqcup_{t\in\mathbb{P}^1} G(1,H_t)$. We pull-back to G' the projective bundle $\mathbb{P}(\mathcal{K}_d^{n+1*})$, and denote the result by

$$P' := G' \times_{G(1,n+1)} \mathbb{P}(\mathcal{K}_d^{n+1*})$$

$$j : P' \to \mathbb{P}(\mathcal{K}_d^{n+1*})\,,$$

We denote by $\pi_{n+1} : \mathbb{P}(\mathcal{K}_d^{n+1*}) \to G(1,n+1)$ the structural map, and let $\pi'_{n+1} := \pi_{n+1}\circ j : P' \to G(1,n+1)$. We have the following properties:

1. $j_*[P'] = \pi_{n+1}^* l \in H^2(\mathbb{P}(\mathcal{K}_d^{n+1*}),\mathbb{Q})$.
2. If $n' : P' \to \mathbb{P}^1$ is the natural map, and $k = c_1(\mathcal{O}_{\mathbb{P}^1}(1))$,
$$j_*(n'^*k) = \pi_{n+1}^* c_2 \in H^4\big(\mathbb{P}(\mathcal{K}_d^{n+1*}),\mathbb{Q}\big)\,.$$

Observe also that the cohomology of P' is generated by $j^*H^*(\mathbb{P}(\mathcal{K}_d^{n+1*}),\mathbb{Q})$ and n'^*k, because for each fiber P'_t of n', the restriction map

$$j_t^* : H^*\big(\mathbb{P}(\mathcal{K}_d^{n+1*}),\mathbb{Q}\big) \to H^*(P'_t,\mathbb{Q})$$

is surjective.

Introduce

$$P'' := \sqcup_{t\in\mathbb{P}^1}\mathbb{P}(\mathcal{K}_d^{n,t*})$$

where $\mathbb{P}(\mathcal{K}_d^{n,t*})$ is the variety $\mathbb{P}(\mathcal{K}_d^{n*})$ for the hyperplane H_t. There is a natural rational linear projection of projective bundles,

$$\phi : P' \dashrightarrow P''$$

which associates $(\Delta,t,X\cap H_t)$ to (Δ,t,X). Denote the natural maps by

$$\pi''_{n+1} : P'' \to G(1,n+1)\,, \quad n'' : P'' \to \mathbb{P}^1.$$

Then we have

$$\pi''_{n+1}\circ\phi = \pi'_{n+1} = \pi_{n+1}\circ j\,, \quad n' = n''\circ\phi\,.$$

Let h be the class $c_1(\mathcal{O}_{\mathbb{P}(\mathcal{K}_d^{n+1*})}(1))$. The projective bundle $P'' \to G''$ has a natural polarization h'' such that

$$\phi^*h'' = j^*h\,.$$

The induction assumption is that, for each fiber P''_t of n'', a class of the form ${\pi''_{n+1}}^*(-\epsilon l^{n-d} + c_{n-d}(S^{n-d-1}\mathcal{E})) + h''c'$ with $\epsilon > 0$ is effective on P''_t. Putting this in family, we conclude that on P'', a class of the form

$${\pi''_{n+1}}^*\big(-\epsilon l^{n-d} + c_{n-d}(S^{n-d-1}\mathcal{E})\big) + h''c' + n''^*kc''$$

is effective. The fact that the indeterminacy locus of ϕ has high codimension then shows that

$$\begin{aligned}\phi^*\big(\pi''^*_{n+1}(-\epsilon l^{n-d} + c_{n-d}(S^{n-d-1}\mathcal{E})) + h''c' + n''^*kc''\big)\\ = j^*\big(\pi_{n+1}^*(-\epsilon l^{n-d} + c_{n-d}(S^{n-d-1}\mathcal{E}))\big) + j^*h\phi^*c' + n'^*k\phi^*c'' \quad (2.9)\end{aligned}$$

is also effective on P'. As noted above, the class ϕ^*c'' comes from a class on $\mathbb{P}(\mathcal{K}_d^{n+1*})$,

$$\phi^*c'' = j^*c'''.$$

Applying j_* to (2.9), the properties 1 and 2 above, and the projection formula, we conclude that a class of the form

$$l\big(\pi_{n+1}^*(-\epsilon l^{n-d}+c_{n-d}(S^{n-d-1}\mathcal{E}))+hc'\big)+(\pi_{n+1}^*c_2)c'''$$
$$=\pi_{n+1}^*\big(-\epsilon l^{n-d+1}+lc_{n-d}(S^{n-d-1}\mathcal{E})\big)+hlc'+(\pi_{n+1}^*c_2)c'''$$

is effective on $\mathbb{P}(\mathcal{K}_d^{n+1*})$.

As $c_{n-d}(S^{n-d-1}\mathcal{E}))$ is divisible by c_2, we conclude that the assumptions of Lemma 2.10 are satisfied on the fibers of

$$\rho:\mathbb{P}(\mathcal{K}_d^{n+1*})\to\mathbb{P}(S^dV^*)\,,$$

where h vanishes. Thus by Lemma 2.10, Conjecture 2.12 is satisfied for $n+1,d$. $\square$

3 On the Generalized Hodge Conjecture for Coniveau 2 Complete Intersections

We now prove the following result which motivated our interest in Theorem 1.2 and Conjecture 2.5. Let X be a generic complete intersection of multidegree $d_1\leq\ldots\leq d_r$ in $\mathbb{P}^n$, and assume the bound (0.1) holds, that is

$$n\geq\sum_i d_i+d_r\,.$$

By Theorem 1.2, the varieties $F_G\subset F$ are very moving.

Theorem 3.1. *If the varieties F_G satisfy Conjecture 2.5, that is $[F_G]$ is big, then the generalized Hodge conjecture for coniveau 2 is satisfied by X.*

Proof. Assume that $[F_G]$ is big. Then it follows by Lemma 2.2 that for some positive large integer N and for some effective cycle E of codimension $n-\sum_i d_i$ on F, one has

$$N[F_G]=l^{n-\sum_i d_i}+[E]\,. \tag{3.10}$$

(Here we could work as well with real coefficients, but as we want actually to do geometry on E, it is better if E is a true cycle.) Now we recall from Lemma 1.1 that, for $a\in H^{n-r}(X)_{prim}$, $\eta=p_*q^*a\in H^{n-r-2}(F)$ is primitive with respect to l and furthermore vanishes on F_G, with $\dim F_G=n-r-2$. Let us assume that $a\in H^{p,q}(X)_{prim}$ and integrate $(-1)^{\frac{k(k-1)}{2}}i^{p-q}\eta\cup\overline{\eta}$, $k=p+q-2=n-r-2$ over both sides in (3.10). We thus get

$$0=\int_F(-1)^{\frac{k(k-1)}{2}}i^{p-q}l^{n-\sum_i d_i}\cup\eta\cup\overline{\eta}+\int_E(-1)^{\frac{k(k-1)}{2}}i^{p-q}\eta\cup\overline{\eta}\,.$$

As η is primitive, and non-zero if a is non-zero, by the second Hodge–Riemann bilinear relations (cf. [V2, I, 6.3.2]), we have $\int_F(-1)^{\frac{k(k-1)}{2}}i^{p-q}l^{n-\sum_i d_i}\cup\eta\cup\overline{\eta}>0$. It thus follows that

$$\int_E(-1)^{\frac{k(k-1)}{2}}i^{p-q}\eta\cup\overline{\eta}<0\,.$$

Let $\widetilde{E}=\sqcup\widetilde{E_j}$ be a desingularization of the support of $E=\sum_j m_jE_j$, $m_j>0$. Thus we have

$$\sum_j m_j\int_{\widetilde{E_j}}(-1)^{\frac{k(k-1)}{2}}i^{p-q}\eta\cup\overline{\eta}<0\,.$$

It thus follows that there exists one E_j such that

$$\int_{\widetilde{E_j}} (-1)^{\frac{k(k-1)}{2}} i^{p-q} \eta \cup \overline{\eta} < 0 \,. \tag{3.11}$$

Choose an ample divisor H_j on each $\widetilde{E_j}$. By the second Hodge–Riemann bilinear relations, inequality (3.11) implies that $\eta_{|\widetilde{E_j}}$ is not primitive with respect to the polarization given by H_j, that is $\eta \cup [H_j] \neq 0$ and in particular

$$\eta_{|H_j} \neq 0 \,.$$

In conclusion, we proved that the composed map

$$H^{n-r}(X,\mathbb{Q})_{prim} \stackrel{p_*q^*}{\to} H^{n-r-2}(F,\mathbb{Q}) \to \bigoplus H^{n-r-2}(H_j,\mathbb{Q})$$

is injective, where the second map is given by restriction. If we dualize this, recalling that $\dim H_j = n-r-3$, we conclude that

$$\bigoplus H^{n-r-4}(H_j,\mathbb{Q}) \to H^{n-r}(X,\mathbb{Q})_{prim}$$

is surjective, where we consider the pull-backs of the incidence diagrams to H_j

$$\begin{array}{ccc} P_j & \stackrel{q_j}{\longrightarrow} & X \\ {\scriptstyle p_j}\big\downarrow & & \\ H_j & & \end{array}$$

and the map is the sum of the maps $q_{j*}p_j^*$, followed by orthogonal projection onto primitive cohomology. As for $n-r$ even and $n-r \geq 3$, the image of $\sum_j q_{j*}p_j^*$ also contains the class $h^{\frac{n-r}{2}}$ (up to adding if necessary to the H_j the class of a linear section of F) it follows immediately that the map

$$\sum_j q_{j*}p_j^* : \bigoplus H^{n-r-4}(H_j,\mathbb{Q}) \to H^{n-r}(X,\mathbb{Q})$$

is also surjective, if $n-r \geq 3$ (the case $n-r=2$ is trivial). This implies that $H^{n-r}(X,\mathbb{Q})$ is supported on the $n-r-2$-dimensional variety $\cup_j q_j(P_j)$, that is vanishes on $X \setminus \cup_j q_j(P_j)$. The result is proved. $\square$

This theorem combined with Theorem 2.9 allows us to reprove and generalize the previously known results concerning the generalized Hodge conjecture for coniveau 2 hypersurfaces (see [O]). Indeed the bound (2.6) of Theorem 2.9 can also be reinterpreted as follows (in the case of hypersurfaces): the generic hypersurface of degree d in $\mathbb{P}^{n+1}$ is swept-out by plans. In [O], A. Otwinowska proves that this implies the triviality of $CH_1(X)_{\mathbb{Q},\mathrm{hom}}$ for X any hypersurface of degree d in $\mathbb{P}^n$ (note that it is likely that the same can be done for complete intersections as well). From Bloch-Srinivas argument (see [BS], [V2, II, proof of Th. 10.31]) one can also deduce from this that $H^{n-r}(X)_{prim}$ vanishes on the complementary set of a closed algebraic subset of X of codimension 2, that is the generalized Hodge conjecture for coniveau 2 holds for X.

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