On a conjecture of Matsushita

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Abstract

We prove that Lagrangian fibrations on projective hyper-Kähler $2n$-folds with maximal Mumford-Tate group satisfy Matsushita's conjecture, namely the generic rank of the period map for the fibers of such a fibration is either 0 or maximal (that is n). We establish for this a universal property of the Kuga-Satake variety associated to a $K3$-type Hodge structure with maximal Mumford-Tate group.

0 Introduction

Let X be a smooth projective hyper-Kähler manifold of dimension $2n$ admitting a Lagrangian fibration $f : X \to B$. The smooth fibers X_b of f are thus abelian varieties of dimension n. It is proved in [14, Lemma 2.2] (see also [15] when B is smooth) that the restriction map

$$H^2(X, \mathbb{Z}) \to H^2(X_b, \mathbb{Z})$$

has rank 1, so that the fibers X_b are in fact canonically polarized by the restriction of any ample line bundle on X. Denoting by α the type of the polarization, we thus have a moduli morphism

$$m : B^0 \to \mathcal{A}_{n,\alpha}$$

where $B^0 \subset B$ is the open set parameterizing smooth fibers and $\mathcal{A}_{n,\alpha}$ is the moduli space of n-dimensional abelian varieties with a polarization of type α. It has been conjectured by Matsushita that m is either generically finite on its image or constant (the second case being the case of isotrivial fibrations). This conjecture was communicated to us by Ljudmila Kamenova and Misha Verbitsky. Our goal in this note is to prove the following weakened form of Matsushita's conjecture. Let $P \subset H^2(X, \mathbb{Z})$ be a subgroup of the Néron-Severi group of X containing one ample class and the pull-back of the generator of $\operatorname{Pic} B$ (one has $\operatorname{Pic} B = \mathbb{Z}$ by [10]). One can construct the universal family $\mathcal{M}_P$ of marked deformations of X with fixed Picard group P, that is deformations X_t for which all the classes in P remain Hodge on X_t. It follows from [14] (see also the proof of Corollary 11 for an alternative argument) that such deformations locally preserve the Lagrangian fibration on X. So deformations parameterized by $\mathcal{M}_P$ automatically induce a deformation of the triple (X, f, B), at least on a dense Zariski open set of $\mathcal{M}_P$. Furthermore, up to shrinking this Zariski open set, the deformed variety X_t remains projective.

Theorem 1. *Let X be a projective hyper-Kähler manifold of dimension $2n$ admitting a Lagrangian fibration $f : X \to B$. Assume $b_{2,P}(X) := b_2(X) - \operatorname{rank} P \geq 5$. Then the deformation (X', f', B') of the triple (X, f, B) parameterized by a very general point of $\mathcal{M}_P$ satisfies Matsushita's conjecture, that is the moduli map $m' : B' \dashrightarrow \mathcal{A}_{n,\alpha}$ is either constant or generically of maximal rank n.*

Corollary 2. *In the space $\mathcal{M}_P$ of deformations of X with Néron-Severi group containing P, either there is a dense Zariski open set of points parameterizing triples (X', f', B') for which the moduli map has maximal rank n, or for any point of $\mathcal{M}_P$, the moduli map is constant.*

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This follows indeed from the fact that the condition that m is generically of maximal rank is Zariski open.

Remark 3. The assumption $b_2(X) - \operatorname{rank} P \geq 5$ (hence in particular $b_2(X) - 2 \geq 5$) in Theorem 1 is presumably not essential here, but some more arguments would be needed otherwise. It is related to the simplicity of the orthogonal groups. Note also that no compact hyper-Kähler manifold with $b_2 < 7$ is known, so in practice, this does not seem to be very restrictive.

Remark 4. We had originally proved Theorem 1 and Corollary 2 under the assumption that B is smooth, (and in fact it is believed that this condition always holds). Matsushita [10], [11] proved a number of results on the geometry and topology of the base B suggesting that it must be isomorphic to $\mathbb{P}^n$, and Hwang [7] proved this is the case if it is smooth. The only property that we actually use is the deformation result from [14], which does not need the smoothness of the base (see also the proof of Corollary 11). Note that this deformation result had been proved earlier in [13, Corollary 1.7] when the base is $\mathbb{P}^n$, or equivalently is smooth.

Our proof will use the fact that the very general point of $\mathcal{M}_P$ parameterizes a deformation X' of X for which the Mumford-Tate group of the Hodge structure on

$$H^2(X', \mathbb{Q})_{tr} = H^2(X', \mathbb{Q})^{\perp P} \cong H^2(X, \mathbb{Q})^{\perp P}$$

is the full special orthogonal group of the $\mathbb{Q}$-vector space $H^2(X', \mathbb{Q})_{tr}$ equipped with the Beauville-Bogomolov intersection form q (see Section 1). Theorem 1 will be then a consequence of the following more precise result:

Theorem 5. *Let X be a projective hyper-Kähler manifold of dimension $2n$ admitting a Lagrangian fibration $f : X \to B$. Assume $b_{2,tr}(X) := b_2(X) - \rho(X) \geq 5$ and the Mumford-Tate group of the Hodge structure on $H^2(X, \mathbb{Q})$ is the group $SO(H^2(X, \mathbb{Q})_{tr}, q)$. Then the pair (X, f) satisfies Matsushita's conjecture.*

The proof of Theorem 5 will be obtained as a consequence of the following proposition (cf. Proposition 12) establishing a universal property of the Kuga-Satake construction (see [9], [3], [4]):

Proposition 6. *Let $(H, q, H^{p,q})$ be a weight 2 polarized rational Hodge structure of $K3$ type, that is, such that $h^{2,0} = 1$. Assume that $\dim H \geq 5$ and the Mumford-Tate group of $(H, H^{p,q})$ is the special orthogonal group of of (H, q). Then for any irreducible weight 1 polarized rational Hodge structure H_1 such that, for some weight 1 Hodge structure H_2, there is an embedding of weight 2 Hodge structures*

$$H \subset H_1 \otimes H_2,$$

H_1 is isomorphic to an irreducible weight 1 sub-Hodge structure of $H^1(A_{KS}(H), \mathbb{Q})$, where $A_{KS}(H)$ is the Kuga-Satake variety of $(H, q, H^{p,q})$.

Remark 7. This implies that there is a finite and in particular discrete set of such Hodge structures H_1. The condition on the Mumford-Tate group of H is quite essential here. We will give in the last section an example of a $K3$ type polarized Hodge structure H for which there is a continuous family of irreducible weight 1 Hodge structures H_1 satisfying the conditions above.

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1 Mumford-Tate groups and the Kuga-Satake construction

Let $(H, H^{p,q})$ be a rational Hodge structure of weight k. The group $\mathbb{S}^1$ acts on $H_{\mathbb{R}}$ by the following rule: $z \cdot \alpha^{p,q} = z^p \overline{z}^q \alpha^{p,q}$ for $z \in \mathbb{S}^1$ and $\alpha^{p,q} \in H^{p,q} \subset H_{\mathbb{C}}$.

Definition 8. *The Mumford-Tate group of H is the smallest algebraic subgroup of $GL(H)$ which is defined over $\mathbb{Q}$ and contains the image of $\mathbb{S}^1$.*

Let X be a compact hyper-Kähler manifold. Consider the Hodge structure of weight 2 on $H^2(X, \mathbb{Q})$. It is compatible with the Beauville-Bogomolov intersection form q (by the first Hodge-Riemann bilinear relations), so that its Mumford-Tate group is contained in $SO(q)$. We now have:

Lemma 9. *Let $P \subset \mathrm{NS}(X) \subset H^2(X, \mathbb{Q})$ be a subspace which contains an ample class (so that the Beauville-Bogomolov form is nondegenerate of signature $(1, \mathbf{rank}\, P - 1)$ on P). Then for a very general marked deformation X' of X for which $P \subset \mathrm{NS}(X')$, the Mumford-Tate group of the Hodge structure on $H^2(X', \mathbb{Q})_{tr}$ is the whole special orthogonal group $SO(H^2(X', \mathbb{Q})_{tr}, q)$.*

Remark 10. Note that the fact that the period map for hyper-Kähler manifolds is open implies that for X' as above, $H^2(X', \mathbb{Q})_{tr}$ is nothing but the orthogonal complement of P in $H^2(X', \mathbb{Q})$ with respect to q.

Proof of lemma 9. Via the period map, the marked deformations X_t of X for which $P \subset \mathrm{NS}(X_t)$ are parameterized by an open set $\mathcal{D}_P^0$ in the period domain

$$\mathcal{D}_P = \{\sigma_t \in \mathbb{P}(H^2(X, \mathbb{C})^{\perp P}),\, q(\eta_t) = 0,\, q(\eta_t, \overline{\eta}_t) > 0\}.$$

For such a period point σ_t, the Mumford-Tate group $MT(H^2(X_t, \mathbb{Q}))$ is the subgroup leaving invariant all the Hodge classes in the induced Hodge structures on the tensor powers $\bigotimes H^2(X_t, \mathbb{Q})$. For each such class α, either α remains a Hodge class everywhere on the family, or the locus where it is a Hodge class is a closed proper analytic subset of the period domain. As there are countably many such Hodge classes, it follows that the Mumford-Tate group for the very general fiber X' of the family contains the Mumford-Tate groups of $H^2(X_t, \mathbb{Q})$ for all $t \in \mathcal{D}_P^0$. We then argue by induction on $\dim H^2(X, \mathbb{Q})^{\perp P}$. If $\dim H^2(X, \mathbb{Q})^{\perp P} = 2$, then it is immediate to check that $MT(H^2(X, \mathbb{Q}))$ is the Deligne torus itself, which is equal to $SO(H^2(X_t, \mathbb{Q})^{\perp P})$. Suppose now that we proved the result for $\dim H^2(X, \mathbb{Q})^{\perp P} = k - 1$ and assume $\dim H^2(X, \mathbb{Q})^{\perp P} = k \geq 3$. First of all, we easily see that the strong form of Green's theorem on the density of the Noether-Lefschetz locus holds, by which we mean the following statement:

There exists a non-empty open set $V \subset H^2(X, \mathbb{R})^{\perp P}$ such that for any $\lambda \in V \cap H^2(X, \mathbb{Q})^{\perp P}$, the Noether-Lefschetz locus

$$NL_\lambda \cap \mathcal{D}_P^0 =: \{t \in \mathcal{D}_P^0,\, \lambda \in H^{1,1}(X_t)\} = \{t \in \mathcal{D}_P^0,\, q(\sigma_t, \lambda) = 0\}$$

is nonempty.

Letting $\sigma_t \in \mathcal{D}_P^0$ be the period point of X_t, for any $\lambda \in H^2(X, \mathbb{Q})^{\perp P} \cong H^2(X_t, \mathbb{Q})^{\perp P}$, one has

$$\mathcal{D}_P^0 \cap NL_\lambda = \{\sigma_t \in \mathcal{D}_P^0,\, q(\sigma_t, \lambda) = 0\},$$

which can be rephrased by saying that $\sigma_t \in \mathcal{D}_P^0 \cap NL_\lambda$ if and only if $\lambda \in H^2(X, \mathbb{Q})^{\perp <P, \sigma_t>}$ and by taking complex conjugates,

$$\lambda \in H^2(X, \mathbb{Q})^{\perp <P, \sigma_t, \overline{\sigma_t}>}.$$

But clearly, $\cup_{\sigma_t \in \mathcal{D}_P^0} H^2(X, \mathbb{R})^{\perp \langle P, \sigma_t, \overline{\sigma_t} \rangle}$ is an open non-empty subset of $H^2(X, \mathbb{R})$ since the real 2-plane $\langle \sigma_t, \overline{\sigma_t} \rangle$ runs through a non-empty open subset of the Grassmannian of real 2-planes of $H^2(X, \mathbb{R})^{\perp P}$. We thus can take for V this open set.

For any $t \in NL_\lambda \cap \mathcal{D}_P^0$, the rational subspace $< P, \lambda > \subset H^2(X_t, \mathbb{Q})$ is contained in $\mathrm{NS}(X_t)_\mathbb{Q}$ and applying the induction hypothesis, we conclude that for the very general point t of $NL_\lambda \cap \mathcal{D}_P^0$, the Mumford-Tate group of $H^2(X_t, \mathbb{Q})$ is equal to $SO(H^2(X'_\lambda, \mathbb{Q})^{\perp <\lambda, P>}, q)$ (and acts as the identity on $< \lambda, P >$).

By the previous argument, we then conclude that for the very general point X' of $\mathcal{D}_P^0$, the Mumford-Tate group $MT(H^2(X', \mathbb{Q}))$ contains the orthogonal groups $SO(H^2(X'_\lambda, \mathbb{Q})^{\perp <\lambda, P>}, q)$ for any $\lambda \in V \cap H^2(X, \mathbb{Q})^{\perp P}$. As V is open in $H^2(X, \mathbb{R})^{\perp P}$, it immediately follows that $MT(H^2(X', \mathbb{Q}))$ is equal to the orthogonal group $SO(H^2(X', \mathbb{Q})^{\perp P}, q)$. □

Let now X be a hyper-Kähler manifold admitting a Lagrangian fibration $f : X \to B$. Let $P \subset \mathrm{NS}(X)$ be a sublattice containing $l := f^* c_1(\mathcal{L})$, where $\mathcal{L}$ generates $\mathrm{Pic}\, B$. We get the following:

Corollary 11. *There exists a (small) deformation X' of X which is projective with Néron-Severi group P, admits a Lagrangian fibration $X' \to B'$ deforming the Lagrangian fibration of X, and such that the Mumford-Tate group of $H^2(X', \mathbb{Q})$ is equal to $SO(H^2(X', \mathbb{Q})^{\perp P}, q)$.*

Proof. By Lemma 9, the very general X' in the family $\mathcal{M}_P$ of deformations of X with Néron-Severi group containing P has Mumford-Tate group $SO(H^2(X', \mathbb{Q})^{\perp P}, q)$. Furthermore, as P contains an ample class, X' is also projective, at least on a dense open set of the deformation family. On the other hand, it follows from the stability result of [14] that deformations of X preserving the Hodge class $l \in \mathrm{NS}(X)$ locally preserve the given Lagrangian fibration on X. The following alternative argument was also shown to us by Ch. Lehn: Matsushita proves in [14, Lemma 2.2] that denoting J_b the general fiber of our Lagrangian fibration $f : X \to B$, the rank of the restriction map $H^2(X, \mathbb{Q}) \to H^2(J_b, \mathbb{Q})$ is 1, and its kernel is equal to $H^2(X, \mathbb{Q})^{\perp l}$. By [17], the locus of deformations of X preserving one Lagrangian smooth torus J_b is open in the locus of deformations of X preserving the Hodge class l. By [8] or [5, Lemmas 3.2 and 2.9], any projective deformation X_t of X containing a deformation J' of J_b has a deformed Lagrangian almost holomorphic fibration with fiber J'. By [12], almost holomorphic fibration are holomorphic. This allows to conclude that if the deformation is small enough and satisfies $P \subset \mathrm{NS}(X')$, X' admits a Lagrangian fibration deforming the one of X. □

Recall [2], [9], [4] that a polarized integral Hodge structure H of weight 2 with $h^{2,0} = 1$ has an associated Kuga-Satake variety $A_{KS}(H)$, which is an abelian variety with the property that the Hodge structure H can be realized (up to a shift) as a sub-Hodge structure of the weight 0 Hodge structure on $\mathrm{End}\,(H^1(A_{KS}(H), \mathbb{Z}))$. If H is a rational polarized Hodge structure, $A_{KS}(H)$ is defined only up to isogeny. The Kuga-Satake variety is essentially constructed by putting, using the Hodge structure on H, a complex structure on the underlying vector space of the Clifford algebra $C(H_\mathbb{R}, q)$, which provides a complex structure on the real torus $C(H_\mathbb{R}, q)/C(H)$. In general, the Kuga-Satake is not a simple abelian variety, because it has a big endomorphism algebra given by right Clifford multiplication of $C(H)$ on this torus. The main ingredient in our proof of Theorem 5 will be the following result:

Proposition 12. *Let (H, q) be a weight 2 polarized Hodge structure with Mumford-Tate group equal to $SO(q)$. Let A, B be polarized weight 1 rational Hodge structures such that $H \subset A \otimes B$ as weight 2 Hodge structures. Then if A is simple (as a Hodge structure) and $\dim H \geq 5$, A is isomorphic as a rational Hodge structure to $H^1(M, \mathbb{Q})$, where M is an abelian subvariety of the Kuga-Satake variety of H.*

Proof. The Mumford-Tate group $MT(A \otimes B)$ maps onto $MT(H)$. As $\dim H \geq 5$, the Lie algebra $mt(H) = so(q)$ is simple, so it is a summand of $mt(A \otimes B)$. As $MT(A \otimes B) \subset MT(A) \times MT(B)$, the Lie algebra $mt(A \otimes B)$ is contained in $mt(A) \times mt(B)$.

If the projection of the simple Lie algebra $mt(H) = so(q)$ to $mt(A)$ and to $mt(B)$ is injective and $mt(A \otimes B)$ contains both these copies of $so(q)$, then the Lie algebra of the Mumford Tate group of the tensor product of the corresponding weight one sub-Hodge structures of A and B has $so(q) \times so(q)$ as sub-Lie algebra. This contradicts that $A \otimes B$

has a sub-Hodge structure with $h^{2,0} = 1$. Using Proposition 1.7 of [6], one concludes that $mt(A \otimes B)$ contains one copy of $so(q)$ which maps onto $mt(H)$ and whose projections to $mt(A)$ and to $mt(B)$ are injective. The Hodge structures on H and the sub-Hodge structures of A and B defined by this copy of $so(q)$ in $mt(A) \times mt(B)$ are obtained from one map of the Lie algebra of $\mathbb{S}^1$ to $so(q)_{\mathbb{R}}$.

Now one considers the classification of the cases where the complex Lie algebra $so(q)_{\mathbb{C}}$ is a (simple) factor of the complexified Lie algebra of the Mumford-Tate group of a weight 1 polarized Hodge structure A and then one finds all the possible representations of $so(q)_{\mathbb{C}}$ on $A_{\mathbb{C}}$. This was done by Deligne [3].

The case where $\dim H$ is odd is the easiest one: in that case the Lie algebra $so(q)_{\mathbb{C}}$ has a unique such representation, which is the spin representation. This spin representation also occurs on $H^1(A_{KS}(H), \mathbb{C})$, with the same map of the Lie algebra of $\mathbb{S}^1$ to $so(q)_{\mathbb{R}}$. Thus there is non-trivial $so(q)_{\mathbb{C}}$-equivariant map, respecting the Hodge structures, from $A_{\mathbb{C}}$ to $H^1(A_{KS}(H), \mathbb{C})$. As the complex vector space of such maps is the complexification of the rational vector space of $so(q)$-equivariant maps from A to $H^1(A_{KS}(H), \mathbb{Q})$, there is such a map from A to $H^1(A_{KS}(H), \mathbb{Q})$. It follows that A is a simple factor of the Hodge structure on $H^1(A_{KS}(H), \mathbb{Q})$.

In the case where $\dim H$ is even, the representations of $so(q)_{\mathbb{C}}$ that can occur are the standard representation and the two half spin representations. However, the tensor product of the standard representation with any of these three cannot have a subrepresentation which is again the standard representation. Thus H cannot be a summand of $A \otimes B$ if $A_{\mathbb{C}}$ is the standard representation of $so(q)_{\mathbb{C}}$. Therefore $A_{\mathbb{C}}$ must have a half-spin representation of $so(q)_{\mathbb{C}}$ as summand. As before, it follows that A is a summand of the H^1 of the Kuga-Satake variety of H. □

2 Proof of the theorems

We first prove that Theorem 1 is a consequence of Theorem 5. Let X be a projective hyper-Kähler manifold of dimension $2n$ with a Lagrangian fibration $f : X \to B$ and let $P \subset H^2(X, \mathbb{Z})$ be a sublattice containing the class $l = f^*c_1(\mathcal{L})$ and an ample class. Then by Corollary 11, there exists a point (in fact many!) in the space $\mathcal{M}_P$ of deformations of X with Picard group containing P, which parameterizes a projective hyper-Kähler manifold X' such that $\mathrm{NS}(X') = P$ and the Mumford-Tate group of the Hodge structure on $H^2(X', \mathbb{Q})$ is the orthogonal group of $(H^2(X', \mathbb{Q})_{tr}, q) = (H^2(X, \mathbb{Q})^{\perp P}, q)$. As we assumed that $b_2(X) - \operatorname{rank} P \geq 5$, Theorem 5 applies to X', which proves Theorem 1.

We now assume that $X = X'$ satisfies the assumption in Theorem 5 and turn to the proof of Theorem 5.

Proof of Theorem 5. Let $f : X \to B$ be a Lagrangian fibration with $\dim H^2(X, \mathbb{Q})_{tr} \geq 5$ and $MT(H^2(X, \mathbb{Q})_{tr}) = SO(H^2(X, \mathbb{Q})_{tr}, q)$. We have to prove that f satisfies Matsushita's conjecture, that is, if the general fiber of the moduli map m is positive dimensional, then the moduli map is constant. Let $b \in B$ be a general point and assume the fiber F_b of the moduli map m passing through b is positive dimensional. Over the Zariski open set $U = F_b \cap B^0$ of F_b, the Lagrangian fibration restricts to an isotrivial fibration $X_U \to U$. As we are in the projective setting, it follows that after passing to a generically finite cover U' of U, the base-changed family $X_{U'} \to U'$ splits as a product $J_b \times U'$, where the abelian variety J_b is the typical fiber $f^{-1}(b)$, for $b \in U$. Let F'_b be a smooth projective completion of U' and $X_{F'_b}$ be a smooth projective completion of $X_{U'}$. The natural rational map $X_{F'_b} \dashrightarrow X$ induces a rational map $f_b : J_b \times F'_b \dashrightarrow X$. Consider the induced morphism of Hodge structures

$$f_b^* : H^2(X, \mathbb{Q}) \to H^2(J_b \times F'_b, \mathbb{Q}).$$

We claim that the composite map

$$\alpha : H^2(X, \mathbb{Q}) \stackrel{f_b^*}{\to} H^2(J_b \times F'_b, \mathbb{Q}) \to H^1(J_b, \mathbb{Q}) \otimes H^1(F'_b, \mathbb{Q}),$$

where the second map is given by Künneth decomposition, has an injective restriction to $H^2(X,\mathbb{Q})_{tr}$.

This indeed follows from the following facts :

a) The Hodge structure on $H^2(X,\mathbb{Q})_{tr}$ is simple. Indeed, it is polarized with $h^{2,0}$-number equal to 1 and it does not contain nonzero Hodge classes. Hence if there is a nontrivial sub-Hodge structure $H\subset H^2(X,\mathbb{Q})_{tr}$, it must have $H^{2,0}\neq 0$. But then the orthogonal complement $H^\perp\subset H^2(X,\mathbb{Q})_{tr}$ is either trivial or with nonzero $(2,0)$-part, which contradicts the fact that $H^{2,0}(X)$ is of dimension 1.

b) The $(2,0)$-form σ on X has a nonzero image in $H^0(\Omega_{J_b})\otimes H^0(\Omega_{F'_b})$. To see this last point, we recall that J_b is Lagrangian, that is, the form σ restricts to zero on J_b. If it vanished also in $H^0(\Omega_{J_b})\otimes H^0(\Omega_{F'_b})$, its pull-back to $J_b\times F'_b$ would lie in $H^0(\Omega^2_{F'_b})$. But as $\dim F_b>0$, this contradicts the fact that σ is nondegenerate and $\dim J_b=n=\frac{1}{2}\dim X$. This proves the claim since by b), the map α is nonzero and thus by a) it is injective.

The abelian variety J_b might not be a simple abelian variety, (or equivalently, the weight 1 Hodge structure on $H^1(J_b,\mathbb{Q})$ might not be simple), but the (polarized) Hodge structure on $H^1(J_b,\mathbb{Q})$ is a direct sum of simple weight 1 Hodge structures

$$H^1(J_b,\mathbb{Q})\cong A_1\oplus\ldots\oplus A_s,$$

and for some $i\in\{1,\ldots s\}$ the induced morphism of Hodge structures

$$\beta: H^2(X,\mathbb{Q})_{tr}\stackrel{\alpha}{\to} H^1(J_b,\mathbb{Q})\otimes H^1(F'_b,\mathbb{Q})\to A_i\otimes H^1(F'_b,\mathbb{Q})$$

must be nonzero, hence again injective by the simplicity of the Hodge structure on $H^2(X,\mathbb{Q})_{tr}$.

We are now in position to apply Proposition 12 because A_i is simple. We thus conclude that A_i is isomorphic to a direct summand of $H^1(A_{KS}(X),\mathbb{Q})$, where $A_{KS}(X)$ is the Kuga-Satake variety built on the Hodge structure on $H^2(X,\mathbb{Q})_{tr}$. Let $f^0: X^0\to B^0$ be the restriction of f to $f^{-1}(B^0)$, where $B^0\subset B$ is the open set of regular values of f. The above reasoning shows that $\mathrm{Hom}_{HS}(H^1(A_{KS}(X),\mathbb{Q}),H^1(X_b,\mathbb{Q}))\neq 0$ for $b\in B^0$. The end of the proof given below is a translation in the language of variations of Hodge structures of weight 1, suggested by one of the referees, of our original argument which was written in terms of abelian fibrations. Let M be the local system

$$\mathrm{Hom}(H^1(A_{KS}(X),\mathbb{Q}),R^1f^0_*\mathbb{Q})$$

on B^0. Then M carries a weight 0 variation of Hodge structure and at the very general point of B^0, M_b contains nontrivial Hodge classes. There is a local subsystem $\mathrm{Hdg}(M)$ of M on B^0 whose stalk at the very general point $b\in B^0$ is $\mathrm{Hom}_{HS}(H^1(A_{KS}(X),\mathbb{Q}),R^1f^0_*\mathbb{Q})$ (see [18]), so this local system is nontrivial, and an obvious nontrivial evaluation morphism of local systems on B^0

$$\alpha:\mathrm{Hdg}(M)\otimes H^1(A_{KS}(X),\mathbb{Q})\to R^1f_*\mathbb{Q}. \tag{1}$$

This morphism is a morphism of variations of Hodge structures, where the left hand side has a locally constant Hodge structure. Thus $R^1f^0_*\mathbb{Q}$ contains a local subsystem A which is defined as the image of α. If α is surjective, then the variation of Hodge structure on $R^1f^0_*\mathbb{Q}$ is trivial, which shows that the fibration f is isotrivial. If α is not surjective, as the variation of Hodge structure on $R^1f^0_*\mathbb{Q}$ is polarized because X is projective, there is a direct sum decomposition

$$R^1f^0_*\mathbb{Q}\cong A\oplus B, \tag{2}$$

where B is a subvariation of Hodge structure of $R^1f^0_*\mathbb{Q}$. This contradicts [14, Lemma 2.2], which (combined with Deligne's global invariant cycle theorem) says that $R^2f^0_*\mathbb{Q}$ has only one global section up to multiples, since the global section of $R^2f^0_*\mathbb{Q}$ polarizes the variation of Hodge structure on $R^1f^0_*\mathbb{Q}$, hence has a nontrivial image in $\bigwedge^2 A$ and $\bigwedge^2 B$.

$\square$

Remark 13. One may wonder if the hypothesis that X is projective has really been used in the proof of Theorem 5. Indeed, even if X is not projective, one knows that the fibers of a Lagrangian fibration are abelian varieties, and even canonically polarized abelian varieties. It is possible that Matsushita's conjecture holds in this context, however our argument would fail at the following point. In the Kähler context, one would work with deformations of X preserving the class of the line bundle coming from the base. This class l is isotropic for the Beauville-Bogomolov form but we can work with the Hodge structure on $H^2(X,\mathbb{Q})^{\perp l}/\mathbb{Q}l$ which certainly has Mumford-Tate group equal to the special orthogonal group for a very general deformation of (X,l). Unfortunately, with the notation introduced in the proof of Theorem 5, this Hodge structure does not map to $H^2(X_{F'_b},\mathbb{Q})$, since the class l does not vanish in $H^2(X_{F'_b},\mathbb{Q})$.

3 An example

In this section, we construct an example of a projective $K3$ surface S, such that the Hodge structure H on $H^2(S,\mathbb{Q})_{tr}$ can be realized as a sub-Hodge structure of a tensor product $H_1\otimes H_2$, with H_1 and H_2 of weight 1, for a continuous family of weight 1 polarized Hodge structures H_1.

We start with a projective $K3$ surface S admitting a non-symplectic automorphism ϕ of prime order $p\geq 5$ (see [1], [16] for construction and classification). Let $H=H^2(S,\mathbb{Q})_{prim}$.

Proposition 14. *There is a continuous family of polarized Hodge structures H_1 of weight 1 such that for some weight 1 Hodge structure H_2, one has*

$$H\subset H_1\otimes H_2$$

as Hodge structures.

Proof. Let $\lambda\neq 1$ be the eigenvalue of $\psi=\phi^*$ acting on $H^{2,0}(S)$. Let H_1 be any weight 1 polarized Hodge structure admitting an automorphism ψ' of order p such that

1. λ^{-1} is not an eigenvalue of ψ' acting on $H_1^{1,0}$.
2. λ^{-1} is an eigenvalue of ψ' acting on $H_1^{0,1}$.

For such H_1, we find that the weight 3 Hodge structure

$$H_2 := (H_1\otimes H)^G \overset{i}{\hookrightarrow} H_1\otimes H,$$

where G is $\mathbb{Z}/p\mathbb{Z}$ acting on $H\otimes H_1$ via $\psi\otimes\psi'$, is the Tate twist of a weight 1 Hodge structure H_2, since we have

$$((H_1\otimes H)^G)^{3,0}=(H_1^{1,0}\otimes H^{2,0})^G=(H_1^{1,0})^{\lambda^{-1}}\otimes H^{2,0}=0.$$

On the other hand, H_2 is nonzero, since λ^{-1} is an eigenvalue of ψ' acting on $H_1^{0,1}$, which by the same argument as above provides a nonzero element in $(H_1^{0,1}\otimes H_2^{2,0})^G$.

By composing the inclusion $Id_{H_1^*}\otimes i: H_1^*\otimes H_2\hookrightarrow H_1^*\otimes H_1\otimes H$ with the contraction map $c\otimes Id_H: H_1^*\otimes H_1\otimes H\to H$, we get a map $\mu: H_1^*\otimes H_2\to H$. This map is non-trivial, since choosing nonzero $\sigma\in (H_1^{0,1})^{\lambda^{-1}}$ and $\eta\in H^{2,0}$ we have $\sigma\otimes\eta=i(\omega)$ for some $\omega\in H_2$. Next, choosing $u\in H_1^*$ such that $u(\sigma)\neq 0$, we see that, after tensoring with $\mathbb{C}$,

$$\mu(u\otimes\omega)=(c\otimes Id_H)((Id_{H_1^*}\otimes i)(u\otimes\omega))=(c\otimes Id_H)(u\otimes\sigma\otimes\eta)=u(\sigma)\eta\neq 0.$$

Since these Hodge structures are polarized, they are isomorphic to their duals up to Tate twists. Thus there is a nontrivial morphism of Hodge structures

$$H\to H_1^*\otimes H_2$$

that is injective by the simplicity of the Hodge structure H.

We conclude observing that by the assumption $p\geq 5$, the family of weight 1 polarized Hodge structures H_1 satisfying conditions 1 and 2 above has positive dimension. $\square$

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