Torsion points of sections of Lagrangian torus fibrations and the Chow ring of hyper-Kähler manifolds

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Abstract

Let $\phi : X \to B$ be a Lagrangian fibration on a projective irreducible hyper-Kähler manifold. Let $M \in \operatorname{Pic} X$ be a line bundle whose restriction to the general fiber X_b of ϕ is topologically trivial. We prove that if the fibration is isotrivial or has maximal variation and X is of dimension ≤ 8, the set of points b such that the restriction $M_{|X_b}$ is torsion is dense in B. We give an application to the Chow ring of X, providing further evidence for Beauville's weak splitting conjecture.

0 Introduction

0.1 The general problem and main result

Let X be an algebraic variety over $\mathbb{C}$, $X \to B$ a smooth projective morphism and let $M \in \operatorname{Pic} X$ be a line bundle which is topologically trivial on the fibers of ϕ. The line bundle M then determines an algebraic section ν_M of the torus fibration $\operatorname{Pic}^0(X^0/B^0) \to B^0$ (which we will call the normal function associated to M), where $X^0 \to B^0$ is the restriction of ϕ over the open set $B^0 \subset B$ of regular values of ϕ. The group of such sections is called the Mordell-Weil group of the fibration. The problem we investigate in this paper is:

Question 0.1. *When do there exist points $b \in B^0$ where $\nu_M(b)$ is a torsion point? Are these points dense (for the usual or Zariski topology) in B?*

Note that this question can be restated in terms of variations of mixed Hodge structures E_t obtained as an extension

$$0 \to H_t \to E_t \to \mathbb{Z} \to 0$$

of $\mathbb{Z}$ by a pure Hodge structure H_t of weight 1 (see for example [8], [7], [9]). Torsion points then correspond to those points t for which E_t has a nonzero rational element which belongs to $F^1 E_{t,\mathbb{C}}$. In this setting, similar existence or density problems have been investigated in the study of the density of the Noether-Lefschetz locus (see [10], [38]), where variations of weight 2 Hodge structures are considered and special points t are those for which the Hodge structure E_t acquires a Hodge class. More generally, the set of special points is the object of a vast literature related to André-Oort conjecture and "unlikely intersection" theory [42], but more focused on bounding special points than on their existence. It turns out that both problems are related to transversality questions so that there is in fact a certain overlap in the methods (see for example [13]).

Question 0.1 is natural only when the codimension of the locus of torsion points in $\operatorname{Pic}^0(X/B)$, that is $h^{1,0}(X_b) = \dim \operatorname{Pic}^0(X/B) - \dim B$, is not greater than $n = \dim B$, so that the section ν_M is expected to meet this locus. We will consider in this paper the case where $h^{1,0}(X_b) = \dim B$, and more precisely we will work in the following setting: X will be a projective hyper-Kähler manifold of dimension $2n$ and $\phi : X \to B$ will be a Lagrangian fibration. According to Matsushita [27], any morphism $\phi : X \to B$ with $0 < \dim B < \dim X$ and connected fibers provides such a Lagrangian fibration. The smooth fibers X_b are then

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abelian varieties of dimension n, which are in fact canonically polarized, by a result of Matsushita (see [29, Lemma 2.2], see also Proposition 1.3). So in the following definition, the local period map could be replaced in our case by a moduli map from the base to a moduli space of abelian varieties.

Definition 0.2. *We will say that a smooth torus fibration $X^0 \to B^0$ has maximal variation if the associated (locally defined) period map $B^0 \to \mathcal{D}$ is generically of maximal rank $n = \dim B$.*

The fibration is said to satisfy Matsushita's alternative if it has maximal variation or it is locally constant (the isotrivial case) Matsushita conjectured that this alternative holds for Lagrangian fibrations on projective hyper-Kähler manifolds. This conjecture was proved in [18] assuming that the Mumford-Tate group of the Hodge structure on the transcendental cohomology $H^2(X, \mathbb{Q})_{tr} \subset H^2(X, \mathbb{Q})$ is the full special orthogonal group of $H^2(X, \mathbb{Q})_{tr}$ equipped with the Beauville-Bogomolov intersection form, and that $b_2(X)_{tr} \geq 5$. In particular, it is satisfied by general deformations of X with fixed Picard lattice, assuming $b_2(X)_{tr} \geq 5$.

Our main result in this paper is the following:

Theorem 0.3. *Let X, ϕ be as above, and let $M \in \operatorname{Pic} X$ restrict to a topologically trivial line bundle on the smooth fibers X_b of ϕ. Assume that either the fibration is locally isotrivial or $\dim X \leq 8$ and the variation is maximal. Then the set of points $b \in B^0$ such that $\nu_M(b) = M_{|X_b}$ is a torsion point in $\operatorname{Pic} X_b$ is dense in B^0 for the usual topology.*

Applying the result of [18] mentioned above, we conclude:

Corollary 0.4. *Let X, ϕ, M be as in Theorem 0.3. Assume $b_2(X) \geq 8$. Then the very general deformation X_t of X preserving L, M and an ample line bundle on X satisfies the conclusion of Theorem 0.3.*

The restriction to dimension ≤ 8 is certainly not essential here and we believe that the result is true in any dimension although the analysis seems to be very complicated in higher dimension. The restriction to dimension 8 appears in the analysis of the constraints on the infinitesimal variation of Hodge structures imposed by contradicting the conclusion of Theorem 0.3. In dimension 10 we completely classify the possible (although improbable) situation where the conclusion of Theorem 0.3 does not hold (see Proposition 2.8 proved in Section 6).

In order to introduce some of the ingredients used in the proof of Theorem 0.3, we will start with the general case of an elliptic fibration, (and without any hyper-Kähler condition). One gets in this case the following result, which, although we could not find a reference, is presumably not new, but will serve as a toy example for the strategy used.

Theorem 0.5. *Let $\phi : X \to B$ be an elliptic fibration, where B is a smooth projective variety and X is smooth projective, and let $B^0 \subset B$ be the open set of regular values of ϕ. Let M be a line bundle on X which is of degree 0 on the fibers of ϕ. Then either the set of points $b \in B^0$ such that $\sigma_M(b) = M_{|X_b}$ is a torsion point in $\operatorname{Pic} X_b$ is dense in B^0 for the usual topology, or the restriction map $H^1(X, \mathbb{Q}) \to H^1(X_b, \mathbb{Q})$ is surjective. In the second case, the fibration ϕ is locally isotrivial and the associated Jacobian fibration is rationally isogenous over B to the product $J(X_b) \times B$.*

Our approach to Theorem 0.3 is infinitesimal and completed by an analysis of the monodromy. It uses the easy Proposition 2.1 which says that the torsion points of ν are dense in B if the natural locally defined map $f_\nu : B \to H_1(A_{t_0}, \mathbb{R})$ obtained from ν by a real analytic trivialization of the family of complex tori $A \to B$ is generically submersive. This map is called the Betti map in [2], [13]. A key tool for our work is the following very useful result of André-Corvaja-Zannier [2].

Theorem 0.6. *Let $\pi : A \to B$ be a family of abelian varieties of dimension d, where B is quasi-projective over $\mathbb{C}$ and $\dim B \geq d$. Let ν be an algebraic section of π. Assume*

   (i) The family has no locally trivial subfamily.

   (ii) The multiples $m\nu(B)$ are Zariski dense in A.

   Then if the real analytic map f_ν is never of maximal rank, the variation of Hodge structure of weight 1 is degenerate in the following sense:

   For any $b \in B$, for any $\lambda \in H^{1,0}(A_b)$ the map

$$\overline{\nabla}_\lambda : T_{B,b} \to H^{0,1}(A_b)$$

is not of maximal rank.

We refer to Section 3 for the definition of $\overline{\nabla}$. We will also for completeness sketch the proof of this result in that section.

0.2 An application to the Chow ring of hyper-Kähler fourfolds

Let us explain one consequence of Theorem 0.3, which was our motivation for this work. Following the discovery made in [5] that a projective $K3$ surface S has a canonical 0-cycle o_S with the property that for any two divisors D, D' on S, the intersection $D \cdot D'$ is a multiple of o_S in $\mathrm{CH}_0(S)$, Beauville conjectured the following:

Conjecture 0.7. *(See [4]). Let X be a projective hyper-Kähler manifold. Then the cohomological cycle class restricted to the subalgebra of $\mathrm{CH}(X)_\mathbb{Q}$ generated by divisor classes is injective.*

This conjecture has been proved in [39] for varieties of the form $S^{[n]}$, where S is a $K3$ surface and $n \leq 2b_2(S)_{tr} + 4$, and in [17] for generalized Kummer varieties. It is also proved in [39] for Fano varieties of lines of cubic fourfolds, which are well-known to be irreducible hyper-Kähler fourfolds of $K3^{[2]}$ deformation type (see [3]). Finally, Conjecture 0.7 is proved by Riess [35] in the case of irreducible hyper-Kähler varieties of $K3^{[n]}$ or generalized Kummer deformation type admitting a Lagrangian fibration. Here the condition that X is a deformation of a punctual Hilbert scheme of a $K3$ surface guarantees, according to [30], that X satisfies the conjecture that any nef line bundle L on X which is isotropic for the Beauville-Bogomolov quadratic form is the pull-back of a $\mathbb{Q}$-line bundle on the basis B of a Lagrangian fibration on X.

The application to Conjecture 0.7 we give in this paper concerns one natural relation in the cohomology ring of a hyper-Kähler manifold equiped with a Lagrangian fibration. Note that, according to [6], the set of polynomial relations between degree 2 cohomology classes on an irreducible hyper-Kähler $2n$-fold X is generated as an ideal of $\mathrm{Sym}\, H^2(X,\mathbb{R})$ by the relations

$$\alpha^{n+1} = 0 \text{ in } H^{2n+2}(X,\mathbb{R}) \text{ if } q(\alpha) = 0.$$

For the same reason, the cohomological relations between divisor classes $d \in \mathrm{NS}(X)_\mathbb{R} \subset H^2(X,\mathbb{R})$ are generated by the relations

$$d^{n+1} = 0 \text{ in } H^{2n+2}(X,\mathbb{R}) \ \text{ if } q(d) = 0, \tag{1}$$

In particular there are no relations in degree $\leq 2n$. If X is any projective hyper-Kähler manifold, let $l \in H^2(X,\mathbb{Q})$ be such that $q(l) = 0$. One has then $l^{n+1} = 0$ in $H^{2n+2}(X,\mathbb{Q})$. If furthermore X admits a Lagrangian fibration such that $l = c_1(L)$ and L is pulled-back from the base, then one clearly has $L^{n+1} = 0$ in $\mathrm{CH}(X)$ and this is the starting point of Riess' work [34], [35]. Observe next that by differentiation of (1) along the tangent space to Q at l, namely $l^{\perp_q}$, one gets the relations

$$l^n h = 0 \ \text{ in } H^{2n+2}(X,\mathbb{R}) \text{ if } q(l) = 0 \text{ and } q(l,h) = 0. \tag{2}$$

Our application of Theorem 0.3 is the following result, proving that if $l = c_1(L)$ as above comes from a Lagrangian fibration, $d = c_1(D)$ is a divisor class with $q(l,d) = 0$, and the dimension is ≤ 8, then (2) already holds in $\mathrm{CH}(X)_\mathbb{Q}$.

Theorem 0.8. *Let $\phi : X \to B$ be a Lagrangian fibration on a projective irreducible hyper-Kähler manifold of dimension $2n \leq 8$ with $b_2(X) \geq 8$ and let L generate $\phi^*\mathrm{Pic}\, B$. Let furthermore $D \in \mathrm{Pic}\, X$ satisfy the property that $L^n \cdot D$ is cohomologous to 0 on X. Then $L^n \cdot D = 0$ in $\mathrm{CH}^{n+1}(X)_\mathbb{Q}$.*

We refer to [24] for further applications of Theorem 0.8 to Conjecture 0.7. We close this introduction by mentioning that there are other potential applications of the study of Question 0.1 in various geometric contexts. Here is an example: Let S be a $K3$ surface. Huybrechts [21] defined a constant cycle curve $C \subset S$ to be a curve all of whose points are rationally equivalent in S. All points of C are then rationally equivalent to the canonical cycle o_S. Rational curves in S are constant cycles curves, but there are many other constant cycles curves, some of which can be constructed as follows: Choose an ample linear system $|L|$ on S and let $\mathcal{C} \subset |L| \times S$ be the universal curve. Over the regular locus $|L|^0$, we have the Jacobian fibration $\mathcal{J} \to |L|^0$ and we get a section ν of the pull-back $\mathcal{J}_\mathcal{C}$ on $\mathcal{C}$ by defining

$$\nu(c, C) = \mathrm{alb}_C(dc - L_{|C}),$$

where $d = L^2$ so that $dc - L_{|C}$ has degree 0 on C. The locus where this section is of torsion is *expected* to be 1-dimensional and to project via the natural map $\mathcal{C}$ to S to a countable union of constant cycles curves. An analysis of Question 0.1 is necessary here to make this expectation into a statement. Another related example is as follows: Let now $|L_1|, |L_2|$ be two linear systems on the $K3$ surface S, and let $N = L_1 \cdot L_2$. Choose integers $w_1, \ldots, w_N$ such that $\sum_i w_i = 0$. Over the open set $(|L_1| \times |L_2|)^0$ where the curves C_1, C_2 are smooth and meet transversally, there is an étale cover $Z \to (|L_1| \times |L_2|)^0$ parameterizing orderings of the set $C_1 \cap C_2$ and two sections

$$\nu_1 : Z \to \mathcal{J}_{1,Z}, \nu_2 : Z \to \mathcal{J}_{2,Z},$$

where $\mathcal{J}_{1,Z}$, $\mathcal{J}_{2,Z}$ are the pull-backs to Z of the Jacobian fibrations on $|L_1|$, resp. $|L_2|$. They are defined by

$$\nu_1(z) = \mathrm{alb}_{C_{1,z}}(\sum_i w_i x_{i,z}), \nu_2(z) = \mathrm{alb}_{C_{2,z}}(\sum_i w_i x_{i,z}),$$

where $z \in Z$ parameterizes the curves $C_{1,z}$, $C_{2,z}$ and the ordering $C_{1,z} \cap C_{2,z} = (x_{1,z}, \ldots, X_{N,z})$ of their intersections. The vanishing (or torsion) locus of (ν_1, ν_2) is expected to be 0-dimensional. It provides a geometric way of producing elements of the higher Chow group $\mathrm{CH}^2(S, 1)$ (see [37], [12], [31]).

The paper is organized as follows. In the very short section 1, we will show how Theorem 0.8 follows from Theorem 0.3. We will prove Theorem 0.5 in Section 2. Section 3 will be devoted to sketching the proof of the André-Corvaja-Zannier theorem. The proof of Theorem 0.3 will be given in Section 4 in the non-isotrivial case and in Section 5 in the isotrivial case. The subsections 4.1, 5.1 and 5.2 present in detail the local analysis of the nontransversality of a Lagrangian section of a Lagrangian torus fibration and show how it is related to a degenerate real Monge-Ampère equation. In Section 6, we will analyze the case of a 10-dimensional Lagrangian fibration and prove Proposition 2.8.

1 Application of Theorem 0.3 to the Beauville conjecture

We prove in this section Theorem 0.8 assuming Theorem 0.3. Let $\phi : X \to B$ be as in the introduction, with $\dim X = 2n \leq 8$ and $b_2(X) \geq 8$ and let L generate $\phi^* \mathrm{Pic}\, B$ (see [27]). We want to prove that for any line bundle D on X such that $D \cdot L^n$ is cohomologous to 0 on X, then $D \cdot L^n$ is rationally equivalent to 0 on X modulo torsion.

Let us consider a universal family $\mathcal{X} \to T$ of deformations of X preserving the line bundles L, D and an ample line bundle on X. The very general fiber X_t of this family is projective and admits the deformed line bundles D_t and L_t. It also admits a Lagrangian fibration associated to L_t (see [29], and also [18] if the base is not smooth). The condition that $D_t \cdot L_t^n$ is rationally equivalent to 0 on X_t modulo torsion is satisfied on a countable union of closed algebraic subsets of T, so if we prove it is satisfied at the very general point $t \in T$, it will be also satisfied for any $t \in T$, hence for X. As $b_2(X) \geq 8$, we can apply Corollary 0.4 to the very general deformation (X_t, ϕ_t, L_t, D_t). It thus suffices to prove the result when (X, ϕ, D) satisfies the conclusion of Theorem 0.3, which we assume from now on. In the Chow ring $\mathrm{CH}(X)_{\mathbb{Q}}$, the fibers X_b, for $b \in B^0$, are all rationally equivalent (if B is smooth, then $B \cong \mathbb{P}^n$ by [22], so this is obvious; if B is not smooth, we refer to [24] for a proof of this statement). It follows that we have for any $b \in B$

$$L^n = \mu X_b \text{ in } \mathrm{CH}^n(X)_{\mathbb{Q}}, \tag{3}$$

for some nonzero $\mu \in \mathbb{Z}$. Let $b \in B$ be a general point. The kernel of the map

$$[X_b] \cup = \frac{1}{\mu} c_1(L)^n \cup : H^2(X, \mathbb{Q}) \to H^{2n+2}(X, \mathbb{Q})$$

is equal to the kernel of the restriction map $H^2(X, \mathbb{Q}) \to H^2(X_b, \mathbb{Q})$. This is a general fact proved in [40, Lemme 1.5 and Remarque 1.6]:

Lemma 1.1. *(See [40]) Let $j : Y \to W$ be a generically finite morphism, where Y and W are smooth projective varieties (in particular connected), and let $\dim W - \dim Y = k$, $\dim W = m$. Let $\alpha := j_*[Y]_{fund} \in H^{2k}(W, \mathbb{Q})$. Then the two $\mathbb{Q}$-vector subspaces*

$$K_1 := \mathrm{Ker}\,(j^* : H^2(W, \mathbb{Q}) \to H^2(Y, \mathbb{Q})), \quad K_2 := \mathrm{Ker}\,(\alpha \cup : H^2(W, \mathbb{Q}) \to H^{2k+2}(W, \mathbb{Q}))$$

of $H^2(W, \mathbb{Q})$ are equal.

Let now $D \in \mathrm{Pic}\, X$ such that $D \cdot L^n$ is cohomologous to 0. Then $c_1(D) \cup [X_b] = 0$ by (3), and thus $c_1(D)_{|X_b} = 0$ in $H^2(X_b, \mathbb{Q})$, hence also in $H^2(X_b, \mathbb{Z})$. As we assumed that (X, ϕ, D) satisfies the conclusion of Theorem 0.3, there exist points in B^0 such that $D_{|X_b}$ is a torsion point in $\mathrm{Pic}^0(X_b)$, so that $D \cdot X_b$ is a torsion cycle in $\mathrm{CH}^{n+1}(X)$. This implies by (3) that $D \cdot L^n$ vanishes in $\mathrm{CH}^{n+1}(X)_{\mathbb{Q}}$, which proves Theorem 0.8.

Remark 1.2. In our specific case, Lemma 1.1 also follows directly from Proposition 1.3 below, that will be also used later on.

Proposition 1.3. *(Matsushita [29]) In the situation above, the restriction map $H^2(X, \mathbb{Q}) \to H^2(X_b, \mathbb{Q})$ has rank 1.*

Indeed, K_1 is always included in K_2, and in the other direction, if $[X_b] \cup D$ is cohomologous to 0 in X, $D_{|X_b}$ cannot be ample, hence it must have trivial first Chern class by Proposition 1.3.

2 A toy case: elliptic fibrations

We will use the same notation $\phi : X^0 \to B^0$ for the restriction to B^0 of the elliptic fibration $\phi : X \to B$. The associated Jacobian fibration $J \to B^0$ is a complex manifold which is a

fibration into complex tori, and its sheaf of holomorphic sections $\mathcal{J}$ is described in complex analytic terms as the quotient

$$\mathcal{J} = R^1\phi_*\mathcal{O}_{X^0}/R^1\phi_*\mathbb{Z} = R^1\phi_*\mathbb{C} \otimes \mathcal{O}_B/(\mathcal{H}^{1,0} \oplus R^1\phi_*\mathbb{Z}) = \mathcal{H}^{0,1}/R^1\phi_*\mathbb{Z}, \tag{4}$$

where

$$\mathcal{H}^{1,0} = R^0\phi_*\Omega_{X^0/B^0}, \ \ \mathcal{H}^{0,1} := R^1\phi_*\mathcal{O}_{X^0} = (R^1\phi_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_{B^0})/\mathcal{H}^{1,0}.$$

Formula (4) describes J as a holomorphic torus fibration but as a $\mathcal{C}^\omega$ real torus fibration, the right formula is

$$\mathcal{J}_{\mathbb{R}} = \mathcal{H}^1_{\mathbb{R}}/R^1\phi_*\mathbb{Z}, \ \ \mathcal{H}^1_{\mathbb{R}} := R^1\phi_*\mathbb{R} \otimes \mathcal{C}^\omega_{B^0,\mathbb{R}} \tag{5}$$

which uses the natural fiberwise isomorphisms

$$H^1(X_b,\mathbb{R}) \cong H^1(X_b,\mathbb{C})/H^{1,0}(X_b) \cong H^1(X_b,\mathcal{O}_{X_b}) \tag{6}$$

globalizing into the $\mathcal{C}^\omega_{B^0,\mathbb{R}}$ sheaf isomorphism

$$\mathcal{H}^1_{\mathbb{R}} \cong R^1\phi_*\mathcal{O}_{X^0} \otimes_{\mathcal{O}_B} \mathcal{C}^\omega_{B^0,\mathbb{C}} \cong \mathcal{H}^{0,1} \otimes_{\mathcal{O}_B} \mathcal{C}^\omega_{B^0,\mathbb{C}}. \tag{7}$$

Trivializing the locally constant sheaves $R^1\phi_*\mathbb{Z}$, $R^1\phi_*\mathbb{R}$ on simply connected open sets $U \subset B$, (5) is the counterpart, at the level of sheaves of sections, of a local real analytic trivialization

$$J_U = U \times (H^1(X_0,\mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}), \tag{8}$$

where 0 is a given point of U. Let now ν be a holomorphic section of J over U. Using the trivialization (8), the section ν gives a differentiable (in fact real analytic) map

$$f_\nu : U \to H^1(X_0,\mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z} = J(X_0),$$

(which is called the Betti map in [13]) and we will use the following easy criterion:

Proposition 2.1. *(i) For a section ν of a torus fibration with local associated map f_ν as above, the points x of U where $\nu(x)$ is of torsion are dense in U for the usual topology if the differential df_ν is surjective at some point $b \in U$.*

(ii) In the situation of an elliptic fibration over a 1-dimensional base, the differential df_ν is surjective if and only if it is nonzero.

Proof. (i) Observe first that f_ν being real analytic, if its differential is surjective at some point $b \in U$, it is surjective on a dense set of points of U, hence it suffices to prove density near a point b where the differential $df_{\nu,b}$ is surjective. If the differential $df_{\nu,b}$ is surjective, f_ν is an open map in a neighborhood U_b of b. As the torsion points are dense in $J(X_{b_0})$, their preimages under f_ν are then dense in U_b.

(ii) This is a consequence of the following fact:

Lemma 2.2. *The kernel of the differential $df_{\nu,b} : T_{U,b} \to T_{J(X_0)} = H^1(X_0,\mathbb{R})$ is a complex vector subspace of $T_{U,b}$.*

Proof. Let us prove more generally that the fibers of f_ν are analytic subschemes of U. Let $\alpha \in J(X_0) = H^1(X_0,\mathbb{R})/H^1(X_0,\mathbb{Z})$ and let $\tilde{\alpha} \in H^1(X_0,\mathbb{R})$ be a lifting of α (such a lift is defined up to the addition of an element β of $H^1(X_0,\mathbb{Z})$). The class $\tilde{\alpha}$ extends as a constant section also denoted $\tilde{\alpha}$ of the sheaf $R^1\phi_*\mathbb{R}$ on U, which induces a holomorphic section $\tilde{\alpha}^{0,1}$ of the sheaf $\mathcal{H}^{0,1} = R^1\phi^0_*\mathcal{O}_{X^0}$ on U. On the other hand, the holomorphic section ν of J lifts to a holomorphic section η of $\mathcal{H}^{0,1}$ and it is clear that

$$f_\nu^{-1}(\alpha) = \{b \in U, \ \tilde{\alpha}^{0,1}_b = \eta_b \text{ in } \mathcal{H}^{0,1}(X_b)/H^1(X_b,\mathbb{Z})\}.$$

It follows that $f_\nu^{-1}(\alpha)$ is the countable locally finite union of the closed analytic subsets defined as zero sets of the holomorphic sections

$$\tilde{\alpha}^{0,1} - \eta - \beta^{0,1} \in \Gamma(U,\mathcal{H}^{0,1})$$

of the bundle $\mathcal{H}^{0,1}$, over all sections β of $H^1_{\mathbb{Z}}$ on U. $\qquad\square$

In the situation of (ii), the base U and the fiber $J(X_0)$ are both of real dimension 2 and Lemma 2.2 implies that the kernel of $df_{\nu,b}$ is of real dimension 0 or 2. So either $df_{\nu,b} = 0$ or $df_{\nu,b}$ is surjective. $\qquad\square$

Proof of Theorem 0.5. Let $\phi : X^0 \to B^0$ be our elliptic fibration and ν a section of $J(X^0/B)$. Assume that the points of B where ν is a torsion point are not dense for the usual topology of B. Then by Proposition 2.1, it follows that df_ν vanishes everywhere on any open set U of B^0 where it is defined. Coming back to the sheaf theoretic language, this means equivalently that ν, seen as a section of $\mathcal{H}^1_{\mathbb{R}}/H^1_{\mathbb{Z}}$ via the isomorphism (5), is locally constant, or that our section $\nu \in \Gamma(B^0, \mathcal{H}^{0,1}/H^1_{\mathbb{Z}})$ comes from a locally constant section $\tilde\nu_{\mathbb{R}} \in \Gamma(B^0, H^1_{\mathbb{R}}/H^1_{\mathbb{Z}})$. We are thus in position to apply Manin's theorem on the kernel and conclude. For further use, we give the complete argument here. Note that, by assumption, $\tilde\nu_{\mathbb{R}}$ is not of torsion and thus, fixing a base point $0 \in B$, corresponds to an element $\alpha_0 \in H^1(X_0, \mathbb{R})$ which is not rational but has the property that for any $\gamma \in \pi_1(B^0, 0)$,

$$\rho(\gamma)(\alpha_0) - \alpha_0 \in H^1(X_0, \mathbb{Z}), \tag{9}$$

where $\rho : \pi_1(B^0, 0) \to \operatorname{Aut} H^1(X_0, \mathbb{Z})$ denotes the monodromy representation of the smooth fibration $\phi : X^0 \to B^0$. We use the following easy lemma.

Lemma 2.3. *Let $\rho : \Gamma \to \operatorname{Aut} V_{\mathbb{Q}}$ be a finite dimensional rational representation of a group Γ. Then if the invariant space $V_{\mathbb{Q}}^{inv}$ is trivial, the set*

$$\{v \in V_{\mathbb{R}}, \ \rho(\gamma)(v) - v \in V_{\mathbb{Q}} \text{ for any } \gamma \in \Gamma \}$$

is equal to $V_{\mathbb{Q}}$.

As the monodromy representation is rational, this lemma tells us that the set of classes $\alpha \in H^1(X_0, \mathbb{R})$ satisfying property (9) contains a nonrational class if and only if the set

$$H^1(X_0, \mathbb{Q})^{inv} := \{\alpha \in H^1(X_0, \mathbb{Q}), \rho(\gamma)(\alpha) - \alpha = 0, \ \forall\gamma \in \pi_1(B^0, 0)\}$$

of monodromy invariant elements is nontrivial. By Deligne's invariant cycles theorem [14], it then follows from the existence of α_0 that the restriction map

$$H^1(X, \mathbb{Q}) \to H^1(X_0, \mathbb{Q})$$

is nontrivial, hence surjective since this a morphism of Hodge structures of weight 1 and the right hand side has dimension 2. The conclusion that X is rationally isogenous to $J(X_0) \times B$ is then immediate since X is rationally isogenous to a projective completion of the Jacobian fibration $J(X^0/B^0)$ and the later is isogenous to $J(X_0) \times B^0$ if the restriction map $H^1(X, \mathbb{Q}) \to H^1(X_0, \mathbb{Q})$ is surjective. $\qquad\square$

2.1 Some examples with higher dimensional fiber dimension

Looking at Theorem 0.5, one may wonder what makes hard and wrong a generalization to higher fiber dimension. Let us give some examples explaining the main difficulties encountered: Consider more generally any complex torus fibration $\phi_A : A \to B$ with a section ν_A. The torus fibration is canonically isomorphic, as a real torus fibration, to the locally constant fibration $H_{1,\mathbb{R},\phi_A}/H_{1,\mathbb{Z},\phi_A}$, where $H_{1,\mathbb{R},\phi_A} := (R^1\phi_{A*}\mathbb{R})^*$. A section ν_A thus admits local liftings $\tilde\nu_{A,\mathbb{R}}$ which are $\mathcal{C}^\infty$ (in fact real analytic) sections of the flat vector bundle associated with the local system $H_{1,\mathbb{R},\phi_A}$.

Example 2.4. *Assume that the local lifts $\tilde\nu_{A,\mathbb{R}}$ of the section ν_A are constant sections $\tilde\nu_{A,\mathbb{R}} \in \Gamma(H_{1,\mathbb{R},\phi_A})$. Then if $\tilde\nu_{A,\mathbb{R}}$ is not rational, that is, does not belong to $\Gamma(H_{1,\mathbb{Q},\phi_A})$, ν_A has no torsion point.*

This case, which corresponds to the situation where the local maps f_{ν_A} of the previous section are constant, can be in general excluded by a monodromy argument. The following example is slightly more subtle:

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Example 2.5. *Assume that the torus fibration $\phi_A : A \to B$ is a fibered product*

$$A = A' \times_B A'',$$

where $\phi' : A' \to B$, $\phi'' : A'' \to B$ are torus fibrations. Choose a section $\nu_{A''} : B \to A''$ of ϕ'' which is as in Example 2.4, that is locally lifts to a constant nonrational section of $(R^1\phi''_*\mathbb{R})^*$. Then for any section $\nu_{A'} : B \to A'$ of ϕ', $\nu_A = (\nu_{A'}, \nu_{A''})$ does not have any torsion point.

Another slightly more involved situation where we can avoid torsion points is as follows: Recall from the introduction that if $\dim B < g := \dim A_b$ a "generic" section of the abelian scheme $\phi_A : A \to B$ avoids the torsion points. Of course, for a general family $A \to B$, there might be no non trivial section, but there are always multisections and they can be chosen to pass through any point with arbitrary differential. By "generic" we mean generic section of a base-changed family $A' \to B'$. The next example exhibiting nontransversality is as follows:

Example 2.6. *Choose $A_1 \to B_1$ with $\dim B_1 < g_1 := \dim A_{1,b}$ and a section $\nu_1 : B_1 \to A_1$ with no torsion points. For any $A_2 \to B_2$, such that $\dim B_1 + \dim B_2 = g_1 + g_2$, and for any section $\nu_2 : B_2 \to A_2$, the section $(\nu_1, \nu_2) : B_1 \times B_2 \to A_1 \times A_2$ of the product family $A_1 \times A_2 \to B_1 \times B_2$ has no torsion point.*

Let us conclude with the following abstract situation where we do not have transversality of the normal function (or rather of its local real analytic representation as in Section 2, so we do not expect the normal function to have torsion points.

Example 2.7. *Assume the abelian scheme $A \to B$ satisfies $\dim B = g = \dim A_b$ and has the following property: B admits a foliation given by an algebraic distribution which is analytically integrable with holomorphic leaves $\mathcal{F}_t$ of dimension d, such that along the leaves, the real variation of Hodge structures on $H_{1|\mathcal{F}_t}$ splits as $H_{1,\mathbb{R}|\mathcal{F}_t} = L_{\mathbb{R},t} \oplus L'_{\mathbb{R},t}$, with rank $L'_{\mathbb{R}} = 2d'$, and $d' < d$. Then if furthermore, the normal function $\nu : B \to A$ or rather its local real analytic lift $\tilde{\nu}_{\mathbb{R}}$ decomposes along the leaves as $\nu_{\mathbb{R}|\mathcal{F}_t} = \nu_{L,t} + \nu_{L',t}$ with $\nu_{L,t}$ locally constant, then $\nu_{\mathbb{R}}$ is never of maximal rank.*

Indeed, the differential $d\nu_{\mathbb{R}}$ cannot be injective at any point since its restriction to the tangent space of the leaf is equal to $d\nu_{L',t} : T_{\mathcal{F}_t,\mathbb{R}} \to L'_{\mathbb{R},t}$, the two spaces being of dimensions $2d > 2d'$.

This situation seems to be improbable if the $L_{\mathbb{R},t}$ furthermore varies with t (that is, does not come from a local system on B). However this is the only case that we could not exclude for normal sections associated with divisors on hyper-Kähler manifolds of dimension 10. Let $X \to B$ be a projective hyper-Kähler manifold of dimension 10 equipped with a Lagrangian fibration with maximal variation, and let $M \in \text{Pic}\,X$ be a divisor which is cohomologous to 0 on fibers. The following result will be proved in Section 6:

Proposition 2.8. *(i) If the torsion points of $\nu_M : B^0 \to X^0$ are not dense in B for the usual topology, a Zariski dense open set B^0 has a foliation with 3-dimensional leaves $\mathcal{F}_t$, and the variation of Hodge structure along the leaves*

$$H_{1,\mathbb{R}|\mathcal{F}_t} = L_{\mathbb{R},t} \oplus L'_{\mathbb{R},t}$$

where $\text{rank}\,L_{\mathbb{R},t} = 6$, $\text{rank}\,L'_{\mathbb{R},t} = 4$. Furthermore, the real variation of Hodge structure on $L_{\mathbb{R},t}$ is constant along $\mathcal{F}_t$.
(ii) Along each leaf $\mathcal{F}_t$, the normal function ν_M has the L-component $\tilde{\nu}_{M,\mathbb{R},L}$ of its real lift constant.

3   The André-Corvaja-Zannier theorem

We describe in this section following [2] the proof of Theorem 0.6. The reason for doing so is not only the fact that this result is important and the arguments beautiful, but also the fact that their paper is written with notations that are not familiar to Hodge theoretists.

We first comment on the meaning of the conclusion of the theorem and the notation used there: consider the local system $H_{1,\mathbb{Z}} = (R^1\pi_*\mathbb{Z})^*$ on B. It generates the flat holomorphic vector bundle $\mathcal{H}_1 := H_{1,\mathbb{Z}} \otimes \mathcal{O}_B$, which carries the Hodge filtration

$$\mathcal{H}_{1,0} \subset \mathcal{H}_1$$

with quotient $\mathcal{H}_{0,1}$. The Griffiths $\overline{\nabla}$ map is defined as the composite

$$\mathcal{H}_{1,0} \xrightarrow{\nabla} \mathcal{H}_1 \otimes \Omega_B \to \mathcal{H}_{0,1} \otimes \Omega_B,$$

where ∇ is the Gauss-Manin connection. It is $\mathcal{O}_B$-linear, hence induces, for each $b \in B$, $\lambda \in \mathcal{H}_{1,0,b}$, a linear map: $\overline{\nabla}_\lambda : T_{B,b} \to \mathcal{H}_{0,1,b}$.

A crucial ingredient in the proof of Theorem 0.6 is the following result due to André [1]: In the situation of the theorem, let $\Gamma_0 \subset \pi_1(B,b)$ be the subgroup acting trivially on $H^1(A_b,\mathbb{Z})$. On the cover $B_{\Gamma_0} \to B$ equipped with a point $\tilde{b}$ over $b \in B$, the base-changed fibration $A_{\Gamma_0} \to B_{\Gamma_0}$ has a natural globally defined real analytic trivialization $A_{\Gamma_0} \cong B \times H_1(A_{\tilde{b}},\mathbb{R})/H_1(A_{\tilde{b}},\mathbb{Z})$, so that the section ν becomes a well-defined real analytic map $f_\nu : B_{\Gamma_0} \to H_1(A_{\tilde{b}},\mathbb{R})/H_1(A_{\tilde{b}},\mathbb{Z})$.

Theorem 3.1. *(André [1]) Under the assumptions (i) and (ii), the image of the monodromy*

$$\Gamma_0 \to H_1(A_{\tilde{b}},\mathbb{Z})$$

of f_ν is Zariski dense in $H_1(A_{\tilde{b}},\mathbb{C})$.

Proof of Theorem 0.6. We work on B_{Γ_0}. The real analytic map f_ν is constructed as follows: The holomorphic section ν on B is a section of the sheaf $\mathcal{H}_{0,1}/H_{1,\mathbb{Z}}$. Choose a local lift $\nu_{0,1} \in \Gamma(\mathcal{H}_{0,1})$. Due to Hodge symmetry, the real analytic flat vector bundle $\mathcal{H}_{1,\mathbb{R}}$ is isomorphic to $\mathcal{H}_{0,1}$ as a real analytic vector bundle. This way, the holomorphic section $\nu_{0,1}$ of $\mathcal{H}_{0,1}$ provides a real analytic section $\tilde{\nu}_\mathbb{R}$ of $\mathcal{H}_{1,\mathbb{R}}$. We need to describe more explicitly how $\tilde{\nu}_\mathbb{R}$ is deduced from $\nu_{0,1}$. Choose a local holomorphic lift $\tilde{\nu}_{hol}$ of ν to a holomorphic section of the bundle $\mathcal{H}_1$. We use now the real analytic decomposition

$$\mathcal{H}_{1,an} \cong \mathcal{H}_{1,0,an} \oplus \mathcal{H}_{0,1,an}, \tag{10}$$

with $\mathcal{H}_{0,1} = \overline{\mathcal{H}_{1,0}}$. Here the index "an" indicates that we consider the associated real analytic complex vector bundle. We can thus write $\tilde{\nu}_{hol} - \overline{\tilde{\nu}_{hol}} = \eta_1 - \overline{\eta_1}$ for some real analytic section η_1 of $\mathcal{H}_{1,0}$. Then we conclude that

$$\tilde{\nu}_{hol} - \eta_1 = \overline{\tilde{\nu}_{hol}} - \overline{\eta_1} \tag{11}$$

is real and maps to $\nu_{0,1}$ via the quotient map $\mathcal{H}_{1,an} \to \mathcal{H}_{0,1,an}$. Thus $\tilde{\nu}_{hol} - \eta_1$ gives the desired section $\tilde{\nu}_\mathbb{R}$ of $\mathcal{H}_{1,\mathbb{R},an}$ (which is then made into a real analytic map $f_\nu : B \to H_1(A_b,\mathbb{R})$ by local flat trivialization of $\mathcal{H}_{1,\mathbb{R},an}$). Note that it is clearly independent of the choice of holomorphic lifting $\tilde{\nu}_{hol}$ of $\nu_{0,1}$.

Our assumption is that f_ν is nowhere of maximal rank. What is unpleasant about f_ν is its real analytic, as opposed to holomorphic, character. The first step in the André-Corvaja-Zannier proof is the separation of holomorphic and antiholomorphic variables so as to make this assumption into a holomorphic equation. Let us work locally on $B \times \overline{B}$ (later on, we will rather consider $B_{\Gamma_0} \times \overline{B_{\Gamma_0}}$). Here $\overline{B}$ is B equipped with its conjugate complex structure, for which the holomorphic functions are the complex conjugates of holomorphic functions on B.

Over $\overline{B}$ we have the holomorphic bundle $\overline{\mathcal{H}_{1,0}}$ and this provides us with two holomorphic vector bundles

$$\mathcal{K}_1 := pr_1^* \mathcal{H}_{1,0}, \ \mathcal{K}_2 := pr_2^* \overline{\mathcal{H}_{1,0}}$$

on $B \times \overline{B}$. Near the diagonal of B, we have the flat holomorphic bundle $\mathcal{H}_1$ associated to the local system $H_{1,\mathbb{C}}$ on B. The inclusions of holomorphic subbundles

$$\mathcal{H}_{1,0} \subset \mathcal{H}_1, \ \text{resp.} \ \overline{\mathcal{H}_{1,0}} \subset \mathcal{H}_1$$

on B, resp. $\overline{B}$ gives on $B \times \overline{B}$ two holomorphic vector subbundles

$$\mathcal{K}_1 \subset \mathcal{H}_1, \ \mathcal{K}_2 \subset \mathcal{H}_1,$$

which restricted to the diagonal produce (10). Thus we have

$$\mathcal{H}_1 = \mathcal{K}_1 \oplus \mathcal{K}_2 \tag{12}$$

near the diagonal of B in $B \times \overline{B}$. We now produce a holomorphic version of $\tilde{\nu}_\mathbb{R}$ on $B \times \overline{B}$ near the diagonal as follows: starting from local lifts $\tilde{\nu}_{hol}$ of $\nu_{0,1}$ to a holomorphic section of $\mathcal{H}_1$ on B, we consider

$$\tilde{\nu}_1 := pr_1^* \tilde{\nu}_{hol}, \ \tilde{\nu}_2 = pr_2^* \overline{\tilde{\nu}_{hol}}$$

as holomorphic sections of $\mathcal{H}_1$ defined on $B \times \overline{B}$ near the diagonal and write $\tilde{\nu}_1 - \tilde{\nu}_2 = \eta_1 - \eta_2$, where η_1, η_2 are holomorphic sections of $\mathcal{K}_1, \mathcal{K}_2$ respectively. We then consider the holomorphic section $\tilde{\nu}_1 - \eta_1$ of $\mathcal{H}_1$ and observe that its restriction to the diagonal of B is exactly our real lifting $\tilde{\nu}_\mathbb{R}$. Similarly, the locally defined holomorphic map $F_\nu : B \times \overline{B} \to H^1(A_b, \mathbb{C})$ deduced from $\tilde{\nu}_1 - \eta_1$ by locally trivializing the flat vector bundle $\mathcal{H}_1$, restricts to $f_\nu : B \to H^1(A_b, \mathbb{R})$ on the diagonal Δ_B.

One easily shows (see [2, Lemma 4.2.1]) that $F_\nu : B \times \overline{B} \to H^1(A_b, \mathbb{C})$ is generically of maximal rank if and only if $f_\nu : B \to H^1(A_b, \mathbb{R})$ is generically of maximal rank. Thus the assumption of the theorem is that F_ν is nowhere of maximal rank where it is defined, namely near the diagonal of B.

It is important to observe that $\tilde{\nu}_1 - \eta_1$ and F_ν in fact do not depend on the choice of lift $\tilde{\nu}$ of $\nu_{0,1}$. Indeed, if we add to $\tilde{\nu}$ a section λ of $\mathcal{H}_{1,0}$ without changing $\overline{\nu}$, then we have $\tilde{\nu}_1' = \tilde{\nu}_1 + \lambda_1$, with $\lambda_1 = pr_1^* \lambda$, and $\tilde{\nu}_2' = \tilde{\nu}_2$, so that $\tilde{\nu}_1' - \tilde{\nu}_2' = \tilde{\nu}_1 - \tilde{\nu}_2 + \lambda_1$ and $\eta_1' = \eta_1 + \lambda_1$. Thus $\tilde{\nu}_1' - \eta_1' = \tilde{\nu}_1 - \eta_1$. Similarly, if we change now $\overline{\nu}$ by adding to it a section $\overline{\lambda}$ of $\mathcal{H}_{0,1}$, then we do not change $\tilde{\nu}_1'$ and we have $\tilde{\nu}_2' = \tilde{\nu}_2 + \lambda_2$ where $\lambda_2 = pr_2^* \overline{\lambda}$. Then $\eta_1' = \eta_1$ and $\eta_2' = \eta_2 + \lambda_2$, so finally $\tilde{\nu}_1' - \eta_1' = \tilde{\nu}_1 - \eta_1$.

In order to use the monodromy Theorem 3.1, we need to make the above construction more global, and this can be done by working on $B' := B_{\Gamma_0} \times \overline{B_{\Gamma_0}}$. On B_{Γ_0}, the local system H_1 is trivial, and thus we have a global isomorphism $pr_1^* H_1 \cong pr_2^* H_1 = H$, so that $\mathcal{K}_1 = pr_1^* \mathcal{H}_{1,0}$ and $\mathcal{K}_2 = pr_2^* \overline{\mathcal{H}_{1,0}}$ are holomorphic subbundles of the trivial bundle $\mathcal{H} := H \otimes \mathcal{O}_{B'}$. On a dense Zariski open set B_0' of B', we have $\mathcal{H} \cong \mathcal{K}_1 \oplus \mathcal{K}_2$. Furthermore $pr_1^* \mathcal{H}_{0,1} \cong \mathcal{H}/\mathcal{K}_1$. We thus get a multivalued holomorphic map on B_0'

$$F_\nu : B_0' \to H_1(A_{\tilde{b}}, \mathbb{C}) \tag{13}$$

which coincides with the previously defined map near the diagonal of B_{Γ_0}. The fact that F_ν is multivalued follows from the fact that it is well defined once the lifts $\nu_{0,1}, \overline{\nu_{0,1}}$ are chosen, but it depends on these lifts.

We now use Theorem 3.1 on B_{Γ_0}. It says that the couple $(\nu_{0,1}, \overline{\nu_{0,1}})$ (or their lifts $\tilde{\nu}, \overline{\tilde{\nu}}$) changes under monodromy along B' (or B_0') by the addition of couples $(u, v) \in H_1(A_{\tilde{b}}, \mathbb{Z})^2$ with u, v in a Zariski dense open set of $H_1(A_{\tilde{b}}, \mathbb{Z})^2 \subset H_1(A_{\tilde{b}}, \mathbb{C})^2$. Write now

$$\tilde{\nu}_u = \tilde{\nu} + u, \ \overline{\tilde{\nu}}_v = \overline{\tilde{\nu}} + v.$$

Then, using (12), we can write $u - v = \lambda_1 - \lambda_2$ on B_0', where λ_1 is a holomorphic section of $\mathcal{K}_1$ and λ_2 is a holomorphic section of $\mathcal{K}_2$, and we get

$$pr_1^* \tilde{\nu}_u - pr_2^* \overline{\tilde{\nu}}_v = pr_1^* \tilde{\nu}_{0,1} - pr_2^* \overline{\tilde{\nu}_{0,1}} + u - v$$

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$$= pr_1^* \tilde{\nu}_{0,1} - pr_2^* \overline{\tilde{\nu}_{0,1}} + \lambda_1 - \lambda_2 = \eta_1 - \eta_2 + \lambda_1 - \lambda_2,$$

so that $\tilde{\nu}_{1,u,v} = \tilde{\nu}_1 + u$ and $\eta_{1,u,v} = \eta_1 + \lambda_1$. According to the recipe described above, the map F_ν thus becomes $F_\nu + u - \lambda_1$. A translation by u does not change the differential of F_ν, so we can do here $u = 0$ and choose v in a Zariski open set of $H^1(A_{\tilde{b}}, \mathbb{Z})$. As the holomorphic map F_ν is nowhere of maximal rank, we conclude that for any v in a Zariski dense set of $H^1(A_{\tilde{b}}, \mathbb{Z}) \subset H^1(A_{\tilde{b}}, \mathbb{C})$, the holomorphic map $F_\nu - \lambda_1 : B_0' \to H^1(A_{\tilde{b}}, \mathbb{C})$ is nowhere of maximal rank, where λ_1 is the holomorphic section of $\mathcal{K}_1 \subset \mathcal{H}_1$ defined by

$$v = \lambda_1 - \lambda_2.$$

We now apply the following easy lemma: for any $\mu \in H^1(A_{\tilde{b}}, \mathbb{C})$, we can write as before, using (12)

$$\mu = \mu_1 + \mu_2$$

as sections of $\mathcal{H}$ on B_0', with μ_i a holomorphic section of $\mathcal{K}_i$, $i = 1, 2$. We can see μ_1 as a holomorphic map $B_0' \to H = H_1(A_{\tilde{b}}, \mathbb{C})$.

Lemma 3.2. *Fix $b \in B_0'$. Then the set of $\mu \in H_1(A_{b'}, \mathbb{C})$ such that the holomorphic map $F_\nu - \mu_1$ is not of maximal rank at b is Zariski closed in $H_1(A_{b'}, \mathbb{C})$.*

Proof. Indeed, the assigment $\mu \mapsto \mu_1$ is $\mathbb{C}$-linear in μ, hence for fixed b, μ_1, $d\mu_{1,b}$ depend $\mathbb{C}$-linearly on μ, which concludes the proof. $\square$

Lemma 3.2 and the fact that v above can be taken in a Zariski dense set of $H^1(A_{\tilde{b}}, \mathbb{Z}) \subset H^1(A_{\tilde{b}}, \mathbb{C})$ allow now to conclude that for any $b' \in B'$ and any $\mu \in H_1(A_b, \mathbb{C})$, the holomorphic map $F_\nu - \mu_1$ is not of maximal rank at b' so that μ_1 is not of maximal rank at b'. The proof of Theorem 0.6 concludes now with the following lemma:

Lemma 3.3. *Assume that for any $b' \in B_0'$, and any $\mu \in H_1(A_{b'}, \mathbb{C})$, the differential $d\mu_1 : T_{B',b'} \to H_1(A_{b'}, \mathbb{C})$ is not of maximal rank. Then for any $b \in B$, and any $\lambda \in H_{1,0}(A_b)$ the map $\overline{\nabla}_\lambda : T_{B,b} \to H_{0,1,b}$ is not of maximal rank.*

Proof. The subbundle $\mathcal{K}_1 \subset \mathcal{H}$, where $\mathcal{H}$ is trivial, has a variation $\delta_1 : \mathcal{K}_1 \to \mathcal{K}_2 \otimes \Omega_{B'}$ and similarly for $\mathcal{K}_2$. Along $B \subset B \times \overline{B}$, $\delta_1 = \overline{\nabla}$ and δ_2 is its complex conjugate. The differential $d\mu_1$ is computed as follows: we have

$$d\mu = 0 = d\mu_1 + d\mu_2, \tag{14}$$

with

$$d\mu_1 = d_1\mu_1 + \delta_1(\mu_1), \; d\mu_2 = d_2\mu_2 + \delta_2(\mu_2)$$

for some differentials $d_i\mu_i \in \Omega_{B'} \otimes \mathcal{K}_i$. It follows from (14) that $d_1\mu_1 + \delta_2(\mu_2) = 0$, $d_2\mu_2 + \delta_1(\mu_1) = 0$, so that

$$d\mu_1 = -\delta_2(\mu_2) + \delta_1(\mu_1). \tag{15}$$

Suppose there exist $b \in B$ and $\lambda \in H^{1,0}(A_b)$ such that the map $\overline{\nabla}(\lambda) : T_{B,b} \to H_{0,1,b}$ is of maximal rank. Let $\mu = \lambda + \overline{\lambda}$. Equation (15) at $b \in B \subset B \times \overline{B}$ gives then $d\mu_1 = -\overline{\overline{\nabla}(\lambda)} + \overline{\nabla}(\lambda)$ and this sum is the direct sum of the two isomorphisms

$$\overline{\nabla}(\lambda) : T_B \cong H_{0,1}(A_b), \; -\overline{\overline{\nabla}(\lambda)} : T_{\overline{B}} \cong H_{1,0}(A_b),$$

hence it is an isomorphism

$$d\mu_1 : T_{B',b} = T_B \oplus T_{\overline{B}} \cong H_1(A_b, \mathbb{C}),$$

contradicting our assumption. $\square$

This concludes the proof of Theorem 0.6. $\square$

We finish this section by observing that the proof given above allows to prove in fact a statement slightly stronger than Theorem 0.6, namely the following variant, for which we introduce a notation: associated to our normal function ν, which is a section of the sheaf

$$\mathcal{J} = \mathcal{H}_{0,1}/H_{1,\mathbb{Z}}$$

and given a local lift of ν to a holomorphic section $\nu_{0,1}$ of $\mathcal{H}_{0,1}$, we get an affine subbundle $\mathcal{H}_{1,0,\nu}$ of $\mathcal{H}_1 = H_{1,\mathbb{Z}} \otimes \mathcal{O}_B$ defined as

$$\mathcal{H}_{1,0,\nu} := q^{-1}(\nu_{0,1}),$$

where $q : \mathcal{H}_1 \to \mathcal{H}_{0,1}$ is the quotient map. Composing the inclusion $\mathcal{H}_{1,0,\nu} \subset \mathcal{H}_1$ of the affine subbundle $\mathcal{H}_{1,0,\nu}$ with a local flat trivialization $\Phi : \mathcal{H}_1 \to H_{1,b,\mathbb{C}}$, we get a holomorphic map

$$\Phi_\nu : \mathcal{H}_{1,0,\nu} \to H_{1,b,\mathbb{C}}. \tag{16}$$

Theorem 3.4. *Let $\pi : A \to B$ be a family of abelian varieties of dimension d over $\mathbb{C}$, where B is quasi-projective and $\dim B \geq d$. Let ν be an algebraic section of π. Assume*
(i) The family has no locally trivial subfamily.
(ii) The multiples $m\nu(B)$ are Zariski dense in A.
Then if the associated real analytic map f_ν is nowhere of maximal rank, the map Φ_ν is nowhere submersive.

Note that this strengthens Theorem 0.6 since the conclusion of Theorem 0.6 is the fact that the holomorphic map $\Phi_{1,0} := \Phi_{|\mathcal{H}_{1,0}} : \mathcal{H}_{1,0} \to H_{1,b,\mathbb{C}}$ is nowhere of maximal rank. Choosing a holomorphic lift $\tilde{\nu}$ of ν in $\mathcal{H}_1$, the map Φ_ν of (16) identifies to $\Phi \circ \tilde{\nu} + \Phi_{1,0} : \mathcal{H}_{1,0} \to H_{1,b,\mathbb{C}}$ and it is clear, by passing to the linear part, that if $\Phi \circ \tilde{\nu} + \Phi_{0,1}$ is nowhere of maximal rank on $\mathcal{H}_{1,0}$, so is $\Phi_{1,0}$.

4 Lagrangian fibrations on hyper-Kähler manifolds; the non-isotrivial case of Theorem 0.3

4.1 Local structure results

We recall in this section, following [15] or [20], the local data determining a holomorphic Lagrangian polarized torus fibration. The holomorphic torus fibration $\phi : A \to U$ is determined by a weight 1 (or rather -1) variation of Hodge structure, that is the data of a local system $H_{1,\mathbb{Z}} = (R^1\phi_*\mathbb{Z})^*$ and a holomorphic subbundle

$$\mathcal{H}_{1,0} \subset \mathcal{H}_1 := H_{1,\mathbb{Z}} \otimes \mathcal{O}_U$$

determining a weight 1 Hodge structure at any point of U. The sheaf $\mathcal{A}$ of holomorphic sections of A identifies to

$$\mathcal{H}_{0,1}/H_{1,\mathbb{Z}}, \ \mathcal{H}_{0,1} := \mathcal{H}_1/\mathcal{H}_{1,0}. \tag{17}$$

The holomorphic 2-form σ_A on A for which the fibration $\phi : A \to U$ is a Lagrangian fibration provides by interior product an isomorphism of vector bundles

$$T_U = R^0\phi_*(T_A/T_{A/U}) \cong R^0\phi_*\Omega_{A/U}, \tag{18}$$

which by dualization provides an isomorphism (which is canonical, given the choice of σ_A)

$$\mathcal{H}_{0,1} \cong \Omega_U. \tag{19}$$

Using the isomorphism of (19), the surjective map of holomorphic vector bundles

$$\mathcal{H}_1 \to \mathcal{H}_{0,1}$$

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is thus given, on any simply connected open set U' where the flat vector bundle $\mathcal{H}_1$ is trivialized, by the evaluation morphism of $2n$ holomorphic 1-forms α_i on U', which must have the property that their real parts $\operatorname{Re}\alpha_i$ form a basis of $\Omega_{U',\mathbb{R}}$ at any point. We have (see [15]):

Lemma 4.1. *The forms α_i are closed on U', hence we have, up to shrinking our local open set U' if necessary, $\alpha_i = df_i$, where the f_i's are holomorphic and defined up to a constant.*

Proof. The proof will use a different description of the forms α_i. For any locally constant section γ of $H_{1,\mathbb{C}} \cong H_{\mathbb{C}}^{2g-1}$, $g = \dim A_b$, we get using the closed 2-form σ_A a closed 1-form $\phi_*(\gamma \cup \sigma_A)$ on U' (when γ is a section of $H_{1,\mathbb{Z}}$, γ is a combination of classes γ_i of oriented circle bundles $\Gamma_i \subset A$, and $\phi_*(\gamma \cup \sigma_A)$ is defined as $(\phi_{|\Gamma_i})_*(\sigma_{A|\Gamma_i})$). We conclude observing that for any locally constant section γ of $H_{1,\mathbb{C}}$ on an open set U' of U, with induced section $\gamma_{0,1}$ of $\mathcal{H}_{0,1}$, providing a holomorphic form $\alpha_{\gamma_{0,1}}$ via the isomorphism (19), we have

$$\alpha_{\gamma_{0,1}} = \phi_*(\gamma \cup \sigma_A) \text{ in } \Gamma(U', \Omega_{U'}),$$

thus proving that the forms α_i are closed. $\qquad\square$

If we now choose the α_i to form a basis of $H_{1,\mathbb{R}}$, the corresponding $2n$ holomorphic 1-forms on U' have the properties that at any point $b \in U'$, the forms $\alpha_{i,b}$ are independent over $\mathbb{R}$. Another way to say it is that the functions $\operatorname{Re} f_i$ give local real coordinates on U'.

Remark 4.2. By definition, the functions $\operatorname{Re} f_i$, which are defined up to a constant and depend only on the choice of σ_A and of local basis α_i of $H_{1,\mathbb{R}}$, provide a local real analytic identification

$$U \cong H^1(A_{b_0}, \mathbb{R}). \tag{20}$$

Globally, they provide an affine flat structure on U.

The last piece of information we need concerning the torus fibration $\phi : A \to U$ is the polarization ω on the fibers. It is given by a monodromy invariant skew-symmetric pairing $\langle\,,\rangle$ on $H_{1,\mathbb{R}}$ (we do not need here the fact that the polarization is integral), which has to polarize the Hodge structure at any point, so that for any $b \in U$

$$\langle\,,\rangle = 0 \text{ on } \mathcal{H}_{1,0,b} \subset H_{1,\mathbb{C},b} \tag{21}$$
$$i\langle\alpha,\overline{\alpha}\rangle > 0, \quad \forall 0 \neq \alpha \in \mathcal{H}_{1,0,b}.$$

This can be viewed as a nondegenerate skew-symmetric form $\omega \in \bigwedge^2 H^1(A_{b_0},\mathbb{R})$ which produces via the diffeomorphism (20) a closed 2-form ω^* on the open set U of the trivialization introduced above. The 2-form ω^* is constant in the affine coordinates introduced above. We have the following lemma whose proof is a formal computation (see [20]):

Lemma 4.3. *The Hodge-Riemann bilinear relations (21) satisfied by $\langle\,,\rangle$ and the Hodge structure on $H_1(A_b,\mathbb{R})$ at any point $b \in U$ are equivalent to the fact that ω^* is a Kähler form on U.*

As a corollary of the above descrition, we get the following result due to Donagi and Markman [15]:

Theorem 4.4. *(Donagi-Markman) Let $X \to B$ be a Lagrangian fibration. Then locally on B_{reg}, there exist a holomorphic function g and holomorphic coordinates $z_1,\ldots,z_n$ such that the infinitesimal variation of Hodge structures $T_{B,b} \to \operatorname{Hom}(H_{1,0,b}, H_{0,1,b})$ at $b \in B_{reg}$ is induced by the cubic form $g^{(3)}$ defined by the third partial derivatives of g, using the identifications $T_{B,b} \cong H_{1,0,b} \cong H^*_{0,1,b}$.*

Proof. Indeed, let f_i be the local holomorphic functions appearing above. The indices correspond to a local flat basis $e_1, \ldots, e_{2n}$ of $H_{1,\mathbb{Z}}$. We can assume that the basis is chosen so that $\langle e_1, \ldots, e_n \rangle$ is totally isotropic for the pairing giving the polarization. We choose for coordinates $z_1 = f_1, \ldots, z_n = f_n$. We have

$$df_{n+i} = \sum_{j=1}^{n} g_{ij} dz_j,$$

with $g_{ij} = \frac{\partial f_{n+i}}{\partial z_j}$, so that the kernel of the evaluation map

$$\mathbb{C}^{2n} \otimes \mathcal{O}_B \to \Omega_B$$

is generated by $e_{n+i} - \sum_{j=1}^{n} g_{ij} e_j$ for $j = 1, \ldots, n$. As the subspace generated by these elements is totally isotropic for the pairing, we get that $g_{ij} = g_{ji}$. Thus we have $\frac{\partial f_{n+i}}{\partial z_j} = \frac{\partial f_{n+j}}{\partial z_i}$ for any i, j, and this implies that locally there is a holomorphic function g such that $f_{n+i} = \frac{\partial g}{\partial z_i}$.

We finally compute the infinitesimal variation of Hodge structures by differentiating the generators $e_i' := e_{n+i} - \sum_{j=1}^{n} g_{ij} e_j$ of $\mathcal{H}_{1,0}$: we get

$$\frac{\partial e_i'}{\partial z_k} = -\sum_j \frac{\partial g_{ij}}{\partial z_k} e_j = -\sum_j \frac{\partial^3 g}{\partial z_i \partial z_j \partial z_k} e_j.$$

This proves the claim as the basis e_i', $i \geq n + 1$ is dual to the basis e_i for $i \leq n$, which itself corresponds to the basis $\frac{\partial}{\partial z_i}$ of $T_{B,b}$. $\qquad\square$

4.2 Proof of Theorem 0.3 when $\dim X \leq 8$ and the variation is maximal

Let X be a very general hyper-Kähler manifold with Lagrangian fibration $\pi : X \to B$ and $M \in \operatorname{Pic} X$ a divisor topologically trivial on fibers of π. Assume π is not locally isotrivial. Let us prove that the assumptions of Theorem 0.6 are satisfied. This follows from

Lemma 4.5. *Let $\phi : X \to B$ be a Lagrangian fibration on a hyper-Kähler manifold. Then there is no proper nontrivial real subvariation of Hodge structure of $H_{1,\mathbb{R}}$ on B^0.*

Proof. According to Matsushita's Proposition 1.3, the restriction map $H^2(X, \mathbb{Q}) \to H^2(X_b, \mathbb{Q})$ has rank 1. By Deligne's invariant cycle theorem, this implies that the local system $R^2 \phi_* \mathbb{R}$ on B^0 has only one global section (provided by the polarization ω and its real multiples). If the local system $(R^1 \phi_* \mathbb{R})^*$ contains a nontrivial local subsystem L which carries a real variation of Hodge structures, the restriction of ω to L is nondegenerate and the orthogonal complement $L^{\perp \omega}$ with respect to ω is also a local subsystem of $(R^1 \phi_* \mathbb{R})^*$ on which ω is nondegenerate. Thus we have a decomposition

$$(R^1 \phi_* \mathbb{R})^* = L \oplus L^{\perp} \tag{22}$$

and if both L and $L^{\perp}$ were nontrivial, we would get two independent sections of $R^2 \phi_* \mathbb{R} = \bigwedge^2 ((R^1 \phi_* \mathbb{R})^*)^*$, namely $\pi_L^* \omega_{|L}$ and $\pi_{L^{\perp}}^* (\omega_{|L^{\perp}})$, where π_L and $\pi_{L^{\perp}}$ are the two projections associated with the decomposition (22). This is a contradiction, hence a proper nontrivial such L does not exist. $\qquad\square$

If ν_M is not of torsion, the Zariski closure of $\mathbb{Z}\nu_M(B)$ has an irreducible component which is a subfamily of abelian varieties (of positive dimension) of $\operatorname{Pic}^0(X_{reg}/B_{reg})$. Such a family must be equal to $\operatorname{Pic}^0(X_{reg}/B_{reg})$ by Lemma 4.5. This establishes assumption (ii) of Theorem 0.6. For the same reason, $\operatorname{Pic}^0(X_{reg}/B_{reg})$ has no locally trivial subfamily because we are in the nonisotrivial case and this establishes assumption (i) of Theorem 0.6.

Proof of Theorem 0.3 when $\dim X \leq 8$ *in the case of maximal variation.* The maximal variation assumption tells us that at a general point $b \in B_{reg}$ the map

$$d\mathcal{P} : T_{B,b} \to \mathrm{Hom}(H_{1,0,b}, H_{0,1,b})$$

is injective.

Reasoning by contradiction, if ν_M has the property that its torsion points are not dense in B_{reg}, then as explained above, Theorem 0.6 applies and tells that the map $\overline{\nabla} : H_{1,0,b} \to \mathrm{Hom}(T_{B,b}, H_{0,1,b})$ associated to $d\mathcal{P}$ above has the property that $\overline{\nabla}_\lambda \in \mathrm{Hom}(T_{B,b}, H_{0,1,b})$ is not an isomorphism for any $\lambda \in H_{1,0,b}$. According to Theorem 4.4, these two maps are in fact identical and the $\overline{\nabla}_\lambda$'s are the partial derivatives $\partial_\lambda g^{(3)}$ of a cubic form $g^{(3)}$ on $T_{B,b} = H_{1,0,b}$. The homogeneous polynomials of degree 3 in n variables satisfying the condition that all of their partial derivatives are degenerate quadrics have been classified by Hesse and Gordan-Noether if $n \leq 4$, and by Lossen [25] if $n \leq 5$. We refer to section 6 for the latter. In the case where $n \leq 4$, one has the following

Theorem 4.6. *(see [25] Any cubic homogeneous polynomial in $n \leq 4$ variables all of whose partial derivatives are degenerate is a cone.*

Being a cone means that one partial derivative derivative is identically 0, which contradicts the injectivity of the map $T_{B,b} \to \mathrm{Hom}(H_{1,0,b}, H_{0,1,b})$. $\qquad\square$

Remark 4.7. Part of the arguments described in this section also appear in Lin's thesis [23].

5   Proof of Theorem 0.3 when the fibration is isotrivial

5.1   Further general facts about sections of lagrangian fibrations

Let $\phi : X \to B$ be a Lagrangian fibration, where X is a projective irreducible hyper-Kähler manifold of dimension $2n$, and let $B^0 \subset X^0$ be the Zariski open set of regular values of ϕ. Denote by σ the holomorphic 2-form on X and by ω the first Chern class of an ample line bundle H on X. Over B^0, the fibration $\phi : X^0 \to B^0$ is a fibration into abelian varieties or rather a torsor on the corresponding Albanese fibration $\mathrm{Alb}(X^0/B^0)$, which admits a multisection $Z \to B^0$ of degree d. Using these data, $X^0 \to B^0$ is isogenous to the associated Albanese fibration $\mathrm{Alb}(X^0/B^0)$ via the map

$$i_Z : X^0 \to \mathrm{Alb}(X^0/B^0)$$

defined by $i_Z(u) = \mathrm{alb}_{X_b}(d\{u\} - Z_b)$, $b = \phi(u)$. Here the notation $\{u\}$ is used to denote u as a 0-cycle in the fiber X_b (in order to avoid the confusion between addition of cycles and addition of points), so $d\{u\} - Z_b$ should be thought as a 0-cycle of degree 0 on X_b, giving rise to an element $\mathrm{alb}_{X_b}(d\{u\} - Z_b) \in \mathrm{Alb}(X_b)$. Next, using the polarisation H, the abelian fibration $\mathrm{Alb}(X^0/B^0)$ and its dual fibration $\mathrm{Pic}^0(X^0/B^0)$ are isogenous, the isogeny

$$i_H : \mathrm{Alb}(X^0/B^0) \to \mathrm{Pic}^0(X^0/B^0) \tag{23}$$

being given by

$$i_H(b, v_b) = t_{v_b}^*(H_{|X_b}) - H_{|X_b}$$

for any $b \in B^0$, $v_b \in \mathrm{Alb}(X_b)$. Here the translation t_{v_b} acts on X_b. The cycle $H^{n-1} \in \mathrm{CH}^{n-1}(X)$ also gives an isogeny

$$\mathrm{alb}_{X^0/B^0} \circ H^{n-1} : \mathrm{Pic}^0(X^0/B^0) \to \mathrm{Alb}(X^0/B^0) \tag{24}$$

which is easily shown to provides an inverse of (23) up to a nonzero coefficient.

Note now that the $(2,0)$-form σ on X is the pull-back of a $(2,0)$-form σ_A on $\mathrm{Alb}(X^0/B^0)$ (or on $\mathrm{Pic}^0(X^0/B^0)$) via the rational map $i_Z : X \dashrightarrow \mathrm{Alb}(X^0/B^0)$. This indeed follows easily

from Mumford's theorem [32], since for any two points x, y of X^0 such that $i_Z(x) = i_Z(y)$ the difference $\text{alb}_{X_b}(\{x\} - \{y\})$ is a torsion point in $\text{Alb}(X_b)$, $b = \phi(x) = \phi(y)$, hence the cycle $\{x\} - \{y\}$ is a torsion cycle in $\text{CH}_0(X_b)$ and a fortiori in $\text{CH}_0(X)$ (in fact it actually vanishes in $\text{CH}_0(X)$ since the later group has no torsion). Using the maps above, we get as well a nondegenerate $(2,0)$-form σ_A on $A := \text{Pic}^0(X^0/B^0)$, extending to a smooth projective completion, and for which the fibration is Lagrangian. Furthermore $\sigma = ((i_H \circ i_Z)^* \sigma_A$.

Our analysis of the properties of the holomorphic section ν of the torus fibration will be infinitesimal hence local, but we want first to add one restriction which is global: The Lagrangian fibration $\phi : X \to B$ being as above, let $M \in \text{Pic} \, X$ be a line bundle on X which is topologically trivial on the fibers and let $\nu = \nu_M : B^0 \to A$ be the associated normal function.

Lemma 5.1. *The section ν is Lagrangian for the holomorphic 2-form σ_A.*

Proof. Let $\widetilde{B}$ be a desingularization of B. Then $H^0(\widetilde{B}, \Omega^2_{\widetilde{B}}) = 0$ since a nonzero holomorphic 2-form on $\widetilde{B}$ would provide by pull-back via the rational map $\phi : X \dashrightarrow \widetilde{B}$ a nonzero holomorphic form on X which is not proportional to σ. So we just have to prove that $\nu^*(\sigma_A)$ (which is a priori only defined as a holomorphic 2-form on B^0) extends to a holomorphic 2-form on $\widetilde{B}$. However, we can provide another construction of $\nu^*(\sigma_A)$ which will make clear that it extends to $\widetilde{B}$: Looking at the definitions of i_H, i_Z, the subvariety $\Gamma := (i_H \circ i_Z)^{-1}(\text{Im} \, \nu)$ is algebraic in X^0 and it is a multisection of ϕ over B^0. Its Zariski closure in $\widetilde{B} \times X$ will provide a correspondence $\overline{\Gamma} \subset \widetilde{B} \times X$. Then as $\sigma = i_Z^* \sigma_A$ on X^0, we get that

$$\overline{\Gamma}^* \sigma = N \nu^*(\sigma_A) \tag{25}$$

on B^0, where N is the degree of $i_H \circ i_Z$. This gives the desired extension of $\nu^*(\sigma_A)$. $\quad\square$

5.2 Degenerate normal function and degenerate Monge-Ampère equation

We complete the Donagi-Markman analysis in section 4.1 by adding to the data described there the normal function $\nu : U \to A$, which satisfies the property that $\nu^* \sigma_A = 0$, which is the global restriction coming from Lemma 5.1. This normal function $\nu \in \Gamma(U, \mathcal{A}) = \Gamma(U, \mathcal{H}_{0,1}/H_{1,\mathbb{Z}})$ lifts locally to a section $\tilde{\nu}$ of $\mathcal{H}_{0,1} \cong \Omega_U$, hence provides locally a holomorphic 1-form α on U.

Lemma 5.2. *The form α is closed.*

Proof. It is a general fact of the theory of integrable systems (see [15]) that the isomorphism (19) is also constructed in such a way that the pull-back $\tilde{\sigma}_A$ of the holomorphic 2-form σ_A to the total space of the bundle $\mathcal{H}_{0,1}$ identifies with the canonical 2-form σ_{can} on the cotangent bundle Ω_U, which has the property that for any holomorphic 1-form β on U, seen as a section of Ω_U, one has

$$d\beta = \beta^*(\sigma_{can}). \tag{26}$$

As the section $\tilde{\nu}$ corresponds via this isomorphism to the 1-form α, the assumption that $\nu^*(\sigma_A) = 0$ says equivalently that

$$0 = \tilde{\nu}^*(\tilde{\sigma}_A) = \alpha^*(\sigma_{can}) = d\alpha.$$

$\square$

The closed holomorphic 1-form α thus provides us on simply connected open sets U' of U with a holomorphic function f such that $df = \alpha$.

Remark 5.3. From now on, the holomorphic 2-form on our Lagrangian fibration will disappear. It is hidden in the fact that we work with the cotangent bundle of the base, which has a canonical holomorphic 2-form.

We are in the local setting where the holomorphic functions f_i, f constructed in the previous section exist. Let thus V be a complex manifold, and let $f_1, \ldots, f_{2n}$ be holomorphic functions on V such that the functions $x_i := \operatorname{Re} f_i$ provide real coordinates on V, giving rise to a real variation of Hodge structures

$$\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathcal{O}_V \overset{ev}{\to} \Omega_V \to 0, \tag{27}$$
$$e_i \mapsto df_i$$

with underlying local system $\mathbb{R}^{2n} = T_{V,\mathbb{R}}$ and Hodge bundles $\mathcal{H}_{1,0} = \operatorname{Ker} ev$, $\mathcal{H}_{0,1} = \Omega_V$. Let f be a holomorphic function on V; as the df_i's are independent over $\mathbb{R}$ in $\Omega_{V,b}$ at any point $b \in V$, we can write the equality

$$df = \sum_i a_i df_i, \tag{28}$$

of $\mathcal{C}^\omega$ forms of type $(1,0)$ on V^0, where the a_i's are real $\mathcal{C}^\omega$ functions on V, uniquely determined by (28). Denoting $x_i := \operatorname{Re} f_i$, the x_i's thus provide locally real coordinates on V^0. Note that (28) is obviously equivalent, since the a_i's are real and the considered forms are of type $(1,0)$, to the equality of real 1-forms:

$$d(\operatorname{Re} f) = \sum_i a_i dx_i. \tag{29}$$

We assume that the map $a. = (a_1, \ldots, a_{2n}) : V^0 \to \mathbb{R}^{2n}$ is of constant rank $2n - 2k$ (the rank is even by Lemma 2.2). The condition that $a.$ is not of maximal rank at the general point is a degenerate real Monge-Ampère equation (see [33]).

Lemma 5.4. *The fibers of $a.$ are affine in the coordinates x_i.*

Proof. As the map $a.$ has locally constant rank near $b \in V^0$, it factors through a submersion $V^0 \to \Sigma$ and an immersion $\Sigma \hookrightarrow \mathbb{R}^{2n}$, where Σ is real manifold of dimension $2n - 2k$. Equation (29) says equivalently that for fixed $\sigma = (\lambda_1, \ldots, \lambda_{2n}) \in \Sigma$, the function $\operatorname{Re} f - \sum_i \lambda_i x_i$ has vanishing differential on V^0 at any point of the fiber $V_\sigma^0 = a_.^{-1}(\lambda_1, \ldots, \lambda_{2n})$. The function $\operatorname{Re} f - \sum_i a_i x_i$ on V^0 is thus the pull-back $a_.^* \psi$ of a function ψ on Σ, that is

$$\operatorname{Re} f - \sum_i a_i x_i - a_.^* \psi = 0 \tag{30}$$

on V^0. If $u = (u_1, \ldots, u_{2n}) \in T_{\Sigma,\sigma}$, we get for any $b \in V_\sigma^0$, by differentiating (30) along any $v \in T_{V^0,b}$ such that $a_{.*}(v) = u$ and applying (29)

$$0 = -\sum_i u_i x_i - d\psi(u),$$

which says that the function $\sum_i u_i x_i$ is constant along the fiber V_σ^0. This provides $2n - 2k$ real equations, affine in the x_i's, vanishing along V_σ^0 and independent over $\mathbb{R}$, and they must define V_σ^0 in V^0 since V_σ^0 is a real submanifold of V of codimension $2n - 2k$. $\qquad\square$

The following lemma was partially proved in the proof of Lemma 2.2.

Lemma 5.5. *The fiber V_σ^0 is a complex submanifold of V^0, along which the functions $\sum_i u_i f_i$ are constant, for any $u. \in T_{\Sigma,\sigma}$.*

Proof. The fact that the functions $\sum_i u_i f_i$ are constant along V_σ^0 for any $u. \in T_{\Sigma,\sigma}$ is proved exactly as the previous lemma, replacing (29) by (28). This implies that V_σ^0 is a complex submanifold of V^0 since the real parts of these functions define V_σ^0 as a real submanifold of V^0 by Lemma 5.4. $\qquad\square$

Theorem 0.3 when the fibration is isotrivial works in any dimension. It will be obtained as a consequence of the following Proposition 5.6. Let $\phi : A \to B$ be a locally isotrivial Lagrangian torus fibration, where A, B are quasiprojective and A has a generically nondegenerate closed holomorphic 2-form σ_A extending to a projective completion $\overline{A}$ and generating $H^{2,0}(\overline{A})$. Let $\nu : B \to A$ be an algebraic section such that $\nu^* \sigma_A = 0$.

Proposition 5.6. *If the torsion points of ν are not dense in B^0 for the usual topology, either the section ν is locally constant, or the local system*

$$H_{1,\mathbb{R},A} := (R^1 \phi_* \mathbb{R})^*$$

admits a proper nontrivial local subsystem L which underlies a real subvariation of Hodge structures. Furthermore, writing the real lift (see Section 1) of the section ν as a sum $\tilde{\nu}_{\mathbb{R},L} + \tilde{\nu}_{\mathbb{R},L^\perp}$, where $L^\perp$ is the orthogonal variation of Hodge structures with respect to ω, then $\tilde{\nu}_{\mathbb{R},L}$ is a locally constant section of L.

Proof. The local data described in Section 4.1 take the following form: there are locally $2n$ holomorphic functions $f_1, \ldots, f_{2n}$ on simply connected open subsets $U \subset B$, which are independent over $\mathbb{R}$ and provide real coordinates $x_1, \ldots, x_{2n}$, $x_i = \operatorname{Re} f_i$. By isotriviality, modulo the constants, the f_i's provide only n independent holomorphic functions independent over $\mathbb{C}$. In particular the open set B^0 over which ϕ is smooth and σ_A is nondegenerate is endowed with a flat holomorphic structure.

Lemma 5.7. *This flat holomorphic structure, after passing to a generically finite cover B' of B^0, is induced by the choice of n holomorphic 1-forms on a smooth projective completion $\overline{B'}$ of B'.*

Proof. Indeed, the fibration $\phi : A \to B$ is trivialized over a finite cover B' of a Zariski open set of B^0 as

$$A' := A \times_B B' \cong B' \times J_0,$$

where $0 \in B'$ is a given point, and J_0 is the fiber of $\phi' : A' \to B'$ over it. The local flat holomorphic coordinates on B^0, pulled back to B', come from the holomorphic 2-form on $B' \times J_0$ which is of the form

$$\sum_i pr_{B'}^* \alpha_i \otimes pr_{J_0}^* \beta_i,$$

for a basis β_i of $H^0(\Omega_{J_0})$. The flat structure on B^0 is given by the holomorphic forms α_i. We now claim that the forms α_i extend to holomorphic 1-forms on $\overline{B'}$. This is seen as follows: As the fibration is Lagragian and σ_A generates $H^{2,0}(\overline{A})$, the transcendental cohomology $H^2(\overline{A}, \mathbb{Q})_{tr}$, pulled-back to $B' \times J_0$, falls in the $(1,1)$-Künneth component $H^1(B', \mathbb{Q}) \otimes H^1(J_0, \mathbb{Q})$ of the Künneth decomposition of $H^2(B' \times J_0, \mathbb{Q})$, and this implies that it is contained in the image of $H^1(\overline{B'}, \mathbb{Q}) \otimes H^1(J_0, \mathbb{Q})$ in $H^1(B', \mathbb{Q}) \otimes H^1(J_0, \mathbb{Q})$ by Deligne's strictness theorem for morphisms of mixed Hodge structures [14]: indeed, we get from the above a morphism of mixed Hodge structures

$$\alpha : H^2(\overline{A}, \mathbb{Q})_{tr} \otimes H^1(J_0, \mathbb{Q})^* \to H^1(B', \mathbb{Q})$$

with image $K \subset H^1(B', \mathbb{Q})$ such that $H^2(\overline{A}, \mathbb{Q})_{tr}$, pulled-back to $B' \times J_0$ is contained in $K \otimes H^1(J_0, \mathbb{Q})$. By Deligne strictness theorem [14], this morphism α has its image contained in the pure (smallest weight) part of $H^1(B', \mathbb{Q})$, namely $H^1(\overline{B'}, \mathbb{Q})$. This finishes the proof. $\square$

As a consequence, we get the following:

Corollary 5.8. *Let $(F_t)_{t \in \Sigma}$ be a continuous family of complex submanifolds of an open set V^0 of B^0. Assume that each F_t is flat for the flat holomorphic structure described above, and is algebraic, that is a connected component of the intersection of an algebraic subvariety C_t of B with V^0. Then the F_t's are parallel, that is, F_t being affine in the flat coordinates, the linear part of their defining equations is constant.*

Proof. The linear part of the defining equations for F_t is determined, using Lemma 5.7, by the vector space of holomorphic forms on $\overline{B'}$ vanishing on F_t, or equivalently on C_t. On the other hand, by desingularization and up to shrinking our family, we can assume that the C_t form a family of smooth projective varieties mapping to $\overline{B'}$. It is then clear that the space of holomorphic forms on $\overline{B'}$ vanishing on C_t does not depend on t. $\square$

The proof of Proposition 5.6 will finally use the following Lemma 5.9. We add to the data of our locally isotrivial fibration $\phi : A \to B$ the section ν, which satisfies the condition $\tilde{\nu}^* \sigma_A = 0$. This section lifts locally on B^0 to a section $\tilde{\nu}_M$ of the sheaf $H_{1,\mathbb{R}} \otimes \mathcal{C}_\mathbb{R}^\infty$ where $H_{1,\mathbb{R}}$ is the the real local system $(R^1 \phi_* \mathbb{R})^*$. This section is well-defined up to translation by a constant (integral) section of $H_{1,\mathbb{R}}$, and by Proposition 2.1, if the torsion points of ν_M are not dense in B^0 for the usual topology, then the map (defined on simply connected open sets $V \subset B^0$)

$$\tilde{\nu}_M : V \to H_1(J_0, \mathbb{R})$$

via a trivialization of $H_{1,\mathbb{R}}$ has positive dimensional fibers F_t.

Lemma 5.9. *In the isotrivial case, the fibers F_t are algebraic.*

Proof. Indeed, after passing to the finite cover B' already introduced above, the normal function ν_M gives a morphism of algebraic varieties

$$\nu'_M : B' \to \mathrm{Pic}^0(J_0) \cong_{isog} J_0,$$

which to $b' \in B'$ associates $M_{|X'_{b'}}$ with $X'_{b'} \cong J_0$ canonically. It is imediate that the fibers of $\tilde{\nu}_M$ coincide with the fibers of ν'_M, via the morphism $B' \to B^0$ and after restricting to the adequate open sets. $\square$

We can now apply Corollary 5.8 to the fibers $V_\sigma^0 \subset V^0$ of the local map a. restricted to the open subset $V^0 \subset V$ where a. has constant rank. These fibers are both complex and affine by Lemmas 5.4 and 5.5. We thus conclude that they vary in a parallel way. The tangent bundle to the fibers is thus a local subsystem of $H_{1,\mathbb{R}}$, and as the fibers are complex submanifolds, this local system is a subvariation of Hodge structure. This subvariation of Hodge structures is nontrivial by assumption. Saying that it is equal to $H_{1,\mathbb{R}}$ is equivalent to saying that ν is locally constant. This finishes the proof. $\square$

Proof of Theorem 0.3 in the isotrivial case. Let $\phi : X \to B$ be a Lagrangian isotrivial fibration, X projective hyper-Kähler, and let ν_M be the normal function associated to a line bundle M on X which is topologically trivial on the smooth fibers of ϕ. Proposition 5.6 applies and shows that if the set of points $b \in B$ where $\nu_M(b)$ is a torsion point is not dense in B, then either the normal function is locally constant or there is a proper nontrivial real subvariation of Hodge structure L of $H_{1,\mathbb{R}}$. We now use Lemma 4.5 which says that the second case is excluded and thus $\tilde{\nu}$ is locally constant, so that ν_M is a section of $R^1 \phi_* \mathbb{R} / R^1 \phi_* \mathbb{Z}$. But then Lemma 2.3 says that a nontorsion section $R^1 \phi_* \mathbb{R} / R^1 \phi_* \mathbb{Z}$ can exist only if the invariant part of $H^1(X_b, \mathbb{R})$ is not trivial, which provides a contradiction by Deligne's global invariant cycles theorem since $H^1(X, \mathbb{R}) = 0$. $\square$

6 The case of dimension 10 : proof of Proposition 2.8

We prove in this section Proposition 2.8. We start with the proof of (i). We consider a projective hyper-Kähler 10-fold X admitting a Lagrangian fibration $\pi : X \to B$ with maximal variation. Let $M \in \mathrm{Pic}\, X$ be cohomologous to 0 on the fibers X_b. We assume that the associated normal function $\nu_M : B^0 \to \mathrm{Pic}\, X^0/B^0$ does not have a dense set of torsion points in B^0. We use the induced Lagrangian fibration structure on the dual fibration $\mathrm{Pic}\,(X^0/B^0) \to B^0$ discussed in Subsection 5.1, that we denote by $A^0 \to B^0$. We perform the same analysis as in Section 4. By Theorem 4.4, there are locally on B^0

holomorphic coordinates $z_1, \ldots, z_5$ and a holomorphic function (a potential) g with the following properties: Let $g_i = \frac{\partial g}{\partial z_i}$. Then the holomorphic forms $dz_1, \ldots, dz_5, dg_1, \ldots, dg_5$ are pointwise independent over $\mathbb{R}$ and the real variation of Hodge structure on $H_{1,A}$ is given by the evaluation map

$$\text{ev} : \mathbb{R}^{10} \otimes \mathcal{O}_B \to \Omega_B, \ e_i \mapsto dz_i, \ e_{i+5} \mapsto dg_i, \ i = 1, \ldots, 5.$$

Thus the infinitesimal variation of Hodge structure

$$T_B \to \text{Hom}\left(\mathcal{H}_{1,0}, \mathcal{H}_{0,1}\right)$$

is given, using the natural identifications $T_B \cong \mathcal{H}_{0,1}^* \cong \mathcal{H}_{1,0}$, by the homogeneous cubic form $g^{(3)}$ on T_B determined at each point by the partial derivatives $\frac{\partial^3 g}{\partial z_i \partial z_j \partial z_k}$. The maximal variation assumption says that the cubic form $g_b^{(3)}$ is not a cone at a general point b, while the non-density assumption for the torsion points implies by the André-Corvaja-Zannier theorem 0.6 that it has the property that all of its partial derivatives are degenerate quadrics. For cubic forms in 5 variables, such cubics are classified in [25], who proves that such a cubic form on $\mathbb{P}^4$ has to be singular along a plane (and conversely, if a cubic in $\mathbb{P}^4$ is singular along a plane, then all of its partial derivatives vanish along a plane, hence are degenerate quadrics). We now prove the following result:

Lemma 6.1. *Let $g(z_1, \ldots, z_5)$ be a holomorphic function defined on an open set U of $\mathbb{C}^5$ with the property that at any point $b \in U$, the cubic form $g_b^{(3)}$ is not a cone and is singular along a projective plane $P_b \subset \mathbb{P}(T_{U,b})$. Then there is locally on U a holomorphic map $\phi : U \to \mathbb{C}^2$ with fibers S_b affine in the coordinates z_i, such that the partial derivatives $\frac{\partial g}{\partial z_i}$ are affine on each S_b. In particular, g restricted to the fibers S_b is quadratic.*

Proof. Let us consider the Taylor expansion of g in the coordinates z_i centered at $b \in B$:

$$g = q_b + g_b^{(3)} + g_b^{(4)} + \ldots, \tag{31}$$

where q_b is quadratic nonhomogeneous in the coordinates, and the homogeneous forms $g_b^{(k)}$ in the variables $X_i = z_i - b$ depend polynomially on b. We observe that

$$\frac{\partial g_b^{(3)}}{\partial z_i} = \pm \frac{\partial g_b^{(4)}}{\partial X_i}. \tag{32}$$

By assumption, the cubic form $g_b^{(3)}$ is singular along a projective plane $P_b \subset \mathbb{P}^4$ which is uniquely determined by b hence varies holomorphically with b (indeed, otherwise the cubic form $g_b^{(3)}$ is a cone, as one checks easily). It follows that $\frac{\partial g_b^{(3)}}{\partial z_i}$ vanishes on P_b, which means by (32) that the cubic $\frac{\partial g_b^{(4)}}{\partial X_i}$ vanishes on P_b. Hence the quartic $g_b^{(4)}$ is also singular along P_b. We can then continue and show inductively that all $g_b^{(k)}$ for $k \geq 3$ are singular along P_b. In conclusion, fixing b, we have by (31)

$$g = q_b + g_b', \tag{33}$$

where q_b is quadratic and g_b' vanishes doubly along the affine 3-space $S_b := \mathbb{C}^3 \cap U \subset U$ corresponding to P_b. This is saying that the map $b \mapsto P_b$ is constant along S_b, which provides us with a foliation on U with affine fibers S_b, with the property that for each b, (33) holds along S_b with q_b constant along S_b. Differentiating (33) along S_b gives $\frac{\partial g}{\partial z_i} = \frac{\partial q_b}{\partial z_i}$ along S_b which proves that $\frac{\partial g}{\partial z_i}$ is affine on S_b. $\qquad \square$

Lemma 6.1 tells us that for our data (X, B, d) as above, B^0 (or rather the open set where the cubic is not a cone) is foliated by the locally defined fibers S_b, which clearly are both

complex submanifolds and affine submanifolds of B^0 for the real affine structure. Indeed, the partial derivatives $g_i := \frac{\partial g}{\partial z_i}$ are affine along S_b, so that each S_b maps to an affine $\mathbb{C}^3 \subset \mathbb{C}^{10}$ via the functions $z_1, \ldots, z_5, g_1, \ldots, g_5$, hence to a real affine subspace $\mathbb{R}^6 \subset \mathbb{R}^{10}$ via the real coordinates $\operatorname{Re} z_1, \ldots, \operatorname{Re} z_5, g_1, \ldots, \operatorname{Re} g_5$.

The proof of Proposition 2.8 (i) concludes with the following :

Corollary 6.2. *Along the leaves $\mathcal{F}_b$ of the foliation, the real variation of Hodge structure is a direct sum $L_{b,\mathbb{R}} \oplus L'_{b,\mathbb{R}}$, where the real Hodge structure is locally constant on the first summand.*

Proof. By Lemma 6.1, the leaf $\mathcal{F}_b$, which identifies locally to S_b, locally maps, via the real affine structure given by $\operatorname{Re} z_i$, $\operatorname{Re} g_i$, to an affine $\mathbb{R}^6 \subset \mathbb{R}^{10}$ which will define $L'_{b,\mathbb{R}} \subset H_{1,\mathbb{R}|\mathcal{F}_b}$ as the four dimensional subspace of real affine forms vanishing on S_b. The decomposition is then obtained using the polarization, $L_{b,\mathbb{R}}$ being defined as the orthogonal complement of $L'_{b,\mathbb{R}}$. We have to prove that $L'_{b,\mathbb{R}}$ and $L_{b,\mathbb{R}}$ are real subvariations of Hodge structure of $H_{1,\mathbb{R}|\mathcal{F}_b}$, and that the variation of Hodge structre on $L_{b,\mathbb{R}}$ is constant. The first point follows from the fact that $S_b \subset U$ is also a complex submanifold of U so that the evaluation map $L'_{b,\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{O}_B \to \Omega_U$ maps to complex linear forms vanishing on S_b, which is a rank 2 vector subbundle of $\Omega_{U|S_b}$. The fact that the real variation of Hodge structure on $L_{b,\mathbb{R}}$ is constant is due to the fact that it identifies to the evaluation map $L_{b,\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{O}_{S_b} \to \Omega_{S_b}$, $e_i \mapsto df_i$, associated with a 6-dimensional real vector space of functions f_i on S_b, where the f_i's form only a 3-dimensional space of holomorphic functions on S_b since S_b maps via the f_i's to an open subset of an affine $\mathbb{C}^3 \subset \mathbb{C}^{10}$). $\square$

Proof of Proposition 2.8, (ii). We only have to prove that the normal function ν_M (or rather its lift $\tilde{\nu}_{M,0,1} \in \mathcal{H}_{0,1}$) decomposes locally along the leaves, according to the decomposition of Corollary 6.2, as $\nu^v_{M,0,1} + \nu^h_{M,0,1}$, where $\nu^v_{M,0,1}$ is constant. We use the local description introduced in section 5.2: the lift $\tilde{\nu}_{M,0,1} \in \mathcal{H}_{0,1} \cong \Omega_B$ provides a holomorphic 1-form which is closed hence locally exact df. The variation of Hodge structure is given in local coordinates $z_1, \ldots, z_5$ by the evaluation map

$$\mathbb{C}^{10} \otimes \mathcal{O}_B \to \Omega_B,$$

$$e_i \mapsto dz_i, \; e_{5+i} \mapsto dg_i, \; i = 1, \ldots, 5,$$

where $g_i = \frac{\partial g}{\partial z_i}$ and g is as above. Writing $df = \sum_i \frac{\partial f}{\partial z_i} dz_i$, the affine bundle $\mathcal{H}_{0,1,\nu} \subset \mathbb{C}^{10} \otimes \mathcal{O}_B$ is the set of elements $\sum_{i=1}^{5} \frac{\partial f}{\partial z_i} e_i + \sum_{i=1}^{5} \lambda_i (e_{i+5} - \sum_{j=1}^{5} g_{ij} e_j)$. We apply now Theorem 3.4 which tells us that the map $B \times \mathbb{C}^5 \to \mathbb{C}^{10}$

$$(b, \lambda_i) \mapsto \sum_{i=1}^{5} \frac{\partial f}{\partial z_i} e_i + \sum_{i=1}^{5} \lambda_i \left(e_{i+5} - \sum_{j=1}^{5} g_{ij} e_j\right) \tag{34}$$

is nowhere of maximal rank. Computing its differential at a point $(b, \lambda.)$, we conclude that the quadric

$$f_b^{(2)} - \partial_{\lambda.}(g_b^{(3)})$$

is degenerate on $T_{B,b}$ for all $\lambda.$. As the cubic form $g_b^{(3)}$ at the general point b is not a cone but is singular along the plane $P_b \subset \mathbb{P}(T_{B,b})$ defined by two homogeneous equations X_1, X_2, the cubic form has for equation $X_1^2 A + X_2^2 B + X_1 X_2 C$ where the linear forms A, B, C are independent on P_b, hence the general quadric $\partial_{\lambda.}(g_b^{(3)})$ has rank 4 and its vertex is a general point of P_b. The condition that $f_b^{(2)} - \partial_{\lambda.}(g_b^{(3)})$ is degenerate for general $\lambda.$ then implies that the Hessian $f_b^{(2)}$ of f at b vanishes on the vertex of the quadric defined by $\partial_{\lambda.}(g_b^{(3)})$, hence $f^{(2)}$ has to vanish on P_b. Thus the restricted function $f_{|S_b}$ has trivial Hessian, that is, f is affine on S_b. Thus $d(f_{|S_b})$ is a constant 1-form for the affine structure given by the z_i's, which concludes the proof. $\square$

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