Schiffer variations and the generic Torelli theorem for hypersurfaces

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Abstract

We show how to recover a general hypersurface in $\mathbb{P}^n$ of sufficiently large degree d dividing $n+1$, from its finite order variation of Hodge structure. We also analyze the two other series of cases not covered by Donagi's generic Torelli theorem. Combined with Donagi's theorem, this shows that the generic Torelli theorem for hypersurfaces holds with finitely many exceptions.

0 Introduction

We will consider in this paper smooth hypersurfaces $X_f \subset \mathbb{P}^n$ of degree d defined by a homogeneous polynomial equation f. The generic Torelli theorem for such varieties is the following statement:

Let X_f be a very general smooth hypersurface of degree d in $\mathbb{P}^n$. Then any smooth hypersurface $X_{f'}$ of degree d in $\mathbb{P}^n$ such that there exists an isomorphism of Hodge structures

$$H^{n-1}(X_f, \mathbb{Q})_{prim} \cong H^{n-1}(X_{f'}, \mathbb{Q})_{prim} \tag{1}$$

is isomorphic to X_f.

Remark 0.1. The Torelli theorem is usually stated for the polarized period map. However, by Mumford-Tate group considerations, we can see that, except in the case of cubic surfaces, the Hodge structure on the primitive cohomology of X_f has a unique polarization (up to a scalar) for very general X_f. So the polarized and nonpolarized statements are equivalent.

Remark 0.2. The Torelli theorem is usually stated for integral Hodge structures. The fact that we can work with rational Hodge structures is related to the nature of the proof which relies on the study of the (complex!) variation of Hodge structures for hypersurfaces of given degree and dimension and its local invariants.

The reason for the "very general" assumption in this statement is the fact that, by Cattani-Deligne-Kaplan [3], the set of pairs (X, X') such that an isomorphism as in (1) exists, is a countable union of closed algebraic subsets in $U_{d,n} \times U_{d,n}$, where $U_{d,n}$ is the moduli space of smooth hypersurfaces of degree d in $\mathbb{P}^n$. Th generic Torelli theorem thus says that, among these closed algebraic subsets, only the diagonal dominates $U_{d,n}$ by the first projection.

Donagi proved in [4] the following beautiful result.

Theorem 0.3. *The generic Torelli theorem holds for smooth hypersurfaces of degree d in $\mathbb{P}^n$, with $(d,n) \neq (3,3)$ or $(4,2)$ and the following possible exceptions:*

1. *d divides $n+1$,*
2. *$d = 4$, $n = 4m+1$, with $m \geq 1$,*
3. *$d = 6$, $n = 6m+2$, with $m \geq 1$.*

The starting point of Donagi's proof is the description due to Griffiths and Carlson-Griffiths of the infinitesimal variation of Hodge structure of a smooth hypersurface. Denote by $S^*=\mathbb{C}[X_0,\ldots,X_n]$ the graded polynomial ring of $\mathbb{P}^n$ and by $R^*_f=S^*/J^*_f$ the Jacobian ring of f, where

$$J^*_f=S^{*-d+1}\langle\frac{\partial f}{\partial X_i}\rangle\subset S^* \tag{2}$$

is the Jacobian ideal of f, generated by the partial derivatives of f. The infinitesimal variation of Hodge structure on the primitive cohomology of degree $n-1$ of X_f is given, according to Griffiths [6], see also [10, 6.1.3], by linear maps

$$R^d_f\to \mathrm{Hom}\,(H^{p,q}(X_f)_{prim},H^{p-1,q+1}(X_f)_{prim}) \tag{3}$$

for $p+q=n-1$. Here, the space R^d_f is naturally identified with the first order deformations of X_f in $\mathbb{P}^n$ modulo the action of $\mathrm{PGl}(n+1)$. It also identifies via the Kodaira-Spencer map to the subspace $H^1(X_f,T_{X_f})_0\subset H^1(X_f,T_{X_f})$ of deformations of X_f induced by a deformation of f. Griffiths constructs residue isomorphisms

$$\mathrm{Res}_{X_f}:R_f^{(q+1)d-n-1}\stackrel{\cong}{\to}H^{n-q-1,q}(X_f)_{prim} \tag{4}$$

and the paper [2] in turn describes (3) using the isomorphisms (4) as follows:

Theorem 0.4. *Via the isomorphisms (4), the maps (3) identify up to a coefficient to the map*

$$R^d_f\to\mathrm{Hom}(R_f^{(q+1)d-n-1},R_f^{(q+2)d-n-1}). \tag{5}$$

induced by multiplication in R^*_f. In other words, the following diagram is commutative up to a coefficient

$$\begin{array}{ccc} R^d_f & \longrightarrow & \mathrm{Hom}\,(R_f^{(q+1)d-n-1},R_f^{(q+2)d-n-1}) \\ \downarrow\cong & & \downarrow\cong \\ H^1(X_f,T_{X_f})_0 & \longrightarrow & \mathrm{Hom}\,(H^{n-1-q,q}(X_f)_{prim},H^{n-2-q,q+1}(X_f)_{prim}) \end{array}\quad . \tag{6}$$

Donagi's proof starts with the observation that Theorem 0.3 is implied by the following result:

Theorem 0.5. *Let X be a smooth hypersurface of degree d in $\mathbb{P}^n$. Assume $(d,n)\neq(3,3)$, $(4,2)$ and we are not in the cases 1, 2, 3 listed in Theorem 0.3. Then X is determined by its polarized infinitesimal variation of Hodge structures (5).*

Concretely, this says that if X_f and $X_{f'}$ are two smooth hypersurfaces of degree d and dimension $n-1$ such that there exist isomorphisms

$$R^d_f\cong R^d_{f'},\ R_f^{kd-n-1}\cong R_{f'}^{kd-n-1},\ \forall k,$$

compatible with the Macaulay pairing between $R_f^{(q+1)d-n-1}$ and $R_f^{(n-q)d-n-1}$ (corresponding to the Serre pairing between $H^{n-1-q,q}(X_f)_{prim}$ and $H^{q,n-1-q}(X_f)_{prim}$, see [10, 6.2.2]), and such that the following diagram commutes:

$$\begin{array}{ccc} R^d_f & \longrightarrow & \bigoplus_k\mathrm{Hom}\,(R_f^{kd-n-1},R_f^{(k+1)d-n-1}) \\ \downarrow & & \downarrow \\ R^d_{f'} & \longrightarrow & \bigoplus_k\mathrm{Hom}\,(R_{f'}^{kd-n-1},R_{f'}^{(k+1)d-n-1}) \end{array}\ , \tag{7}$$

then X_f is isomorphic to $X_{f'}$.

The reason why Theorem 0.5 implies Theorem 0.3 is the fact that, as an easy consequence of Macaulay theorem [10, Theorem 6.19], the considered hypersurfaces satisfy the infinitesimal Torelli theorem. It follows that, if X is very general and X' has an isomorphic polarized Hodge structure on H^{n-1}_{prim}, X' is also very general and there is an isomorphism of variations of Hodge structures on respective neighborhoods of $[X]$ and $[X']$ in their moduli space $U_{d,n}$. Taking the differential of this isomorphism provides a commutative diagram where the vertical maps are isomorphisms

$$\begin{array}{ccc} R^d_f & \longrightarrow & \bigoplus_{p+q=n-1} \mathrm{Hom}\,(H^{p,q}(X_f)_{prim}, H^{p-1,q+1}(X_f)_{prim}) \\ \downarrow & & \downarrow \\ R^d_{f'} & \longrightarrow & \bigoplus_{p+q=n-1} \mathrm{Hom}\,(H^{p,q}(X_{f'})_{prim}, H^{p-1,q+1}(X_{f'})_{prim}) \end{array} \quad . \tag{8}$$

By theorem 0.4, we then get a commutative diagram (7) to which Theorem 0.5 applies.

Donagi's method does not work in the case where $(d,n) = (4,3)$, that is, quartic $K3$ surfaces because Theorem 0.5 is clearly wrong in this case, while Theorem 0.3 is true by Piateski-Shapiro-Shafarevich [8]. In the case of cubic surfaces, the generic Torelli theorem is clearly wrong, since they have moduli, but their variation of Hodge structure is trivial. The case of plane quartics is also a counterexample to the generic Torelli theorem with rational coefficients, since in genus 3, a general curve is not determined by the isogeny class of its Jacobian.

Donagi's proof of Theorem 0.5 consists in recovering from the data of the IVHS (3) its *polynomial structure* (see Section 1) given by Theorem 0.4, and more precisely, reconstructing the whole Jacobian ring of f from its partial data appearing in (5). His method, based on the use of the symmetrizer lemma (Proposition 1.1), gives nothing more, when d divides $n+1$, than the subring $R^{*d}_f \subset R^*_f$ defined as the sum of the graded pieces of R^*_f of degree divisible by d. This is why Donagi's method fails to give the result in that case. The goal of this paper is to extend Theorem 0.3 to most families of hypersurfaces not covered by Donagi's theorem.

Theorem 0.6. *(1) The generic Torelli theorem holds for smooth hypersurfaces of degree d in $\mathbb{P}^n$ such that d divides $n+1$, and d large enough. In particular, it holds for Calabi-Yau hypersurfaces of degree d large enough.*

(2) The generic Torelli theorem holds for smooth hypersurfaces of degree 4 in $\mathbb{P}^{4m+1}$, and for smooth hypersurfaces of degree 6 in $\mathbb{P}^{6m+2}$ for m sufficiently large.

These results combined with Donagi's theorem 0.3 imply the following result:

Corollary 0.7. *The generic Torelli theorem holds for hypersurfaces of degree d in $\mathbb{P}^n$ with finitely many exceptions.*

The proof of Theorem 0.6 (2) will be given in Section 2. We will give there an effective estimate for m, which can probably be improved by refining the method. In that case, the method of proof follows closely Donagi's ideas, and in particular passes through a proof of Theorem 0.5, at least for X generic.

The case (1) of the theorem had been also proved in [11] in the case of quintic threefolds, the first case which is not covered by Theorem 0.3, by extending Theorem 0.5 to that case. It is quite possible that the same strategy works similarly in all cases not covered by Theorem 0.3 and of sufficiently large degree, but the proof given in [11] is very technical and specific, hence is not encouraging. The proof given in the present paper also rests on the algebraic analysis of the finite order variation of Hodge structure, but it does not pass through a proof of Theorem 0.5. Instead, it introduces a main new ingredient, which is the notion of *Schiffer variation of a hypersurface* (see Section 3). These Schiffer variations are of the form

$$f_t = f + tx^d \tag{9}$$

and we believe they are interesting for their own. The terminology comes from the notion of Schiffer variations for a smooth curve C. They consist in deforming the complex structure of C in a way that is supported on a point p of C. First order Schiffer variations are the elements $u_p \in \mathbb{P}(H^1(C, T_C))$ given by

$$[H^0(C, 2K_C(-p))] \in \mathbb{P}(H^0(C, 2K_C)^*) = \mathbb{P}(H^1(C, T_C)).$$

First order Schiffer variations (9) of hypersurfaces X_f are supported on a linear section. They are parameterized by the d-th Veronese embedding of $\mathbb{P}^n$ projected to $\mathbb{P}(R_f^d)$ via the linear projection $\mathbb{P}(S^d) \dashrightarrow \mathbb{P}(R_f^d)$, so recovering them intrinsically will allow to reconstruct the polynomial structure of the Jacobian ring, or rather its degree divisible by d part, which, according to Theorem 0.4, is given by the infinitesimal variation of Hodge structure of X_f. Our strategy consists in characterizing Schiffer variations by the formal properties of the variation of Hodge structure of the considered family of hypersurfaces along them. An obvious but key point (see Lemma 3.8) is the fact that the structure of the Jacobian ring does not change much along them. Compared to Donagi's method which involves the first order properties of the period map, this is a higher order argument. It would be nice to have a better understanding and a more Hodge-theoretic, less formal, characterization of Schiffer variations.

The paper is organized as follows. In Section 1, we discuss the notion of polynomial structure on the data of an infinitesimal variation of Hodge structure of an hypersurface. We discuss alternative recipes toward proving uniqueness of the polynomial structure. For example, we exhibit a very simple recipe to show that the natural polynomial structure for most hypersurfaces of degree d dividing $n+1$ is rigid. In Section 2, we prove the case (2), that is degrees 4 and 6, of Theorem 0.6. This proof follows Donagi's argument but provides a different recipe to prove the uniqueness of the polynomial structure of the infinitesimal variation of Hodge structures in these cases.

The main new ideas and results of the paper appear starting from Section 3 where we introduce Schiffer variations of hypersurfaces and discuss their formal properties. The proof of Case (1) of Theorem 0.6 is given in Section 4.2, where we give a characterization of Schiffer variations based on the local analysis of the infinitesimal variation of Hodge structure of the considered hypersurfaces.

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1 Polynomial structure

The method used by Donagi to prove Theorem 0.5 consists in applying the "symmetrizer lemma" (Proposition 1.1 below), in order to recover from the data (5) the whole Jacobian ring in degrees divisible by l, where l is the g.c.d. of $n+1$ and d. This result proved first in [4] for the Jacobian ring of generic hypersurfaces, and reproved in [5] for any smooth hypersurface (and more generally quotients $R_{f_\bullet}$ of the polynomial ring $S = \mathbb{C}[X_0, \ldots, X_n]$ by a regular sequence $f_\bullet = (f_0, \ldots, f_n)$ with $\deg f_i = d-1$), is the following statement. Consider the multiplication map

$$R_{f_\bullet}^k \otimes R_{f_\bullet}^{k'} \to R_{f_\bullet}^{k+k'}, \qquad a \otimes b \mapsto ab. \tag{10}$$

Proposition 1.1. *Let* $N = (n+1)(d-2)$*. Then, if* $\mathrm{Max}\,(k, N-k') \geq d-1$ *and* $N-k-k' > 0$*, the multiplication map*

$$R_{f_\bullet}^{k'-k} \otimes R_{f_\bullet}^k \to R_{f_\bullet}^{k'}$$

is determined by the multiplication map (10) as follows

$$R_{f_\bullet}^{k'-k} = \{h \in \mathrm{Hom}\,(R_{f_\bullet}^{k}, R_{f_\bullet}^{k'}),\, bh(a) = ah(b) \text{ in } R_{f_\bullet}^{k+k'},\, \forall a,\, b \in R_{f_\bullet}^{k}\}. \tag{11}$$

Coming back to the case of a Jacobian ring R_f, when d divides $n+1$, the infinitesimal variation of Hodge structure (3) of X_f, translated in the form (5), involves only pieces R_f^k of the Jacobian ring of degree k divisible by d. Hence the symmetrizer lemma allows at best, starting from the IVHS of the hypersurface, to reconstruct the Jacobian ring in degrees divisible by d. At the opposite, when d and $n+1$ are coprime, repeated applications of the symmetrizer lemma allow to reconstruct the whole Jacobian ring. In degree $< d-1$, the Jacobian ring coincides with the polynomial ring, hence we directly recover in that case the multiplication map

$$\mathrm{Sym}^d(S^1) \to R_f^d$$

and its kernel J_f^d. The proof of Donagi is then finished by applying Mather-Yau's theorem [7] (see also Proposition 3.2).

This leads us to the following definition. Suppose that we have two integers d, n and the partial data of a graded ring structure R^*, namely finite dimensional vector spaces R^d, $R^{-(n+1)+id}$, $-(n+1)+id \geq 0$ with multiplication maps

$$\mu_i : R^d \otimes R^{-(n+1)+id} \to R^{-(n+1)+(i+1)d}. \tag{12}$$

When d divides $n+1$, we get all the upper-indices divisible by d, and an actual ring structure R^{d*}, but in general (12) is the sort of data provided by the infinitesimal variation of Hodge structure of a hypersurface of degree d in $\mathbb{P}^n$. Let S^k be the degree k part of the polynomial ring in $n+1$ variables.

Definition 1.2. *A polynomial structure in $n+1$ variables for*

$$(R^d,\, R^{-(n+1)+id}, \mu_i)$$

is the data of a rank $n+1$ base-point free linear subspace $J \subset S^{d-1}$ generating a graded ideal $J^* \subset S^*$, of a linear isomorphism $S^d/J^d \cong R^d$ and, for all i, of linear isomorphisms

$$S^{-(n+1)+id}/J^{-(n+1)+id} \cong R^{-(n+1)+id},$$

compatible with the multiplication maps, that is, making the following diagrams commutative:

$$\begin{array}{ccc} S^d \otimes S^{-(n+1)+id} & \longrightarrow & S^{-(n+1)+(i+1)d} \\ \downarrow \cong & & \downarrow \cong \\ R^d \otimes R^{-(n+1)+id} & \xrightarrow{\mu_i} & R^{-(n+1)+(i+1)d} \end{array} \tag{13}$$

The group $Gl(n+1)$ acts in the obvious way on the set of polynomial structures. We will say that the polynomial structure of $(R^d,\, R^{-(n+1)+id},\, \mu_i)$ is unique if all its polynomial structures are conjugate under $Gl(n+1)$. As explained above, Donagi's Theorem 0.5 has the more precise form that, under some assumptions on (d,n), the polynomial structure of the infinitesimal variation of Hodge structures $(R_f^d,\, R_f^{-(n+1)+id},\, \mu_i)$ of a smooth hypersurface X_f is unique. This statement applies as well to the polynomial structure of $(R_{f_\bullet}^d,\, R_{f_\bullet}^{-(n+1)+id},\, \mu_i)$ for any regular sequence $f_\bullet$ of polynomials of degree $d-1$. We will prove a similar statement in the case (2) (degrees $d=4$ and $d=6$) of Theorem 0.6, at least for generic f.

For the main series of cases not covered by Donagi's theorem, namely when d divides $n+1$, we have not been able to prove the uniqueness of the polynomial structure of R_f^{d*} (even for generic f), although it is likely to be true (and it is proved in [11] for $d=5$, $n=4$). We conclude this section by the proof of a weaker statement that provides evidence for the uniqueness. We will say that a polynomial structure is rigid if its small deformations are given by its orbit under $Gl(n+1)$. We have the following

Proposition 1.3. *Assume $n+1 \geq 8$ and $d \geq 6$. Let $f \in S^d$ be a generic homogeneous polynomial of degree d in $n+1$ variables and R_f^{d*} be its Jacobian ring in degrees divisible by d. Then the natural polynomial structure*

$$S^{d*} \to R_f^{d*}$$

given by the quotient map is rigid.

Remark 1.4. The case where $d = 4$, resp. $d = 6$, will be studied in next section. We will prove there, using a different recipe, that the polynomial structure on R^{2*}, resp.R^{3*}, is unique for n large enough.

Remark 1.5. Proposition 1.3 implies that the natural polynomial structure of $R_{f_\bullet}^{d*}$ for a generic rank $n+1$ regular sequence $f_\bullet$ of degree $d-1$ homogeneous polynomials is rigid.

We will use in fact only the multiplication map in degree d

$$\mu : R_f^d \times R_f^d \to R_f^{2d}.$$

Proposition 1.3 will be implied by Proposition 1.8 below. For our original polynomial structure on R_f^{d*}, and for each $x \in S^1$, we get a pair of vector subspaces

$$I_x^d := xR_f^{d-1} \subset R_f^d,\ I_x^{2d} := xR_f^{2d-1} \subset R_f^{2d}, \tag{14}$$

which form an ideal in the sense that

$$R_f^d I_x^d \subset I_x^{2d}. \tag{15}$$

It is not hard to see that the multiplication map by x, from R_f^{d-1} to R_f^d, is injective for a generic $x \in S^1$ when f is generic with $d \geq 4$ and $n \geq 3$ (or $d \geq 3$ and $n \geq 5$). In fact, we even have (statement (ii) will be used only later on)

Lemma 1.6. *(i) The multiplication map by x is injective on R_f^{2d-1} when f is generic, $x \in S^1$ is generic and*

$$2(2d-1) < (d-2)(n+1) \tag{16}$$

(for example, $n+1 \geq 5$ and $d > 8$, or $n+1 \geq 6$ and $d > 4$ work).

(ii) The multiplication map by x is injective on R_f^{3d-1} when f is generic, $x \in S^1$ is generic and

$$2(3d-1) < (d-2)(n+1)$$

(for example, $n+1 \geq 5$ and $d > 8$, or $n+1 \geq 6$ and $d > 4$ work).

(iii) The multiplication map by x^l is injective on R_f^k when f is generic, $x \in S^1$ is generic and

$$2k + l \leq (d-2)(n+1).$$

Proof. Take for f the Fermat polynomial $f_{Fermat} = \sum_{i=0}^n X_i^d$. Then $R_{f_{Fermat}}^*$ identifies with the polynomial ring $H^{2*}((\mathbb{P}^{d-2})^{n+1}, \mathbb{C})$ and $x = \sum_i x_i$ corresponds to an ample class in $H^2((\mathbb{P}^{d-2})^{n+1}, \mathbb{C})$. By hard Leschetz theorem, the multiplication by x is thus injective on $R_{f_{Fermat}}^{2d-1}$ if $2(2d-1) < (d-2)(n+1)$, and injective on $R_{f_{Fermat}}^{3d-1}$ if $2(3d-1) < (d-2)(n+1)$. More generally, the Lefschetz isomorphism for the power x^l gives the injectivity of x^l on R_f^k when $2k + l \leq (d-2)(n+1)$. □

Remark 1.7. This estimate is optimal for dimension reasons. Indeed, the dimensions of the graded pieces R_f^k are increasing in the interval $k \leq \frac{(d-2)(n+1)}{2}$, and decreasing in the interval $\frac{(d-2)(n+1)}{2} \leq k \leq (d-2)(n+1)$.

It follows from Lemma 1.6 that, assuming inequality (16), the space I_x^d defined in (14) has dimension $r_{d-1} := \dim R_f^{d-1}$, while I_x^{2d} has generic dimension $r_{2d-1} := \dim R_f^{2d-1}$.

Proposition 1.8. *If f is generic of degree d in $n+1$ variables, and $d \geq 6,\ n \geq 9$, then the set of ideals $Z_{ideal} = \{[I_x^d] \in G(r_{d-1}, R_f^d),\ x \in S^1\}$ is a (reduced) component of the closed algebraic subset $Z \subset G(r_{d-1}, R_f^d)$ defined as*

$$Z = \{[W] \in G(r_{d-1}, R_f^d),\ \dim R_f^d \cdot W \leq r_{2d-1}\}. \tag{17}$$

Proof. The tangent space to Z_{ideal} at the point $[I_x^d] \in G(r_{d-1}, R_f^d)$ is the image of $S^1/\langle x\rangle$ in $\mathrm{Hom}\,(R_f^{d-1}, R_f^d/xR_f^{d-1}) = T_{G(r_{d-1}, R_f^d), [I_x^d]}$ given by multiplication by $y \in S^1/\langle x\rangle$, where we identify I_x^d with R_f^{d-1} via multiplication by x. Let us now compute the Zariski tangent space to Z at $[I_x^d]$ for f and x generic. As $\dim I_x^{2d} = r_{2d-1}$ is maximal by the claim above, the condition (17) provides the following infinitesimal conditions:

$$T_{Z,[I_x^d]} = \{h \in \mathrm{Hom}\,(R_f^{d-1}, R_f^d/xR_f^{d-1}),\ \sum_i A_i h(B_i) = 0 \text{ in } R_f^{2d}/xR_f^{d-1}, \tag{18}$$

$$\text{for any } K = \sum_i A_i \otimes B_i \in R_f^d \otimes R_f^{d-1} \text{ such that } \sum_i A_i B_i = 0 \text{ in } R_f^{2d-1}\}.$$

Equation (18) says that $h : R_f^{d-1} \to R_f^d/xR_f^{d-1}$ is a "morphism of R_f^d-modules", the set of which we will denote by $\mathrm{Mor}_{R_f^d}(R_f^{d-1}, R_f^d/xR_f^{d-1})$, in the sense that we have a commutative diagram for some $h' \in \mathrm{Hom}\,(R_f^{2d-1}, R_f^{2d}/\langle x\rangle)$

$$\begin{array}{ccc} R_f^d \otimes R_f^{d-1} & \longrightarrow & R_f^{2d-1} \\ \downarrow{\scriptstyle Id\otimes h} & & \downarrow{\scriptstyle h'} \\ R_f^d \otimes R_f^d/\langle x\rangle & \longrightarrow & R_f^{2d}/\langle x\rangle \end{array}, \tag{19}$$

where the horizontal maps are given by multiplication. The equality $T_{Z_{ideal}} = T_Z$ at the point $[I_x^d]$ is thus equivalent to the fact that all the "R_f^d-modules morphisms" $h : R_f^{d-1} \to R_f^d/xR_f^{d-1}$, are given by multiplication by some $y \in S^1$, followed by reduction mod x. This is the statement of the following

Lemma 1.9. *Let f be a generic homogeneous degree d polynomial in $n+1$ variables with $d \geq 5$, $n \geq 9$ (or $d \geq 6$ and $n \geq 7$), and let $x \in S^1$ be generic. Then the natural map $S^1/\langle x\rangle \to \mathrm{Mor}_{R_f^d}(R_f^{d-1}, R_f^d/\langle x\rangle)$ is surjective.*

Proof. The existence of h' as in (19) says that for any tensor $\sum_i A_i \otimes B_i \in R_f^d \otimes R_f^{d-1}$ such that $\sum_i A_i B_i = 0$ in R_f^{2d-1}, $\sum_i A_i h(B_i) = 0$ in $R_f^{2d}/\langle x\rangle$.

Claim 1.10. *Under the same assumptions as in Lemma 1.9, for a generic $q \in R_f^{d-1}$, the multiplication map*

$$q : R_f^{d+1}/\langle x\rangle \to R_f^{2d}/\langle x\rangle$$

is injective.

Proof. This is proved again by looking at the Fermat polynomial $f_{Fermat} = \sum_i X_i^d$ and choosing carefully x so that multiplication by x is injective on $R_{f_{Fermat}}^{2d-1}$, and multiplication by q is injective on $R_{f_{Fermat}}^{d+1}/\langle x\rangle$. We write $f_{Fermat} = f'_{Fermat} + f''_{Fermat}$, where $f'_{Fermat} = \sum_{i=0}^4 X_i^d$ and $f''_{Fermat} = \sum_{i=5}^n X_i^d$. We take $x = \sum_{i=0}^4 X_i$ and $q = (\sum_{i=5}^N X_i)^{d-1}$. We observe that

$$R_{f_{Fermat}}^* \cong R_{f'_{Fermat}}^* \otimes R_{f''_{Fermat}}^*,$$

as graded rings, and that x acts by multiplication on the left term $R^*_{f'_{Fermat}}$, while q acts by multiplication on the right term $R^*_{f''_{Fermat}}$. So it suffices to show that multiplication by x is injective on $R^k_{f'_{Fermat}}$ for $k \leq 2d-1$ and multiplication by q is injective on $R^k_{f''_{Fermat}}$ for $k \leq d+1$. The first statement follows from Lemma 1.6 (i) when $2(d+1) < 5(d-2)$, hence when $d \geq 5$. The second statement holds by Lemma 1.6 (iii) when $2(d+1)+d-1 \leq (n-4)(d-2)$, and in particular if $d \geq 6$ and $n \geq 9$. □

We deduce from Claim 1.10 that for any tensor $\sum_i A_i \otimes B_i \in S^1 \otimes R_f^{d-1}$ such that $\sum_i A_i B_i = 0$ in R_f^d, we have

$$\sum_i A_i h(B_i) = 0 \text{ in } R_f^{d+1}/\langle x\rangle, \tag{20}$$

since this becomes true after multiplication by q. It follows now that h vanishes on $\langle x\rangle$. Indeed, let $b = xb'$. Then for any $y \in S^1$, we have $yb = xb''$ with $b'' = yb'$. Hence by (20), we get $yh(b) = xh(b'') = 0$. Hence $yh(b) = 0$ in $R_f^{d+1}/\langle x\rangle$ for any $y \in S^1$, and it follows, by choosing y such that multiplication by y is injective on $R_f^d/\langle x\rangle$, that $h(b) = 0$ in $R_f^d/\langle x\rangle$. Thus h induces a morphism

$$\overline{h} : R_f^{d-1}/\langle x\rangle \to R_f^d/\langle x\rangle,$$

which also satisfies (20). Assuming $d \geq 6$, $n \geq 9$, we now show by similar arguments as above that for generic z, $y \in S^1/\langle x\rangle$, the following holds. For any p, $q \in R_f^d/\langle x\rangle$,

$$yp + zq = 0 \text{ in } R_f^{d+1}/\langle x\rangle \Rightarrow p = zr,\ q = -yr, \tag{21}$$

for some $r \in R_f^{d-1}/\langle x\rangle$. Furthermore we already know that the multiplication map by z from $R_f^d/\langle x\rangle$ to $R_f^{d+1}/\langle x\rangle$ is injective. It follows that there exists

$$\overline{h}'' : R_f^{d-2}/\langle x\rangle \to R_f^{d-1}/\langle x\rangle$$

inducing $\overline{h}$, that is,

$$\overline{h}(ap) = a\overline{h}''(p) \tag{22}$$

for any $p \in R_f^{d-2}/\langle x\rangle$, and any $a \in S^1/\langle x\rangle$. Indeed, y and z being as above, we have for any $p \in R_f^{d-2}/\langle x\rangle$

$$y(zp) - z(yp) = 0 \text{ in } R_f^d,$$

hence by (20), we get that $y\overline{h}(zp) - z\overline{h}(yp) = 0$ in $R_f^{d+1}/\langle x\rangle$, and by (21), this gives $\overline{h}(zp) = z\overline{h}''(p)$, which defines $\overline{h}''$. One then shows that the map $\overline{h}''$ so defined does not depend on z and satisfies (22), which is easy. To finish the proof, we construct similarly $\overline{h}''' : R_f^{d-3}/\langle x\rangle \to R_f^{d-2}/\langle x\rangle$ inducing $\overline{h}''$ and $\overline{h}^{iv} : R_f^{d-4}/\langle x\rangle \to R_f^{d-3}/\langle x\rangle$ inducing $\overline{h}'''$. As $R_f^i/\langle x\rangle = S^i/\langle x\rangle$ for $i \leq d-2$, it is immediate to show that $\overline{h}^{iv}$ is multiplication by some element of S^1, hence also $\overline{h}$. □

The proof of Proposition 1.8 is thus complete. □

Proof of Proposition 1.3. Let f be generic of degree $d \geq 6$ in $n+1 \geq 10$ variables. We first claim that for *any* $x \in S^1$, the multiplication map by $x : R_f^{d-1} \to R_f^d$ is injective, and that the morphism

$$\Phi : \mathbb{P}(S^1) \to G(r_{d-1}, R_f^d),\ x \mapsto xR_f^{d-1} \tag{23}$$

so constructed is an embedding. None of these statements is difficult to prove. The first statement says that if f is a generic homogeneous degree d polynomials in $n+1$ variables, f

does not satisfy an equation $\partial_u(f)_{|H} = 0$ for some hyperplane $H \subset \mathbb{P}^n$ and vector field u on $\mathbb{P}^n$. The obvious dimension count shows that this holds if $h^0(\mathbb{P}^{n-1}, \mathcal{O}(d)) > n - 1 + \frac{(n+1)^2}{2}$, which holds if $d \geq 4$, $n \geq 3$. As for the second statement, suppose that $xR_f^{d-1} = yR_f^{d-1}$ for some non-proportional x, $y \in S^1$. Then there is a subspace of dimension $\geq \dim S^{d-1}$ of pairs $(p, q) \in S^{d-1} \times S^{d-1}$ such that $xp = yq$ in R_f^d, that is $xp - yq \in J_f^d$. As the kernel of the map $x - y : S^{d-1} \times S^{d-1} \to S^d$ is of dimension $\dim S^{d-2}$, this would imply that

$$\dim J_f^d \cap \mathrm{Im}\,(x + y) \geq \dim S^{d-1} - \dim S^{d-2} = h^0(\mathbb{P}^{n-1}, \mathcal{O}(d-1)). \tag{24}$$

As $\dim J_f^d = (n+1)^2$, (24) is impossible if $h^0(\mathbb{P}^{n-1}, \mathcal{O}(d-1)) > (n+1)^2$, which holds if $n \geq 5$, $d \geq 4$. We thus proved that the map Φ of (23) is injective. That it is an immersion follows in the same way because the differential at x is given by the multiplication map

$$y \mapsto \mu_y : R_f^{d-1} \to R_f^d / xR_f^{d-1},$$

and μ_y is zero if and only if $yR_f^{d-1} \subset xR_f^{d-1}$, which has just been excluded. The claim is thus proved.

It follows from the claim and from Proposition 1.8 that, if we have a family of polynomial structures

$$\phi_t : S^{d*} \to R_f^{d*},$$

with ϕ_0 the natural one, then there is an isomorphism

$$\psi_t : \mathbb{P}(S^1) \cong \mathbb{P}(S^1),$$

such that for any $x \in S^1$,

$$\phi_t(xS^{d-1}) = \psi_t(x) R_f^{d-1}.$$

Such a projective isomorphism is induced by a linear isomorphism

$$\tilde{\psi}_t : S^1 \cong S^1,$$

and composing ϕ_t with the automorphism of S^{d*} induced by $\tilde{\psi}_t^{-1}$, we conclude that we may assume that for any $x \in S^1$,

$$\phi_t(xS^{d-1}) = xR_f^{d-1}. \tag{25}$$

We claim that this implies $\phi_t : S^d \to R_f^d$ is the natural map of reduction mod J_f. To see this, choose a general x, so that the multiplication map by x is injective on R_f^{2d-1}. The polynomial structure given by ϕ_t and satisfying (25) provides two linear maps

$$\phi'_t : S^{d-1} \to R_f^{d-1},\ \phi''_t : S^{2d-1} \to R_f^{2d-1},$$

such that $x\phi'_t = \phi_t \circ x : S^{d-1} \to R_f^d$, $x\phi''_t = \phi_t \circ x : S^{2d-1} \to R_f^{2d}$, and the injectivity of the map of multiplication by x on R_f^{2d-1} implies that the following diagram commutes, since it commutes after multiplying the maps by x.

$$\begin{array}{ccc} S^{d-1} \otimes S^d & \longrightarrow & S^{2d-1} \\ \downarrow{\scriptstyle \phi'_t \otimes \phi_t} & & \downarrow{\scriptstyle \phi''_t} \\ R_f^{d-1} \otimes R_f^d & \longrightarrow & R_f^{2d-1} \end{array} . \tag{26}$$

The horizontal maps in the diagram above are the multiplication maps. Following Donagi [4], the multiplication map on the bottom line determines the polynomial structure of R_f^*, because it determines (for $d \geq 3$) S^1 and the multiplication map $S^1 \otimes R_f^{d-1} \to R_f^d$ by the symmetrizer lemma 1.1. The diagram (26) then says that up to the action of an automorphism g of S^*, the polynomial structure given by (ϕ'_t, ϕ_t) is the standard one. Finally, as g must act trivially on the space Z_{ideal} of ideals by (25), g is proportional to the identity. $\square$

2 The cases of degree 4 or 6

We explain in this section how to recover the polynomial structure of a generic hypersurface of degree 4 or 6 so as to prove Theorem 0.6 (2), namely the cases where $d=4$, $n=4m+1$, or $d=6$, $n=6m+2$, with m large. Note that the methods of Schiffer variations that we will develop later would presumably also apply to this case, but it is much more difficult and does not prove Theorem 0.5 (saying that one can recover a hypersurface from its IVHS).

The congruence conditions are equivalent in both cases to the fact that we have $d=2d'$ and $gcd(d,n+1)=d'$, with $d=4$ or 6. The infinitesimal variation of Hodge structure

$$R_f^d \to \oplus_l \mathrm{Hom}(R_f^{ld-n-1}, R_f^{(l+1)d-n-1}), \tag{27}$$

has for smallest degree term the multiplication map

$$R_f^d \otimes R_f^{d'} \to R_f^{d+d'}$$

and the symmetrizer lemma (see Proposition 1.1) allows to reconstruct in these cases the whole ring $R_f^{d'*}$, and in particular the multiplication map

$$R_f^{d'} \otimes R_f^{d'} \to R_f^d. \tag{28}$$

(Note that $R_f^{d'}=S^{d'}$.) We thus only have to explain in both cases how to recover the polynomial structure of (27) from (28), at least for a generic polynomial f. We use the notation $\mathcal{S}q_f^{2l} \subset R_f^{2l}$ for the set of squares

$$\mathcal{S}q_f^{2l} = \{A^2,\, A\in R_f^l\} \subset R_f^{2l}.$$

This is a closed algebraic subset which is a cone in R_f^{2l} and we will denote by $\mathbb{P}(\mathcal{S}q^{2l})$ the corresponding closed algebraic subset of $\mathbb{P}(R_f^{2l})$. When $d=4$, $d'=2$, (28) determines $\mathcal{S}q_f^4$ and we observe that $\mathcal{S}q_f^2 \subset R_f^{d'} = R_f^2 = S^2$ determines the desired polynomial structure, since, passing to the projectivization of these affine cones, $\mathbb{P}(\mathcal{S}q_f^2)$ is the second Veronese embedding of $\mathbb{P}(S^1)$ in $\mathbb{P}(S^2)$. Thus the positive generator H of $\mathrm{Pic}\,(\mathbb{P}(\mathcal{S}q_f^2))$ satisfies the property that $H^0(\mathbb{P}(\mathcal{S}q_f^2),H) =: V$ has dimension $n+1$ and the restriction map $(S^2)^* \to \mathrm{Sym}^2 V$ is an isomorphism. The dual isomorphism gives the desired isomorphism $\mathrm{Sym}^2 S^1 \cong S^2$, with $S^1 := V^*$.

When $d=6$, $d'=3$, (28) determines $\mathcal{S}q_f^6$. Next, for any l and any fixed $0\neq K\in S^1$, denote by $K\mathcal{S}q_f^{2l} \subset R_f^{2l+1}$ the set of polynomials of the form KA^2, for some $A\in R_f^l$. We have $\dim \mathcal{S}q^2 = n+1$, $\dim K\mathcal{S}q^2 = n+1$ and for $d=6$, the data of the subspaces $K\mathcal{S}q^2 \subset R_f^3 = S^3$ determines the isomorphism $S^3 \cong \mathrm{Sym}^3(S^1)$, hence the polynomial structure. Indeed, the singular locus of the variety $\bigcup_K K\mathcal{S}q^2$ is the variety of cubes $\mathcal{C}u^3 \subset S^3$, and the same Veronese argument as above shows that it determines the polynomial structure $S^3 = R_f^3 \cong \mathrm{Sym}^3 S^1$, with $S^1 = V^*$.

We observe now that the spaces $\mathcal{S}q_f^2 \subset R_f^2$, resp. $K\mathcal{S}q_f^2 \subset R_f^3$, have the following property

$$\forall A,\, B\in \mathcal{S}q_f^2,\, AB\in \mathcal{S}q_f^4, \tag{29}$$

resp.

$$\forall A,\, B\in K\mathcal{S}q_f^2,\, AB\in \mathcal{S}q_f^6, \tag{30}$$

We prove now the following result, which concludes the proof of Theorem 0.6 (2).

Proposition 2.1. *(1) Let f be a generic homogeneous polynomial of degree 4 in $n+1$ variables, with $n\geq 599$. Then the only subvariety $T\subset R_f^2 = S^2$ of dimension $\geq n+1$ satisfying the condition*

$$AB\in \mathcal{S}q_f^4 \text{ for any } A,\, B\in T$$

is $\mathcal{S}q^2_f$.

(2) Let f *be a generic homogeneous polynomial of degree* 6 *in* $n+1$ *variables, with* $n \geq 159$. *Then the only subvarieties* $T \subset R^3_f = S^3$ *of dimension* $\geq n+1$ *satisfying the condition*

$$AB \in \mathcal{S}q^6_f \text{ for any } A,\, B \in T$$

are the varieties $K\mathcal{S}q^2_f$ *for* $0 \neq K \in S^1$.

Note that in this statement, we can clearly assume that T is a cone, since the conditions are homogeneous.

Proof of Proposition 2.1. We observe that by a proper specialization argument, the schematic version of the statement (namely that T satisfying condition (29 or (30) must be reduced) is an open condition on the set of polynomials f for which R^d_f, or equivalently J^d_f, has the right dimension. This will happen in the case $d = 4$, where we will prove the schematic version of the statements above for one specific f, for which J^d_f has the right dimension. In the case $d = 6$, the schematic statement is not true anymore but it is neither true for the generic f, so the schematic analysis will also allow to conclude by specialization.

Let us first explain the specific polynomials we will use. In the case of degree 4, we will first choose general linear sections $\mathbb{P} \cap Pf_4$ of the Pfaffian quartic $\mathcal{P}f_4 \subset \mathbb{P}(\bigwedge^2 V_8)$, where $\mathbb{P} \subset \mathbb{P}(\bigwedge^2 V_8)$ is a linear subspace of dimension 23, or 24. We get this way polynomials f_i of degree 4 in 24 or 25 variables. In higher dimension, we will then consider polynomials of the form

$$f = f_1(X_{1,1}, \ldots, X_{1,i_1}) + \ldots + f_l(X_{l,1}, \ldots, X_{l,i_l}),$$

with $i_1, \ldots, i_l \in \{24, 25\}$, which allows to construct degree 4 polynomials with any number $n+1$ of variables starting from 600.

In degree 6, we will first choose, for $39 \leq n \leq 62$, a general linear section $\mathbb{P} \cap Pf_6$, where $\mathbb{P} \subset \mathbb{P}(\bigwedge^2 V_{12})$ is a linear subspace of dimension n and $\mathcal{P}f_6 \subset \mathbb{P}(\bigwedge^2 V_{12})$ is the Pfaffian sextic hypersurface. We get this way polynomials f_i of degree 6 in $n+1$ variables, where $40 \leq n+1 \leq 63$. In higher dimension, we will then consider polynomials of the form

$$f = f_1(X_{1,1}, \ldots, X_{1,i_1}) + \ldots + f_l(X_{l,1}, \ldots, X_{l,i_l}),$$

with $i_1, \ldots, i_l \in \{40, \ldots, 63\}$, which allows to construct degree 6 polynomials with any number $n+1$ of variables starting from 160.

Lemma 2.2. *For a polynomial* f *of the form above,* J^d_f *has the right dimension* $(n+1)^2$.

Proof. One has $f = \sum_j f_j$, where each f_j involves variables $X_{j,1}, \ldots, X^{j,i_j}$. It is immediate to check that the statement for each f_j implies the statement for f. Turning to the f_j, they are either general linear sections of the quartic Pfaffian hypersurface in $\mathbb{P}^{27}$ by a $\mathbb{P}^n$, $n = 23$ or 24, or of the sextic Pfaffian hypersurface in $\mathbb{P}^{65}$ by a $\mathbb{P}^n$, for $39 \leq n \leq 62$. Let us show that each of them has no infinitesimal automorphism. The automorphism group of the general Pfaffian hypersurface $\mathcal{P}f_k \subset \mathbb{P}(\bigwedge^2 V_{2k})$ is the group $PGl(2k)$. We claim that the automorphism group of a general linear section of dimension $> 2(2k-2) = \dim G(2, V_{2k})$ is also contained in $PGl(2k)$. This follows from the fact that after blowing-up in $\mathcal{P}f_{2k}$ its singular locus, which parameterizes forms of rank $< 2k-2$, we get a dominant morphism $\widetilde{Pf_{2k}} \to G(2, V_{2k})$, which to a denerate form associates its kernel. If we consider a general linear section X_l of $\mathcal{P}f_{2k}$ of dimension $> \dim G(2, V_{2k})$ defined by a r-dimensional vector subspace $W \subset \bigwedge^2 V^*_{2k}$, the same remains true and we get a morphism $\widetilde{X}_l \to G(2, V_{2k})$ which is dominant with connected fiber of positive dimension. Thus the automorphism group of X_l has to act on $G(2, V_{2k})$ and it has to identify with the group of automorphisms of $G(2, V_{2k})$, or automorphisms of $\mathbb{P}(V_{2k})$ preserving the space $W \subset \bigwedge^2 V^*_{2k}$. It is easy to check that this space is zero once $r \geq 3$. Coming back to our situation where $k = d = 4$ or 6, our choices of r are $r = 3$ or 4 for $k = 4$, and $3 \leq r \leq 26$ for $k = 6$. In all cases, the variety X_l has dimension $> 2(2k-2)$ so the analysis above applies. □

We now prove Proposition 2.1 for f as above. Let us first assume $d = 4$.

Lemma 2.3. *Let $f = \sum_j f_j$ be a polynomial of degree 4 in $n+1$ variables as constructed above. Then if $T \subset S^2$ is an algebraic subvariety of dimension $\geq n+1$ such that*

$$AB \in \mathcal{S}q^4_f \subset R^4_f$$

for any A, $B \in T$, $T = \mathcal{S}q^2$.

Proof. Let as above $f = \sum_l f_j$. The singular locus Z_f of $X_f = V(f)$ is the join of the singular loci Z_j of $V(f_j)$ in $\mathbb{P}^{i_j}$. This means that, introducing the natural rational projection map $\pi : \mathbb{P}^{\sum_j i_j - 1} \dashrightarrow \prod_l \mathbb{P}^{i_j - 1}$, one has $Z_f = \pi^{-1}(\prod_j Z_j)$.

Claim 2.4. *The varieties Z_f are not contained in any quadric.*

Proof. Consider first the case of the Pfaffian linear sections Z_j. The claim follows in this case because they are general linear sections of the singular locus Z of the quartic universal Pfaffian $\mathcal{P}f_4$ in $\mathbb{P}(\bigwedge^2 V_8)$, which is defined by the equations $\omega^3 = 0$, that is, by cubics, and is not contained in any quadric. The last point can be seen by looking at the singular locus of Z, which consists of forms of rank 2, that is the Grassmannian $G(2, V_8^*)$. Along this locus, the Zariski tangent space of Z is the full Zariski tangent space of $\mathbb{P}(\bigwedge^2 V_8)$. A quadric containing Z should thus be singular along $\operatorname{Sing} Z$. But $\operatorname{Sing} Z = G(2, V_8^*)$ is not contained in any proper linear subspace of $\mathbb{P}(\bigwedge^2 V_8)$. It follows that Z is not contained in any quadric. It remains to conclude that the same statement is true for the general linear section $\mathbb{P}^{i_l - 1} \cap Pf_4$, with $i_l = 24, 25$. Its singular locus Z_l is the general linear section $\mathbb{P}^{i_l - 1} \cap Z$, and we show inductively that any quadric containing Z_l is the restriction of a quadric containing Z. This statement only needs that Z_l is non-empty (it has dimension ≥ 18 in our case) and that all the successive linear sections $\mathbb{P}^j \cap Z$, with $j \geq i_l$, are linearly normal in $\mathbb{P}^j$, which is not hard to prove. Finally we have to show that the same is true for a general $f = \sum f_j$. In that case, Z_f is a join $Z_1 * \ldots * Z_l$ and a join of varieties not contained in any quadrics is not contained in any quadric. □

We also prove the following

Claim 2.5. *(a) The restriction map $S^1 \to H^0(Z_f, \mathcal{O}_{Z_f}(1))$ is an isomorphism.*

(b) The only n-dimensional family $\{D_A\}$ of divisors on Z_f such that for some fixed effective divisor D_0,

$$2D_A + D_0 \in |\mathcal{O}_{Z_f}(2)|$$

is the family of hyperplane sections of Z_f.

Proof. We recall for this that Z_f is the join of the $Z_j \subset \mathbb{P}^{i_j}$, and that each Z_j is a smooth linear section of the singular locus Z of $\mathcal{P}f_4$ by either a $\mathbb{P}^{24}$ or a $\mathbb{P}^{23}$. Let us first conclude when f is one of the f_j, so Z_f is one of the Z_j. We observe that $Z \subset \mathbb{P}(\bigwedge^2 V_8)$ is the set of 2-forms of rank ≤ 4 (the generic element of Z being of rank exactly 4), and has the natural resolution

$$\widetilde{Z} \subset G(4, V_8^*) \times \mathbb{P}(\bigwedge^2 V_8),\ \ \widetilde{Z} = \{([W_4], \omega),\, W_4 \subset \operatorname{Ker} \omega\}.$$

The statement (a) easily follows from the above description of $\widetilde{Z}$ and the fact that the Z_j are general linear sections of codimension 3 or 4 of Z. Let us prove (b). The variety $\widetilde{Z}$ is smooth and, being a projective bundle fibration over $G(4, V_8^*)$, has Picard rank 2. Its effective cone is very easy to compute: indeed, the line bundle l which is pulled-back from the Plücker line bundle on the Grassmannian via the first projection pr_1 is clearly one extremal ray of the effective cone since the corresponding morphism has positive dimensional fibers. There is a second extremal ray of the effective cone, which is the class of the divisor D contracted by the birational map $\widetilde{Z} \to Z$ (induced by the second projection pr_2). One easily computes that this class is $2h - l$, where h is the pull-back of hyperplane class on $\mathbb{P}(\bigwedge^2 V_8)$ by pr_2.

We observe that the fibers of pr_1 are of dimension 5, so that, when we take a general linear section of Z by a codimension 3 or 4 projective subspace, getting the singular locus Z_j of X_j, all the properties above remain satisfied, and thus Pic $\widetilde{Z}_j = \mathbb{Z}h_j + \mathbb{Z}l_j$, with effective cone generated by l_j and $2h_j - l_j$. We now finish the argument for Z_j: we lift our data $\{D_A\}$, D_0 to $\widetilde{Z}_j$. We then have

$$2h_j = \widetilde{D}_0 + 2\widetilde{D}_A$$

in Pic $\widetilde{Z}_j$, with $\widetilde{D}_0$ effective and $\dim|\widetilde{D}_A| \geq n$, where n is $\dim|h_{|Z_j}|$ by (a). Let us write

$$\widetilde{D}_0 = \alpha h_j + \beta l_j = \frac{\alpha}{2}(2h_j - l_j) + (\beta + \frac{\alpha}{2})l_j \text{ in Pic } \widetilde{Z}_j,$$

with α, $\beta \in 2\mathbb{Z}$. Then the analysis above shows that

$$\alpha \geq 0,\ \beta + \frac{\alpha}{2} \geq 0.$$

As $2h_j - \widetilde{D}_0$ is effective, we also have

$$2 - \alpha \geq 0,\ -\beta + 1 - \frac{\alpha}{2} \geq 0.$$

As α is even, we only get the possibilities $\alpha = 0, 2$. If $\alpha = 2$, we get $\beta = 0$ and thus $2h_j - \widetilde{D}_0 = 0$, contradicting the assumption $\dim|2h - \widetilde{D}_0| \geq n$. If $\alpha = 0$, we get from the inequalities above $\beta = 0$ since β has to be even, and $2h = 2\widetilde{D}_A$, which proves statement (b) (using the fact that Pic Z_j has no 2-torsion and statement (a)).

We now have to prove the same result for the join $Z_f = Z_1 * Z_2 \ldots * Z_l$, which is done inductively on the number l, assuming, as this is satisfied in our situation, that the Z_i's are simply connected. This way we are reduced to consider only the join $Z_1 * Z_2 \subset \mathbb{P}^n$, with $n = n_1 + n_2 + 1$, of two linearly normal varieties $Z_1 \subset \mathbb{P}^{n_1}$, $Z_2 \subset \mathbb{P}^{n_2}$, which satisfy the properties (a) and (b). We observe that $Z_1 * Z_2$ is dominated by a $\mathbb{P}^1$-bundle over $Z_1 \times Z_2$, namely

$$\widetilde{Z_1 * Z_2} := \mathbb{P}(\mathcal{O}_{Z_1}(1) \oplus \mathcal{O}_{Z_2}(1)) \xrightarrow{\pi} Z_1 \times Z_2, \tag{31}$$

the two sections being contracted to Z_1, resp. Z_2, by the natural morphism to $Z_1 * Z_2 \subset \mathbb{P}^n$. The description (31) of the join immediately proves (a) for $\widetilde{Z_1 * Z_2}$ once we have it for Z_1 and Z_2. We now turn to (b). Let $h = \mathcal{O}_{\mathbb{P}(\mathcal{O}_{Z_1}(1)\oplus\mathcal{O}_{Z_2}(1))}$ on $\widetilde{Z_1 * Z_2}$ and let D_0 be a fixed effective divisor and $\{D_A\}$ be a mobile family of divisors on $\widetilde{Z_1 * Z_2}$ such that

$$D_0 + 2D_A = 2h, \tag{32}$$

$$\dim\{D_A\} \geq n. \tag{33}$$

Then either D_0 or D_A is vertical for π. Indeed, they otherwise both restrict to a degree ≥ 1 divisor on the fibers of π contradicting (32). Assume D_0 is vertical for π, that is, $D_0 = \pi^{-1}(D'_0)$. The equality $2h - D'_0 = 2D_A$ says that $D'_0 = 2D''_0$ as divisors on $Z_1 \times Z_2$ and, as Z_1 and Z_2 are simply connected, $D''_0 \in |pr_1^* D''_{0,1} + pr_2^* D''_{0,2}|$ and both $2D''_{0,1}$, $2D''_{0,2}$ are effective. The divisors $D''_{0,i}$ on Z_i have the property that the linear system $|h - pr_1^* D''_{0,1} - pr_2^* D''_{0,2}|$ on $\mathbb{P}(\mathcal{O}_{Z_1}(1) \oplus \mathcal{O}_{Z_2}(1))$ has dimension $\geq n$, which says that

$$\dim|\mathcal{O}_{Z_1}(1)(-D''_{0,1})| + \dim|\mathcal{O}_{Z_2}(1)(-D''_{0,2})| \geq n_1 + n_2.$$

As $2D''_{0,1}$ is effective on Z_1 and $2D''_{0,2}$ is effective on Z_2, we conclude that $D''_{0,i} = 0$ and that the D_A's belong to $|\mathcal{O}_{Z_1 * Z_2}(1)|$, so (b) is proved in this case.

In the case where D_0 is not vertical, then restricting again to the fibers of π, $D_0 = 2h - D'_0$ whereD'_0 is effective and comes from $Z_1 \times Z_2$, and D_A is vertical, $D_A = \pi^{-1}(D'_A)$. Hence we have again $D'_0 = 2D''_0$, and $\dim|h - D'_0 - D'_A| \geq n$, so the proof concludes as before that $D'_0 = 0$ and $D_A = 0$ which contradicts (33). □

We now conclude the proof of Lemma 2.3. We have $J_f \subset I_{Z_f}$, since Z_f is contained in $\operatorname{Sing} X_f$. Let $T \subset S^2$ be a closed algebraic subset satisfying the assumptions of Lemma 2.3 for f. Then for any $A,\, B \in T$, $AB_{|Z_f} = M^2_{|Z_f}$, for some $M \in S^2$. This implies that the variable part of $\operatorname{div} A_{|Z_f}$ appears with multiplicity 2. Hence $\operatorname{div} A_{|Z_f} = D_0 + 2D_A$. We now use Claim 2.4 which implies that the family of divisors $\operatorname{div} A_{|Z_f}$ is of dimension $\geq n$, hence also the family $\{D_A\}$ of divisors. We then conclude from Claim 2.5 that $T \subset \mathcal{S}q^2$. □

We next turn to the case where $d = 6$. We will not give the full proofs of the intermediate statements, since they are tedious and work as before.

Lemma 2.6. *Let f be a polynomial of degree 6 in $n+1$ variables as constructed above. Then if $T \subset S^3$ is an algebraic subvariety of dimension $\geq n+1$ such that*

$$AB \in \mathcal{S}q^6_f \subset R^6_f$$

for any $A,\, B \in T$, there exists $K \in S^1$, such that $T = K\mathcal{S}q^2_f$.

Proof. The proof works essentially as before so we will just indicate what has to be changed. First of all we have

Claim 2.7. *There is no cubic vanishing on Z_f.*

As before, we first reduce to the case of a single Z_i, which is the singular locus of a general linear section X_i of the Pfaffian sextic by a linear space of dimension ≥ 39 (that is, codimension at most 26). Then Z_i is a corresponding general linear section of the singular locus Z of $\mathcal{P}f_6 \subset \mathbb{P}^{65}$, and Z is defined by equations of degree 5, namely $\omega^5 = 0$ in $\bigwedge^{10} V_{12}$. One shows as before that Z is not contained in any cubic, and that the same remains true for a general linear section Z_i as above, by using the same presentation of the desingularization $\widetilde{Z}$ of Z contained in $G(4, V_{12}^*) \times \mathbb{P}(\bigwedge^2 V_{12})$. Finally we show that this implies the result for Z_f , with $f = \sum_k f_{i_k}$.

Second, we show

Claim 2.8. *(a) The restriction map $S^1 \to H^0(Z_f, \mathcal{O}_{Z_f}(1))$ is surjective (hence an isomorphism).*

(b) A linear system $|D|$ on Z_f with $\dim|D| \geq n$ and $H^0(\mathcal{O}_{Z_f}(3)(-2D)) \neq 0$ must be $|\mathcal{O}_{Z_f}(1)|$.

The first statement is proved as above by a cohomological analysis on $\widetilde{Z}$. As for (b), we observe that the Picard group of the induced desingularization $\widetilde{Z}_i \subset \widetilde{Z}$ of $Z_i \subset Z$ is $\mathbb{Z}^2$, generated by l and h. The dimensions have been chosen in such a way that the natural morphism $\widetilde{Z}_i \to G(4, V_{12}^*)$ have positive dimensional fibers. On the other hand, the second projection map $\widetilde{Z}_i \to Z_i \subset \mathbb{P}^n$ contracts a divisor D_i which is easily computed to be in the linear system $|4h_i - l|$ on $\widetilde{Z}_i$. It follow that the effective cone of $\widetilde{Z}_i$ is the cone generated by $4h_i - l$ and l. We then try to write

$$3h = D_0 + 2D_1$$

with $D_0 = \alpha(4h_i - l) + \beta l$, $D_1 = \alpha'(4h_i - l) + \beta' l$, and $\alpha,\, \beta,\, \alpha',\, \beta'$ nonnegative numbers such that the corresponding divisors are integral on $\widetilde{Z}_i$, that is, belong to $\mathbb{Z}h_i + \mathbb{Z}l$, and we find that the only solution is $D_0 = h_i$, $D_1 = h$. This proves the result for Z_i, and we extend it to $Z_f = Z_{i_1} * \ldots Z_{i_l}$ by similar arguments as before.

The end of the proof of Lemma 2.6 is then as before. A closed algebraic subset $T \subset S^3$ with $AB \in \mathcal{S}q^6$ for any $A,\, B \in T$ provides by restriction to Z_f divisors $D \in \mathcal{O}_{Z_f}(3)|$ with the property that the moving part D_m of D has multiplicity 2, that is $D_m = 2D'_m$. Furthermore, if $\dim T = n+1$, by the Claim 2.7, this family of divisors has dimension at least n. Denoting by D_0 its constant part, we have $D_0 + 2D'_m = \mathcal{O}_{Z_f}(3)$ in $\operatorname{Pic} Z_f$. By the Claim 2.8, we conclude that the divisors D'_m belong to the linear system $|\mathcal{O}_{Z_f}(1)|$ and $D_0 \in |\mathcal{O}_{Z_f}(1)|$. Using Claim 2.8, (a), D_0 and the D'_m are the divisors of sections $K_{|Z_f}$, $A_{m|Z_f}$, for some $K,\, A_m \in S^1$. Finally, using Claim 2.7 again, we conclude that $T = \{KA^2,\, A \in S^1\}$. □

In order to conclude the proof of Proposition 2.1, it suffices now to make Lemmas 2.3 and 2.6 more precise by analyzing the schematic structure of a closed algebraic subset $T \subset S^2$ (resp. $T \subset S^3$) satisfying the assumptions of these lemmas. We first observe the following:

Lemma 2.9. *(i) Let f be as in Lemma 2.3 and let $A,\, B \in S^1$ be general. Let $M := AB$ and consider the subspace $MS^2 = MR^2_f \subset R^4_f$. Then*

$$[MS^2 : A^2] = BS^1 \subset S^2, \tag{34}$$

where as usual, the notation $[MS^2 : A^2]$ is used for $\{S \in S^2,\, SA^2 \in \langle M\rangle\}$.

(ii) Let f be as in Lemma 2.6 and let $K,\, A,\, B \in S^1$ be general. Let $M = KAB$ and consider the subspace $MR^3_f \subset R^6_f$. Then

$$[MR^3_f : KA^2] = BS^2 \subset S^3, \tag{35}$$

The lemma is obvious, using restriction to Z_f and using Claim 2.8. We now have to separate again the cases of degree 4 and 6.

Case of degree 4, end of the proof. Equality (34) implies Proposition 2.1 in the case of degree 4 because MS^2 is the tangent space to $\mathcal{S}q^4$ at M^2, while $BS^1 \subset S^2$ is the tangent space to $\mathcal{S}q^2$ at B^2. Equation (34) thus says that a space $T \subset S^2$ satisfying the assumptions of Lemma 2.3 must be generically the *reduced* $\mathcal{S}q^2$. The conclusion of the proof then follows by a specialization and cycle-theoretic argument, using the fact that these sets T above are cones, hence come from closed algebraic subsets of $\mathbb{P}(T) \subset \mathbb{P}(S^2)$.

Case of degree 6, end of the proof. In this case, we cannot conclude as above using Lemma 2.9(ii), because the tangent space to $K\mathcal{S}q^2$ at KB^2 is KBS^1, while equation (35) gives us as Zariski tangent space to T at KB^2 the subspace $BS^2 \subset S^3$.

But we observe that the same phenomenon holds for a generic f. Namely, the equations

$$AB \in \mathcal{S}q^6_f,\ \forall A,\, B \in T \tag{36}$$

do not define sets of the form $K\mathcal{S}q^2 \subset S^3$ with their reduced structure, but with the larger Zariski tangent space BS^2 at KB^2. In order to conclude, we have to perform a second order analysis. This is done in the following

Lemma 2.10. *For the special f as above, the maximal scheme theoretic structure on a closed algebraic subset $T \supset K\mathcal{S}q^2$ satisfying (36), at its generic point, is the nonreduced structure on $K\mathcal{S}q^2$ obtained by putting the Zariski tangent space $BS^2 \supset KBS^1$ at the general point $KB^2 \in T$.*

Proof. We have to show that for K and B generic, a tangent vector $u \in K\mathcal{S}q^2$ which is not contained in BS^1 does not satisfy equations (36) at second order. We fix $A,\, B,\, K \in S^1$ generic, and examine for fixed A the equations

$$UKA^2 \in \mathcal{S}q^6 \tag{37}$$

imposed on $U \in S^3$ near KB^2. Write

$$U = KB^2 + \epsilon U_1 + \epsilon^2 U_2 \tag{38}$$

and let again $M = KAB$. We easily show that the morphism $P \mapsto P^2$ is an embedding at $M = KAB$, which is equivalent to the injectivity of the map of multiplication by M from S^3 to R^6_f. Thus the equations (37) to second order with U as in (38) provide the equation

$$(KB^2 + \epsilon U_1 + \epsilon^2 U_2)KA^2 = (KAB + \epsilon M_1 + \epsilon^2 M_2)^2 \bmod \epsilon^2, \tag{39}$$

where the equality holds in R_f^6. Lemma 2.9 (ii) says that $U_1 = BV_1$ for some $V_1 \in S^2$. The injectivity of the map of multiplication by M shows that $M_1 = \frac{AV_1}{2}$, so that (39) becomes

$$(KB^2 + \epsilon BV_1 + \epsilon^2 U_2)KA^2 = (KAB + \epsilon \frac{AV_1}{2} + \epsilon^2 M_2)^2 \text{ mod } \epsilon^2,$$

that is

$$U_2 KA^2 = 2KABM_2 + \frac{A^2V_1^2}{4} \text{ in } R_f^6. \tag{40}$$

This equation says that $\frac{A^2V_1^2}{4}$ belongs to $KAS^4 \subset R_f^6$ so the proof of the lemma is concluded by proving that the condition $A^2V_1^2 \in KAS^4 \subset R_f^6$ for any $A \in S^1$ implies that $V_1 \in KS^1$, which is not hard. □

We now conclude the proof of Proposition 2.1 by a specialization and cycle theoretic argument. Again, we recall that we assumed that the spaces KS^2, T as above are cones over the corresponding subspaces of $\mathbb{P}(S^3)$. By Lemma 2.6 and the last claim, the equations (36) have for the special f, as maximal scheme theoretic solutions, the subvarieties $K\mathcal{S}q^2$ with a generic scheme-theoretic structure of length $N = \dim(BS^2/KBS^1) + 1$. For the general f, the subspaces $K\mathcal{S}q^2$ with a scheme theoretic structure of length N also provide such a subset. It then follows that for a generic f, the subspaces $K\mathcal{S}q^2$ (with a scheme theoretic structure of length N) are the only solutions to equations (36). □

3 Schiffer variations and Jacobian ideals

Definition 3.1. *A Schiffer variation of a homogeneous polynomial f of degree d in $n+1$ variables $X_0, \ldots, X_n$ is an affine line $f + tx^d$, $t \in \mathbb{C}$, where $x \in S^1$ is a linear form of the variables $X_0, \ldots, X_n$.*

In the definition above, we are not interested in the linear character of the parameterization, as this does not make sense anymore after projection of this line to the moduli space. We should thus consider more generally 1-parameter families of polynomials supported (up to the action of $Gl(n+1)$) on a line as above. We can also speak of finite order Schiffer variations, which consist in looking at a finite order arc in an affine line as above passing through f. Observe that if $g = f + x^d$, then for any $u \in H^0(\mathbb{P}^n, T_{\mathbb{P}^n}(-1)) = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))^*$ such that $\partial_u(x) = 0$, one has $\partial_u(f) = \partial_u(g)$. It follows that the Jacobian ideals J_f, resp. J_g generated by the partial derivatives of f, resp. g, satisfy the condition

$$\dim \langle J_f^{d-1}, J_g^{d-1} \rangle \leq n + 2. \tag{41}$$

We will describe later on other properties of Schiffer variations but (41) is already very restrictive, as shows Proposition 3.3 below. A well-known result due (in various forms) to Carlson-Griffiths [2], Donagi [4] and Mather-Yau [7] says the following:

Proposition 3.2. *Let f, g be two homogeneous polynomials in $n+1$ variables, defining smooth hypersurfaces in $\mathbb{P}^n$. If the Jacobian ideals J_f^{d-1} and J_g^{d-1} coincide, then f and g are in the same orbit under the group $PGl(n+1)$.*

A nice proof of this statement is given in [4]. The example of the Fermat equation $f = \sum_i X_i^d$ and its variations $g = \sum_i \alpha_i X_i^d$ shows that one does not always have $f = \mu g$ for some coefficient μ, under the assumptions of Theorem 3.2. The Mather-Yau is the following variant which is more precise but works only for d large enough and f generic.

Proposition 3.3. *Let f, g be two homogeneous polynomials of degree d in $n+1$ variables, defining smooth hypersurfaces in $\mathbb{P}^n$. Assume $d \geq 4$, $n \geq 4$. If f is generic and the Jacobian ideals J_f and J_g coincide, then $f = \mu g$ for some coefficient μ.*

Let us prove a closely related statement concerning the case where J_f^{d-1} and J_g^{d-1} are not equal but almost equal, that is, satisfy equation (41).

Proposition 3.4. *Let d, n be such that*

$$4(n-3)+10 \leq h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-2)). \tag{42}$$

Then for a generic polynomial $f \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$, the equation (41) holds if and only if g belongs to a Schiffer variation of λf for some coefficient λ.

Note that (42) holds if $d \geq 4$ and $n \geq 4$.

Proof of Proposition 3.4. As f is generic, its Jacobian ideal J_f^{d-1} has dimension $n+1$. The equation (42) can thus be written in the following form, for adequate choices of linear coordinates $X_0, \dots, X_n$ and $Y_0, \dots, Y_n$ on $\mathbb{P}^n$

$$\frac{\partial g}{\partial X_i} = \frac{\partial f}{\partial Y_i} \text{ for } i = 0, \dots, n-1. \tag{43}$$

Let us now use the symmetry of partial derivatives:

$$\frac{\partial^2 g}{\partial X_i \partial X_j} = \frac{\partial^2 g}{\partial X_j \partial X_i}.$$

Combined with (43), it provides, for any i, j between 0 and $n-1$, the following second order equations

$$\frac{\partial^2 f}{\partial X_i \partial Y_j} = \frac{\partial^2 f}{\partial X_j \partial Y_i}. \tag{44}$$

Lemma 3.5. *Under the numerical assumption (42), a generic polynomial f of degree d in $n+1$ variables does not satisfy a nontrivial second order partial differential equation of the type (44).*

Proof. This is a dimension count. The differential equations appearing in (44) are linear second order equations determined by elements U of rank ≤ 4 in $\mathrm{Sym}^2 H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))^*$. The nonzero elements U are parameterized by a variety of dimension $4(n-3)+9$. Given a nonzero U, the differential equation $\partial_U^2 \phi = 0$ determines a linear subspace H_U of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ of codimension $h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-2))$ since

$$\partial_U^2 : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-2))$$

is surjective. If (42) holds, the union of the spaces H_U does not fill-in a Zariski open set of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$. $\square$

It follows that all the equations appearing in (44) are trivial, which says equivalently that for any i, j

$$\frac{\partial}{\partial X_i}\frac{\partial}{\partial Y_j} - \frac{\partial}{\partial X_j}\frac{\partial}{\partial Y_i} = 0 \text{ in } \mathrm{Sym}^2(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))^*).$$

These equations exactly say that for some $\lambda \in \mathbb{C}$,

$$\frac{\partial}{\partial Y_i} = \lambda \frac{\partial}{\partial X_i}$$

for $i = 0, \dots, n_1$. Thus $g - \lambda f$ satisfies $\frac{\partial}{\partial X_i}(g - \lambda f) = 0$ for $i = 0, \dots, n_1$, hence $g = \lambda f + \alpha X_n^d$ for some coefficient α, which concludes the proof. $\square$

We conclude this section by the following lemma which shows the relevance of first order Schiffer variations to our subject.

Proposition 3.6. *Let f, g be two degree homogeneous d polynomials in $n+1$ variables defining smooth hypersurfaces X_f, X_g. Assume $d \geq 4$, $n \geq 4$ and f is generic. Then if there exists a linear isomorphism $i : R_f^d \cong R_g^d$ mapping the set of first order Schiffer variations of f to the set of first order Schiffer variations of g, X_f is isomorphic to X_g.*

Proof. The condition on d, n are used in the following

Lemma 3.7. *For f generic with $d \geq 4, n \geq 2$, if $x \in S^1$ is nonzero, then $x^d \neq 0$ in R_f^d. Furthermore the map $x \mapsto x^d$ induces a (incomplete) Veronese imbedding $v_f : \mathbb{P}(S^1) \hookrightarrow \mathbb{P}(R_f^d)$.*

Proof. The statement is equivalent to proving that, if f is a generic homogeneous polynomial of degree d in $n+1$ variables, $J_f^d \subset S^d$ does not contain any power x^d of a linear form, or any sum $x^d - y^d$ of two such powers. In the first case, we get that $x \cdot x^{d-1} = 0$ in R_f^d and in the second case, we get that $(x-y)(x^{d-1} + x^{d-2}y + \ldots + y^{d-1}) = 0$ in R_f^d. An easy dimension count shows that for generic f, the multiplication map $x : R_f^{d-1} \to R_f^d$ by any nonzero linear form $x \in S^1$ is injective, so in the first case we conclude that $x^{d-1} = 0$ in R_f^{d-1} and in the second case, $x^{d-1} + x^{d-2}y + \ldots + y^{d-1} = 0$ in R_f^{d-1}, or equivalently

$$x^{d-1} \in J_f^{d-1}, \text{ or } x^{d-1} + x^{d-2}y + \ldots + y^{d-1} \in J_f^{d-1}.$$

Using the fact that $n \geq 2$, and choosing a coordinate system $x = X_0$, $y = X_1, \ldots$, we get in both cases a nontrivial second order equation

$$\sum_{i=0,1} \alpha_i \frac{\partial^2 f}{\partial X_2 \partial X_i} = 0$$

which is excluded by Lemma 3.5. □

We now conclude the proof. Using the lemma, the isomorphism i induces an isomorphism $i_1 : \mathbb{P}^n(S^1) \cong \mathbb{P}^n(S^1)$ between the two projected Veronese $v_f(\mathbb{P}(S^1))$ and $v_g(\mathbb{P}(S^1))$, which lifts to a linear isomorphism $\tilde{i}_1 : S^1 \cong S^1$, such that $v_g \circ i_1 = v_f$. The incomplete Veronese embeddings v_f, resp. v_g factor canonically through the complete Veronese embeddings

$$\mathbb{P}^n(S^1) \stackrel{V_d}{\to} \mathbb{P}(S^d)) \dashrightarrow \mathbb{P}(R_f^d),$$

(resp. $\mathbb{P}^n(S^1) \to \mathbb{P}(S^d)) \dashrightarrow \mathbb{P}(R_f^d)$), which implies that we have a commutative diagram, where $\tilde{i}_d : S^s \cong S^d$ is induced by $\tilde{i}_1$

$$\begin{array}{ccc} S^d & \xrightarrow{\tilde{i}_d} & S^d \\ \downarrow & & \downarrow \\ R_f^d & \xrightarrow{i_*} & R_g^d. \end{array} \tag{45}$$

The quotient maps have for respective kernels J_f^d, J_g^d. We thus conclude that $\tilde{i_d}(J_f^d) = J_g^d$, and thus by Proposition 3.2, X_f is isomorphic to X_g. □

3.1 Formal properties of Schiffer variations

Our strategy for the proof of Theorem 0.6 when d divides $n+1$ consists in finding a characterization of the set of Schiffer variations of a hypersurface X_f that can be read from its local variation of Hodge structure. In fact we will need not only the infinitesimal variation of Hodge structure (IVHS) of X_f but also the "deformation of the IVHS" along the Schiffer variation, which is a higher order argument. The IVHS itself provides the first order invariants of the variation of Hodge structure of X_f at the point $[f]$, hence part of the multiplicative structure of the Jacobian ring R_f, by (6). We wish in this section analyze the specifities of the first order Schiffer variations $\phi \in R_f^d$ as elements of the Jacobian ring R_f^* and also analyze, using (41), the way the Jacobian ring deforms along them.

Recall that a first order Schiffer variation of f is an element $\phi = x^d \in R_f^d$, where $x \in S^1$. We will consider only the subring R_f^{d*} of R_f^*, because this is, when d divides $n+1$ the data that we get from the IVHS of f. As R_f^{d*} contains only the graded pieces of degree divisible by d, it does not contain the linear form x and by Proposition 3.6, recovering the polynomial structure of R_f^d precisely means recognizing the set of powers $\phi = x^d$. However the ideal of R_f^{*d} generated by such a $\phi \in R_f^d$ has some numerical particularities, even if we only use the multiplication map $R_f^{kd} \otimes R_f^{ld} \to R_f^{(k+l)d}$ for $k + l \leq 3$.

Indeed, let $x \in S^1$ be generic and, for $1 \leq k \leq d-1$, let

$$I_{x,k}^{*d} := x^k R_f^{*d-k} \subset R_f^{*d}.$$

Then, the following condition (*) holds, at least if d is large enough.

$$\dim I_{x,k}^{id} = \dim R_f^{id-k} \text{ for } i = 1,\, 2,\, 3. \tag{46}$$

$$I_{x,k}^{d} I_{x,d-k}^{d} \subset \phi R_f^d,\ I_{x,k}^{d} I_{x,d-k}^{2d} \subset \phi R_f^{2d}, \tag{47}$$

$$R_f^d I_{x,k}^{id} \subset I_{x,k}^{(i+1)d}. \tag{48}$$

$$I_{x,k}^{d} \cdot I_{x,l}^{d} \subset I_{x,k+l}^{2d} \text{ for } k + l \leq d-1. \tag{49}$$

Condition (46) follows from the fact that the multiplication by x is injective in the relevant degrees (see Lemma 1.6), at least if d or n are large enough.

The second obvious property of a Schiffer variation is described in the following lemma.

Lemma 3.8. *Let $x \in S^1$ determine a first order Schiffer variation $f_t = f + tx^d$ of f with tangent vector $\phi = x^d \in R_f^d$, and let $I_{x,d-1,t}^{*d} := x^{d-1} R_{f_t}^{*d-d+1} \subset R_{f_t}^{*d}$ be defined as above. Then the quotient ring $R_{f_t}^{*d}/I_{x,d-1,t}^{*d}$ does not deform (as a ring) along the Schiffer variation (f_t).*

Proof. Indeed, if $f_t = f + tx^d$, then $J_{f_t} = J_f$ modulo x^{d-1}, hence the quotient

$$S^{*d}/(J_{f_t}^{*d} + x^{d-1} S^{(*-1)d+1})$$

is constant. A fortiori, its isomorphism class as a graded ring does not depend on t. □

4 Proof of Theorem 0.6 when d divides $n + 1$

4.1 Specialization and Schiffer variations

We will consider in this section singular hypersurfaces X_f of degree d in $\mathbb{P}^n$ defined by a polynomial of the form $f = \sum_{i=1}^m f_i g_i$, with $n - 2m \geq 0$. If $d = 2d'$ is even, we will choose the f_i and g_i to be of degree d' and if $d = 2d' + 1$ we will choose the f_i of degree d' and the g_i of degree $d' + 1$. The hypersurface X_f is then singular along the variety Z defined by the polynomials f_i and g_i which are all of degree $\leq \frac{d+1}{2}$, and when they are generically chosen, it is of dimension $n - 2m$. Let us start with the following result of independent interest, which will be important below and in the next section.

Proposition 4.1. *Let f be a homogeneous polynomial of degree d in $n+1$ variables, defining a hypersurface X_f singular along a smooth subvariety Z defined by homogeneous polynomial equations of degree $\leq \frac{d+1}{2}$. Then the dimension of the space R_f^k is equal to the dimension of the space $R_{f_{gen}}^k$ for a generic polynomial f_{gen}, assuming*

$$k < (n - \dim Z + 1)\frac{d-3}{2}. \tag{50}$$

Proof of Proposition 4.1. Recall that J_f is generated by the partial derivatives $\frac{\partial f}{\partial X_i}$ for $i = 0, \ldots, n$. For the generic polynomial f_{gen}, these partial derivatives form a linear system W of degree $d-1$ polynomials with no base-point on $\mathbb{P}^n$, and the associated Koszul resolution

$$0 \to \bigwedge^{n+1} W \otimes \mathcal{O}_{\mathbb{P}^n}(-(n+1)(d-1)) \to \ldots \to W \otimes \mathcal{O}_{\mathbb{P}^n}(-(d-1)) \stackrel{\alpha}{\to} \mathcal{O}_{\mathbb{P}^n} \to 0, \qquad (51)$$

twisted by $\mathcal{O}_{\mathbb{P}^n}(k)$, allows to compute the dimension of $R^k_{f_{gen}}$, or equivalently $J^k_{f_{gen}} = \operatorname{Im} H^0(\alpha(k))$, using the fact that this twisted Koszul complex, that we will denote by $\mathcal{K}^*_{\mathbb{P}^n,k}$, remains exact at the level of global sections, at least in strictly negative degrees, where we put the last term $\mathcal{O}_{\mathbb{P}^n}(k)$ in degree 0), which is what we need to compute $\dim J^k_{f_{gen}}$. We thus get an equality

$$\dim J^k_{f_{gen}} = (n+1) \dim S^{k-d+1} - \frac{n(n+1)}{2} \dim S^{k-2d+2} + \binom{n+1}{3} \dim S^{k-3d+3} \ldots. \qquad (52)$$

In our special case, the partial derivatives $\frac{\partial f}{\partial X_i}$ form a linear system W of degree $d-1$ polynomials on $\mathbb{P}^n$ with base-locus Z and we have to understand how this affects the computation. Clearly, the Koszul complex (51) is no more exact. Let $\tau : Y \to \mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ along Z, and let E be the exceptional divisor of τ. Then W provides a base-point free linear system, that we also denote by W, of sections of the line bundle $L := \tau^* \mathcal{O}_{\mathbb{P}^n}(d-1)(-E)$ on Y. Then we have an exact Koszul complex on Y associated with W, which has the following form:

$$0 \to \bigwedge^{n+1} W \otimes \mathcal{O}_Y(-(n+1)L) \to \ldots \to W \otimes \mathcal{O}_Y(-L) \stackrel{\alpha'}{\to} \mathcal{O}_Y \to 0. \qquad (53)$$

We now twist by $\tau^* \mathcal{O}_{\mathbb{P}^n}(k)$ so that $\operatorname{Im} H^0(\alpha'(k)) = \dim J^k_f$. Let $\mathcal{K}^*_{Y,k}$ be this twisted Koszul complex. We observe that, as only positive twists of E appear in $\mathcal{K}^*_{Y,k}$,(at least in negative degrees,) the global sections of $\mathcal{K}^*_{Y,k}$ are $\bigwedge^i W \otimes S^{k-i(d-1)}$ in degree $-i$. So our problem is actually to prove that if the inequality (50) holds, the complex $K^*_{Y,k}$ of global sections of $\mathcal{K}^*_{Y,k}$ is as before exact in strictly negative degrees. To prove this, we have to analyze the hypercohomology spectral sequence

$$E_1^{p,q} = H^q(Y, \mathcal{K}^p_{Y,k}) \Rightarrow \mathbb{H}^{p+q}(Y, \mathcal{K}^*_{Y,k}). \qquad (54)$$

of $\mathcal{K}^*_{Y,k}$. As $\mathcal{K}^*_{Y,k}$ is exact, one has $\mathbb{H}^{p+q}(Y, \mathcal{K}_{Y,k}) = 0$, hence

$$E_\infty^{p,q} = 0. \qquad (55)$$

We have

$$H^q(Y, \mathcal{K}^p_{Y,k}) = \bigwedge^{-p} W \otimes H^q(Y, \mathcal{O}_Y(pL(k)),$$

where

$$pL(k) = \tau^* \mathcal{O}_{\mathbb{P}^n}(p(d-1)+k)(-pE). \qquad (56)$$

We now observe that, when $-p < n - \dim Z$, we have $H^q(Y, \mathcal{K}^p_{Y,k}) = 0$ for $q \neq 0$, n. This is because one then has $R^i \pi_* \mathcal{K}^p_{Y,k} = 0$ for $i > 0$, and

$$R^0 \pi_* \mathcal{K}^p_{Y,k} \cong \mathcal{O}_{\mathbb{P}^n}(p(d-1)k).$$

When $-p \geq n - \dim Z$, we have $H^q(Y, \mathcal{K}^p_{Y,k}) = 0$ for $q < n - \dim Z - 1$. Indeed, this follows from the Leray spectral sequence of $pL(k)$ with respect to the map τ and the fact that $R^0 \tau_* pL(k)$ has nonzero cohomology only in degree n, and the only nonzero other higher direct images is $R^{n-\dim Z-1} \tau_*(pL(k))$, which contributes to cohomology of degree $\geq n - \dim Z - 1$.

A second source of vanishing comes from Kodaira vanishing applied to the line bundle $pL(k)$ on Y. We observe that Z is by assumption defined by equations of degree $\leq \frac{d+1}{2}$, so that the line bundle $\tau^*\mathcal{O}_{\mathbb{P}^n}(l)(-mE)$ on Y is nef and big when $m \geq 0$ and $l > m\frac{d+1}{2}$. Using (56), Kodaira vanishing thus tells us that $H^q(Y, \mathcal{K}^p_{Y,k}) = 0$ for $q < n$ if $-p(d-1)-k > -p\frac{d+1}{2}$.

Summarizing, we thus proved that the spectral sequence (54) has vanishing as follows

$$E_1^{p,q} = 0 \text{ if } p < n - \dim Z, \text{ and } \ n \neq q, \tag{57}$$

$$E_1^{p,q} = 0 \text{ if, } -p \geq n - \dim Z, \text{ and } q < n - \dim Z - 1, \tag{58}$$

$$E_1^{p,q} = 0 \text{ if } q < n \text{ and } -p(d-1) - k > -p\frac{d+1}{2}. \tag{59}$$

We now conclude the proof. The complex $K^*_{Y,k}$ of global sections of the complex $\mathcal{K}^*_{Y,k}$ is the complex $E_1^{*,0}$ of our spectral sequence, hence its cohomology is the complex $E_2^{*,0}$. Recall that we have to prove the vanishing of the cohomology of $K^*_{Y,k}$ in strictly negative degrees. Let $a < 0$ be a fixed negative integer. There is clearly no differential d_r with $r \geq 2$ starting from $E_2^{a,0}$. As $E_\infty^{a,0} = 0$ (see 55), it follows that, if $E_2^{a,0}$ is nonzero, there must be a nonzero differential

$$d_r : E_r^{a-r,r-1} \to E_r^{a,0} = E_2^{a,0}.$$

Let $p = a - r$. By the vanishing (57), (58), we must have $r - 1 = n$ or, if $-p < n - \dim Z$, $r - 1 \geq n - \dim Z - 1$. If $r - 1 = n$, then, as $a < 0$, $p = a - r < -n - 1$. The term $\mathcal{K}^p_{Y,k}$ is then 0. Hence the only nontrivial differential appears when $r - 1 \neq n$. But then, by (58), $r \geq n - \dim Z$ and thus

$$p = a - r < -n + \dim Z. \tag{60}$$

Furthermore, using (59), $-p(d-1) - k \leq -p\frac{d+1}{2}$, that is,

$$-p\frac{d-3}{2} \leq k. \tag{61}$$

Combining (60) and (61), we proved that the existence of a nonzero $E_2^{a,0}$ for some $a < 0$ implies

$$(n - \dim Z + 1)\frac{d-3}{2} \leq k. \tag{62}$$

which contradicts inequality (50). □

Imposing the dimension of Z to be at most 4 (we will later choose dimension of Z to be equal to 3 if n is even and 4 if n is odd), we get

Corollary 4.2. *For a polynomial f as above with d dividing $n+1$ and $\dim Z \leq 4$, the dimension of the spaces R_f^d, R_f^{2d} and R_f^{3d} are respectively equal to the dimensions of the corresponding spaces $R_{f_{gen}}^d$, $R_{f_{gen}}^{2d}$ and $R_{f_{gen}}^{3d}$ for generic f_{gen}, assuming $d \geq 13$.*

Proof. Indeed, if $\dim Z \leq 4$ and $k \leq 3d$, (50) is satisfied if

$$3d < (n-3)\frac{d-3}{2}. \tag{63}$$

As $n \geq d - 1$ and $d \geq 3$, (63) is satisfied if $3d < (d-4)\frac{d-3}{2}$, hence if $d \geq 13$. □

Remark 4.3. The estimate of Corollary 4.2 is sharp only when $d = n + 1$.

We now assume $f = \sum_{i=1}^m f_i g_i$ with f_i, g_i generic and d, n are such that the conclusion of Corollary 4.2 holds. We observe that, with the same notation as above, as f is singular along Z, one has $J_f \subset I_Z$, hence the Jacobian ring R_f^{d*} has $H^0(Z, \mathcal{O}(d*))$ as a quotient. We will use the notation $g_{|Z}$ for the image of an element g in this quotient. With the same notation I_l^k as in (46)-(49), let us denote $\overline{I}_l^k := I^k_{l|Z}$. Let us prove the following.

Lemma 4.4. *Let $\phi \in R^d_f$, and $I^{*d}_k \subset R^{*d}_f$ satisfy condition (*) (see (46)-(49)). Then either $\overline{I}^d_{d-1} = 0$ and $\overline{I}^{2d}_{d-1} = 0$ or there exists an element g of R^k_f with $k \geq d-1$, such that $g_{|Z} \neq 0$ and*

$$\overline{I}^d_{d-1} \subset gH^0(Z, \mathcal{O}_Z(d-k)),\ \overline{I}^{2d}_{d-1} \subset gH^0(Z, \mathcal{O}_Z(2d-k)). \tag{64}$$

Proof. For a nonzero linear system W on Z, let us denote by $FL(W)$ (for the "fixed locus") the divisorial part of the base-locus of W. We now observe that, as Z is a smooth complete intersection of dimension at least 3, one has $\mathrm{CH}^1(Z) = \mathrm{Pic}\, Z = \mathbb{Z}\mathcal{O}_Z(1)$ by Grothendieck-Lefschetz theorem. In particular, if $\overline{I}^k_l \neq 0$, we have

$$FL(\overline{I}^k_l) = D_l \in |\mathcal{O}_Z(d_{l,k})|,$$

for some nonnegative integers $d_{l,k}$.

We first make the following

Claim 4.5. *For d large enough, one has $\overline{I}^d_1 \neq 0$, and $d_{1,d} \leq 1$.*

Proof. This is proved by a dimension argument. Indeed, it suffices to prove that

$$\dim \overline{I}^d_1 > h^0(Z, \mathcal{O}_Z(d-2)). \tag{65}$$

As $\dim I^d_1 = \dim R^{d-1}_f$ by (46), one has $\dim \overline{I}^d_1 \geq \dim S^{d-1} - \dim I_Z(d)$ and it thus suffices to prove that

$$h^0(Z, \mathcal{O}_Z(d-2)) < \dim S^{d-1} - \dim I_Z(d). \tag{66}$$

Recalling that Z is a complete intersection of $2m < n$ hypersurfaces defined by equations f_i of degree d' and g_i of degree d'', with $d''-1 \leq d' \leq d''$, and $d' + d'' = d$, we conclude that

$$h^0(Z, \mathcal{O}_Z(d-2)) = \dim S^{d-2} - m(\dim S^{d''-2} + \dim S^{d'-2}),$$

and that

$$\dim I_Z(d) \leq m(\dim S^{d''} + \dim S^{d'}).$$

Inequality (66) will thus be a consequence of

$$\dim S^{d-2} - m(\dim S^{d''-2} + \dim S^{d'-2}) < \dim S^{d-1} - m(\dim S^{d''} + \dim S^{d'}). \tag{67}$$

Inequality (67) easily follows (at least for d large enough) from our conditions $n > 2m$ and $d' = d'' = d/2$ if d is even, $d' = d''_1 = (d-1)/2$ if d is odd. ☐

We now use the fact that $\overline{I}^d_1 \overline{I}^d_{d-1} \subset \phi_{|Z} \cdot H^0(Z, \mathcal{O}_Z(d))$ (see (47)). Combined with Claim 4.5, this implies that, either $\overline{I}_{d-1} = 0$, or $d_{d-1,d} \geq d-1$. Similarly, as

$$\overline{I}^d_1 \overline{I}^{2d}_{d-1} \subset \phi_{|Z} \cdot H^0(Z, \mathcal{O}_Z(2d)),$$

we conclude that $d_{d-1,2d} =: k \geq d-1$. Finally we use the fact that $H^0(Z, \mathcal{O}_Z(d)) \cdot \overline{I}^d_{d-1} \subset \overline{I}^{2d}_{d-1}$ (see (48)) to deduce that the same g of degree $k \geq d-1$ works for both $\overline{I}^d_{d-1}$ and $\overline{I}^{2d}_{d-1}$. ☐

We now want to study, for a generic polynomial f of the form $\sum_{i=1}^m f_i g_i$ as above, the elements $\phi \in R^d_f$ which both satisfy condition (*) and the property described in Lemma 3.8. As Lemma 4.1 and Corollary 4.2 hold only for R^d_f, R^{2d}_f and R^{3d}_f and not for the whole R^{*d}_f when f is singular, we are going to use only the data of the multiplication map of R^*_f in degree d, which is described by a triple (R^d_f, R^{2d}_f, μ) consisting of (isomorphism class of) two vector spaces of dimensions $\dim R^d_{f_{gen}}$, resp. $\dim R^{2d}_{f_{gen}}$, and a symmetric linear map

$$\mu_f : R^d_f \otimes R^d_f \to R^{2d}_f.$$

We will also consider similar data $\overline{\mu}_f : \overline{R}^d \otimes \overline{R}^d \to \overline{R}^{2d}$ for quotients of R^{d*}_f and $\mu : S^d \otimes S^d \to S^{2d}$ for the multiplication in the polynomial ring itself. We will call such data a "partial ring".

We study now elements $\phi \in R^d_f$, I^d_1 satisfying the following condition (**) (satisfied by Schiffer variations, see Section 3.1)

(**) *(i) There exist $I^d_k \subset R^d_f$, $I^{2d}_k \subset R^{2d}_f$, $I^{3d}_k \subset R^{3d}_f$ satisfying condition (*) (see 46)-(49).*

(ii) Along a one-parameter family f_t, with $f_0 = f$ and $\frac{d}{dt}(f_t)_{|t=0} = \phi$, there exist data $I^{*d}_{k,t} \subset R^{*d}_{f_t}$, $* = 1, 2, 3$, associated to $\phi_t = \frac{d}{dt}(f_t) \in R^d_{f_t}$, and also satisfying condition (*).

(iii) The (isomorphism class of the) partial ring $(R^d_{f_t}/I^d_{d-1,t}, R^{2d}_{f_t}/I^{2d}_{d-1,t}, \overline{\mu})$ does not deform with t.

Proposition 4.6. *For d sufficiently large and for a generic $f = \sum_{i=1}^m f_i g_i$ as above, any $\phi \in R^d_f$ satisfying (**) is a first order Schiffer variations of f.*

The proof of Proposition 4.6 will use several preliminary lemmas.

Lemma 4.7. *The assumptions being the same as in Proposition 4.6, then*

(a) If $\overline{I}^{2d}_{d-1} = 0$, f_t remains singular along Z (or rather, a variety isomorphic to Z). In particular $\phi_{|Z} = 0$.

(b) If $FL(\overline{I}^{2d}_{d-1})$ is defined by $g \in H^0(Z, \mathcal{O}_Z(k))$, f_t remains singular along the locus $Z_g := \{g = 0\} \subset Z$.

(c) If $k \geq d$ in (b), f_t remains singular along Z.

Proof. (a) If $\overline{I}^{2d}_{d-1} = 0$, the partial ring $(R^d_f/I^d_{d-1}, R^{2d}_f/I^{2d}_{d-1}, \overline{\mu}_f)$ admits the partial ring $(H^0(Z, \mathcal{O}_Z(d)), H^0(Z, \mathcal{O}_Z(2d)), \mu_Z)$ as a quotient. As by assumption, the quotient

$$(R^d_{f_t}/I^d_{d-1,t}, R^{2d}_{f_t}/I^{2d}_{d-1,t}, \overline{\mu}_{f_t})$$

of $(R^d_{f_t}, R^{2d}_{f_t}, \mu_{f_t})$ is isomorphic to $(R^d_f/I^d_{d-1}, R^{2d}_f/I^{2d}_{d-1}, \overline{\mu})$, we conclude that the partial ring $(R^d_{f_t}, R^{2d}_{f_t}, \mu_{f_t})$ also admits the partial ring $(H^0(Z, \mathcal{O}_Z(d)), H^0(Z, \mathcal{O}_Z(2d)), \mu_Z)$ as a quotient. Denoting by $\alpha_t : S^{*d} \to H^0(Z, \mathcal{O}_Z(*d))$ the quotient map for $* = 1, 2$, this means that we have a commutative diagram

$$\begin{array}{ccc} S^d \otimes S^d & \xrightarrow{\mu} & S^{2d} \\ \downarrow{\scriptstyle \alpha_t \otimes \alpha_t} & & \downarrow{\scriptstyle \alpha_t} \\ H^0(Z, \mathcal{O}_Z(d)) \otimes H^0(Z, \mathcal{O}_Z(d)) & \xrightarrow{\mu_Z} & H^0(Z, \mathcal{O}_Z(2d)), \end{array} \tag{68}$$

where α_t is surjective with kernel containing J_{f_t}, since it factors through R_{f_t}. The map α_t gives an embedding j_t of Z in $\mathbb{P}((S^d)^*)$. As the quadrics in $\operatorname{Ker} \mu$ are the defining equations for the d-th Veronese embedding $V_d(\mathbb{P}^n)$ in $\mathbb{P}((S^d)^*)$, one concludes that j_t factors through an embedding j'_t of Z in $\mathbb{P}^n = \mathbb{P}((S^1)^*)$, that is $j_t = V_d \circ j'_t$. As $\operatorname{Pic} Z$ has no torsion because $\dim Z \geq 2$ and Z is a complete intersection in $\mathbb{P}^n$, j'_t is given by sections of $\mathcal{O}_Z(1)$. For $t = 0$, j'_0 is the original embedding, hence is linearly normal. It follows that this is also true for t close to 0. Hence j'_t is (up to the action of $PGl(n+1)$) the original embedding for t close to 0. Finally, as the map $\alpha_t = (j'_t)^*$ contains J_{f_t} in its kernel, J_{f_t} vanishes on $j'_t(Z)$, which means that f_t is singular along $j'_t(Z)$.

(b) We know that for $t = 0$, and $* = 1, 2$, $(I^{*d}_{d-1})_{|Z}$ is contained in the ideal generated by g. It follows that the partial ring $(R^d_f/I^d_{d-1}, R^{2d}_f/I^{2d}_{d-1}, \overline{\mu}_f)$ has the partial ring

$$(H^0(Z_g, \mathcal{O}_{Z_g}(d)), H^0(Z_g, \mathcal{O}_{Z_g}(2d)), \mu_{Z_g})$$

as a quotient, where $Z_g := \{g = 0\} \subset Z$. We can then argue exactly as before, using the fact that $\mathrm{Pic}\, Z_g$ has no torsion (we use again the fact that $\dim Z \geq 3$). We then conclude that f_t is singular along $j_t(Z_g)$. $\square$

We next make the following observation

Lemma 4.8. *Let Z be a smooth complete intersection of dimension ≥ 3 of hypersurfaces X_{h_j} of degrees $d_j \geq 2$ and let f_t be a polynomial of degree d such that f_t is singular along Z_g for some $0 \neq g \in H^0(\mathcal{O}_Z(k))$ with $k \geq d-1$. Then either f_t is singular along Z or there exists a $x \in S^1$ such that $f - x^d$ is singular along Z.*

Proof. We first claim that if $f_{t|Z} = 0$, then f_t is singular along Z. This is proved as follows: As $f_{t|Z} = 0$, we can write $f_t = \sum_j a_j h_j$, with $\deg a_j = d - d_j$. The differential of f_t vanishes at a point $z \in Z$ if and only if all a_j vanish at z. As the a_j's are of degree $< d-1$, the vanishing of df_t along Z_g implies the vanishing of df_t along Z.

Next, if $k \geq d$, we conclude that the partial derivatives of f vanish identically along Z, since they vanish along Z_g, so f is singular along Z. We can thus assume that $f_{|Z} \neq 0$ and $k = d_1$.

We then claim that there exists an $x \in S^1$ such that $(f_t - x^d)_{|Z} = 0$. We use here the fact that $\dim Z \geq 3$ so that $\mathrm{Pic}\, Z = \mathbb{Z}\mathcal{O}_Z(1)$. We decompose $g \in H^0(Z, \mathcal{O}_Z(d-1))$ into irreducible factors as

$$g = \prod_j \gamma_j^{a_j},$$

where $\gamma_j \in H^0(Z, \mathcal{O}_Z(d_j))$ and $\sum_j a_j d_j = d-1$. Now if $f_{t|Z}$ vanishes to order b_j along $\{\gamma_j = 0\}$, df_t vanishes to order $\leq b_j - 1$ along $\{\gamma_j = 0\}$. We thus conclude that $b_j \geq a_j + 1$. As $\sum_j b_j \leq d$ and $\sum_j a_j = d-1$, we conclude that there is a single j and the corresponding a_j equals $d-1$, which proves the second claim.

The second claim finally implies Lemma 4.8 since $f_t - x^d$ vanishes along Z and is singular along Z_g, with $g = x^{d-1}_{|Z}$, so that the first claim applies to show that $f_t - x^d$ is singular along Z. $\square$

Proof of Proposition 4.6. With the notation and assumptions of Proposition 4.6, Lemma 4.7 tells us that, modulo the action of $Gl(n+1)$, we can assume f_t is singular along Z or f_t is singular along Z_g. Lemma 4.8 then says that for some $x \in S^1$, $f_t - x^d$ is singular along Z, and the same is true for $\phi = \frac{\partial f_t}{\partial t}_{|t=0}$. It follows that either (i) $\phi \in I^2_Z(d)$ or (ii) $\phi - x^d \in I^2_Z(d)$.

We use now the fact (this is (47) in condition (*)) that

$$I^d_k \cdot I^d_{d-k} \subset \phi R^d_f, \tag{69}$$

for $1 \leq i \leq d$, with $I^d_k \subset R^d_f$ of dimension equal to $\dim R^{d-k}_f$ (this is (46 in condition (*)). We previously used this condition only for $k = 1$. We are going to use it for $k = 3$ to prove the following claim which excludes case (i).

Claim 4.9. *For d large enough and f, Z generic, a nonzero $\phi \in R^d_f$ satisfying the condition (*) cannot belong to I^2_Z.*

Proof. First of all, we use the same dimension arguments as in the proof of Claim 4.5 to show that $\overline{I}^d_3 := (I^d_3)_{|Z} \neq 0$. More precisely, we can show that it is of dimension $> h^0(Z, \mathcal{O}_Z(d-4))$, at least if d is large enough. If $\phi_{|Z} = 0$, we then have $I^d_{d-3} \subset I_Z(d)$ since

$$I^d_3 \cdot I^d_{d-3} \subset \phi R^d_f \subset I_Z(2d) \bmod J_f. \tag{70}$$

On the one hand, as $\dim I^d_{d-3} = \dim S^3$ for $d > 4$, and $\dim I^2_Z(d) < \dim S^3$, I^d_{d-3} is not contained in $I^2_Z(d)$. On the other hand, if we look at the image of I^d_{d-3} in $I_Z(d)/(I^2_Z(d)+J^d_f)$, it is annihilated by multiplication by elements of $\overline{I}^d_3$ acting by

$$H^0(Z,\mathcal{O}_Z(d)) \supseteq \overline{I}^d_3 \ni \alpha : I_Z(d)/(I^2_Z(d)+J^d_f) \to I_Z(2d)/(I^2_Z(2d)+J^{2d}_f).$$

This follows indeed from the condition $\phi \in I^2_Z(d)$ and (70). Now, writing $f = \sum_j f_j g_j$ with $\deg f_j = d'$ and $\deg g_j = d''$, we have a graded isomorphism (given by differentiation along Z)

$$(I_Z/I^2_Z)(*) \cong \oplus^m_{j=1} H^0(Z,\mathcal{O}_Z(*-d')) \oplus^m_{j=1} H^0(Z,\mathcal{O}_Z(*-d'')),$$

which to $\sum_j a_j f_j + b_j g_j$ associates $(a_j,b_j)_{j=1,\dots,m}$. By the Leibniz rule, this isomorphism maps $\frac{\partial f}{\partial X_i} \in J_f$ to the $2m$-uple $(\frac{\partial g_j}{\partial X_i}, \frac{\partial f_j}{\partial X_i})_{j=1,\dots,m}$. In other words, observing that we have a natural isomorphism $N_{Z/\mathbb{P}^n} \cong N^*_{Z/\mathbb{P}^n}(d)$ given by the quadratic form defined as the Hessian of f along Z, we have on the one hand the composite morphism $\mathcal{I}_Z \to N^*_Z \cong N_Z(-d)$ and on the other hand the normal bundle sequence of Z

$$0 \to T_Z \to T_{\mathbb{P}^n|Z} \stackrel{\beta}{\to} N_Z \to 0. \tag{71}$$

Then the computation above shows that

$$I_Z(*)/(I^2_Z(*)+J^*_f) \cong H^0(Z,N_Z(*-d))/\mathrm{Im}\, H^0(\beta(*-d)), \tag{72}$$

and these isomorphisms are compatible with the multiplication map by $b \in H^0(Z,\mathcal{O}_Z(d))$. Finally, the exact sequence (71) together with the fact that $\dim Z \geq 3$ show that the right hand side in (72) is isomorphic to $H^1(Z,T_Z)$. Let now $w \in I^d_{d-3} \subset I_Z(d)$ such that $w \neq 0$ in $I_Z(d)/(J^d_f + I^2_Z(d))$. Then w has a nonzero image $\overline{w} \in H^1(Z,T_Z)$ and $\overline{w}$ is annihilated by multiplication by any $b \in \overline{I}^d_3 \subset H^0(Z,\mathcal{O}_Z(d))$, that is,

$$b\overline{w} = 0 \text{ in } H^1(Z,T_Z(d)) \tag{73}$$

for any $b \in \overline{I}^{d-1}_3$. The extension class $\overline{w} \in H^1(Z,T_Z)$ determines a vector bundle F on Z which fits in an exact sequence

$$0 \to T_Z \to F \to \mathcal{O}_Z \to 0, \tag{74}$$

and the condition (73) says equivalently that $\overline{I}^d_3 \subset H^0(Z,\mathcal{O}_Z(d))$ lifts to sections of $F(d)$. Let $\mathcal{G} \subset F(d)$ be the coherent subsheaf generated by the global sections of $F(d)$. Observe that $\det \mathcal{G} = \mathcal{O}_Z(k)$ with $k \geq 0$. Assume first that $\mathcal{G}$ has rank 1. Then we have

$$h^0(Z,\mathcal{O}_Z(k)) \geq \dim \overline{I}^d_3$$

and we already noted that the right hand side is $> h^0(Z,\mathcal{O}_Z(d-4))$. It follows that $k \geq d-3$, and that for some $0 \neq \sigma \in h^0(Z,\mathcal{O}_Z(3))$, one has

$$\sigma\overline{w} = 0 \text{ in } H^1(Z,T_Z(3)). \tag{75}$$

Equation (75) says that $\overline{w}$ is coming from a section of $H^0(Z_\sigma,T_{Z|Z_\sigma}(3))$, where $Z_\sigma := \{\sigma = 0\} \subset Z$. A dimension count shows that for d large enough and Z generic as above, there does not exist a cubic section Z_σ of Z and a nonzero section of $T_{Z|Z_\sigma}(3)$. This case is thus ruled-out. We thus conclude that the rank of $\mathcal{G}$ is at least 2. We then get a contradiction as follows. Let now $\mathcal{G}' := \mathrm{Ker}\,(\mathcal{G} \to \mathcal{O}_Z(d))$. We have $\det \mathcal{G}' = \mathcal{O}_Z(k')$ with $k' \geq -d$. Thus the slope of $\mathcal{G}'$ is at least $-d$. Recall that Z is a complete intersection of m hypersurfaces of degree d' and m hypersurfaces of degree d'' with $d'+d'' = d$ and that $s := \dim Z$ is equal to 3 or 4. It follows that $n = 2m+s$ and

$$K_Z = \mathcal{O}_Z(-n-1+md) = \mathcal{O}_Z(-2m-s-1+md) = \mathcal{O}_Z(m(d-2)-s-1).$$

Thus the slope of T_Z is at most $\frac{-m(d-2)+5}{4}$. Hence we have slope $\mathcal{G}' >$ slope T_Z if

$$-d > \frac{-m(d-2)+5}{4},$$

which holds if $m \geq 6$, $d \geq 9$. This gives a contradiction for d large enough since Z is a variety with ample canonical bundle, hence has stable tangent bundle by [1], [12] or [9]. The claim is thus proved. □

We are thus in case (ii), that is,

$$\phi = x^d + \alpha \text{ mod } J_f^d \tag{76}$$

for some $\alpha \in I_Z^2$, and we need to show that, in fact, $\phi = x^d$ mod J_f^d. We start with the following lemma, where we use again the notation $\overline{I}_k^d := (I_k^d)_{|Z}$.

Lemma 4.10. *One has* $\overline{I}_1^d \subset xH^0(Z, \mathcal{O}_Z(d-1))$.

Proof. We have

$$\begin{aligned} \overline{I}_1^d \cdot \overline{I}_{d-1}^d &\subset x^d H^0(Z, \mathcal{O}_Z(d)), \\ \overline{I}_1^d \cdot \overline{I}_{d-1}^{2d} &\subset x^d H^0(Z, \mathcal{O}_Z(2d)) \end{aligned} \tag{77}$$

If $\overline{I}_1^d \not\subset xH^0(Z, \mathcal{O}_Z(d-1))$, then (77) imply that

$$\overline{I}_{d-1}^d \subset \mathbb{C}x^d,\ \overline{I}_{d-1}^{2d} \subset x^d H^0(Z, \mathcal{O}_Z(d)). \tag{78}$$

By Lemma 4.7, (c), this implies that f_t remains singular along Z. Thus $f_t \in I_Z^2(d)$ and $\phi \in I_Z^2(d)$, contradicting (76).

Corollary 4.11. *Let* $I_1' \subset I_1^d$ *be defined by* $I_1' = I_1^d \cap xR_f^{d-1}$, *and let* $\overline{I}_1' := (I_1')_{|Z} \subset xH^0(Z, \mathcal{O}_Z(d-1))$. *Then for d (hence also n) large enough*

$$\dim \overline{I}_1' > h^0(Z, \mathcal{O}_Z(d-2)).$$

Proof. Indeed, as $\overline{I}_1^d \subset xH^0(Z, \mathcal{O}_Z(d-1))$, we have $I_d^1 \subset xR_f^{d-1} + I_Z(d)$, hence

$$\text{codim}\,(I_1' \subset I_d^1) \leq \dim I_Z(d),$$

which implies a fortiori

$$\text{codim}\,(\overline{I}_1' \subset \overline{I}_d^1) \leq \dim I_Z(d).$$

The inequality $\dim \overline{I}_1' > h^0(Z, \mathcal{O}_Z(d-2))$ is then proved for d large enough in the same way as the inequality (65) proved in Claim 4.5. □

We come back to our $\phi = x^d + \alpha$ satisfying the conditions (*), with $0 \neq x \in S^1$, and $\alpha \in I_Z^2(d)$. By (77) and using the fact that $\dim \overline{I}_1^d > h^0(Z, \mathcal{O}_Z(d-2))$, we conclude that

$$\overline{I}_{d-1}^d \subset x^{d-1} H^0(Z, \mathcal{O}_Z(1)),$$

which we write $w = x^{d-1}y + k_y$ for any $w \in I_{d-1}^d$, where $y \in S^1$ and $k_y \in I_Z(d)$ mod J_f^d. For $a = xa' \in I_1' \subset R_f^d$, and $w \in I_{d-1}^d$, we then have

$$xa'w = xa'(x^{d-1}y + k_y) = (x^d + \alpha)\gamma_{a',w} \text{ in } R_f^{2d}. \tag{79}$$

Restricting to Z, we get $(\gamma_{a',w})_{|Z} = a'_{|Z} y_{|Z}$, which we write

$$\gamma_{a',w} = a'y + \gamma'_{a',w}$$

for some $\gamma'_{a',w} \in I_Z(d)$ which depends linearly on a', for w fixed.

We now use again the observation that $\dim I_Z(d)$ is (asymptotically) small compared to $h^0(Z, \mathcal{O}_Z(d-1))$ and conclude that for a subspace $I''_1 \subset I'_1$ such that $\dim (I''_1)_{|Z} > h^0(Z, \mathcal{O}_Z(d-2))$, one can take $\gamma'_{a',w} = 0$ in R^d_f, so that (79) becomes $xa'(x^{d-1}y + k_y) = (x^d + \alpha)a'y$ in R^{2d}_f, that is,

$$xa'k_y = \alpha a'y \text{ in } R^{2d}_f. \tag{80}$$

The right hand side belongs to $(I^2_Z(2d) + J^{2d}_f)/J^{2d}_f$. We argue now as in the proof of Claim 4.9 to deduce that $k_y \in (I^2_Z(d) + J^d_f)/J^d_f$. Indeed, we consider the image $\overline{k}_y$ of k_y in $I_Z(d)/(I^2_Z + J^d_f)$ and (82) says that it is annihilated by multiplication by xa' for $xa' \in I'_1$, which is of large dimension. Then we conclude that $\overline{k}_y = 0$.

The equations (82) are thus relations in $(I^2_Z + J_f)/J_f$. We claim that

$$I^2_Z(2d) \cap J^{2d}_f = I_Z(d+1) \cdot J^{d-1}_f. \tag{81}$$

Indeed, recall from the proof of Claim 4.9 that the image of J^*_f in $I_Z(*)/I^2_Z$ identifies naturally with the image of $H^0(\mathbb{P}^n, T_{\mathbb{P}^n}(*-d+1))$ in $H^0(Z, N_Z(*-d+1))$. We have the exact sequence

$$0 \to T_Z \to T_{\mathbb{P}^n|Z} \to N_Z \to 0$$

and we observe as in the proof of Claim 4.9 that the stability of the tangent bundle of Z implies that $h^0(Z, T_Z(d)) = 0$ for d (hence n) large enough. It follows that the map $H^0(Z, T_{\mathbb{P}^n|Z}(d) \to H^0(Z, N_Z(d))$ is injective, and thus

$$I^2_Z(2d) \cap J^{2d}_f = \mathrm{Ker}\,(J^{2d}_f \to I_Z/I^2_Z(2d))$$

comes from $H^0(\mathbb{P}^n, T_{\mathbb{P}^n} \otimes \mathcal{I}_Z(d))$, which proves (81). We thus conclude that

$$\dim I^2_Z(2d) \cap J^{2d}_f \leq (n+1) \dim I_Z(d+1)$$

which is, for d (hence n) large enough, much smaller than $\dim I'_1$. It follows that, taking representatives of k_y, α in $I^2_Z(d)$, we have an actual vanishing

$$xa''k_y = \alpha a''y \text{ in } I^2_Z(2d) \tag{82}$$

for a nonzero $a'' \in S^d$. Thus $xk_y = \alpha y$ in $I^2_Z(d+1)$. Using the fact that the space of (y, k_y) satisfying this property has dimension $\geq n+1$, we conclude that $k_y = 0$ for generic (y, k_y) and thus $\alpha = 0$. □

Proposition 4.6 is now proved. □

Remark 4.12. Note that, in turn, $\alpha = 0$ and equation (82) imply that $k_y = 0$, so that we also proved that $I^d_{d-1} = x^{d-1}S^1 \mod J^d_f$. This will be used below.

4.2 Proof of Theorem 0.6

We conclude in this section the proof of Theorem 0.6. We start by establishing the following.

Proposition 4.13. *Let f be a generic homogeneous polynomial of degree d in $n+1$ variables with d dividing $n+1$ and d large enough. Let $\phi \in R^d_f$, $I^{*d}_k \subset R^{*d}_f$, for $* \leq 3$ and $1 \leq k \leq d-1$ satisfy condition (**) of section 4.1. Then ϕ is a (first order) Schiffer variation of f.*

Proof. Proposition 4.6 proves Proposition 4.13 when $f = \sum_{j=1}^m f_j g_j$ is the singular polynomial used in previous section. It thus remains to see that this implies the same result for the generic f. This almost follows because the condition is closed on f once the dimensions of the spaces R^d_f, R^{2d}_f, R^{3d}_f remain respectively equal to the dimensions of the spaces $R^d_{f_{gen}}$, $R^{2d}_{f_{gen}}$, $R^{3d}_{f_{gen}}$ for the generic f_{gen}, which is guaranteed for d large enough by Lemma 4.1. This is not completely true because we did not prove the statement of Proposition 4.6 schematically for the special f. In fact, what we have to do in order to conclude is to prove the following complement to Proposition 4.6.

Lemma 4.14. *Let the notation and assumption on f, ϕ be as in Proposition 4.6. Assume moreover that $\phi = x^d$ in R_f^d, with x generic in S^1. Then $I_l^d = x^l R_f^{d-l}$ for $1 \leq l \leq d-1$. Furthermore the Veronese image $v_f(\mathbb{P}(S^1)) \subset \mathbb{P}(R_f^d)$ is schematically defined near $\phi = x^d$ by the condition (*).*

Remark 4.15. We did not use up to now equation (49) of Condition (*). We will need it for the proof of this lemma.

Proof of Lemma 4.14. We already noted in Remark 4.12 that $I_{d-1}^d = x^{d-1} R_f^{d-1}$. We study again the equations

$$aw = x^d \gamma \text{ in } R_f^{2d}, \tag{83}$$

for a in a subspace $I_1^d \subset R_f^d$ of dimension $\dim R_f^{d-1}$, and w in $I_{d-1}^d = x^{d-1} S^1$. We already proved in Lemma 4.10 that $\overline{I}_1^d = I_{1|Z}^d \subset x H^0(Z, \mathcal{O}_Z(d-1))$. We thus conclude that elements $a \in I_1^d$ can be written as

$$a = xa' + k_a \text{ mod } J_f^d,$$

with $a' \in S^{d-1}$, $k_a \in I_Z(d)$. Restricting (83) to Z, we also get

$$\gamma = a'w' + \gamma' \text{ mod } J_f^d,$$

for some $\gamma' \in I_Z(d)$. The equation (83) then becomes

$$x^{d-1} w' k_a = x^d \gamma' \text{ in } R_f^{2d}, \tag{84}$$

where w' is generic in S^1. One then easily concludes that $k_a = 0 \text{ mod } \langle x S^{d-1}, J_f \rangle$, that is, $I_1^d \subset x R_f^{d-1}$. Hence we proved (by dimension reasons) that

$$I_1^d = x R_f^{d-1}. \tag{85}$$

We now use (49). We get

$$I_1^d \cdot I_1^d \subset I_2^{2d},$$

which provides, using (85) $x^2 R_f^{2d-2} \subset I_2^{2d}$. Using (46), this inclusion gives in turn, by dimension reasons,

$$x^2 R_f^{2d-2} = I_2^{2d}. \tag{86}$$

Here, in order to apply the dimension argument, we need to know that multiplication by x^2 is injective on R_f^{2d-2}. More generally we will need to know that multiplication by x^i is injective on R_f^{d+i} for $1 \leq i \leq d$, which is not hard to prove since x is generic. We next use (48)

$$R_f^d I_2^d \subset I_2^{2d},$$

that is,

$$I_2^d \subset [I_2^{2d} : R_f^d] = [x^2 R_f^{2d-2} : R_f^d],$$

and easily conclude that $I_2^d \subset x^2 R_f^{d-2}$, hence $I_2^d = x^2 R_f^{d-2}$ by dimension reasons. We continue this way and prove that $I_k^d = x^k R_f^{d-k}$ for all $1 \leq k \leq d-1$. Thus the first statement is proved.

In order to prove the schematic statement, we consider a first order variation $(h, h_1, \ldots, h_{d-1})$ of $(x^d, I_1^d, \ldots, I_{d-1}^d)$ satisfying conditions (83) at first order. We thus have a first order deformation $x^d + \epsilon h \in R_f^d$ of x^d and

$$h_1 \in \text{Hom}\,(I_1^d, R_f^d / I_1^d),\ \ldots,\ h_{d-1} \in \text{Hom}\,(I_{d-1}^d, R_f^d / I_{d-1}^d),$$

satisfying the infinitesimal version of the equations (47)-(49). We have to prove that there is a $y \in S^1/x$ such that

$$h_l : I_1^d \cong R_f^{d-l} \to R_f^d / x^l R_f^d$$

is given by multiplication by lyx^{l-1}. We first observe that it suffices to prove the result for $l = 1$, because the reasoning above, which deduces the equality $I_k^d = x^k R_f^{d-k}$ for all $1 \leq k \leq d-1$ from the equality (85) using equations (48) and (49) work as well schematically.

We thus have a first order deformation $x^d + \epsilon h \in R_f^d$ of x^d and

$$h_1 \in \mathrm{Hom}\,(I_1^d, R_f^d/I_1^d),\ h_{d-1} \in \mathrm{Hom}\,(I_{d-1}^d, R_f^d/I_{d-1}^d),$$

satisfying the equations

$$(a + \epsilon h_1(a))(w + \epsilon h_{d-1}(w)) = (x^d + \epsilon h)\gamma_\epsilon \text{ in } R_f^{2d} \otimes \mathbb{C}[\epsilon]/(\epsilon^2), \tag{87}$$

for any $a = xa' \in xR_f^{d-1}$, $w = x^{d-1}w' \in x^{d-1}S^1$, and for $\gamma_\epsilon = \gamma + \epsilon\gamma_1$, where γ is as in (83). We want to prove that there exists $y \in S^1/\langle x \rangle$, such that for any $a = xa'$, the following holds in R_f^d

$$h_1(a) = ya' \bmod xR_f^{d-1}. \tag{88}$$

Looking at the previous proof, we deduce from (84) with $k_a = 0$ that $\gamma' = 0$ (using injectivity of the multiplication by x^d), so $\gamma = a'w'$ in R_f^d. We thus have $\gamma_\epsilon = a'w' + \epsilon\gamma_1$. Equation (87) then gives

$$h_1(a)x^{d-1}w' + h_{d-1}(w)xa' = x^d\gamma_1 + ha'w' \text{ in } R_f^{2d}, \tag{89}$$

for any $a = xa' \in xR_f^{d-1}$, $w = x^{d-1}w' \in x^{d-1}S^1$. We claim that

$$h_1(a)_{|Z} \in \langle a' \rangle \bmod \langle x \rangle. \tag{90}$$

Indeed, (89) first implies that $h = xh'$ since it becomes divisible by x after multiplication by any element of R_f^d, and then, after simplification by x, that

$$h_1(a)x^{d-2}w' + h_{d-1}(w)a' = x^{d-1}\gamma_1 + h'a'w' \text{ in } R_f^{2d-1}. \tag{91}$$

We rewrite (91) in the form

$$x^{d-2}(h_1(a)w' - x\gamma_1) + a'(h_{d-1}(w) - h'w') = 0. \tag{92}$$

We now restrict (92) to Z. As x^{d-2} and a' have no common divisor on Z for a' generic, it follows that

$$(h_1(a)w' - x\gamma_1)_{|Z} \in \langle a' \rangle,$$

which proves (90) since $w' \in S^1$ is generic.

We can even conclude by similar arguments that

$$h_1(a)_{|Z} = m_1 a' \bmod \langle x \rangle,$$

for some $m_1 \in H^0(Z, \mathcal{O}_Z(1))$. We can see m_1 as an element $y \in S^1$ because the map of restriction to Z is an isomorphism in degree 1, and we can thus write in R_f^d

$$h_1(a) = ya' + k_1(a') \text{ in } R_f^d/xR_f^{d-1}, \tag{93}$$

where $k_1(a') \in I_Z(d)$ for any $a' \in R_f^{d-1}$. Equation (92) then gives $x^{d-2}((ya' + k_1(a'))w' - x\gamma_1) + a'(h_{d-1}(w) - h'w') = 0$ in R_f^{2d-1}, that is

$$x^{d-2}(k_1(a')w' - x\gamma_1) + a'(yx^{d-2}h_{d-1}(w) - h'w') = 0 \text{ in } R_f^{2d-1}. \tag{94}$$

The term $x^{d-2}(k_1(a')w' - x\gamma_1)$ belongs by (94) to $x^{d-2}I_Z(d+1) \cap \langle a' \rangle$. For a' generic, it is easy to show that it implies that it belongs to $a'x^{d-2}I_Z(2) = 0$. Thus $x^{d-2}(k_1(a')w' - x\gamma_1) = 0$ in R_f^{2d-1}, hence $k_1(a')w' - x\gamma_1 = 0$, and $k_1(a') = 0 \bmod \langle x \rangle$. This is true for a' generic in R_f^{d-1}, hence for all a'. Thus (88) is proved. $\square$

Lemma 4.6 is a schematic version of Proposition 4.6 that guarantees that the Veronese image $v_f(\mathbb{P}(S^1)) \subset \mathbb{P}(R_f^d)$ is characterized not only set theoretically but also schematically (at the generic point) by condition (**) (in fact, we can see from the proof above that condition (*) even suffices for the scheme-theoretic statement, but condition (**) was needed to prove the set-theoretic statement for the special f). It follows that for generic f_{gen}, the Veronese image $v_f(\mathbb{P}(S^1)) \subset \mathbb{P}(R_f^d)$ is also characterized by condition (**). □

Proof of Theorem 0.6 (1). Fix integers d, n with d dividing $n+1$, and for which the conclusion of Proposition 4.13 holds. We want to show that if X_f is a very general hypersurface of degree d in $\mathbb{P}^n$, then any smooth hypersurface X_g of degree d in $\mathbb{P}^n$ such that there exists an isomorphism

$$\phi : H^{n-1}(X_g, \mathbb{Q})_{\rm prim} \cong H^{n-1}(X_f, \mathbb{Q})_{\rm prim}$$

as rational Hodge structures, is isomorphic to X_f.

We first argue as in Donagi's paper (see the introduction). As f is very general, our assumption provides simply connected Euclidean open neighborhoods $U \subset U_{d,n}$, $V \subset U_{d,n}$ of $[f]$, $[g]$ respectively, a holomorphic diffeomorphism $i : U \cong V$ with $i([f]) = [g]$, and an isomorphism of complex variations of Hodge structures

$$(H_{\mathbb{C}}^{n-1}, F^{\cdot}\mathcal{H}^{n-1}) \cong i^*(H_{\mathbb{C}}^{n-1}, F^{\cdot}\mathcal{H}^{n-1})$$

on U. Here, if $\pi : \mathcal{X}_{d,n} \to U_{d,n}$ is the universal hypersurface, $H_{\mathbb{C}}^{n-1}$ is the local system $R^{n-1}\pi_*\mathbb{C}_{\rm prim}$ on $U_{d,n}$, and $F^{\cdot}\mathcal{H}^{n-1}$ is the Hodge filtration on the associated flat holomorphic vector bundle $\mathcal{H}^{n-1} = H_{\mathbb{C}}^{n-1} \otimes \mathcal{O}_{U_{d,n}}$. Indeed, the Hodge locus $\Gamma_\phi \subset U \times V \subset U_{d,n} \times U_{d,n}$ locally defined by the condition that ϕ is an isomorphism of Hodge structures is algebraic by [3], and as $[f]$ is very general, and the period map is an immersion at the points $[f]$, $[g]$ of the moduli space $U_{d,n}$ once $(d, n) \neq (3, 3)$, Γ_ϕ must dominate U, hence also V and it has to induces a local holomorphic diffeomorphism between U and V.

Choosing representatives f, g, the differential $i_* : T_{U,[f]} \to T_{V,[g]}$ is a linear isomorphism

$$i_* : R_f^d \cong R_g^d.$$

Claim 4.16. *In the situation described above, the differential i_* sends the set of first order Schiffer variations of f to the set of first order Schiffer variations of g.*

Proof. Indeed, the local diffeomeorphism i induces an isomorphism of variations of Hodge structures. It thus sends a 1-parameter Schiffer variation $(f_t)_{t\in\Delta}$ of f to a 1-parameter variation $(g_t)_{t\in\Delta}$, $g_t := i(f_t)$, of g, which satisfies the assumptions of Proposition 4.13. Proposition 4.13 then tells us that $\psi := \frac{\partial g_t}{\partial t}_{|t=0}$ is a first order Schiffer variation of g. But $\phi := \frac{\partial f_t}{\partial t}_{|t=0}$ is an arbitrary first order Schiffer variation of f and we have $\psi = i_*(\phi)$. □

Having the claim, the proof of the theorem is finished using Proposition 3.6. □

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