REMARKS ON ZERO-CYCLES OF SELF-PRODUCTS OF VARIETIES

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1. Introduction

1.1 This paper proposes a few results and rises many questions on algebraic cycles on self-products X^k of a variety X. If X is a curve, it is well known that $CH_0(X^{(k)}) = CH_0(X^{(k+1)})$ for $k \geq g(X)$ where $X^{(k)}$ denotes the k^{th} symmetric product of X, and the isomorphism is given by the map μ_{x_0}: $X^{(k)} \to X^{(k+1)}$, $\mu_{x_0}(z) = z + x_0$, for any point $x_0 \in X$. Correspondingly, one has: $\forall k \geq g(X)$, $\forall l \in \mathbb{N}$, $H^0(\Omega^l_{X^{(k+1)}}) = H^0(\Omega^l_{X^{(k)}})$, which is the effect on holomorphic forms of the previous equality, using Mumford-Roitman theorem ([16], [22]). This last fact generalizes to higher dimensional varieties as follows: Assume $H^0(\Omega^l_X) = 0$ for l even, $l \neq 0$; then for $k \geq \sum_{i=1}^{\dim X} h^{i,0}$ and any l, $H^0(\Omega^l_{X^k})^{\mathfrak{S}_k} = H^0(\Omega^l_{X^{k+1}})^{\mathfrak{S}_{k+1}}$, where $()^{\mathfrak{S}}$ means the invariant part under the action of the symmetric group $\mathfrak{S}$.. According to Bloch-Beilinson conjectures [11], this should imply that $CH_0(X^k)^{\mathfrak{S}_k} = CH_0(X^{k+1})^{\mathfrak{S}_{k+1}}$ for $k \geq \sum_{i=1}^{\dim X} h^{i,0}$. I have no general result on this but I will construct in section 3 families of threefolds with no $H^{2,0}, H^{1,0}$ and which satisfy: $CH_0(X^k)^{\mathfrak{S}_k} = CH_0(X^{k+1})^{\mathfrak{S}_{k+1}}$ for $k \geq h^{3,0}(X)$.

1.2 Now, let us consider surfaces: If we have the equality $CH_0(S^k)^{\mathfrak{S}_k} = CH_0(S^{k+1})^{\mathfrak{S}_{k+1}}$ for some k, then, again by Mumford-Roitman theorem, this implies that the symmetric holomorphic forms of degree $2(k+1)$ on S^{k+1} vanish. Since the space of such forms is equal to $S^{k+1}H^{2,0}(S)$ it follows that $H^{2,0}(S) = 0$. Now Bloch's conjecture for surfaces asserts that if $H^{2,0}(S) = 0, CH_0(S)_0 \cong Alb(S)$, where $CH_0(S)_0$ denotes the group of zero cycles of degree zero, and the isomorphism is given by the Albanese map. So one can hope to split Bloch's conjecture into steps:

1) Show that $H^{2,0}(S) = 0 \Rightarrow CH_0(S^k)^{\mathfrak{S}_k} = CH_0(S^{k+1})^{\mathfrak{S}_{k+1}}$ for some $k \geq 0$.

2) Show that $CH_0(S^k)^{\mathfrak{S}_k} = CH_0(S^{k+1})^{\mathfrak{S}_{k+1}}$ for some $k \geq 0$ implies $CH_0(S)_0 \cong Alb(S)$.

We will show in section 2 the following first non trivial case of 2):

Theorem.(See 2.2) *Suppose* $CH_0(S^2)^{\mathfrak{S}_2} = CH_0(S)$; *then* $CH_0(S)_0 = Alb(S)$.

The remainder of this section is devoted to some speculations on how the equality $CH_0(X^k)^{\mathfrak{S}_k} = CH_0(X^{k+1})^{\mathfrak{S}_{k+1}}$ for a threefold (we will construct such threefolds with $k = h^{3,0}$ arbitrary, $h^{2,0} = h^{1,0} = 0$ in section 3) could lead to a proof of the representability of $CH_1(X)_{alg}$.

1.3 In the third section, we will turn to the study of skew zero-cycles on self-products of a surface S, that is zero-cycles $z \in CH_0(S^k)$ such that $\sigma_*(z) = \epsilon(\sigma)z$, $\forall \sigma \in \mathfrak{S}_k$.

Suppose S is a smooth projective surface with $h^{1,0} = 0$; then the space of skew holomorphic forms on S^k, that is the set $\{\omega \in \bigoplus_l H^0(\Omega^l_{S^k})/\sigma^*(\omega) = \epsilon(\sigma)\omega, \forall\sigma \in \mathfrak{S}_k\}$ is equal to $\bigwedge^k H^0(\Omega^2_S) \oplus \bigwedge^{k-1} H^0(\Omega^2_S)$. It follows that for $k \geq h^{2,0}(S)$, one has: the map $\widetilde{pr^*_{1...k}} : H^0(\Omega^{\cdot}_{S^k})^{skew} \to H^0(\Omega^{\cdot}_{S^{k+1}})^{skew}$, obtained as the composite of $pr^*_{1...k}$ and antisymmetrization, is surjective. Again according to Bloch-Beilinson conjectures, one expects the injectivity of the map $pr_{1...k*} : CH_0(S^{k+1})^{skew} \to CH_0(S^k)^{skew}$ for $k \geq h^{2,0}(S)$. This is clearly equivalent to the following:
(*) for $z_1, \ldots, z_k$, zero-cycles of degree zero on S, for $k \geq h^{2,0}(S)+1$, one has $\sum_{\sigma\in\mathfrak{S}_k} \epsilon(\sigma)\sigma^*(z_1 * \ldots * z_k) = 0$ in $CH_0(S^k)$. This section provides several constructions of surfaces satisfying (*). This property is especially beautiful in the case of $K3$-surfaces, where it simply means: Let z, z' be zero-cycles of degree zero on S; then $z * z' = z' * z$ in $CH_0(S \times S)$. We prove this for Kummer surfaces, and for a certain ten dimensional family of $K3$ surfaces with generic Picard number equal to 10 (3.3, 3.4). We use this to show the existence of threefolds with arbitrary $h^{3,0} \neq 0$ (hence CH_0 non representable) satisfying the equality $CH_0(X^k)^{\mathfrak{S}_k} = CH_0(X^{k+1})^{\mathfrak{S}_{k+1}}$ for $k \geq h^{3,0}(X)$ which contrasts very much with Theorem 2.2 for $k = h^{3,0} = 1$. We study also higher dimensional skew cycles on self-products of Kummer surfaces, in order to illustrate the conjecture 3.8, which generalizes the property mentioned above. In section 4, we describe the Kuga-Satake construction and try to show that it should be related to this conjecture.

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2. Symmetric zero-cycles

2.1 Let S be a smooth projective variety over $\mathbb{C}$ and let $x_0 \in S$. The following properties are equivalent:

2.1.1 *$CH_0(S^k)^{\mathfrak{S}_k}$ is supported on $\bigcup_{i=1}^k x_0^i(S^{k-1})$* , where $x_0^i : S^{k-1} \to S^k$ is the inclusion with x_0 on the i^{th} factor.
2.1.2 *The map $pr_{1...k-1} : S^k \to S^{k-1}$ induces a bijective map $(pr_{1...k-1})_* : CH_0(S^k)^{\mathfrak{S}_k} \to CH_0(S^{k-1})^{\mathfrak{S}_{k-1}}$.*
2.1.3 *For $z \in CH_0(S)_0$, one has $z^{*k} = 0$ in $CH_0(S^k)$, where for $z = \sum_i n_i x_i$, $z' = \sum_j m_j y_j$, $z * z' = \sum_{i,j} n_i m_j (x_i, y_j)$.*

To see this, one notes first that it suffices to prove these equivalences for Chow groups with rational coefficients, using the divisibility of ${CH_0}^0$ and Roitman's theorem on torsion [5]. Next one has the following fact:
2.1.4 *For any k, one has a direct sum decomposition:*

$$CH_0(S^k)^{\mathfrak{S}_k}{}_{\mathbb{Q}} = (\bigoplus_{l=0}^{k} {x_0}^{*l} * {CH_0}^0(S)^{*(k-l)})^{\mathfrak{S}_k}{}_{\mathbb{Q}}.$$

Proof of 2.1.4. The surjectivity of the natural map μ from the right hand side to the left hand side of this equality comes from the fact that $CH_0(S^k)^{\mathfrak{S}_k}{}_{\mathbb{Q}}$ is generated by 0-cycles z^{*k}, $z \in CH_0(S)$: any such z can be written as $z' + (d^0(z)x_0$, where $z' \in {CH_0}^0(S)$, so

$z^{*k} = (z' + (d^0(z)x_0)^{*k}$, and the development of the right hand side in powers of x_0 gives the surjectivity of μ.

To see the injectivity, assume:

(*) $0 = \sum_{l=0}^{k} z_l$, where $z_l \in (x_0{}^{*l} * CH_0{}^0(S)^{*(k-l)})^{\mathfrak{S}_k}{}_{\mathbb{Q}}$ are not all zero, and let k_0 be the greatest integer such that $z_{k_0} \neq 0$.

Then applying the projection $pr_{1\ldots k-k_0} =: p$ on the first factor S^{k-k_0} of S^k, one gets:

(**) $0 = p_*(\sum_{l=0}^{k_0} z_l)$.

But each z_l can be written by definition as $z_l = (\sum_{d^0(I)=k-l} j_{I*})(z_l')$, where $I \subset \{1, \ldots, k\}$, $j_I : S^{k-l} \to S^k$ is the inclusion with x_0 on the j-th factor for $j \notin I$, and $z_l' \in (CH_0{}^0(S)^{*(k-l)})^{\mathfrak{S}_{k-l}} \subset CH_0(S^{k-l})$. It follows that $p_*(z_l) = 0$ for $l < k_0$, and $p_*(z_{k_0}) = z_{k_0}'$. So (**) gives $z_{k_0}' = 0$, hence $z_{k_0} = 0$, a contradiction.

2.1.4 implies the equivalence of 2.1.1 and 2.1.3, because it shows that the quotient

$$CH_0(S^k)^{\mathfrak{S}_k}{}_{\mathbb{Q}}/(\sum_{i=1}^{k} x^i_{0*}(CH_0(S^{k-1})))^{\mathfrak{S}_k}{}_{\mathbb{Q}}$$

is isomorphic to $CH_0{}^0(S)^{*k})^{\mathfrak{S}_k}{}_{\mathbb{Q}}$.

2.1.2 $\Rightarrow$ 2.1.3 follows from the fact that $(CH_0{}^0(S)^{*k})^{\mathfrak{S}_k}{}_{\mathbb{Q}}$ is contained in $\mathrm{Ker}((pr_{1\ldots k-1})_*)$.

Finally the proof of 2.1.4 gives the implication 2.1.3 $\Rightarrow$ 2.1.2 as follows: under the assumption 2.1.3, any $z \in CH_0(S^k)^{\mathfrak{S}_k}{}_{\mathbb{Q}}$ is written as $z = \sum_{l=1}^{k_0} z_l$, with $1 \leq k_0 \leq k$ and $z_{k_0} \neq 0$. Then we have shown: $pr_{1\ldots k-k_0*}(z) \neq 0$; so a fortiori $pr_{1\ldots k-1*}(z) \neq 0$ and $pr_{1\ldots k-1*}$ is injective. Its surjectivity is always true, so 2.1.3 $\Rightarrow$ 2.1.2 is proved.

Now we will prove:

2.2 Theorem. *Let S be a surface satisfying one of these properties for $k = 2$; then $CH_0(S)_0 = Alb(S)$.*

Proof: Note first of all that the hypothesis implies $h^{2,0}(S) = 0$ (1.2). Consider now the Chow-Künneth decomposition of the diagonal Δ ([18]): $\Delta = \Delta_0 + \ldots + \Delta_4$, where the Δ_i's are algebraic cycles on $S \times S$, the classes of which are the projectors $H^{\cdot}(S) \to H^i(S)$. The action of Δ_1 and Δ_0 on $CH_0(S)$ is trivial. The image of Δ_4 is isomorphic to $\mathbb{Z}$, and the image of Δ_3 is isomorphic to $Alb(S)$. Δ_2 is symmetric and its cohomology class is the projector on $H^2(S)$. Since $H^{2,0}(S) = \{0\}$, one has an equality of cohomology classes: $cl(\Delta_2) = \sum_{i,j} cl(Z_i) \otimes cl(Z_j')$ where Z_i, Z_j' are divisors on S. One may assume that the cycle $\sum Z_i * Z_j'$ is a symmetric cycle of $S \times S$. Now consider on $S \times S \times S \times S$ the cycle $\Gamma = graph(Identity) + graph(\tau)$, where τ is the involution exchanging factors. By assumption, for $(s, s') \in S \times S, \Gamma((s, s'))$ is supported on $x_0 \times S \cup S \times x_0$ hence, following Bloch and Srinivas [6], we conclude that a multiple of Γ is rationally equivalent to a sum $\Gamma_1 + \Gamma_2$ where Γ_1 is supported on $D \times S \times S$ for some divisor D of $S \times S$ and Γ_2 is supported on $S \times S \times x_0 \times S \cup S \times S \times S \times x_0$.

Then, since $\Delta_2 - \sum Z_i * Z_j'$ is symmetric, we have ${}^t\Gamma(\Delta_2 - \sum Z_i * Z_j') = 2(\Delta_2 - \sum Z_i * Z_j')$, and from the decomposition above for $N\Gamma$ one concludes as in [6] that $2N(\Delta_2 - \sum Z_i * Z_j')$ is algebraically equivalent to zero in $S \times S$.

Now one has the following lemma:

2.3 Lemma. *Let S be a variety, and Z a cycle of codimension $n = \dim S$ in $S \times S$ which is algebraically equivalent to zero. Then the morphism: $[Z] : CH_0(S) \to CH_0(S)$ induced by Z is nilpotent.*

The lemma implies the theorem because $\Delta_2 - \sum Z_i * Z'_j$ induces by construction the identity map on Ker(alb). Since a multiple of it is nilpotent by 2.2 and 2.3, Ker(alb) must be of torsion; but it has no torsion by Roitman's theorem [5] so Ker(alb) = 0.

Proof of 2.3. Let $\circ$ denote the composition of correspondences in $S \times S$; we will prove in fact the following stronger statement:

2.3.1 *Z is nilpotent in $CH^n(S \times S)$ for the multiplication $\circ$.*

Proof. There exists a smooth curve C, a correspondence $\Gamma \subset C \times S \times S$ of codimension n, and a zero-cycle z of degree zero on C such that $Z = \Gamma(z) = pr_{23*}(\Gamma \cdot pr_1{}^*(z))$. Consider then the correspondence $\Gamma^{(k)}$ on $C^k \times S \times S$ defined as follows: Start with the product $\Gamma^{*k} \subset C^k \times (S \times S)^k$ and let Δ be the cycle of codimension $n(k-1)$ defined as $\{(x_1, \ldots, x_{2k}) \in (S \times S)^k / x_2 = x_3, \ldots, x_{2k-2} = x_{2k-1}\}$; then define $\Gamma^{(k)} = pr_{C^k \times S \times S*}(\Gamma^{*k} \cdot pr_{(S\times S)^k}{}^*(\Delta))$. In the projection $pr_{C^k \times S \times S}$ we are considering the first and last factors of $(S \times S)^k$. The meaning of $\Gamma^{(k)}$ is simply the following: Let $(c_1, \ldots, c_k) \in C^k$; then $\Gamma^{(k)}((c_1, \ldots, c_k)) = \Gamma(c_1) \circ \ldots \circ \Gamma(c_k)$.

Now let $z = \sum_{i=1}^m n_i c_i, c_i \in C, \sum n_i = 0$. Then:

$$z^{*k} = \sum_{f:\{1,\ldots,k\} \to \{1,\ldots,m\}} n_{f(1)} \ldots n_{f(k)} (c_{f(1)}, \ldots, c_{f(k)}),$$

so $\Gamma^{(k)}(z^{*k}) = \sum_{f:\{1,\ldots,k\} \to \{1,\ldots,m\}} n_{f(1)} \ldots n_{f(k)} \Gamma(c_{f(1)}) \circ \ldots \circ \Gamma(c_{f(k)}) = \Gamma(z)^{\circ k}$. Now this is finished because we know that $z^{*k} = 0$ in $CH_0(C^k)$ for $k \geq g(C) + 1$ (1.1, 2.1), hence $\Gamma(z)^{\circ k} = \Gamma^{(k)}(z^{*k}) = 0$ in $CH^n(S \times S)$ for $k \geq g(C) + 1$.

2.4 Remark. In fact, one can prove Theorem 2.2 without Lemma 2.3, just applying Bloch-Srinivas argument with more precision; However, we have proved in fact the following strengthening of 2.2:

2.4.1 *Assume the surface S satisfies: there exists a subset $\Sigma \subset S \times S$ of dimension at most two, such that symmetric zero-cycles of $S \times S$ are supported on Σ; Then $CH_0(S)_0 \cong Alb(S)$.*

2.5 Unfortunately, the proof of Theorem 2.2 does not extend to more general situations. There are the following generalizations of Theorem 2.2 that would be very interesting to know:

2.5.1 Conjecture. *Let S be a surface and suppose that for some $k, CH_0(S^k)^{\mathfrak{S}_k} = CH_0(S^{k+1})^{\mathfrak{S}_{k+1}}$. Then $CH_0(S)_0 \cong Alb(S)$.*

The next question concerns correspondences between any variety X and a surface Σ. Suppose we have a codimension two cycle $\Gamma \subset X \times \Sigma$; then we can consider Γ^{*k} in $X^k \times \Sigma^k$ and its effect on symmetric zero-cycles: $[\Gamma^{*k}] : CH_0(X^k)^{\mathfrak{S}_k} \to CH_0(\Sigma^k)^{\mathfrak{S}_k}$. Now the effect of Γ^{*k} on symmetric holomorphic $2k$-forms: $[\Gamma^{*k}]^{2k,0} : H^{2k,0}(\Sigma^k) \to H^{2k,0}(X^k)$ is just the map $S^k[\Gamma^{2,0}] : S^k H^{2,0}(\Sigma) \to S^k H^{2,0}(X)$ induced by $[\Gamma^{2,0}] : H^{2,0}(\Sigma) \to H^{2,0}(X)$. So if $[\Gamma^{*k}]^{2k,0} = 0$, $[\Gamma^{2,0}] = 0$ and Bloch's conjecture implies that $[\Gamma] : CH_0(X)_0 \to CH_0(\Sigma)_0$ factors through $Alb(X)$. Since $[\Gamma^{*k}]^{2k,0} = 0$ when $[\Gamma^{*k}] : CH_0(X^k)^{\mathfrak{S}_k} \to CH_0(\Sigma^k)^{\mathfrak{S}_k}$ vanishes on the subgroup $((CH_0(X)_0)^{*k})^{\mathfrak{S}_k}$, we have the following:

2.5.2 Conjecture. *Let $\Gamma \subset X \times \Sigma$ be a codimension two-cycle; suppose that for some k, $[\Gamma^{*k}] : CH_0(X^k)^{\mathfrak{S}_k} \to CH_0(\Sigma^k)^{\mathfrak{S}_k}$ vanishes on the subgroup $((CH_0(X)_0)^{*k})^{\mathfrak{S}_k}$. Then $[\Gamma] : CH_0(X)_0 \to CH_0(\Sigma)_0$ factors through $Alb(X)$.*

Of course 2.5.1, 2.5.2 are just particular cases of Bloch's conjecture, but the assumptions are much stronger so they could be more accessible. Concerning 2.5.2, Bloch-Srinivas argument gives the following proposition in the case $k=2$. In 3.6 we construct threefolds with $h^{3,0}=1$, and satisfying:

2.5.3 $CH_0(X^2)^{\mathfrak{S}_2} \cong CH_0(X)$, (so, as in 2.1 $((CH_0(X)_0)^{*2})^{\mathfrak{S}_2} = \{0\}$).

For them we have:

2.5.4 Proposition *Assume X satisfies $h^{1,0}=0$ and the property 2.5.3; Then, for any correspondence $\Gamma \subset X \times \Sigma$, the composite ${}^t\Gamma \circ \Gamma : CH_0(X)_0 \to CH_1(X)_{alg}$ is equal to zero.*

We just sketch the proof of 2.5.4, since it is essentially an application of Bloch-Srinivas argument, as in the proof of 2.2; the point is that the assumptions imply that the codimension two cycle ${}^t\Gamma \circ \Gamma \subset X \times X$ is symmetric and homologous to zero modulo cycles coming from X via the projections. Now, also by assumption, symmetric zero-cycles on $X \times X$ are supported on the diagonal. Then, by [6], adapted as in the proof of 2.2, we conclude that a codimension two cycle Z symmetric and homologous to zero in $X \times X$ is rationally equivalent to zero iff its restriction to the diagonal is rationally equivalent to zero, and its Abel-Jacobi invariant vanishes. Finally, because $H^1(X)=\{0\}$, the invariant part under $\mathfrak{S}_2$ of $J^3(X \times X)$ identifies with $J^3(X)$ and the Abel-Jacobi map on codimension two symmetric cycles of $X \times X$ identifies with the restriction to the diagonal, followed by the Abel-Jacobi map of X. But then it is clear that ${}^t\Gamma \circ \Gamma$ is rationally equivalent to a cycle of the form $pr_1^*(Z) + pr_2^*(Z)$, which implies immediately 2.5.4.

2.6 2.5.2 could be a step for the study of the kernel of the Abel-Jacobi map for threefolds. We have the following:

2.6.1 Conjecture. *Let X be a threefold, and suppose that $h^{1,0}(X)=h^{2,0}(X)=0$. Then the Abel-Jacobi map*

$$\Phi_X : CH_1(X)_{alg} \to J^3(X) = H^3(X,\mathbb{C})/(F^2H^3(X) + H^3(X,\mathbb{Z}))$$

is injective.

2.7 We will construct in the next section (3.6) threefolds satisfying the assumptions above, and such that the subgroup $((CH_0(X)_0)^{*k})^{\mathfrak{S}_k}$ of $CH_0(X^k)$ is zero for $k \geq h^{3,0}(X)+1$. (This property should be satisfied by all threefolds with $h^{1,0}=h^{2,0}=0$). These threefolds have $h^{3,0} \neq 0$, so they have a huge CH_0 group and are interesting examples to study 2.6.1. (Note that 2.6.1 is true if $CH_0(X)$ is representable [6]).

2.8 Consider such a threefold, and suppose we have a family of one-cycles in X parametrized by a surface Σ. So we have a correspondence $\Gamma \subset X \times \Sigma$ of codimension two; if 2.5.2 is true and X is as in 2.6, 2.7, the map $[\Gamma] : CH_0(X) \to CH_0(\Sigma)$ factors through the degree map. One concludes then as in Bloch-Srinivas [6], that a multiple $N\Gamma$ of Γ is supported on $D \times \Sigma \cup X \times \sigma_0$ for some divisor D of X and zero-cycle σ_0 of Σ, and then, that the map $N{}^t\Gamma : CH_0(\Sigma) \to CH_1(X)$ factors through a map: $CH_0(\Sigma) \to CH_1(D)$. Since D is a surface $CH_1(D)_{alg}$ is an abelian variety, so one concludes that $N{}^t\Gamma : CH_0(\Sigma)_0 \to CH_1(X)_{alg}$ factors through $Alb(\Sigma)$, hence also ${}^t\Gamma : CH_0(\Sigma)_0 \to CH_1(X)_{alg}$, since $\mathrm{Ker}(alb)$ is divisible. So the next interesting question to solve would be the following:

2.9 Conjecture. *Let X be a threefold, and assume that for any codimension two correspondence $\Gamma \subset X \times Y$, the map ${}^t\Gamma : CH_0(Y)_0 \to CH_1(X)$ factors through $Alb(Y)$. Then*

$CH_1(X)_{alg}$ is representable, and $\Phi_X : CH_1(X)_{alg} \to J^3(X)$ is injective.

3. Skew-cycles on surfaces

We want to construct examples of surfaces with $h^{1,0} = 0$ and which satisfy the expected property:

(*) the map $CH_0(S)_0^{\otimes k} \to CH_0(S^k)$ given by $*$-product vanishes on $(CH_0(S)_0^{\otimes k})^{skew}$ for $k \geq h^{2,0}(S)+1$ (see 1.3).

We use first 2.1 applied to curves, which satisfy the assumptions of 2.1 by 1.1; so we have:

3.0.1. *Let C be a curve of genus g; then for $z_1, \dots, z_k$, zero-cycles of degree zero on C, $k \geq g+1$, one has: $\sum_{\sigma \in \mathfrak{S}_k} \sigma^*(z_1 * \dots * z_k) = 0$ in $CH_0(C^k)$.*

This gives the first example as follows:

3.1 Proposition. *Let C_1 be an elliptic curve, C_2 a hyperelliptic curve of genus g_2, and denote by i_1, i_2 the hyperelliptic involutions of C_1, C_2. Then the surface $S = \widetilde{C_1 \times C_2/(i_1,i_2)}$ satisfies the property (*). Here $\widetilde{C_1 \times C_2/(i_1,i_2)}$ is the desingularization of $C_1 \times C_2$ obtained by blowing up the fixed points of (i_1, i_2).*

Proof: We have essentially to prove: let $z_1, \dots, z_k$ be zero-cycles on $C_1 \times C_2$, of degree zero and invariant under (i_1, i_2). Then some multiple of $\sum_{\sigma \in \mathfrak{S}_k} \epsilon(\sigma)\sigma^*(z_1 * \dots * z_k)$ vanishes for $k \geq g_2 + 1 = h^{2,0}(S)+1$. This is sufficient by [5] which says that the Chow groups considered have no torsion. Now write $z_i = \sum_j n_i^j(c_i^j, d_i^j)$; then one verifies that $4z_i = 2(z_i + (i_1,i_2)z_i) = \sum_j n_i^j(c_i^j - i_1c_i^j) * (d_i^j - i_2d_i^j)$. So $4^k\, z_1 * \dots * z_k = \sum_{(j_1,\dots,j_k)} n_1^{j_1} \dots n_k^{j_k}(c_1^{j_1} - i_1c_1^{j_1}) * \dots * (c_k^{j_k} - i_1c_k^{j_k}) * (d_1^{j_1} - i_2d_1^{j_1}) * \dots * (d_k^{j_k} - i_2d_k^{j_k})$. Now 3.0.1 applied to the elliptic curve C_1 shows that the $*$-product between zero-cycles of degree zero of C_1 is skew-symmetric. It follows that $4^k \sum_{\sigma \in \mathfrak{S}_k} \epsilon(\sigma)\sigma^*(z_1 * \dots * z_k) = 4^k \sum_{(j_1,\dots,j_k)} n_1^{j_1} \dots n_k^{j_k}(c_1^{j_1} - i_1c_1^{j_1}) * \dots * (c_k^{j_k} - i_1c_k^{j_k}) * \sum_{\sigma \in \mathfrak{S}_k} \sigma^*((d_1^{j_1} - i_2d_1^{j_1}) * \dots * (d_k^{j_k} - i_2\, d_k^{j_k}))$. Finally the second factor in each term of the sum over $(j_1, \dots, j_k)$ vanishes by 3.0.1 applied to C_2, because $k \geq g_2 + 1$. So 3.1 is proved.

Now consider surfaces with $h^{2,0} = 1$. In this case, property (*) means simply that the $*$-product between zero-cycles of degree zero of S is symmetric, which is a rather intriguing property.

Let us prove first of all that (*) holds for Kummer surfaces:

3.2 Proposition. *Let A be an abelian surface, and $S = \widetilde{A/\pm 1}$ be its Kummer surface. Then S satisfies property (*).*

Proof: We may assume that A is principally polarized. Consider the map $\phi : A \times A \to A \times A$ given by $\phi((x,y)) = (x+y, x-y)$. Since ϕ is an isogeny, the kernel of ϕ_* is made of torsion elements of $CH_0(A \times A)$ (See [3] or [7]), so it suffices to show: let $x, y \in A$, then $\phi_*((x + \{-x\} - 2\{0\}) * ((y + \{-y\} - 2\{0\}) - (y + \{-y\} - 2\{0\}) * ((x + \{-x\} - 2\{0\})) = 0$ in $CH_0(A \times A)$, that is:

3.2.1 $\phi_*((x,y) - (y,x) + (x,\{-y\}) - (\{-y\},x) + (\{-x\},y) - (y,\{-x\}) - 2(\{0\},y) + 2(y,\{0\}) - 2(\{0\},\{-y\}) + 2(\{-y\},\{0\}) - 2(x,\{0\}) + 2(\{0\},x) - 2(\{-x\},\{0\}) + 2(\{0\},\{-x\}) + (\{-x\},\{-y\}) - (\{-y\},\{-x\})) = 0$.

Now let $(u,v) \in A \times A$; then $\phi_*((u,v)-(v,u)) = \{u+v\}*(\{u-v\}-\{v-u\})$. But we know by [7] that for any $z \in CH_0(A)_0$ one has: $z-(-1)_*(z) = j_*(\tilde{z})$, for $j : C \hookrightarrow A$ the inclusion of a symmetric thêta divisor and $\tilde{z}$ the unique zero-cycle on C, such that $alb_C(\tilde{z}) = 2alb_A(z) \in A$. So $\phi_*((u,v)-(v,u)) = \{u+v\} * j_*(z_{u,v})$, where $alb(z_{u,v}) = 2(u-v) \in A$. It follows that the left hand side in 3.2.1 is equal to:

3.2.2 $\{x+y\} * j_*(z_{x,y}) + \{x-y\} * j_*(z_{x,-y}) + \{-x+y\} * j_*(z_{-x,y}) + \{-x-y\} * j_*(z_{-x,-y}) - 2y * j_*(z_{0,y}) - 2\{-y\} * j_*(z_{o,-y}) - 2x * j_*(z_{x,0}) - 2\{-x\} * j_*(z_{-x,0})$.

Now $z_{x,y} = -z_{-x,-y}, z_{x,-y} = -z_{-x,y}, z_{0,y} = -z_{0,-y}$ and $z_{x,0} = -z_{-x,0}$. So 3.2.2 is equal to:

3.2.3 $(\{x+y\} - \{-x-y\}) * j_*(z_{x,y}) + (\{x-y\} - \{-x+y\}) * j_*(z_{x,-y}) - 2(\{y\} - \{-y\}) * j_*(z_{0,y}) - 2(x - \{-x\}) * z_{x,0})$.

Applying again the formula $z-(-1)_*(z) = j_*(\tilde{z})$ one finds:

3.2.4 $j_*(z_{x+y}) * j_*(z_{x,y}) + j_*(z_{x-y}) * j_*(z_{x,-y}) - 2j_*(z_y) * j_*(z_{0,y}) - 2j_*(z_x) * j_*(z_{x,0})$, where $alb(z_u) = 2u,\ \forall u \in A$.

We have the following obvious relations between the cycles $z_., z_{.,.}$ on C: $z_{x+y} = z_x + z_y$, $z_{x,y} = z_x - z_y, z_{x-y} = z_x - z_y, z_{x,-y} = z_x + z_y, z_{0,y} = -z_y, z_{x,0} = z_x$. So 3.2.4 is in fact equal to:

3.2.5 $(j,j)_*((z_x+z_y)*(z_x-z_y) + (z_x-z_y)*(z_x+z_y) + 2(z_y*z_y) - 2(z_x*z_x))$ which is clearly 0 as we wanted.

3.3 We will consider now the following much more general family of K3-surfaces (it is ten dimensional and the general member has Picard group of rank ten), and prove property (*) for the general member. We simply consider two degree three curves E_1, E_2 in $\mathbb{P}^2$, smooth and meeting transversally. Let $S \to \mathbb{P}^2$, be obtained by blowing up the double cover of $\mathbb{P}^2$ ramified along $E_1 \cup E_2$. Then we have:

3.4 Theorem. *Property (*) holds for S.*

Proof: Let F_1, F_2 be the equations for E_1, E_2; consider in $\mathbb{P}^5$ the sextic fourfold W with equation: $F_1(X_0,X_1,X_2)F_2(X_0,X_1,X_2) - F_1(Y_0,Y_1,Y_2)F_2(Y_0,Y_1,Y_2)$, where $X_i, Y_i, i = 0,1,2$ are homogeneous coordinates for $\mathbb{P}^5$. W has non degenerate quadratic singularities along the locus $\{F_i(X) = 0 = F_i(Y), i = 1,2\}$ and by blowing up this sublocus, we get a variety $\tilde{W}$ which is smooth and clearly invariant under the natural lifting $\tilde{i}$ of $i_{|W}$ where i is the involution on $\mathbb{P}^5$, defined by $i((x_0,\dots,x_2,y_0,\dots,y_2)) = (y_0,\dots,y_2,x_0,\dots,x_2)$ which leaves W invariant.

The first step is to prove:

3.4.1 Lemma. *There is a natural correspondence Γ between $S \times S$ and $\tilde{W}$, equivariant w.r.t. τ and $\tilde{i}$ (τ is the involution exchanging factors on $S \times S$), inducing an inclusion:* $[\Gamma] : \mathrm{Ker}(CH_0(S \times S)^- \xrightarrow{pr_{1*}} CH_0(S)) \hookrightarrow CH_0(\tilde{W})$.

Proof: Let $\Sigma \subset \mathbb{P}^3$ be defined by the equation: $U^6 = F_1(X)F_2(X)$. As in [26], there is a natural rational map $\phi : \Sigma \times \Sigma \to W$, defined by: $\phi((u,x),(u',x')) = (u'x_0,\dots,u'x_2,ux'_0,\dots,ux'_2)$. Finally using the natural map $\psi : \Sigma \to S$, given by $\psi((u,x)) = (u^3,x)$ (here (u,x) is a point in the total space of $\mathcal{O}(1)$ over $\mathbb{P}^2$, and (u^3,x) is a point in the total space of $\mathcal{O}(3)$ over $\mathbb{P}^2$), we get the correspondence

$$\begin{array}{ccc} \widetilde{\Sigma\times\Sigma} & \xrightarrow{\tilde\phi} & \tilde W \\ {\scriptstyle\psi\times\psi}\downarrow & & \\ S\times S & & \end{array}$$

where $\widetilde{\Sigma\times\Sigma}$ is a smooth blow-up of $\Sigma\times\Sigma$ on which $\tilde\phi$ is everywhere defined. To see that $\tilde\phi_*\circ(\psi\times\psi)^*$ is injective on $\mathrm{Ker}(CH_0(S\times S)^- \overset{pr_{1*}}{\to} CH_0(S))$, one notes simply that $\tilde\phi$ is essentially the quotient map for the diagonal action of $\mathbb{Z}/6\mathbb{Z}$ on $\widetilde{\Sigma\times\Sigma}$ (this is only a birational action but this is not a problem since we want to study zero-cycles which are birational invariants). Now on $(\psi\times\psi)^*(CH_0(S\times S))$ the diagonal action of $\mathbb{Z}/6\mathbb{Z}$ reduces to the diagonal action of $\mathbb{Z}/2\mathbb{Z}$ (that is of the involution j of S over $\mathbb{P}^2$). Since j acts by -1 on $CH_0(S)_0$, we see that the diagonal action of j is the identity on $CH_0(S)_0 * CH_0(S)_0$, hence we get:

$$CH_0(S)_0 * CH_0(S)_0 \overset{(\psi\times\psi)^*}{\hookrightarrow} CH_0(\Sigma\times\Sigma)_0^{\mathbb{Z}/6\mathbb{Z}} \overset{\tilde\phi_*}{\cong} CH_0(\tilde W)_0.$$

Now, Theorem 3.4 will follow from:

3.4.2 Proposition. *$CH_0(\tilde W)^- = 0$, where $()^-$ denotes the -1-eigenspace for the action of $\tilde i$.*

Proof: The equation of W is $F_1(X)F_2(X) - F_1(Y)F_2(Y) = 0$. So for each $\alpha\in\mathbb{C}$, W contains the threefold $X\subset\mathbb{P}^5$, with equations: $F_1(X) = \alpha F_2(Y), F_1(Y) = \alpha F_2(X)$. This is the complete intersection of two cubics in $\mathbb{P}^5$, in general smooth, and invariant under i. Since W is covered by such X's, it suffices to prove:

3.4.3 Proposition. *Let $X\subset\mathbb{P}^5$ be a threefold which is defined by $F^+ = F^- = 0$, where F^+ (resp. F^-) are cubic equations invariant (resp. antiinvariant) under i. Then $CH_0(X)^- = 0$.*

We will not give the proof of 3.4.3: one has simply to note that X has trivial canonical bundle and that i^* acts by $+1$ on $H^{3,0}(X)$. The proof works then exactly as in [23], where a similar statement is proved for quintic threefolds in $\mathbb{P}^4$ invariant under an involution satisfying the same condition. So Theorem 3.4 is proved.

3.5 Note the following important consequences of Theorem 3.4, which are obtained using Bloch-Srinivas argument [6].

3.5.1 Corollary. *The generalized Hodge conjecture is true for the sub-Hodge structure of level two: $\bigwedge^2 H^2(S)\subset H^4(S\times S)$, for any S as in 3.3.*

More precisely, there exists a variety W and a family of one cycles

$$\begin{array}{ccc} \Gamma & \xrightarrow{q} & S\times S \\ {\scriptstyle p}\downarrow & & \\ C & & \end{array}$$

inducing an injective map: $\bigwedge^2 H^2(S) \overset{p_*q^*}{\to} H^2(W)$.

We have as well, again by [6], or directly from the corollary above:

3.5.2 Corollary. *The Hodge conjecture is true for Hodge classes in $\bigwedge^2 H^2(S) \subset H^4(S\times S)$; moreover for a skew two-cycle $Z \in CH_2(S\times S)^-$ one has:*

3.5.3 $Z \equiv_{rat} 0 \Leftrightarrow Z_{|s_0\times S} \equiv_{rat} 0$, *and Z is homologous to zero.*

3.6 As mentioned in section 2, Theorem 3.4 or Proposition 3.1 allow us to construct threefolds with $h^{2,0} = h^{1,0} = 0, h^{3,0} \neq 0$ and $CH_0(X^k)^{\mathfrak{S}_k} = CH_0(X^{k+1})^{\mathfrak{S}_{k+1}}$ for $k \geq h^{3,0}(X)$. (As in 2.1, this is equivalent to the following: the $*$-product $CH_0(X)^{\otimes k} \to CH_0(X^k)$ vanishes on $(CH_0(X)_0^{\otimes k})^{\mathfrak{S}_k}$, for $k \geq h^{3,0}(X)+1$ or more simply to: for $z \in CH_0(X)_0, z^{*k} = 0$ in $CH_0(X^k)$ for $k \geq h^{3,0}(X)+1$). The construction is exactly similar to that of 3.1. We start with the $K3$-surfaces S of 3.3, which have an involution i acting by -1 on $CH_0(S)_0$. By Theorem 3.4 one has (*):

For $z, z' \in CH_0(S)_0, z * z' = z' * z$ in $CH_0(S\times S)$.

Then, if C is a hyperelliptic curve of genus g, j the hyperelliptic involution of C, one constructs the threefold $X = \widetilde{C\times S/(j,i)}$ where $\tilde{}$ means desingularization by blowing-up the fixed curves of (j,i). Then X has $h^{3,0} = g, h^{1,0} = h^{2,0} = 0$ and we show:

3.6.1 Lemma. *Let z be a zero-cycle of degree zero on X; Then $z^{*k} = 0$ in $CH_0(X^k)$ for $k \geq g+1$.*

Proof: Since these CH_0 groups have no torsion by Roitman's theorem [5], it suffices to show that for z a zero-cycle of degree zero on $C\times S$, one has: some multiple of $(z+(j,i)(z))^{*k}$ vanishes in $CH_0((C\times S)^k) = CH_0(C^k\times S^k)$ for $k \geq g+1$. Now let $z = \sum_{l=1}^m n_l(c_l, s_l), \sum n_l = 0$. Then from the fact that j acts by -1 on $CH_0(C)_0$, i acts by -1 on $CH_0(S)_0$, one deduces easily: $2(z+(j,i)_*(z)) \equiv \sum_{l=1}^m n_l(c_l - jc_l) * (s_l - is_l)$. So $2^k(z+(j,i)_*(z))^{*k} = \sum_{f:\{1,\dots,k\}\to\{1,\dots,m\}} n_{f(1)}\dots, n_{f(k)}(c_{f(1)} - jc_{f(1)}) * \dots * (c_{f(k)} - jc_{f(k)}) * ((s_{f(1)} - is_{f(1)}) * \dots * (s_{f(k)} - is_{f(k)})$.

Next one knows by 3.4 that the $*$-products $(s_{f(1)} - is_{f(1)}) * \dots * (s_{f(k)} - is_{f(k)})$ are invariant under any permutation of the factors: the common value of these products will be denoted by $*_{l\in I}(s_l - is_l)$ where I denotes the unordered set of indices (with multiplicities) corresponding to f. Then we have: $2^k(z+(j,i)_*(z))^{*k} = \sum_I \prod_{l\in I} n_l[\sum_{f:\{1,\dots,k\}\to I}(c_{f(1)} - jc_{f(1)}) * \dots * (c_{f(k)} - jc_{f(k)})] * *_{l\in I}(s_l - is_l)$, where the summation over f inside the bracket means that if $I = \{l_1,\dots,l_r\}$ with multiplicities $i_1,\dots,i_r$ one considers maps $f:\{1,\dots,k\}\to\{l_1,\dots,l_r\}$ such that l_j has exactly i_j preimages. Now each factor inside the bracket vanishes in $CH_0(C^k)$ for $k \geq g+1$ because it is the symmetrized $*$-product of $k \geq g+1$ 0-cycles of degree 0 of C (3.0.1), hence $2^k(z+(j,i)_*(z))^{*k} = 0$.

Now if we consider the surfaces S of 3.1, which satisfy the property (*): for $k \geq h^{2,0}+1 = g_2+1$, $\forall z_1,\dots,z_k \in CH_0(S)_0$, $\sum_{\sigma\in\mathfrak{S}_k} \epsilon(\sigma)\sigma^*(z_1 * \dots * z_k) = 0$, we can make the same construction with E an elliptic curve: E satisfies $\forall z_1, z_2 \in CH_0(E)_0, z_1 * z_2 = -z_2 * z_1$ in $CH_0(E\times E)$ (3.0.1). S has as before an involution i acting by -1 on $CH_0(S)_0$, and we define $X = \widetilde{E\times S/(j,i)}$ where j is a hyperelliptic involution on E. The same computation then shows:

3.6.2 Lemma *for $k \geq h^{3,0}(X)+1, (CH_0(X)_0^{*k})^{\mathfrak{S}_k} = 0$.*

3.7 The $K3$-surfaces we have considered in 3.2 and 3.3 are special but, as was explained in the introduction, the property (*): $z * z' = z' * z$ in $CH_0(S\times S)$, $\forall z, z' \in CH_0(S)_0$

should hold for all algebraic $K3$-surfaces. In fact, one can even go further: one remarkable consequence of the property $h^{2,0}=1$ is the following: for any k, the Hodge structure on $\bigwedge^k H^2(S) \subset H^{2k}(S^k)$ is of Hodge level two. Now note the following decomposition: $H^{\cdot}(S^k)^{skew} = \bigwedge^k H^2(S) \oplus \bigwedge^{k-1} H^2(S) \oplus \bigwedge^{k-1} H^2(S) \oplus \bigwedge^{k-2} H^2(S)$; In this decomposition, the degrees are respectively $2k, 2k-2, 2k+2, 2k$; According to Bloch-Beilinson's ideas, l-cycles on S^k should be governed by cohomology groups $H^{p,q}$ with $q \leq l$. From the fact that in the decomposition above, all terms are Hodge structures of level two, one concludes that the only piece of $H^{\cdot}(S^k)^{skew}$ having non zero (p,q)-component with $q \leq k-2$ is $\bigwedge^{k-1} H^2(S)$, which comes from $H^{\cdot}(S^{k-1})^{skew}$ via $pr^*_{1,\dots,k-1}$ followed by antisymmetrization. So one should have the following:

3.8 Conjecture. *Let S be an algebraic $K3$-surface. Then for any k, the projection $pr_{1,\dots,k-1}: S^k \to S^{k-1}$ induces an injection $pr_{1,\dots,k-1_*}: CH_l(S^k)^{skew}_{\mathbb{Q}} \hookrightarrow CH_l(S^{k-1})^{skew}_{\mathbb{Q}}$, for $l \leq k-2$.*

Arguing as in Bloch-Srinivas [6] or better as in [20], this conjecture would have as a consequence that $\bigwedge^k H^2(S)$ is parametrized by $(k-1)$-cycles of S^k as is expected by the generalized Hodge conjecture (see 3.5.2). Now, assuming 3.8 is true for $k-1$, one checks easily that the map $\mu_{k-1,l}: CH_l(S^{k-1})^{skew}_{\mathbb{Q}} \otimes CH_0(S)_0 \to CH_l(S^k)^{skew}_{\mathbb{Q}}, \mu_{k-1,l}(Z \otimes z) = \sum_{\sigma \in \mathfrak{S}_k} \epsilon(\sigma)\sigma^*(Z * z)$ induces a surjective map: $\mathrm{Ker}(pr_{1,\dots,k-2_*}) \otimes CH_0(S)_0 \to \mathrm{Ker}(pr_{1,\dots,k-1_*})$ for $l \leq k-2$.

So one has essentially to prove the following generalization of the property (*):

3.9 Conjecture. *For any $k, l \leq k-1$, the map $\mu_{k,l}$ vanishes on* $\mathrm{Ker}(pr_{1,\dots,k-1_*}) \otimes CH_0(S)_0$.

We want now to show 3.9 for $k=2$ and for Kummer surfaces.

3.10 Theorem. *Let A be a principally polarized abelian surface and $S = \widetilde{A/\pm 1}$ be its Kummer surface; then the map $\mu_{2,1}: CH_1(S \times S)^-_{\mathbb{Q}} \otimes CH_0(S) \to CH_1(S \times S \times S)^{skew}_{\mathbb{Q}}, \mu_{2,1}(Z \otimes z) = \sum_{\sigma \in \mathfrak{S}_3} \epsilon(\sigma)\sigma^*(Z * z)$ vanishes on* $\mathrm{Ker}(pr_{1*}) \otimes CH_0(S)_0$.

Proof: The first thing to do is to describe $CH_1(S \times S)^-_{\mathbb{Q}}$. In fact we will work with cycles on $A \times A$ invariant under the action of -1 on each factor, and skew under τ, which is almost the same: (the remaining skew one-cycles on $S \times S$ are generated by $E * z - z * E, E \subset S$ a exceptional curve, $z \in CH_0(S)_0$; For them, Theorem 3.10 follows from 3.2). We will denote these subgroups of $CH.(A \times A)$ by $\widetilde{CH}.(A \times A)$. We have the following:

3.11 Lemma. *Write $A = JC$ and fix a symmetric embedding $C \hookrightarrow A$. For $(u,v) \in C \times C$ denote by $C_{u,v} \subset A \times A$ the curve $\{(x+u, x+v), x \in C\}$. Then $\mathrm{Ker}(\pi_*: \widetilde{CH}_1(A \times A)^-_{\mathbb{Q}} \to \widetilde{CH}_1(A)_{\mathbb{Q}})$ is generated by cycles $Z_{u,v} := C_{u,v} + (-1, Id)_*(C_{u,v}) + (Id, -1)_*(C_{u,v}) + (-1,-1)_*(C_{u,v}) - C_{v,u} - (-1, Id)_*(C_{v,u}) - (Id,-1)_*(C_{v,u}) - (-1,-1)_*(C_{v,u})$, and by cycles $C * z - z * C, z \in CH_0(A)_0$ invariant under -1.*

Proof: Let us consider the map $\sigma: A \times A \to A, \sigma((x,y)) = x+y$. The fiber $\sigma^{-1}(\alpha)$ is isomorphic to A, embedded in $A \times A$ by $j_\alpha(x) = (x, \alpha - x)$. The involution τ acts on the fiber $\sigma^{-1}(\alpha)$ by $\tau_\alpha(x) = \alpha - x$. As we already noted in 3.2, skew 0-cycles on A for τ_α are supported on $C_{\alpha/2} \subset A$, where $\alpha/2$ is any point of A such that $2\alpha/2 = \alpha$. It follows that if $Z \in CH_1(A \times A)^-_{\mathbb{Q}}, Z = Z_1 + Z_2$, where Z_2 is supported on fibers of σ, and Z_1 is supported on the divisor $D = \{(x+\mu, -x+\mu), x \in C, \mu \in A\}$. If now $Z \in \widetilde{CH_1}(A \times A)$, Z is in particular invariant under the multiplication by -1 on $A \times A$, which leaves D invariant. So we may assume that Z_1 is invariant under -1 acting on D. But D is covered by $C \times A$, via the map $(x,\mu) \rightsquigarrow (x+\mu, -x+\mu)$ and -1 on D lifts to the involution $(j,-1)$ on $C \times A$: now cycles on $C \times A$ split into invariants under (j, Id) and skew under (j, Id). Since $C/j \cong \mathbb{P}^1$, the first ones are generated by cycles $C \times \alpha, \alpha \in A$ and $x_0 \times C_\alpha, C_\alpha \in A$ a translate of $C, x_0 \in C$ one of the Weierstrass points of C. The cycles $x_0 \times C_\alpha$ are skew under the involution $(j,-1)$ modulo $x_0 \times C$.

Finally consider skew cycles under (j, Id)): if W is such a cycle, $W + (j,-1)_*(W) = W - (Id,-1)_*(W)$ is skew under $(Id,-1)_*$; So it must be the sum of a cycle supported on fibers of $C \times A \overset{pr_1}{\to} C$, and of a cycle supported on $C \times C \subset C \times A$, which itself decomposes into a sum $N\, diag(C)$+fibers. Summing up, $\widetilde{CH_1}(A \times A)^-_{\mathbb{Q}}$ is generated by the following cycles:

a) $\{(x+u, \alpha - x - u) \mid x \in C,\ \alpha \in A \text{ fixed } u \in A \text{ fixed}\}$, corresponding to cycles Z_2's above.

b) $\{(x+\alpha, x+\alpha) \mid x \in C, \alpha \in A \text{ fixed}\}$, corresponding to cycles $x_0 \times C_\alpha$.

c) $\{(x+\alpha, -x+\alpha) \mid x \in C, \alpha \in A \text{ fixed}\}$, corresponding to $C \times \alpha$.

d) $\{(y+x+\alpha, -y+x+\alpha) \mid y \in C \text{ fixed}, x \in C, \alpha \in A \text{ fixed}\}$, corresponding to $y \times C_\alpha$.

e) $\{(x,-x) \mid x \in C\}$, corresponding to $diag\,(C)$.

f) $\{(x+\alpha, -x+\alpha) \mid x \in C, \alpha \in A \text{ fixed}\}$, corresponding to fibers of $C \times C$.

Obviously b) does not contribute to $\widetilde{CH_1}(A\times A)^-$. As for c), note that $\{(x,-x), x \in C\} \equiv_{rat} -\{(x,x), x \in C\}$ modulo fibers of $C\times C \subset A\times A$. The same argument shows that all the other cycles are contained in the subgroup generated by cycles $(x+u, x+v)\ \ u, v \in A\ fixed, u \in C\}$, modulo cycles supported on fibers of $A \times A$. Note that cycles supported on fibers of $A \times A$ and invariant under $(-1, Id)_*$ and $(Id,-1)_*$ are exactly the cycles $C * z - z * C, z \in CH_0(A)$ invariant under $(-1)_*$. Taking now the projection of these cycles on the invariant subgroup $\widetilde{CH}_1(A \times A)^-_{\mathbb{Q}}$ gives the generators described in the statement of the lemma. (The fact that one may restrict to cycles $Z_{u,v}, u, v \in C$ is easy to prove).

3.12 Now, if one considers a cycle $W_z = C*z - z*C, z \in CH_0(A)_0, (-1)_*(z) = z$, and if one chooses $w \in CH_0(A)_0, (-1)_*(w) = w$, then $\sum_{\sigma\in\mathfrak{S}_3} \epsilon(\sigma)\sigma^*(W_z * w) = 0$ in $CH_1(A \times A \times A)$ by Proposition 3.2. So it remains only to show:

3.12.1 *For* $Z_{u,v} \in \widetilde{CH}_1(A \times A)^-_{\mathbb{Q}}$ *as in 3.11, and* $w \in CH_0(A)_0, (-1)_*(w) = w$, *one has* $\sum_{\sigma\in\mathfrak{S}_3} \epsilon(\sigma)\sigma^*(Z_{u,v} * w) = 0$ *in* $CH_1(A \times A \times A)$,

which is equivalent to:

3.12.2 $Z_{u,v} * w - \tau_{2,3_*}(Z_{u,v} * w) = -w * Z_{u,v}$.

Before we go to the proof of 3.12.2, we need the following lemma concerning the two families of one cycles generating $\widetilde{CH}_1(A \times A)^-_{\mathbb{Q}}$ provided by lemma 3.11. Consider the following two correspondences:

$$\begin{array}{ccc} C \times C \times C & \xrightarrow{q_1} & A \times A \\ \downarrow{\scriptstyle p_1} & & \\ C \times C & & \end{array}$$

where p_1 is the first projection, and $q_1((x,y,z)) = (x+z, y+z)$ and

$$\begin{array}{ccc} A \times C & \xrightarrow{q_2} & A \times A \\ \downarrow{\scriptstyle p_2} & & \\ A & & \end{array}$$

where p_2 is the first projection and q_2 is the identity on the first factors and the inclusion $C \hookrightarrow A$ chosen above on the second factors. We have the following:

3.13 Lemma.

i) $p_{2*}q_2{}^*(Z_{u,v}) = 0$ *in* $CH_0(A)$.

ii) $p_{2*}q_2{}^*(Z_\alpha) = -2\alpha,\ \forall \alpha \in CH_0(A)$.

iii) $p_{1*}q_1{}^*(Z_{u,v}) = -4((u,v) + (ju, jv) - (v,u) - (jv, ju))$, *where* j *is the hyperelliptic involution of* C.

Proof: i) Clearly $p_{2*}q_2{}^*(C_{u,v}) = t_{u-v}(C.C_v) = C_{u-v} \cdot C_u$, where $t_{(\cdot)}$ denotes translation by $(\cdot) \in A$, and $C_v := t_v(C)$. Similarly

$$p_{2*}q_2{}^*((-1, Id)(C_{u,v}) = (-1)_* p_{2*}q_2{}^*(C_{u,v}) = C_{v-u} \cdot C_{-u} \text{ and}$$
$$p_{2*}q_2{}^*((Id, -1)(C_{u,v})) = (-1)_* p_{2*}q_2{}^*((-1,-1)_*(C_{u,v})) = C_{u-v} \cdot C_u.$$

Hence

$$\begin{aligned} p_{2*}q_2{}^*(C_{u,v} + (-1, Id)_*(C_{u,v}) + (Id, -1)_*(C_{u,v}) + (-1,-1)_*(C_{u,v}) \\ &= 2(C_{u-v} \cdot C_u \,+\, C_{v-u} \cdot C_{-u}) \\ &= 2((C_u - C_v + C) \cdot C_u \,+\, (C_v - C_u + C) \cdot (-C_u + 2C)) \\ &= 4((C_u - C_v) \cdot (C_u - C) + C^2)). \end{aligned}$$

So

$$\begin{aligned} p_{2*}q_2{}^*(Z_{u,v}) &= 4(C_u - C_v) \cdot (C_u + C_v - 2C) \\ &= 4(C_u{}^2 - C_v{}^2 - C_u \cdot C + 2C_v \cdot C) \end{aligned}$$

and this is equal to zero since $C_u{}^2 = 2\{u\}, C_u \cdot C = \{u\} + \{0\}$.

ii) $C{*}\alpha$ does not meet $A \times C$ when $Sup\,\alpha$ does not meet C. So $p_{2*}q_2{}^*(Z_\alpha) = -p_{2*}q_2{}^*(\alpha{*}C)$. Since the divisor $A \times C$ of $A \times A$ is smooth, the result follows immediately from the excess intersection formula and from $C^2 = 2$.

iii) One has

$$p_{1*}q_1{}^*(C_{u,v} + (-1,-1)_*(C_{u,v}) - C_{v,u} - (-1,-1)_*(C_{v,u})) = 0,$$

which is proved as follows: Let $\tilde{Z}_{u,v}$ be the cycle above and let $\tilde{Z}^\epsilon_{u,v}$ be its translation by a torsion point $\epsilon \in A$, so $\tilde{Z}_{u,v} \equiv_{rat} \tilde{Z}^\epsilon_{u,v}$ up to torsion. Now $C_{\alpha,\beta} \cap C^\epsilon_{u,v} \neq 0 \Leftrightarrow \alpha + y = u + x + \epsilon$ and $\beta + y = v + x + \epsilon$ for some $x, y \in C$. This is also equivalent to: $\alpha - \beta = u - v$ and if this last condition is satisfied, there are exactly two couples $(x,y) \in C^2$ satisfying the above equalities, at least for generic (u,v). Now $\alpha - \beta = u - v \Leftrightarrow \alpha = u$ and $\beta = v$ or $\alpha = jv$ and $\beta = ju$; So $p_{1*}q_1{}^*(C^\epsilon_{u,v}) = 2((u,v) + (jv, ju))$. Since $(-1,-1)_*(C_{u,v}) = C_{ju,jv}$ it is then immediate that $p_{1*}q_1{}^*(\tilde{Z}^\epsilon_{u,v}) = 0$ in $CH_0(C \times C)$. So we have proved that

$$\begin{aligned} p_{1*}q_1{}^*(Z_{u,v}) =& p_{1*}q_1{}^*((-1, Id)_*(C_{u,v}) + (Id, -1)_*(C_{u,v}) \\ & - (-1, Id)_*(C_{v,u}) - (Id, -1)_*(C_{v,u})) \end{aligned}$$

with $(Id, -1)_*(C_{u,v}) = (-1, Id)_*(C_{ju,jv})$. Now $p_{1*}q_1{}^*((-1, Id)_*(C_{u,v})) = \sum_i (x_i, y_i)$ where the (x_i, y_i)'s are such that: $\exists\, w_i, z_i \in C$ with $(x_i + z_i, y_i + z_i) = (-u - w_i, v + w_i)$; the last equality is realized in the following cases:

a) $x_i = ju, z_i = jw_i, y_i = v, z_i = w_i$, with w_i a Weierstrass point of C.

b) $x_i = jw_i, z_i = ju, y_i = v, z_i = w_i$.

c) $x_i = ju, z_i = jw_i, y_i = w_i, z_i = v$.

d) $w_i = u, x_i = jz_i, y_i + jx_i = v + u$.

e)$w_i = jv, y_i = jz_i, x_i + z_i = -u - v$.

Since there are six Weierstrass points on C we get:

$$p_{1*}q_1{}^*((-1, Id)_*(C_{u,v})) = 6(ju, v) + (u, v) + (ju, jv) + (jv, u) + (ju, v) + (ju, v) + (jv, u),$$

hence:

$$p_{1*}q_1{}^*(Z_{u,v}) = 2((u, v) + (ju, jv) - (v, u) - (jv, ju)) + 6((ju, v) - (jv, u) + (u, jv) - (v, ju)).$$

Let $h = u + ju$, $\forall u \in C$ be the canonical divisor of C; Then $(ju, v) = h * v - (u, v)$, so:

$$p_{1*}q_1{}^*(Z_{u,v}) = -4((u, v) + (ju, jv) - (v, u) - (jv, ju))$$

as announced.

Now, consider a cycle $Z_{u,v} * w - \tau_{23*}(Z_{u,v} * w) \in CH_1(A \times A \times A)_{\mathbb{Q}}$, with $w \in CH_0(A)_{inv}$, i.e. w (-1)-invariant of degree zero, as in 3.12.1. This cycle is invariant under -1 acting on each factor of $A \times A \times A$, and is skew under τ_{23}. So by Proposition 3.2, one concludes that it lies in the image of $CH_0(A)_{inv} \otimes \widetilde{CH}_1(A \times A)^-_{\mathbb{Q}}$ in $CH_1(A \times A \times A)$. Then the proof of 3.12.2 will be finished by the following two lemmas: We consider now the two correspondences:

$$\begin{array}{ccc} A \times C \times C \times C & \xrightarrow{id \times q_1} & A \times A \times A \\ {\scriptstyle id\times p_1}\downarrow & & \\ A \times C \times C & & \end{array}$$

and

$$\begin{array}{ccc} A \times A \times C & \xrightarrow{id \times q_2} & A \times A \times A \\ {\scriptstyle id\times p_2}\downarrow & & \\ A \times A & & \end{array}$$

where the maps $p_i, q_i, i = 1, 2$ are defined in 3.12:

3.14 Lemma. $(id \times p_2)_*(id \times q_2)^*(Z_{u,v} * w - \tau_{23*}(Z_{u,v} * w)) = 0$. *(This is equal to* $-(id \times p_2)_*(id \times q_2)^*(w * Z_{u,v})$ *by* 3.13, i)).

3.15 Lemma. $(id \times p_1)_*(id \times q_1)^*(Z_{u,v} * w - \tau_{23*}(Z_{u,v} * w)) = 4w * ((u, v) + (ju, jv) - (v, u) - (jv, ju))$ *in* $CH_0(A \times C \times C)$. *(This is equal to* $-(id \times p_1)_*(id \times q_1)^*(w * Z_{u,v})$ *by* 3.13, iii)).

Indeed, Lemmas 3.11 and 3.13 show that the map $((id \times p_2)_*(id \times q_2)^*, (id \times p_1)_*(id \times q_1)^*)$ is injective on the image of $CH_0(A)_{inv} \otimes \widetilde{CH}_1(A \times A)^-_{\mathbb{Q}}$.

3.16 Proof of Lemma 3.14. Obviously $Z_{u,v} * w$ is annihilated by $(id \times p_2)_*(id \times q_2)^*$ since one may assume that the support of w does not meet C. Next one has clearly $(id \times p_2)_*(id \times q_2)^*(\tau_{23}(Z_{u,v})) = (p_{2*}q_2{}^*(Z_{u,v})) * w$ which is equal to zero by Lemma 3.13, i).

3.17 Proof of Lemma 3.15. Suppose first of all that w is a point of A. Then $(id \times p_1)_*(id \times q_1)^*(C_{u,v} \times w) = \{(a, \alpha, \beta) \mid \exists y, x \in C, (x + u, x + v, w) = (a, \alpha + y, \beta + y)\}$.

This is also the intersection of $A \times C \times C$ with $\{(x+u, x+v-y, w-y) \mid x, y \in C\}$ in $A \times A \times A$. Let $\sigma_1 : A \times C \times C \to A$ be defined by $\sigma_1((a,\alpha,\beta)) = \alpha - a - \beta$; it follows that:

3.17.1 $$(id \times p_1)_*(id \times q_1)^*(C_{u,v} \times w) = \pi_1{}^*(C_u) \cdot \pi_3^*(C_{w|C}) \cdot \sigma_1{}^*(\{v-u-w\}),$$

where the π's are the projections of $A \times C \times C$ on its factors.

So we get also:

3.17.2 $$\begin{aligned}(id \times p_1)_*&(id \times q_1)^*((-1, Id)_*(C_{u,v} \times w)) \\ &= (-1, Id, Id)_*(id \times p_1)_*(id \times q_1)^*(C_{u,v} \times w) \\ &= \pi_1{}^*(C_{-u}) \cdot \pi_3^*(C_{w|C}) \cdot \sigma_2^*(\{v-u-w\}),\end{aligned}$$

where $\sigma_2((a,\alpha,\beta)) = \alpha + a - \beta$.

In the same way:

3.17.3 $$\begin{aligned}(id \times p_1)_*&(id \times q_1)^*((Id, -1)_*(C_{u,v} \times w)) \\ &= \pi_1{}^*(C_u) \cdot \pi_3^*(C_{w|C}) \cdot \sigma_2^*(\{-v+u-w\}) \quad \text{and} \\ (id \times p_1)_*&(id \times q_1)^*((-1, -1)_*(C_{u,v} \times w)) \\ &= \pi_1{}^*(C_{-u}) \cdot \pi_3^*(C_{w|C}) \cdot \sigma_1^*(\{-v+u-w\}).\end{aligned}$$

Write now $C_u = L_u + C$, with $L_u \in Pic^0(A)$. Then $L_{-u} = -L_u$, and we get:

3.17.4 $$\begin{aligned}(id \times p_1)_*&(id \times q_1)^*(C_{u,v} \times w + (-1, Id)_*(C_{u,v} \times w) \\ &+ (Id, -1)_*(C_{u,v} \times w) + (-1,-1)_*(C_{u,v} \times w)) \\ = \pi_1{}^*(L_u) &\cdot \pi_3^*(C_{w|C} \cdot (\sigma_1{}^*(\{v-u-w\}) - \sigma_2{}^*(\{v-u-w\}) \\ &+ \sigma_2{}^*(\{-v+u-w\}) - \sigma_1{}^*(\{-v+u-w\})) \\ &+ \pi_1{}^*(C) \cdot \pi_3{}^*(C_{w|C}) \cdot (\sigma_1{}^*(\{v-u-w\}) \\ &+ \sigma_2{}^*(\{v-u-w\}) + \sigma_2{}^*(\{-v+u-w\}) \\ &+ \sigma_1{}^*(\{-v+u-w\})).\end{aligned}$$

The last term $(\dots)$ in this sum is clearly symmetric in u and v, so will disappear in the computation of $(id \times p_1)_*(id \times q_1)^*(Z_{u,v} * w)$. Next note the following equality in $CH_0(A)$ (See 3.2.1):

$$\{v-u-w\} - \{-v+u-w\} = 2C_{-w} \cdot (C_v - C_u).$$

So 3.17.4 becomes

3.17.5 $2\pi_1{}^*(L_u) \cdot \pi_3{}^*(C_{w|C})(\sigma_1{}^*(C_{-w}) \cdot \sigma_1{}^*(C_v - C_u) - \sigma_2{}^*(C_{-w}) \cdot \sigma_2{}^*(C_v - C_u)) + (\dots)$.

In the expression above, one replaces C_{-w} by $C + L_{-w}$, and 3.17.5 gives:

3.17.6 $$\begin{aligned}&2\pi_1{}^*(L_u) \cdot \pi_3{}^*(C_{w|C})(\sigma_1{}^*(L_{-w}) \cdot \sigma_1{}^*(C_v - C_u) \\ &\qquad - \sigma_2{}^*(L_{-w}) \cdot \sigma_2{}^*(C_v - C_u)) + (\dots) \\ &+2\pi_1{}^*(L_u) \cdot \pi_3{}^*(C_{w|C})(\sigma_1{}^*(C) \cdot \sigma_1{}^*(C_v - C_u) \\ &\qquad - \sigma_2{}^*(C) \cdot \sigma_2{}^*(C_v - C_u)).\end{aligned}$$

The last term $((\dots))$ in this sum will vanish if we take for w a (-1)-invariant zero-cycle of degree zero on A instead of a point. Finally L_{-w} and $C_v - C_u$ are algebraically equivalent to zero, so we have:

$$\sigma_1{}^*(L_{-w}) = (\pi_2{}^* - \pi_1{}^* - \pi_3{}^*)(L_{-w})$$
$$\sigma_1{}^*(C_v - C_u) = (\pi_2{}^* - \pi_1{}^* - \pi_3{}^*)(C_v - C_u)$$
$$\sigma_2{}^*(L_{-w}) = (\pi_2{}^* + \pi_1{}^* - \pi_3{}^*)(L_{-w})$$
$$\sigma_2{}^*(C_v - C_u) = (\pi_2{}^* + \pi_1{}^* - \pi_3{}^*)(C_v - C_u),$$

and 3.17.6 becomes:

3.17.7
$$-4\pi_1{}^*(L_u) \cdot \pi_3{}^*(C_{w|C}) \cdot (\pi_2{}^*(L_{-w}) \cdot \pi_1{}^*(C_v - C_u) + \pi_1{}^*(L_{-w}) \cdot \pi_2{}^*(C_v - C_u)) + (\dots) + ((\dots)).$$

Finally we want to compute $(id \times p_1)_*(id \times q_1)^*(Z_{u,v} * w - \tau_{23*}(Z_{u,v} * w))$, so we have to subtract $\{(u,v) \to (v,u)\}$ in 3.17.7, apply $Id \times \tau_{23}$ to $A \times C \times C$ and subtract. As mentioned above $(\dots)$ disappears and we get:

3.17.8
$$\begin{aligned}(id{\times}p_1)_*&(id \times q_1)^*(Z_{u,v} * w - \tau_{23*}(Z_{u,v} * w)) \\ =& -4(\pi_1{}^*(L_u + L_v) \cdot \pi_3{}^*(C_{w|C}) \cdot (\pi_2{}^*(L_{-w}) \cdot \pi_1{}^*(C_v - C_u) \\ &+ \pi_1{}^*(L_{-w}) \cdot \pi_2{}^*(C_v - C_u)) \\ &- \pi_1{}^*(L_u + L_v) \cdot \pi_2{}^*(C_{w|C}) \cdot (\pi_3{}^*(L_{-w}) \cdot \pi_1{}^*(C_v - C_u) \\ &+ \pi_1{}^*(L_{-w}) \cdot \pi_3{}^*(C_v - C_u))) + ((\dots)),\end{aligned}$$

where $((\dots))$ will vanish if the point w is replaced by an element of $CH_0(A)_{inv}$ of degree zero.

One may also as in 3.17.5 replace $C_{w|C}$ by $L_{w|C} + C_{|C}$ and 3.17.8 becomes:

3.17.9
$$\begin{aligned}-4(&\pi_1{}^*(L_u + L_v) \cdot \pi_3{}^*(L_{w|C}) \cdot (\pi_2{}^*(L_{-w}) \cdot \pi_1{}^*(C_v - C_u) \\ &+ \pi_1{}^*(L_{-w}) \cdot \pi_2{}^*(C_v - C_u)) \\ &- \pi_1{}^*(L_u + L_v) \cdot \pi_2{}^*(L_{w|C}) \cdot (\pi_3{}^*(L_{-w}) \cdot \pi_1{}^*(C_v - C_u) \\ &+ \pi_1{}^*(L_{-w}) \cdot \pi_3{}^*(C_v - C_u))) + (((\dots))),\end{aligned}$$

where $(((\dots)))$ vanishes when w is replaced by an element of $CH_0(A)_{inv}$ of degree zero.

Let now $\mu : C \times C \to A$ be the sum map: $\mu((c, c')) = c + c'$. The projections of C^4 to C will be denoted by $\pi'_0, \pi'_1, \pi_2, \pi_3$; In the sequel the line bundles $L_{|C}$ will still be denoted by L for $L \in Pic(A)$. Then for $L \in Pic^0(A), \mu^*(\pi_1{}^*(L)) = \pi'_0{}^*(L) + \pi'_1{}^*(L)$ so we get:

3.17.10
$$\begin{aligned}\mu^*(&(id \times p_1)_*(id \times q_1)^*(Z_{u,v} * w - \tau_{23*}(Z_{u,v} * w))) \\ =& -4((\pi'_0{}^*(L_u + L_v) \cdot \pi'_1{}^*(L_{-w}) \\ &+ \pi'_1{}^*(L_u + L_v) \cdot \pi'_0{}^*(L_{-w})) \cdot \pi_3{}^*(L_w) \cdot \pi_2{}^*(L_v - L_u) \\ &- (\pi'_0{}^*(L_u + L_v) \cdot \pi'_1{}^*(L_{-w}) \\ &+ (\pi'_1{}^*(L_u + L_v) \cdot \pi'_0{}^*(L_{-w})) \cdot \pi_2{}^*(L_w) \cdot \pi_3{}^*(L_v - L_u)) \\ &+ (((\dots))).\end{aligned}$$

Now $L_{-w} = -L_w$, and we can apply to C, a curve of genus two, the property 3.0.1, which gives:

$$\begin{aligned}\pi'_0{}^*(L_u + L_v) \cdot \pi'_1{}^*(L_{-w}) \cdot \pi_3{}^*(L_w) + \pi'_1{}^*(L_u + L_v) \cdot \pi'_0{}^*(L_{-w}) \cdot \pi_3{}^*(L_w) \\ = -\pi'_0{}^*(L_{-w}) \cdot \pi'_1{}^*(L_w) \cdot \pi_3{}^*(L_u + L_v).\end{aligned}$$

So 3.17.10 becomes:

3.17.11
$$4(\pi'_0{}^*(L_{-w})\cdot\pi'_1{}^*(L_w)\cdot(\pi_3{}^*(L_u+L_v)\cdot\pi_2{}^*(L_v-L_u) - (\pi_2{}^*(L_u+L_v)\cdot\pi_3{}^*(L_v-L_u))) + (((\dots))).$$

Now note that $\mu_*\mu^*$ = multiplication by 2 on $CH_0(A)$ and that: $\mu_*(\pi'_0{}^*(L_{-w})\cdot\pi'_1{}^*(L_w)) = -(\{w\}+\{-w\}-2\{0\})$ for w a point of A; So we have:

3.17.12
$$\begin{aligned}(id\times p_1)_*(id\times q_1)^*(Z_{u,v}*w-\tau_{23*}(Z_{u,v}*w)) &= -4(\{w\}+\{-w\}-2\{0\})*(\pi_3{}^*(L_u)\cdot\pi_2{}^*(L_v)\\ &\quad -\pi_3{}^*(L_v)\cdot\pi_2{}^*(L_u)) + (((\dots))).\end{aligned}$$

Since the last term vanishes when w is replaced by an element $\tilde{w}$ of $CH_0(A)_{inv}$ of degree zero, we get:

3.17.13
$$\begin{aligned}(id\times p_1)_*(id\times q_1)^*(Z_{u,v}*\tilde{w}-\tau_{23*}(Z_{u,v}*\tilde{w})) &= 8\tilde{w}*(\pi_2{}^*(L_u)\cdot\pi_3{}^*(L_v)-\pi_2{}^*(L_v)\cdot\pi_3{}^*(L_u)),\end{aligned}$$

which proves Lemma 3.15 since

$$2(\pi_2{}^*(L_u)\cdot\pi_3{}^*(L_v)-(\pi_2{}^*(L_v)\cdot\pi_3{}^*(L_u)) = (u,v)+(ju,jv)-(v,u)-(jv,ju)$$

in $CH_0(C\times C)$.

4. Kuga-Satake varieties

To conclude, I want to explain one of the motivations for the study of skew cycles on S^k, for S a K3 surface; There is a beautiful construction, due to Kuga and Satake [14], which associates to a polarized K3 surface S (in fact one needs only a polarized Hodge structure of weight 2 with $h^{2,0}=1$) an abelian variety $KS(S)$ such that $H^2(S)^0$ is recovered as a direct factor Hodge structure of $H^1(KS(S))\otimes H^1(KS(S))$. Here ^{0} means the primitive part with respect to the polarization. In [8] Deligne proved, using the Mumford-Tate group of a Hodge structure, that there are many surfaces S (with $h^{2,0}\geq 2$) for which $H^2(S)^0$ cannot be realized as direct factor in the even dimensional cohomology of an abelian variety. Now the question is: How to realize geometrically the correspondence between $KS(S)\times KS(S)$ and S, that is to solve the Hodge conjecture for the Hodge class on $KS(S)\times KS(S)\times S$ of degree 4 corresponding to the inclusion $H^2(S)^0 \hookrightarrow H^1(KS(S))\otimes H^1(KS(S)) \hookrightarrow H^2(KS(S)\times KS(S))$?

In [15], Morrison has shown that for a Kummer surface $S=\widetilde{A/\pm 1}$, the Kuga-Satake variety is a sum of copies of A and $\check{A}$; In [21], Paranjape considers a 4-dimensional family of K3 surfaces S (so the general one is not a Kummer surface) and constructs geometrically the correspondence between $KS(S)^2$ and S. But the general case is quite mysterious.

4.1 This Kuga-Satake variety is essentially constructed as a complex torus $T^{0,1}/T_{\mathbb{Z}}$, where $T_{\mathbb{Z}}\otimes\mathbb{C}=T^{1,0}\oplus T^{0,1}, T^{1,0}=\overline{T^{0,1}}$. Now, starting from $H^2(S,\mathbb{Z})^0$ one defines

$$T_{\mathbb{Z}}=\bigwedge_{\mathbb{Z}}^{even} H^2(S,\mathbb{Z})^0=\bigoplus_k \bigwedge^{2k} H^2(S,\mathbb{Z})^0,$$

and the space $T^{1,0}$ is defined as follows: Let $\omega \in H^{2,0}$ be such that $<\omega, \overline{\omega}>= 1$; then

$$T_{\mathbb{C}} = \bigwedge_{\mathbb{C}}^{even} H^2(S,\mathbb{C})^0 = \omega \wedge \bigwedge^{odd} H^{1,1} \oplus \overline{\omega} \wedge \bigwedge^{odd} H^{1,1} \oplus \bigwedge^{even} H^{1,1} \oplus i\omega \wedge \overline{\omega} \wedge \bigwedge^{even} H^{1,1}.$$

Then $T^{1,0} = \omega \wedge \bigwedge^{odd} H^{1,1} \oplus T'^{1,0}$, where $T'^{1,0}$ is the subspace of $\bigwedge^{even} H^{1,1} \oplus i\omega\wedge\overline{\omega}\wedge\bigwedge^{even} H^{1,1}$ defined as follows: $\bigwedge^{even} H^{1,1} \oplus i\omega \wedge \overline{\omega} \wedge \bigwedge^{even} H^{1,1}$ has a real structure, since $H^{1,1}$ has a real structure ($H^{1,1} = H^{1,1}_{\mathbb{R}} \otimes \mathbb{C}$), and $i\omega\wedge\overline{\omega}$ is real. Its real part is $\bigwedge^{even}_{\mathbb{R}} H^{1,1}_{\mathbb{R}} \oplus i\omega\wedge\overline{\omega}\wedge\bigwedge^{even}_{\mathbb{R}} H^{1,1}_{\mathbb{R}}$, which has a natural complex structure I given by $I((u, i\omega\wedge\overline{\omega}\wedge v)) = (-v, i\omega\wedge\overline{\omega}\wedge u)$. Then one takes for $T'^{1,0}$ the i-eigenspace of I in $(\bigwedge^{even}_{\mathbb{R}} H^{1,1}_{\mathbb{R}} \oplus i\omega\wedge\overline{\omega}\wedge\bigwedge^{even}_{\mathbb{R}} H^{1,1}_{\mathbb{R}})\otimes\mathbb{C}$. Note finally that this construction does not depend on the choice of ω, because of the normalization $<\omega, \overline{\omega}>= 1$, which determines ω up to multiplication by a complex number of modulus 1, hence determines $\omega \wedge \overline{\omega}$.

4.2 Example: Let us present one example (a four dimensional family of examples where the Kuga-Satake construction is easily constructed: Consider the family of cubic fourfolds in $\mathbb{P}^5$ with equation $U^3 + V^3 + F(X,Y,Z,T) = 0$, where F is of degree 3 and defines a smooth surface in $\mathbb{P}^3$. These cubics W_F contain many planes, so by [25], for each W_F one can find a K3 surface S_F (a double cover of $\mathbb{P}^2$ ramified along a sextic), and a correspondence $\Gamma_F \subset S_F \times W_F$, inducing an isomorphism: $CH_0(S_F)_0 \cong CH_1(W_F)_{alg}$. Note that, following [26], W_F is the quotient of the blow up of $E \times X_F$ along several copies of Σ_F by the diagonal action of $\mathbb{Z}/3\mathbb{Z}$, where E is the elliptic curve with equation $W^3 + V^3 + U^3 = 0$, the action of $\mathbb{Z}/3\mathbb{Z}$ being by multiplication of W by a third root of unity, X_F is the cubic threefold with equation $W'^3 + F(X,Y,Z,T) = 0$, $\mathbb{Z}/3\mathbb{Z}$ acting by multiplication by the same third root of unity on W', and $\Sigma_F \subset X_F$ is defined by $W' = 0$. It follows that the corresponding K3 surfaces S_F have the following property: The Hodge structure on the transcendental part $TH^2(S)$ of $H^2(S)$ identifies with the $\mathbb{Z}/3\mathbb{Z}$-invariant part of $H^1(E) \otimes H^1(A_F)$, where E is an elliptic curve with a $\mathbb{Z}/3\mathbb{Z}$-action, and A_F is a five dimensional abelian variety (the intermediate jacobian of the cubic threefold X_F) with a $\mathbb{Z}/3\mathbb{Z}$-action, satisfying: let g be a generator of $\mathbb{Z}/3\mathbb{Z}$, then for a certain $\zeta, \zeta^3 = 1, \zeta \neq 1, g^*_{|H^{1,0}(E)}$ is multiplication by ζ and $g^*_{|H^{1,0}(A_F)}$ has a one dimensional ζ^2-eigenspace and a four dimensional ζ-eigenspace. Now the transcendental lattice has even rank equal to 10, so one has to distinguish between $KS(S_F)^+$ and $KS(S_F)^-$ which are build as in 4.1 on the even and odd parts of the exterior algebra of $TH^2(S_F)$ respectively. In this case the Kuga-Satake correspondence is the map $TH^2(S_F) \to Hom(H_1(KS(S_F)^+, H_1(KS(S_F)^-)) \cong_{\mathbb{Q}} H^1(KS(S_F)^+) \otimes H^1(KS(S_F)^-)$ given by Clifford multiplication on the left: $TH^2(S_F) \to Hom(\bigwedge^{even} TH^2(S_F), \bigwedge^{odd} TH^2(S_F))$. Note that, although not canonically isogenous, $KS(S)^+$ and $KS(S)^-$ are isogenous in many ways, using Clifford multiplication on the right by elements of $\bigwedge^{odd} TH^2(S_F,\mathbb{Z})$. Now let us check:

4.3 Proposition: *$KS(S_F)^+$ contains A_F, and $KS(S_F)^-$ contains E.*

Notice that $KS(S_F)^+$ and $KS(S_F)^-$ have a big ring of endomorphisms, so in fact $KS(S_F)^+$ contains many copies of A_F and using isogenies $KS(S_F)^+ \cong_{\mathbb{Q}} KS(S_F)^-$, it contains also many copies of E. I do not know whether there are other abelian subvarieties of $KS(S_F)^+$; Also it would remain to verify that restricted to $A_F \times E \subset KS(S_F)^+ \times KS(S_F)^-$ the Kuga-Satake correspondence is actually the isomorphism $TH^2(S_F) \cong (H^1(A_F)\otimes H^1(E))_{inv}$, which is an algebraic correspondence (see 4.2).

Proof of Proposition 4.3: Let K denote the quadratic number field $\mathbb{Q}(\zeta)$. Working with rational cohomology, we have: $H^1(E)$ is a K-vector space of rank one, $H^1(A_F)$ is a K-vector space of rank 5, so $\bigwedge_K^5 H^1(A_F)$ is a K-vector space of rank 1. Let $\beta \in H^1(E)$ be a generator of $H^1(E)$ over K, and $\eta \in \bigwedge_K^5 H^1(A_F), \eta \neq 0$; then β gives an isomorphism $\psi_\beta : (H^1(A_F) \otimes H^1(E))_{inv} \cong H^1(A)$ obtained as the composite:

$$(H^1(A_F) \otimes H^1(E))_{inv} \cong Hom_K(H^1(E), H^1(A_F)) \xrightarrow{\phi_\beta} H^1(A_F),$$

where $\cong$ is given by the polarization of E, and ϕ_β is the evaluation $\psi \rightsquigarrow \psi(\beta)$. By the inclusion of $\mathbb{Q}$-vector spaces $\bigwedge_K^5 H^1(A_F) \hookrightarrow \bigwedge_{\mathbb{Q}}^5 H^1(A_F)$, we can consider η as an element of $\bigwedge_{\mathbb{Q}}^5 H^1(A_F)$, and then it gives two maps $int(\eta)$ (the interior product with $\eta, u \rightsquigarrow int_u(\eta)$ defined using the intersection form of $H^1(A)$) and $\wedge\eta$ (the exterior product with η) from $H^1(A)$ to $\bigwedge^{even} H^1(A)$. Now we show:

4.3.1 Lemma: *For an adequate choice of $\beta \in H^1(E), \eta_1, \eta_2 \in \bigwedge_K^5 H^1(A_F)$, the map $\gamma_{\beta,\eta_1,\eta_2}$: $H^1(A) \to \bigwedge^{even}((H^1(E) \otimes H^1(A_F))_{inv})$ given by the composite:*

$$H^1(A_F) \xrightarrow{int(\eta_1)+\wedge\eta_2} \bigwedge^{even} H^1(A) \xrightarrow{\psi_\beta} \bigwedge^{even} ((H^1(E) \otimes H^1(A_F))_{inv})$$

induces a morphism of abelian varieties: $A_F = \check{A}_F \to KS(S_F)^+$ (which is easily seen to be injective up to torsion).

To prove 4.3.1, we have to check that the map $\gamma_{\beta,\eta_1,\eta_2} \otimes \mathbb{C}$ sends $H^{1,0}(A_F)$ to $T^{1,0} \subset \bigwedge^{even}((H^1(E) \otimes H^1(A_F))_{inv}) \otimes \mathbb{C}$ (notations as in 4.1). Working over $\mathbb{C}$, we have the decomposition: $\beta = \beta^{1,0} + \overline{\beta^{1,0}}$ (in the sequel $\overline{\beta^{1,0}}$ will be denoted by $\beta^{0,1}$), with $g^*(\beta^{1,0}) = \zeta\,\beta^{1,0}$, and

$$H^1(A_F) \otimes \mathbb{C} = H^1(A_F)^\zeta \oplus H^1(A_F)^{\zeta^2} = H^{1,0}(A)^\zeta \oplus H^{0,1}(A_F)^\zeta \oplus H^{1,0}(A_F)^{\zeta^2} \oplus H^{0,1}(A_F)^{\zeta^2},$$

where $H^1(A_F)^\zeta$ is of rank 5, and $H^{1,0}(A_F)^\zeta$ is of rank 4. Let α_1 be a generator of $H^{1,0}(A)^{\zeta^2}$ (which is of rank one (4.2)): Over $\mathbb{C}$ we can write $\eta_1 = \eta'_1 + \overline{\eta'_1}$, where η'_1 generates $\bigwedge^5 H^1(A_F)^\zeta$. Then $\eta'_1 = \overline{\alpha_1} \wedge \eta''$, where η'' generates $\bigwedge^4 H^{1,0}(A_F)^\zeta$. Similarly $\eta'_2 = \overline{\alpha_2} \wedge \eta''$ and α_2 is obtained from α_1 by multiplication by a non zero element of $K \subset \mathbb{C}$.

The isomorphism $\psi_\beta : H^1(A_F) \otimes \mathbb{C} \cong (H^1(E)) \otimes H^1(A_F))_{inv} \otimes \mathbb{C}$ is given by:

$$H^1(A_F)^\zeta \oplus H^1(A_F)^{\zeta^2} \cong 1/c\,\beta^{0,1} \otimes H^1(A_F)^\zeta \oplus (-1/c)\beta^{1,0} \otimes H^1(A_F)^{\zeta^2},$$

where $c =< \beta^{1,0}, \beta^{0,1} >_E$.

On $H^{1,0}(A_F)^\zeta, int(\eta'_1) + \wedge\eta'_2$ vanishes, so for $u \in H^{1,0}(A_F)^\zeta$, we have: $\gamma_{\beta,\eta_1,\eta_2}(u) = \psi_\beta(int_u(\overline{\eta'_1}) + u \wedge (\overline{\eta'_2}))$. Now

$$int_u(\overline{\eta'_1}) = -\alpha_1 \wedge int_u(\overline{\eta''}) \in H^{1,0}(A_F)^{\zeta^2} \otimes \bigwedge^3 H^{0,1}(A_F)^{\zeta^2},$$

so

$$\psi_\beta(int_u(\overline{\eta'_1})) \in (-1/c)\beta^{1,0} \otimes H^{1,0}(A_F)^{\zeta^2} \otimes \bigwedge^3((-1/c)\beta^{1,0} \otimes H^{0,1}(A_F)^{\zeta^2})$$
$$\subset ((H^1(E) \otimes H^1(A_F))_{inv})^{2,0} \otimes \bigwedge^{odd}(((H^1(E) \otimes H^1(A_F))_{inv})^{1,1}) \subset T^{1,0} \quad \text{(see 4.1)}.$$

Similarly

$$u \wedge \overline{\eta'_2} = -\alpha_2 \wedge u \wedge \overline{\eta''} \in H^{1,0}(A)^{\zeta^2} \otimes H^{1,0}(A)^\zeta \otimes \bigwedge^4 H^{0,1}(A)^{\zeta^2},$$

so

$$\psi_\beta(u\wedge\overline{\eta'}_2)\in(-1/c)\beta^{1,0}\otimes H^{1,0}(A)^{\zeta^2}\otimes 1/c\,\beta^{0,1}\otimes H^{1,0}(A)^{\zeta}\otimes\bigwedge^4((-1/c)\beta^{1,0}H^{0,1}(A)^{\zeta^2})$$
$$\subset((H^1(E)\otimes H^1(A_F))_{inv})^{2,0}\otimes\bigwedge^{odd}(((H^1(E)\otimes H^1(A_F))_{inv})^{1,1})\subset T^{1,0}.$$

Next assume $u\in H^{1,0}(A_F)^{\zeta^2}$; Then it is annihilated by $int(\overline{\eta'_1})+\wedge\overline{\eta'}_2$, so

$$\gamma_{\beta,\eta_1,\eta_2}(u)=\psi_\beta(int_u(\eta'_1)+u\wedge(\eta'_2))=\psi_\beta(<u,\overline{\alpha_1}>_{A_F}\eta''+u\wedge\overline{\alpha_2}\wedge\eta'')$$
$$=<u,\overline{\alpha_1}>_{A_F}\psi_\beta(\eta'')+(((-1/c)\beta^{1,0})\otimes u)\wedge((1/c\,\beta^{0,1})\otimes\overline{\alpha_2})\wedge\psi_\beta(\eta''),$$

where $\psi_\beta(\eta'')\in\wedge^4(1/c\,\beta^{0,1}\otimes H^{1,0}(A_F)^{\zeta})\subset\wedge^{even}(((H^1(E)\otimes H^1(A_F))_{inv})^{1,1})$. According to 4.1, to show that $\gamma_{\beta,\eta_1,\eta_2}(u)\in T^{1,0}$, we need the following equality:

4.3.2 $(-1/c^2)(\beta^{1,0}\otimes u)\wedge(\beta^{0,1}\otimes\overline{\alpha_2})=\omega\wedge\overline{\omega}<u,\overline{\alpha_1}>_{A_F}$ where $\omega\in(H^1(E)\otimes H^1(A_F)_{inv})^{2,0}$ satisfies $<\omega,\overline{\omega}>=3$. Here $<,>$ is the intersection form of $(H^1(E)\otimes H^1(A_F))_{inv}$; The reason for the coefficient 3 is that the intersection form on $TH^4(W_F)$ (4.2), which is the opposite of the one on $TH^2(S_F)$ is equal to $-1/3<,>$.

By $\mathbb{C}$-linearity w.r.t. u, and because $u\in H^{1,0}(A_F)^{\zeta^2}$ is proportional to α_1, which was chosen arbitrarily in $H^{1,0}(A_F)^{\zeta^2}$ we may assume $u=\alpha_1$ and $<\alpha_1,\overline{\alpha_1}>_{A_F}=-i$. So what we need is: $-i/c^2(\beta^{1,0}\otimes\alpha_1)\wedge(\beta^{0,1}\otimes\overline{\alpha_2})=\omega\wedge\overline{\omega}$, with $\alpha_2=\mu\alpha_1$, for some $\mu\in K$. For this it is sufficient that: $(-i/c^2)\overline{\mu}<\beta^{1,0},\beta^{0,1}>_E<\alpha_1,\overline{\alpha_1}>_{A_F}=3$, or that $\overline{\mu}=-3c$ with $c=<\beta^{1,0},\beta^{0,1}>$. But clearly $c\in K$, so $\mu=-3\overline{c}\in K$ and we have proved that for β,η_1,η_2 such that $\eta_2=\overline{\mu}\eta_1,\mu=-3\overline{<\beta^{1,0},\beta^{0,1}>}$ a multiple of $\gamma_{\beta,\eta_1,\eta_2}$ gives a map of abelian varieties $\check{A}_F\to KS(S_F)^+$.

4.3.3 It remains now to find copies of E in $KS(S_F)^+$; But this is easy: choose as before $\beta\in H^1(E)$ giving an isomorphism $\psi_\beta:H^1(A_F)\cong(H^1(E)\otimes H^1(A_F))_{inv}$. Now consider the composite map:

$$\bigwedge^5_K H^1(A_F)\subset\bigwedge^5 H^1(A_F)\overset{\psi_\beta}{\cong}\bigwedge^5((H^1(E)\otimes H^1(A_F))_{inv})\subset\bigwedge^{odd}((H^1(E)\otimes H^1(A_F))_{inv}).$$

Now we have:

4.3.4 Lemma: *The weight one Hodge structure of $\wedge^{odd}((H^1(E)\otimes H^1(A_F))_{inv})$ induces a Hodge structure on the rank 2 $\mathbb{Q}$-vector space $\wedge^5_K H^1(A_F)$, and the corresponding elliptic curve is isogenous to E.*

Proof: One has $\wedge^5_K H^1(A_F)\otimes\mathbb{C}=<\{\eta,\overline{\eta}\}>$, where $\eta\in\wedge^5(H^1(A_F)^{\zeta^2}),\overline{\eta}\in\wedge^5(H^1(A_F)^{\zeta})$. Under the map ψ_β,η is sent to an element of $\wedge^5(\beta^{1,0}\otimes H^1(A_F)^{\zeta^2})$, and one has $H^1(A_F)^{\zeta^2}=H^{1,0}(A_F)^{\zeta^2}\oplus H^{0,1}(A_F)^{\zeta^2}$, the first piece being of rank 1, the second one being of rank 4. So

$$\bigwedge^5(\beta^{1,0}\otimes H^1(A_F)^{\zeta^2})\subset(H^1(E)\otimes H^1(A_F)_{inv})^{2,0}\otimes\bigwedge^{even}((H^1(E)\otimes H^1(A_F)_{inv})^{1,1})\subset T^{1,0}.$$

But this means exactly that $\wedge^5_K H^1(A_F)$ has an induced Hodge structure of level one, which is clearly invariant under the natural action of K on $\wedge^5_K H^1(A_F)$ so the corresponding elliptic curve is isogenous to E.

4.4 Now consider for any k the inclusion: $\wedge^{2k}H^2(S)^0\hookrightarrow\wedge^{even}H^2(S)^0=T_{\mathbb{Z}}\otimes\mathbb{C}$. This inclusion is defined over $\mathbb{Z}$, and one has:

$$\bigwedge^{2k}H^2(S)^0\cap T^{1,0}=\omega\wedge\bigwedge^{2k-1}H^{1,1}(S)=F^{2k+1}\bigwedge^{2k}H^2(S)^0.$$

So the Kuga-Satake variety contains naturally the partial torus

$$\bigwedge^{2k} H^2(S)^0/F^{2k+1}\bigwedge^{2k} H^2(S)^0 \oplus \bigwedge^{2k} H^2(S,\mathbb{Z})^0,$$

which is the essential part of the Deligne cohomology group $H_D^{4k+1}(S^k,\mathbb{Z}(2k+1))^{skew}$. Notice that, because rank $(TH^2(S)$ is odd for a general algebraic K3 surface, one may as well construct the Kuga-Satake variety using the odd part of the exterior algebra $\bigwedge^{odd} H^2(S)^0$, which shows that it contains as well the Deligne cohomology groups $H_D^{2k+1}(S^k,\mathbb{Z}(k+1))_0^{skew}$ for k odd ($()_0$ denotes here the part which is build on $\bigwedge^k H^2(S)^0 \hookrightarrow H^{2k}(S^k)^{skew}$). Each of these Deligne cohomology groups is a partial torus which is the target of the first regulator map ([10], or see 4.5). For $k=1$ it was proved by C. Oliva (to appear?), that the regulator map is non trivial, even modulo the obvious subgroup $NS(S)\otimes\mathbb{C}^*\subset H_D^3(S,\mathbb{Z}(2))$ and the torsion. This appears to be a special feature of surfaces with $h^{2,0}=1$ since for $h^{2,0}\geq 2$ one may find smooth projective surfaces for which the regulator map has torsion image modulo $NS(S)\otimes\mathbb{C}^*$(see [24]).

Note that the only continuous part in the image of the regulator map is $NS(S)\otimes\mathbb{C}^*$, so each of the Deligne cohomology groups $H_D^{2k+1}(S^k,\mathbb{Z}(k+1))_0^{skew}$ is a transcendental object. But the Kuga-Satake construction shows that they are part of an abelian variety, the points of which one would like to understand in terms of algebraic cycles on the various S^k's. It seems that Theorem 3.4 and its conjectural generalizations 3.8 would open the way toward this understanding because it shows that there are many equivalence relations between cycles in higher self-products of S, which are necessary to construct data on which a generalization of the regulator map could be defined, interpolating between the various regulator maps of the S^k's, and parametrizing the Kuga-Satake variety.

4.5 To be more precise the regulator map of

$$S^k : H^k(\mathcal{K}_{k+1}(S^k)) \to H_D^{2k+1}(S^k,\mathbb{Z}(k+1))$$

is defined on data $\sum_i(Z_i,\phi_i)$, $Z_i\subset S^k$ irreducible of codimension k, ϕ_i a rational function on Z_i, such that $\sum_i div(\phi_i)=0$ as a codimension $k+1$ cycle on S^k; the group $H^k(\mathcal{K}_{k+1}(S^k))$ is the group of such data modulo a certain subgroup. Let us consider the case $k=1$: the regulator map takes values in $H_D^3(S,\mathbb{Z}(2))$ and its image modulo $NS(S)\otimes\mathbb{C}^*$ is countable. In fact it seems possible that $H^1(\mathcal{K}_2(S))/Pic(S)\otimes\mathbb{C}^*$ itself is countable, or at least, there is the following evidence for that (this was the starting point for C. Oliva's work): if one wants to construct non trivial elements in $H^1(\mathcal{K}_2(S))$, it is natural to make the following construction: Let $|L|,|L'|$ be two linear systems on S; Consider the finite cover W of $|L|\times|L'|$ determined by an ordering of the points of intersection of $C\subset|L|, C'\subset|L'|$. Let N be the cardinal of this intersection, and let $n_1,\ldots,n_N$ be integers such that $\sum_i n_i=0$. Now let $Z=(C,C',p_1,\ldots,p_N)\in W$ and consider the conditions on $Z:\sum_i n_ip_i\equiv 0$ in $CH_0(C)$, and $\sum_i n_ip_i\equiv 0$ in $CH_0(C')$. If these conditions are satisfied one constructs an element α_Z of $H^1(\mathcal{K}_2(S))$ as follows: by assumption, there exists ϕ on C,ϕ' on C', such that: $\sum_i n_ip_i=div(\phi)$ as a zero-cycle of $C,\sum_i n_ip_i=div(\phi')$ as a zero-cycle on C'; Then $\alpha_Z=(C,\phi)+(C',\phi'^{-1})$ gives the desired element, well defined up to $Pic(S)\otimes\mathbb{C}^*$, reflecting the choice of ϕ and ϕ'. Now note that the expected dimension of this sublocus of W is zero, because $\dim(W)=\dim(|L|)+\dim(|L'|)$ and the expected codimension of the sublocus is $g(C)+g(C')$ which is also equal to $\dim(|L|)+\dim(|L'|)$ since we are on a K3 surface. So there are some reasons to expect that $H^1(\mathcal{K}_2(S))/Pic(S)\otimes\mathbb{C}^*$ is discrete. But if we work with the self-product S^2 we can construct positive dimensional families of data $\sum_i(C_i,\phi_i)$ such that $\sum_i div(\phi_i)=0$; For example, if we know the property (*): for

$(x,y) \in S^2, (x,y) - (y,x) \equiv_{rat} -x_0 * (x-y) + (x-y) * x_0$, there is a generically finite cover $G \xrightarrow{\pi} S \times S$ such that $\pi^{-1}((x,y))$ parametrizes curves $C_i \subset S \times S$ with a meromorphic function ϕ_i, such that $\sum_i div\phi_i = (x,y) - (y,x) + x_0 * (x-y) + (x-y) * x_0$. Then the surface $\pi^{-1}(\Delta)$ parametrizes data (c_i, ϕ_i) such that $\sum_i n_i div\phi_i = 0$.

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