VARIATION OF STABLE BIRATIONAL TYPES OF HYPERSURFACES

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Abstract. We introduce and study the question how can stable birational types vary in a smooth proper family. Our starting point is the specialization for stable birational types of Nicaise and the author and our emphasis is on stable birational types of hypersurfaces. Building up on the work of Totaro and Schreieder on stable irrationality of hypersurfaces of high degree, we show that smooth Fano hypersurfaces of large degree over a field of characteristic zero are in general not stably birational to each other. In the appendix Claire Voisin proves a similar result in a different setting using the Chow decomposition of diagonal and unramified cohomology.

1. Introduction

Let k be an uncountable algebraically closed field of characteristic zero. Recall that k-varieties X, Y of the same dimension are called *stably birational* if $X \times \mathbb{P}^m$ and $Y \times \mathbb{P}^m$ are birational for some $m \geq 0$. If in the above definition Y is a projective space, then X is called *stably rational*. There has been recently a lot of progress in showing that for large classes of varieties, including Fano hypersurfaces of high degree very general members are stably irrational [Voi15, CTP16, Tot16, Sch18]. In this paper we introduce and study the following more general question:

Question 1.1. *Given a family of smooth projective varieties, how can we decide if all members are stably birational to each other?*

We answer this question for Fano hypersurfaces of sufficiently high degree. Our main result is the following:

Theorem 1.2 (See Theorem 3.4)**.** *If there exists a stably irrational smooth projective hypersurface of dimension n and degree $d \leq n+1$, then very general hypersurfaces of dimension n and degree d are not stably birational to each other.*

Here by very general hypersurfaces we mean pairs of hypersurfaces corresponding to points in the parameter space $\mathbb{P}(H^0(\mathbb{P}^{n+1}, \mathcal{O}(d)))^2$ lying in the complement of a countable union of divisors.

Thus the only case when smooth hypersurfaces of given degree and dimension are stably birational to each other is when they are all stably rational. This happens in degrees one and two, and for cubic surfaces, and it is widely expected that no other such cases exist.

It has been proved by Totaro [Tot16] that in every dimension $n \geq 3$ very general Fano hypersurfaces of degree $d \geq 2\lceil \frac{n+2}{3} \rceil$ are stably irrational. Schreieder improved Totaro's bound to $d \geq \log_2(n) + 2$ [Sch18]. Using [Sch18, Corollary 1.2] and the Theorem above we deduce the following.

Corollary 1.3. *For $n \geq 3$ and $d \geq \log_2(n) + 2$, very general hypersurfaces of dimension n and degree d are not stably birational to each other.*

In particular we see that there are uncountably many stable birational types of such hypersurfaces. The first interesting case when Corollary applies is that of quartic threefolds ($n = 3$, $d = 4$, here stable irrationality of the very general member follows from [CTP16]).

Under the assumptions of the Theorem every stable birational type is attained at a countable union of Zariski closed subsets in the parameter space of smooth hypersurfaces. A more explicit description of which hypersurfaces of fixed dimension and degree would be stably birational to the given one, seems completely out of reach.

Our approach to stable birational types relies on the Grothendieck ring of varieties, the Larsen-Lunts Theorem [LL03] and the specialization map [NS17, KT17]. Firstly, we reformulate results of [NS17] by introducing the idea of a variation of stable birational types and show that if stable birational type in a family is not constant, then it has to vary in a strong sense (Theorem 3.2). Then, by constructing an appropriate degeneration of smooth hypersurfaces to a hyperplane arrangement, with desingularized total space (Lemma 3.6) and showing that the class of this hyperplane arrangement in the Grothendieck ring is congruent to 1 modulo $\mathbb{L}$ (Lemma 2.1) we deduce that under the conditions of the theorem, stable birational types of hypersurfaces can not be constant (Theorem 3.4). The same method would apply to any family that has a smooth stably irrational member alongside a smooth stably rational member, or more generally, a member with mild singularities and whose class in the Grothendieck modulo $\mathbb{L}$ is equal to one, and provided that the total space of the degeneration is smooth or has mild singularities.

In addition to using the Grothendieck ring of varieties and the specialization map, the main novelty of this work is making use of degeneration of a hypersurface to a hyperplane arrangement; even though such degenerations are ubiquitous in algebraic geometry, starting from computing the genus of a plane curve and all the way to the modern Gross-Siebert program, that's the first time this degeneration appears when studying stable birational types.

In the Appendix to this paper Claire Voisin proves a similar result regarding variation of stable birational types in a slighly different setting using decomposition of diagonal and unramified cohomology. Very soon after appearance of this work, Stefan Schreider gave a different proof of Corollary 1.3 using degeneration to hyperplane arrangement and decomposition of the diagonal, also relying on [Sch19]; in fact Schreieder's proof does not use resolution of singularities and thus generalizes the statement to a field of an arbitrary characteristic.

Finally we note that unlike in Hodge theory, where the term "variation" can be understood using the period map between moduli spaces, our term "variation of stable birational types" has a very naive meaning; it is not at all clear how one could introduce a reasonable moduli space of stable birational types.

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Notation. By a variety we mean a separated irreducible and reduced scheme of finite type over k. By a point of a variety we a mean a closed point. We say that a property holds for very general points of a variety if it holds away from a countable union of divisors.

2. Preliminary results

2.1. **Grothendieck ring of varieties.** Recall that the Grothendieck ring of varieties $K_0(Var/k)$ is generated as an abelian group by isomorphism classes $[X]$ of schemes of finite type X/k modulo the scissor relations

$$[X] = [U] + [Z]$$

for every closed $Z \subset X$ with open complement $U \subset X$. The product structure on $K_0(Var/k)$ is induced by product of schemes. We write $\mathbb{L} \in K_0(Var/k)$ for the class of the affine line $[\mathbb{A}^1]$.

The following lemma is useful when degenerating smooth varieties to hyperplane arrangements.

Lemma 2.1. *Let $H_1, \dots, H_r \subset \mathbb{P}^{n+1}$ be a collection of distinct hyperplanes in $\mathbb{P}^{n+1}$ such that $\bigcup_{i=1}^r H_i$ is a simple normal crossing divisor, that is we assume that any intersection of k hyperplanes is either empty or of codimension k. Then we have*

$$[H_1 \cup \dots \cup H_r] = \sum_{j=0}^{n} (-1)^j \binom{r}{j+1} [\mathbb{P}^{n-j}],$$

and if $r \leq n+1$, then $[H_1 \cup \dots \cup H_r] \equiv 1 \pmod{\mathbb{L}}$.

Proof. Let $P_{r,n} \in K_0(Var/k)$ be the class of the simple normal crossing hyperplane arrangement of r hyperplanes in $\mathbb{P}^{n+1}$ in the Grothendieck ring of varieties. It follows from the inductive argument below that the class $P_{r,n}$ only depends on r and n and not on the relative positions of the hyperplanes.

We prove the formula for $P_{r,n}$ using induction. For the induction base we have for all $r \geq 1$, $P_{r,0} = r$ (r points in $\mathbb{P}^1$). We assume that the formula is true for $P_{r,n-1}$ and $P_{r-1,n-1}$. Given $r \geq 2$ hyperplanes in $\mathbb{P}^{n+1}$, intersecting the first $r-1$ of them with the last one, gives rise to an arrangement of $r-1$ hyperplanes in $\mathbb{P}^n$, which is still simple normal crossing. Using inclusion-exclusion we obtain

$$P_{r,n} = P_{r-1,n} + [\mathbb{P}^n] - P_{r-1,n-1},$$

which by induction hypothesis can be rewritten as

$$P_{r,n} = \sum_{j=0}^{n} (-1)^j \binom{r-1}{j+1} [\mathbb{P}^{n-j}] + [\mathbb{P}^n] - \sum_{i=0}^{n-1} (-1)^i \binom{r-1}{i+1} [\mathbb{P}^{n-i-1}]$$

which easily gives the desired result.

Finally, if $r \leq n+1$, then

$$P_{r,n} \equiv \sum_{j=0}^{n} (-1)^j \binom{r}{j+1} = \sum_{i=1}^{r} (-1)^{i-1} \binom{r}{i} = 1 \pmod{\mathbb{L}}.$$

□

2.2. **Resolution of one toric singularity.** When constructing resolutions of singularities for the total space of a degeneration of hypersurfaces the following result is useful. We refer to [Fu93] for standard facts and constructions from toric geometry.

Lemma 2.2. *Let be $\mathcal{X}$ be a hypersurface in $\mathbb{A}^{n+2}$ defined by equation*

$$t \cdot y = z_1 \cdots z_n$$

and let $\pi : \mathcal{X} \to \mathbb{A}^1$ be the morphism given by the t coordinate.

(1) Let $N = \mathbb{Z}^{n+1}$ with the standard basis $e_1, \dots, e_{n+1}$ and let $N_{\mathbb{R}} = N \otimes \mathbb{R}$. For every $1 \leq i \leq n$ let $f_i = e_i + e_{n+1} \in N$. Let $\sigma \subset N_{\mathbb{R}} = \mathbb{R}^{n+1}$ be the cone generated by the vectors $e_1, \dots, e_n$, $f_1, \dots, f_n$. Then $\mathcal{X}$ is the toric variety corresponding to the cone σ, that is $\mathcal{X} = \operatorname{Spec}(k[N^\vee \cap \sigma^\vee])$.

(2) Subdivision of σ into n cones

$$\sigma_k := \mathbb{R}_{\geq 0} f_1 + \cdots + \mathbb{R}_{\geq 0} f_k + \mathbb{R}_{\geq 0} e_k + \cdots + \mathbb{R}_{\geq 0} e_n \subset N_{\mathbb{R}}, \quad 1 \leq k \leq n$$

provides a resolution of singularities $\tau : \tilde{\mathcal{X}} \to \mathcal{X}$. The composition $\tilde{\pi} := \pi \circ \tau$ has a reduced simple normal crossing fiber over $0 \in \mathbb{A}^1$.

(3) Explicitly desingularization τ is obtained by a sequence of blow ups of proper preimages of $n-1$ Weil divisors $V(t, z_1), \dots, V(t, z_{n-1}) \subset \mathcal{X}$.

Proof. The proof is a standard computation in toric geometry.

(1) Let $M = N^\vee$ be the dual lattice with the dual basis $e_1^*, \dots, e_{n+1}^*$. The dual cone $\sigma^\vee \subset M_{\mathbb{R}}$ is described by the system of inequalities for $(a_1, \dots, a_{n+1}) \in M_{\mathbb{R}}$:

$$a_1 \geq 0, \ \dots, \ a_n \geq 0,$$

$$a_1 + a_{n+1} \geq 0, \ \dots, \ a_n + a_{n+1} \geq 0.$$

It is clear that the $n+2$ vectors

$$e_1^*, \dots, e_n^*, e_{n+1}^*, e_1^* + \cdots + e_n^* - e_{n+1}^* \in M$$

all satisfy these inequalities, and every integral point in $\sigma^\vee$ can be written as a non-negative integer combination of these vectors (indeed, if $a_{n+1} \geq 0$, then we are done, while if $a_{n+1} < 0$, all other coordinates must be positive and a multiple of $e_1^* + \cdots + e_n^* - e_{n+1}^*$ can be subtracted).

If we set

$$z_1, \dots, z_n, t, y$$

to be the monomials corresponding to the vectors above, they satisfy a single relation $ty = z_1 \cdots z_n$.

(2) The cone σ combinatorially is a cone over a prism $\Delta^{n-1} \times [0,1]$ (Δ is a simplex), and the subdivision we consider corresponds to a standard subdivision of this prism into n simplices. To describe this construction in detail note that we have seen that the dual cone $\sigma^\vee$ is generated by $e_1^*, \dots, e_n^*, e_{n+1}^*, e_1^* + \cdots + e_n^* - e_{n+1}^* \in M$, so that the cone $\sigma \subset N_{\mathbb{R}}$ is the set of solutions of

$$x_1 \geq 0, \ \dots, \ x_{n+1} \geq 0, \quad x_{n+1} \leq x_1 + \cdots + x_n.$$

For every $1 \leq k \leq n$ let us consider the cones given by

$$x_1 \geq 0, \ \dots, \ x_{n+1} \geq 0, \quad x_1 + \cdots + x_{k-1} \leq x_{n+1} \leq x_1 + \cdots + x_k.$$

These cones obviously form a partition of σ and it is easy to see that these are precisely the cones σ_k with boundary rays generated by $f_1, \dots, f_k, e_k, \dots, e_n$.

The new fan, consisting of the cones σ_k and all their faces has its cones generated by basis vectors of the lattice N, hence the corresponding morphism $\tau : \widetilde{\mathcal{X}} \to \mathcal{X}$ is a resolution of singularities.

To check that the fiber $\tilde{\pi}$ is reduced simple normal crossing over $0 \in \mathbb{A}^1$, we consider each affine toric chart U_k, corresponding to the cone σ_k. By our choice of coordinates the restriction of $\tilde{\pi}$ to U_k corresponds to the projection onto the last coordinate $N_{\mathbb{R}} = \mathbb{R}^{n+1} \to \mathbb{R}$. Fiber over $0 \in \mathbb{A}^1$ being a reduced simple normal crossing divisor in U_k translates into the fact that every vector $f_1, \dots, f_k, e_k, \dots, e_n$ has zero or one as its last coordinate.

(3) Let us describe the effect of the blow up of $V(t, z_1) \subset \mathcal{X}$ on our toric model. We have two open charts, on the first open chart which we call U_1 we have $z_1 = tz_1'$ so the coordinates are $t, y, z_1', z_2, \dots, z_n$ and the equation is

$$y = z_1' z_2 \cdots z_n.$$

On the other open chart which we call $\mathcal{X}'$ we have $t = z_1 t'$ so coordinates are $t', y, z_1, \dots, z_n$ and the equation is

$$t'y = z_2 \cdots z_n.$$

The gluing between the two open charts is $t' = \frac{1}{z_1'}$. We see that U_1 is the affine space with coordinates $z_1', z_2, \dots, z_n$ which are monomials corresponding to vectors

$$e_1^* - e_{n+1}^*, e_2^*, \dots, e_n^*, e_{n+1}^*,$$

which is precisely the generators for the dual cone to σ_1, while $\mathcal{X}'$ has coordinates corresponding to the monomials

$$e_{n+1}^* - e_1^*, e_1^*, \dots, e_n^*, e_1^* + \cdots + e_n^* - e_{n+1}^*,$$

and it follows easily that $\mathcal{X}'$ is the toric variety corresponding to the cone

$$\sigma' := \sigma_2 \cup \cdots \cup \sigma_n = \sum_{i=2}^{n} \mathbb{R}_{\geq 0} e_i + \sum_{i=1}^{n} \mathbb{R}_{\geq 0} f_i \subset N_{\mathbb{R}}.$$

Furthermore $\mathcal{X}'$ is the product of $\mathbb{A}^1$ (z_1 coordinate) with the same model in $n-1$ variables.

Since U_1 is already smooth, it will not be affected by further blow ups of divisors, while the proper preimage of $V(t, z_k)$ (for $k \geq 2$) in $\mathcal{X}'$ is $V(t', z_k)$, and the same argument can be applied to $\mathcal{X}'$ to get open charts $U_2, \dots, U_n$, corresponding to the subdivision of σ'.

The process terminates after $n-1$ steps, when two smooth charts are produced. $\square$

3. Stable birational types of hypersurfaces

3.1. **Variation of stable birational types.** We recall the following result of Larsen and Lunts which holds over arbitrary fields of characteristic zero and which provides the link between birational geometry and the Grothendieck ring of varieties.

Theorem 3.1. [LL03] *If X and Y are smooth projective varieties with classes $[X], [Y] \in K_0(Var/k)$, then X and Y are stably birational if and only if*

$$[X] \equiv [Y] \pmod{\mathbb{L}}.$$

Thus for a smooth projective variety X the element $[X] \in K_0(Var/k)/(\mathbb{L})$ encodes the stable birational class of X. More generally, even when X is reducible, we may think about $[X] \pmod{\mathbb{L}}$ as an analog of the stable birational class.

We now introduce the idea of the variation of stable birational types in the smooth and simple normal crossing settings. These rely on the results of [NS17].

Theorem 3.2. *Let S be a variety and let $\pi : \mathcal{X} \to S$ be a smooth proper morphism with connected fibers. Then one of the following is true:*

(a) **Constant stable birational type:** *all fibers $\pi^{-1}(t)$, $t \in S$ are stably birational.*
(b) **Variation of stable birational type:** *for very general points $(t, t') \in S \times S$ the fibers $\pi^{-1}(t)$ and $\pi^{-1}(t')$ are not stably birational to each other.*

Proof. For $i = 1, 2$ let us write $p_i : S \times S \to S$ for the two projections. Let $\pi_i : \mathcal{X}_i \to S \times S$ denote the base change of π by p_i. Thus π_1, π_2 are smooth proper morphisms. Let $Z \subset S \times S$ be the set of points where the fibers of π_1 and π_2 are stably birational, in other words Z consists of points (t_1, t_2) such that $\pi^{-1}(t_1)$ and $\pi^{-1}(t_2)$ are stably birational.

Applying the same argument as in the proof of [NS17, Proposition 4.1.5], Z is a countable union of Zariski closed subsets of $S \times S$. Thus either $Z = S \times S$, which corresponds to the case (a), or $Z \subsetneq S \times S$, so that points in $S \times S \setminus Z$ are very general which corresponds to (b). □

The next Proposition provides a generalization of the Theorem above to simple normal crossing singularities. Instead of stable birational types we work with classes in $K_0(Var/k)/(\mathbb{L})$.

Proposition 3.3. *Let C be a smooth connected curve and let $\pi : \mathcal{X} \to C$ be a flat proper morphism with connected fibers and smooth total space $\mathcal{X}$. Let $0 \in C$, and assume that the restriction of π to $C \setminus 0$ is smooth, and that $\pi^{-1}(0)$ is reduced simple normal crossing.*

If all fibers $\pi^{-1}(t)$ for $t \neq 0$ are stably birational to a smooth projective variety X, then the class of the central fiber satisfies

$$[\pi^{-1}(0)] \equiv [X] \pmod{\mathbb{L}}.$$

Proof. Let us form a constant family $\pi' : X \times C \to C$. By assumption the two morphisms π, π' have stably birational fibers for $t \neq 0$.

Thus using [NS17, Proposition 4.1.1] we deduce that fibers over $t = 0$ satisfy

$$[\pi^{-1}(0)] \equiv [\pi'^{-1}(0)] = [X] \pmod{\mathbb{L}}.$$

□

3.2. **Application to hypersurfaces.** In this section we study stably birational types of hypersurfaces $X \subset \mathbb{P}^{n+1}$. The interesting case is the Fano case, that is the case when the degree of X satisfies $d \leq n + 1$.

Theorem 3.4. *Assume that there exists a smooth projective hypersurface of dimension n and degree $d \leq n + 1$ which is stably irrational. Then smooth projective hypersurfaces of dimension n and degree d admit a variation of stable birational types, that is two very general such hypersurfaces are not stably birational to each other.*

Remark 3.5. *By the main result of* [NS17]*, existence of a single stably irrational smooth projective hypersurface of dimension n and degree d is equivalent to very general such hypersurfaces being stably irrational.*

Before we prove the Theorem we need the following Lemma, which provides a convenient degeneration of smooth hypersurfaces.

Lemma 3.6. *For every $n \geq 2$, $1 \leq d \leq n+1$ there exists a smooth connected curve C, with a point $0 \in C$, and a flat proper morphism*

$$\pi : \mathcal{X} \to C$$

with smooth $\mathcal{X}$ such that

1. *All fibers $\pi^{-1}(t)$, for $t \neq 0$ are smooth projective hypersurfaces of dimension n and degree d,*
2. *The fiber $\pi^{-1}(0)$ is reduced simple normal crossing and satisfies*

$$[\pi^{-1}(0)] \equiv 1 \pmod{\mathbb{L}}.$$

Proof. We consider two sections $F_0, F_1 \in H^0(\mathbb{P}^{n+1}, \mathcal{O}(d))$. We take F_0 to be a product of d linearly independent linear forms, and F_1 to be a general section. In particular the hypersurface $V(F_1) \subset \mathbb{P}^{n+1}$ is smooth, and it intersects all the strata of the hyperplane arrangement $V(F_0)$ transversally.

We set $\mathcal{X}$ to denote the zero locus of $F_0 + tF_1$ in $\mathbb{P}^{n+1} \times \mathbb{A}^1$. After restricting to an open subset $C \subset \mathbb{A}^1$ we may assume that the fibers of $\pi : \mathcal{X}' \to C$ for $t \neq 0$ are smooth. The fiber $\pi^{-1}(0)$ is the hyperplane arrangement $V(F_0)$. Since $d \leq n+1$, Lemma 2.1 implies that the fiber $\pi^{-1}(0)$ satisfies

$$[\pi^{-1}(0)] \equiv 1 \pmod{\mathbb{L}}.$$

Thus the morphism $\pi : \mathcal{X} \to C$ satisfies all the requirements of the Lemma except for smoothness of the total space $\mathcal{X}$.

We provide an explicit desingularization of $\mathcal{X}$. Let $E_1, \ldots, E_d$ be the components of $\pi^{-1}(0)$. Locally at every point $P \in \mathcal{X}$ in the central fiber $t = 0$, the model $\mathcal{X}$ is given by equations of the form

$$t \cdot f + l_1 \cdots l_d = 0, \tag{3.1}$$

where l_i are linear polynomials and f is a polynomial of degree d, in $n+1$ variables.

We change coordinates so that $P = 0$, and by our transversality assumptions the equation can be written as

$$t \cdot (x_{n+1} + \text{terms of deg. } \geq 2) + x_1 \cdots x_k \cdot g(x_1, \ldots, x_{n+1}) = 0,$$

where $g(0) \neq 0$ and $k \leq n$. Taking formal completion of $\mathcal{X}$ at P we can change the coordinates again to rewrite the defining local equation as

$$t \cdot x_{n+1} = x_1 \cdots x_k.$$

According to Lemma 2.2 such singularities are resolved by a sequence of blows up of Weil divisors $V(t, x_i)$, and this new model is semistable over $0 \in \mathbb{A}^1$ (and the rest of the fibers are unchanged, so they are smooth hypersurfaces).

Blow ups commute with the completion, and in terms of the original equation (3.1) we have to blow up the sequence of smooth Weil divisors

$$E_1 = V(t, l_1), \ldots, E_{d-1} = V(t, l_{d-1}).$$

Since each open chart is a hypersurface in $\mathbb{A}^{n+2}$, the resulting blow ups only glue in Zariski locally trivial $\mathbb{P}^1$-fibrations. In particular, the class of the central fiber in $K_0(Var/k)/(\mathbb{L})$ does not change at each blow up. $\square$

Proof of Theorem 3.4. Let $U \subset \mathbb{P}(H^0(\mathbb{P}^{n+1}, \mathcal{O}(d)))$ be the open subset parametrizing smooth hypersurfaces. By Theorem 3.2, if stable birational types of hypersurfaces of dimension n and degree d, does NOT vary, it has be constant, that is all such smooth hypersurfaces are stably birational to a smooth projective variety X.

We now consider the family $\pi : \mathcal{X} \to C$ given by Lemma 3.6. From what we explained above, all fibers $\pi^{-1}(t)$, for $t \neq 0$ have to be stably birational to X. By Proposition 3.3, the special fiber has to satisfy

$$1 \equiv [\pi^{-1}(0)] \equiv [X] \pmod{\mathbb{L}}.$$

This is a contradiction, since Larsen-Lunts Theorem 3.1 implies that X is stably rational, contrary to our assumptions. $\square$

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Appendix: Stable birational equivalence and decomposition of the diagonal, by Claire Voisin

We prove in this appendix that, if a family of projective varieties has a mildly singular member with a nonzero unramified cohomology class with given coefficients, while the very general member Y is smooth and has no such class, the stable birational equivalence class of the fibers Y_t is not constant. In particular, quartic and sextic double covers of $\mathbb{P}^3$ do not have a constant stable birational type. This result is inspired by the main theorem of Shinder in this paper. Note however that the assumptions and range of applications in both statements are different. We will work over any algebraically closed field k of infinite transcendence degree over the prime field but the main application (Theorem 8) will assume characteristic 0. We refer to Schreieder recent note [9] for generalizations and a similar statement in nonzero characteristic.

We start with the following decomposition of the diagonal result for stable birational equivalence.

Proposition 1. *Let X, Y be two smooth projective varieties of dimension n. Assume X and Y are stably birational. Then there exist codimension n cycles*

$$\Gamma\in \mathrm{CH}^n(X\times Y),\ \Gamma'\in \mathrm{CH}^n(Y\times X)$$

such that

$$\begin{aligned}\Gamma'\circ\Gamma&=\Delta_X+Z_X \text{ in } \mathrm{CH}^n(X\times X),\\ \Gamma\circ\Gamma'&=\Delta_Y+Z_Y \text{ in } \mathrm{CH}^n(Y\times Y),\end{aligned}\tag{3.2}$$

where Z_X is supported on $D_X\times X$ for some proper closed algebraic subset $D_X\subset X$, and Z_Y is supported on $D_Y\times Y$ for some proper closed algebraic subset $D_Y\subset Y$.

Proof. When X and Y are actually birational, this statement is proved in [4]. In this case, we simply take for Γ the graph of a birational map $\phi: X \dashrightarrow Y$ and for Γ' the graph of ϕ^{-1}. The equality $\Gamma'\circ\Gamma=\Delta_X$ (resp. $\Gamma\circ\Gamma'=\Delta_Y$) is in this case satisfied at the level of cycles on $U\times X$, resp. $V\times Y$, where $U\cong V$ is a Zariski open set of X on which ϕ is an isomorphism onto its image $V\subset Y$. Assume now that

$$\phi: X\times\mathbb{P}^r\dashrightarrow Y\times\mathbb{P}^r$$

is a birational map for some r. Then by the previous step, there exist

$$\Gamma_\phi\in \mathrm{CH}^{n+r}(X\times\mathbb{P}^r\times Y\times\mathbb{P}^r),\ \Gamma'_\phi\in \mathrm{CH}^{n+r}(Y\times\mathbb{P}^r\times X\times\mathbb{P}^r)$$

such that formulas (3.2) hold for some proper closed algebraic subsets $D\subset X\times\mathbb{P}^r$, resp. $D'\subset Y\times\mathbb{P}^r$. For any point $O\in\mathbb{P}^r$, define

$$\begin{aligned}\Gamma&:=p_{XY*}(\Gamma_{\phi|X\times O\times Y\times\mathbb{P}^r})\\ \Gamma'&:=p_{YX*}(\Gamma'_{\phi|Y\times O\times X\times\mathbb{P}^r}),\end{aligned}\tag{3.3}$$

where p_{XY} is the projection from $X\times O\times Y\times\mathbb{P}^r$ to $X\times Y$, and p_{YX} is the projection from $Y\times O\times X\times\mathbb{P}^r$ to $Y\times X$. We have to show that (3.2) holds.

Let us decompose $\mathrm{CH}(X\times\mathbb{P}^r\times Y\times\mathbb{P}^r)$ as polynomials in h_1, h_2 with coefficients in $\mathrm{CH}(X\times Y)$, where $h_1=\mathrm{pr}_2^*c_1(\mathcal{O}_{\mathbb{P}^r}(1))$, $h_2=\mathrm{pr}_4^*c_1(\mathcal{O}_{\mathbb{P}^r}(1))$

$$\mathrm{CH}(X\times\mathbb{P}^r\times Y\times\mathbb{P}^r)=\oplus_{0\leq i\leq r,0\leq j\leq r}h_1^ih_2^j\mathrm{CH}(X\times Y),$$

which gives in particular

$$\Gamma_\phi=\sum_{i,j}h_1^ih_2^j\Gamma_{\phi,i,j},\ \Gamma'_\phi=\sum_{i,j}h_1^ih_2^j\Gamma'_{\phi,i,j},\tag{3.4}$$

with $\Gamma_{\phi,i,j}\in\mathrm{CH}(X\times Y)$, $\Gamma'_{\phi,i,j}\in\mathrm{CH}(Y\times X)$. We obviously have $\Gamma=\Gamma_{\phi,0,r}$, $\Gamma'=\Gamma'_{\phi,0,r}$. With the notation (3.4), we have

$$\Gamma'_\phi\circ\Gamma_\phi=\sum_{i,j,j'}h_1^ih_2^{j'}\Gamma'_{\phi,r-j,j'}\circ\Gamma_{\phi,i,j} \text{ in } \mathrm{CH}(X\times\mathbb{P}^r\times X\times\mathbb{P}^r).\tag{3.5}$$

while $\Delta_{X\times\mathbb{P}^r}=\sum_{i+j=r}h_1^ih_2^j\Delta_X$ in $\mathrm{CH}(X\times\mathbb{P}^r\times X\times\mathbb{P}^r)$. The fact that $\Gamma'_\phi\circ\Gamma_\phi-\Delta_{X\times\mathbb{P}^r}$ is rationally equivalent to a cycle supported via the first projection over a proper closed algebraic subset of $X\times\mathbb{P}^r$ then implies (by taking $i=0$, $j'=r$ in (3.5)) that the cycle

$$\sum_j\Gamma'_{\phi,r-j,r}\circ\Gamma_{\phi,0,j}-\Delta_X\tag{3.6}$$

is supported via the first projection over a proper closed algebraic subset of X. We observe now that $\dim\Gamma=n+r$, $n=\dim X$, so that $\Gamma_{\phi,0,j}$ for $j<r$ has

dimension $< n$, hence does not dominate X via the first projection. It follows that $\sum_{j<r} \Gamma'_{\phi,r-j,r} \circ \Gamma_{\phi,0,j}$ does not dominate X via the first projection, so that the remaining term in (3.6) with $j = r$, namely

$$\Gamma'_{\phi,0,r} \circ \Gamma_{\phi,0,r} - \Delta_X$$

is rationally equivalent to a cycle supported over a proper closed algebraic subset of X via the first projection. Exchanging X and Y concludes the proof. □

Remark 2. In the sequel, we will use a weaker version of Proposition 1, stating only the first decomposition in (3.2). There is in this case no need to assume that X and Y are of the same dimension. Furthermore, as noticed by Shinder, the proof can be then made simpler by observing that the stated existence property for Γ, Γ' holds for pairs of birational varieties, and also for the pair $(X,\, X \times \mathbb{P}^r)$.

We now prove the following version of the specialization theorem for decomposition of the diagonal fist proved in [10], and later improved in [5]. We will say that a variety Z has mild singularities if there exists a desingularization morphism $\tau : \widetilde{Z} \to Z$ which is CH_0-universally trivial in the sense of [5]. This means that τ_* is an isomorphism on CH_0 over any field K containing k. The easiest way to make this condition satisfied is to ask that τ has the following property : for each (irreducible) subvariety $M \subset Z$, the induced morphism $\widetilde{Z}_M \to M$ has generic fiber smooth rational over $k(M)$. For example, ordinary quadratic singularities in dimension ≥ 2 are mild. We refer to [6] for a more general geometric interpretation of the mildness condition.

Theorem 3. *(i) Let $\mathcal{Y} \to B$ be a projective flat morphism of relative dimension n, where B is smooth. Assume the general fiber $\mathcal{Y}_b$ is smooth and stably birational to a fixed smooth projective variety Y of dimension n. Then, for any desingularization $\widetilde{\mathcal{Y}_0}$ of $\mathcal{Y}_0$, there exist codimension n cycles $\Gamma \in \mathrm{CH}^n(\widetilde{\mathcal{Y}_0} \times Y)$, $\Gamma' \in \mathrm{CH}^n(Y \times \widetilde{\mathcal{Y}_0})$ such that*

$$\Gamma' \circ \Gamma = \Delta_{\widetilde{\mathcal{Y}_0}} + Z + Z' \text{ in } \mathrm{CH}^n(\widetilde{\mathcal{Y}_0} \times \widetilde{\mathcal{Y}_0}), \tag{3.7}$$

where Z is supported on $D \times \widetilde{\mathcal{Y}_0}$ for some proper closed algebraic subset D of $\widetilde{\mathcal{Y}_0}$ and Z' is supported over $\widetilde{\mathcal{Y}_0} \times E$, where E is the exceptional locus of τ.

(ii) If the special fiber $\mathcal{Y}_0$ has mild singularities, one can achieve for an adequate choice of desingularization $\widetilde{\mathcal{Y}_0}$ that $Z' = 0$ in (3.7).

Proof. (i) We can assume B is a smooth curve. By assumption and using Proposition 1, there exist for a general point $t \in B$ a divisor $D_t \subset \mathcal{Y}_t$ and codimension n cycles $\Gamma_t \in \mathrm{CH}^n(\mathcal{Y}_t \times Y)$, $\Gamma'_t \in \mathrm{CH}^n(Y \times \mathcal{Y}_t)$, such that

$$\Gamma'_t \circ \Gamma_t = \Delta_{\mathcal{Y}_t} + Z_t \text{ in } \mathrm{CH}^n(\mathcal{Y}_t \times \mathcal{Y}_t), \tag{3.8}$$

where Z_t is supported on $D_t \times \mathcal{Y}_t$. By a countability argument for the Chow varieties parameterizing cycles in fibers, D_t and the cycles Z_t, Γ_t, Γ'_t can be constructed in families after a base change $B' \to B$. We will denote $\mathcal{Y}' := \mathcal{Y} \times_B B'$. This provides us with varieties and cycles

$$\mathcal{D} \subset \mathcal{Y},\ \mathcal{Z} \in \mathrm{CH}(\mathcal{D} \times_{B'} \mathcal{Y}'),\ \Gamma \in \mathrm{CH}(\mathcal{Y}' \times Y),\ \Gamma' \in \mathrm{CH}(Y \times \mathcal{Y}')$$

whose fiber at the general point $t \in B'$ satisfies (3.8) (see [10] for the more details). Restricting to the regular locus of the morphism $\mathcal{Y}' \to B'$, the composition in (3.8) still makes sense as a relative composition because for $\Gamma \in \mathrm{CH}^n(U \times Y)$,

$\Gamma' \in \mathrm{CH}^n(Y \times U')$, the composition $\Gamma' \circ \Gamma$ is well-defined whenever U and Y are smooth, and Y is projective. Furthermore, by specialization of rational equivalence, (3.8) holds in $\mathrm{CH}^n(\mathcal{Y}_{0,reg} \times \mathcal{Y}_0)$ for any $0 \in B$. Here, as we assumed B (hence B') is a curve, the divisor $\mathcal{D}$ can be assumed not to contain any component of the fiber $\mathcal{Y}_0$, hence to restrict to a proper divisor $D_0 \subset \mathcal{Y}_0$. Identifying $\mathcal{Y}_{0,reg}$ with $\widetilde{\mathcal{Y}}_0 \setminus E$, we get as well cycles

$$\widetilde{Z}_0,\ \widetilde{\Gamma}_0 \in \mathrm{CH}^n(\widetilde{\mathcal{Y}}_0 \times Y),\ \widetilde{\Gamma}'_0 \in \mathrm{CH}^n(Y \times \widetilde{\mathcal{Y}}_0)$$

with $\widetilde{Z}_0$ supported on $\widetilde{D}_0$ such that the equality

$$\widetilde{\Gamma}'_0 \circ \widetilde{\Gamma}_0 = \Delta_{\widetilde{\mathcal{Y}}_0} + \widetilde{Z}_0 \tag{3.9}$$

holds in $\mathrm{CH}^n((\widetilde{\mathcal{Y}_0} \setminus E) \times (\widetilde{\mathcal{Y}_0} \setminus E))$. It follows from the localization exact sequence that the cycle $\widetilde{\Gamma}'_0 \circ \widetilde{\Gamma}_0 - \Delta_{\widetilde{\mathcal{Y}}_0} - \widetilde{Z}_0 \in \mathrm{CH}^n(\widetilde{\mathcal{Y}}_0 \times \widetilde{\mathcal{Y}}_0)$ is rationally equivalent to a cycle supported on $E \times \widetilde{\mathcal{Y}}_0 \cup \widetilde{\mathcal{Y}}_0 \times E$. This last cycle is the sum of a cycle Z_1 supported on $E \times \widetilde{\mathcal{Y}}_0$ and a cycle Z_2 supported on $\widetilde{\mathcal{Y}}_0 \times E$. We thus proved (3.7) with $\Gamma = \widetilde{\Gamma}_0$, $\Gamma' = \widetilde{\Gamma}'_0$, $Z = \widetilde{Z}_0 + Z_1$, $Z' = Z_2$.

(ii) As in [5], and using the fact that (3.8) holds in $\mathrm{CH}^n(\mathcal{Y}_{0,reg} \times \mathcal{Y}_0)$, we observe that the cycle $Z' \in \mathrm{CH}_n(\widetilde{\mathcal{Y}}_0 \times E)$ vanishes by construction in $\mathrm{CH}_n(U \times \mathcal{Y}_0)$ for some dense Zariski open subset U of $\widetilde{\mathcal{Y}}_0$. On the other hand, we can work by assumption with the resolution $\tau : \widetilde{\mathcal{Y}}_0 \to \mathcal{Y}_0$ for which the morphism τ is universally CH_0-trivial. It follows that the cycle Z', seen over the generic point of $\widetilde{\mathcal{Y}}_0$ as a 0-cycle of $\widetilde{\mathcal{Y}}_0$ defined on the field $k(\mathcal{Y}_0)$, vanishes in $\mathrm{CH}_0((\widetilde{\mathcal{Y}}_0)_{k(\mathcal{Y}_0)})$. Hence Z' vanishes in $\mathrm{CH}^n(U \times \widetilde{\mathcal{Y}}_0)$ for some dense Zariski open set U of $\widetilde{\mathcal{Y}}_0$. By the localization exact sequence, it is thus supported on $D \times \widetilde{\mathcal{Y}}_0$, where $D = \widetilde{\mathcal{Y}}_0 \setminus U$, and thus can be absorbed in the term Z. □

Corollary 4. *Under the same assumptions as in Theorem 3 (ii), assume that $H^i_{nr}(Y, A) = 0$ for some integer i and abelian group A. Then $H^i_{nr}(\widetilde{\mathcal{Y}_0}, A) = 0$.*

Proof. We let both sides of formula (3.7) with $Z' = 0$ act on $H^i_{nr}(\widetilde{\mathcal{Y}_0}, A)$ (see [4] for a construction of the action). The action of $\Gamma' \circ \Gamma$ factors through $H^i_{nr}(Y, A)$ hence it is 0. Moreover the diagonal acts by the identity map. We thus conclude that for any $\alpha \in H^i_{nr}(\widetilde{\mathcal{Y}_0}, A)$,

$$\alpha = Z^*\alpha.$$

On the other hand, as Z is supported on $D \times \widetilde{\mathcal{Y}}_0$, the class $Z^*\alpha$ vanishes on $U \times X$, where $U := X \setminus D$. Hence $\alpha_{|U} = 0$, which implies $\alpha = 0$ by [3]. □

We are now in position to prove the following result.

Theorem 5. *Let $\mathcal{Y} \to B$ be a projective flat morphism of relative dimension n, where B is smooth, the generic fiber is smooth, and the special fiber $\mathcal{Y}_0$ has mild singularities. Assume*

(i) the very general fiber $\mathcal{Y}_b$ satisfies $H^i_{nr}(\mathcal{Y}_b, A) = 0$,

(ii) $H^i_{nr}(\widetilde{\mathcal{Y}_0}, A) \neq 0$ for some (equivalently, any) desingularization $\widetilde{\mathcal{Y}_0}$ of $\mathcal{Y}_0$.

Then two very general fibers $\mathcal{Y}_b$, $\mathcal{Y}_{b'}$ are not stably birational.

Proof. Fix one very general fiber $\mathcal{Y}_{b'}$ and denote it by Y. We want to show that the general fiber $\mathcal{Y}_b$ is not stably birational to Y. If it is, Corollary 4 and the vanishing

$H^i_{nr}(\mathcal{Y}_{b'}, A) = 0$ given by (i) imply that $H^i_{nr}(\widetilde{\mathcal{Y}_0}, A) = 0$, contradicting assumption (ii). $\square$

The following variant of Theorem 5 is proved as above, using Corollary 7 below instead of Corollary 4.

Theorem 6. *Let $\mathcal{Y} \to B$ be a projective flat morphism of relative dimension n, where B is smooth. Assume*

(i) the very general fiber $\mathcal{Y}_b$ satisfies $H^i_{nr}(\mathcal{Y}_b, A) = 0$,

(ii) The central fiber admits a desingularization $\widetilde{\mathcal{Y}_0}$ with exceptional divisor $E = \cup_j E_j$ with E_j smooth, and $\widetilde{\mathcal{Y}_0}$ has a nonzero class $\alpha \in H^i_{nr}(\widetilde{\mathcal{Y}_0}, A)$ which vanishes on all the divisors E_j.

Then two very general fibers $\mathcal{Y}_b$, $\mathcal{Y}_{b'}$ are not stably birational.

The proof uses the following variant of Corollary 4 based on Schreieder's criterion [7].

Corollary 7. *Under the same assumptions as in Theorem 3 (i), assume that $H^i_{nr}(Y, A) = 0$ for some integer i and abelian group A, and that $\mathcal{Y}_0$ has a desingularization $\widetilde{\mathcal{Y}_0}$ with exceptional divisor $E = \cup_j E_j$ with E_j smooth. Then any unramified cohomology class $\alpha \in H^i_{nr}(\widetilde{\mathcal{Y}_0}, A)$ which vanishes on each component E_j of E is identically* 0*.*

Proof. We let both sides of formula (3.7) act on $H^i_{nr}(\widetilde{\mathcal{Y}_0}, A)$ (see [4] for a construction of the action). The action of $\Gamma' \circ \Gamma$ factors through $H^i_{nr}(Y, A)$ hence it is 0 since this group is assumed to be 0. We conclude as before that for any $\alpha \in H^i_{nr}(\widetilde{\mathcal{Y}_0}, A)$,

$$\alpha = Z^* \alpha + {Z'}^* \alpha.$$

If α vanishes on all the components E_j of the exceptional divisor E, we have ${Z'}^* \alpha = 0$. We thus have $\alpha = Z^* \alpha$ and we conclude as before that $\alpha = 0$. $\square$

The families to which Theorem 5 applies are essentiall, in characteristic 0, all the families of weighted Fano hypersurfaces for which the stable irrationality has been proved by a degeneration argument to a mildly singular member having a nonzero unramified cohomology class of degree ≤ 3. For example, we have

Theorem 8. *(i) Two very general quartic or sextic double solids or quartic hypersurfaces of dimension* 3 *or* 4 *over $\mathbb{C}$ are not stably birational.*

(ii) Two very general hypersurfaces over $\mathbb{C}$ of degree ≥ 5 and dimension n with $5 \leq n \leq 9$ are not stably birational.

Proof. The case (i) uses Theorem 5. We know by [1] in case of quartic double solids, by Beauville [2] in case of sextic double solids, by Colliot-Thélène-Pirutka [5] in case of quartic threefolds and Schreieder [8] in the case of quartic fourfolds, that they admit degenerations with mild singularities having a nonzero unramified cohomology class of degree 2, which is given by a nonzero torsion class in $H^3_B(\widetilde{\mathcal{X}}_0, \mathbb{Z})$. On the other hand, for all these classes of varieties, the smooth member X does not have torsion in $H^3_B(X, \mathbb{Z})$. Theorem 5 thus applies.

For case (ii), we use Schreieder's degeneration, which in the numerical range above produces a desingularized central fiber with a nonzero unramified cohomology class of degree 3 with torsion coefficients on the desingularized central fiber, vanishing on the exceptional divisor. The very general hypersurface X on the other hand has trivial unramified cohomology of degree 3. Indeed, it is proved in [4] that such a class measures the defect of the Hodge conjecture for degree 4 integral Hodge classes on X. But the smooth hypersurface of degree ≥ 5 in $\mathbb{P}^{n+1}$ for $5 \leq n \leq 9$ has no integral Hodge class of degree 4 not coming from $\mathbb{P}^n$ by the Lefschetz theorem on hyperplane sections. $\square$

The reason we can not a priori extend Theorem 5 to all hypersurfaces shown by Schreieder [8] not to be stably rational is the fact that we do not know how to compute unramified cohomology of degree ≥ 4 for general hypersurfaces, so we are not able to check Condition (i) in Theorem 5. Note that the case of smooth hypersurfaces is covered by Shinder's main theorem.

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