Birational invariants and decomposition of the diagonal

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0 Introduction

This paper is a set of expanded notes for lectures I gave in Miami, Sienne, Udine and Gargnano. The Lüroth problem is very simple to state, namely can one distinguish rational varieties from unirational ones? Here the definitions are the following:

Definition 0.1. *A smooth projective variety X over a field K is unirational if there exist an integer N and a dominant rational map $\Phi : \mathbb{P}^N_K \dashrightarrow X$.*

Note that one can always (at least if K is infinite) reduce to the case $N = n = \dim X$ by restricting Φ to a general linear subspace $\mathbb{P}^n_K \subset \mathbb{P}^N_K$.

Definition 0.2. *A smooth projective variety X over a field K is rational if there exists a birational map $\mathbb{P}^n_K \dashrightarrow X$. The variety X is stably rational if $X \times \mathbb{P}^r$ is rational for some integer r.*

More generally, we will say that X and Y are stably birational if $X \times \mathbb{P}^r \overset{birat}{\cong} Y \times \mathbb{P}^s$ for some integers r, s. This is an equivalence relation on the set of irreducible algebraic varieties over K. Of course, all these notions can be reformulated using only the function field $K(X)$ of X, so that the smoothness or projectivity of X is not important here. However, it is very important in practice to work with smooth projective models in order to exhibit stable birational invariants. The simplest example is the case of algebraic differential forms (see Section 1.1): For the space of algebraic differential forms on X of a given degree to be a stable birational invariant of X, one needs to take X to be smooth and projective (or at least complete).

The above mentioned problem had a classical satisfactory solution for curves and surfaces over an algebraically closed field of characteristic 0, namely they are rational once they are rationally connected, that is contain plenty of rational curves. However, after some delicate episodes (we refer to [10] for a precise history of the subject), it was found that in dimension 3, these two notions do not coincide. The three contributions leading to this conclusion were very different. We refer to Kollár's paper in this book for an account of one of the methods, namely "birational rigidity" which in its simple form proposed by Iskovskikh and Manin [31], consisted in proving that the considered variety (they were considering smooth quartic hypersurfaces in $\mathbb{P}^4$) has a very small birational automorphisms group, unlike projective space which has a huge group of birational automorphisms, called the Cremona group. The other approach, proposed by Clemens and Griffiths, has been extremely efficient in dimension 3, starting with the celebrated example of the cubic threefold hypersurface that they had solved. It involves the geometry of the intermediate Jacobian and its theta divisor. The relationships with birational geometry in dimension 3 is the fact that under the blow-up of a smooth curve, this Jacobian gets an extra summand added, which is the Jacobian of a curve.

The Clemens-Griffiths method works a priori only in dimension 3, although the developments of categorical methods might lead to higher dimensional variants. We refer for such developments to the notes of Macrì and Stellari in this volume. Both the Iskovskikh-Manin method and the Clemens-Griffiths method deal with rationality but not stable rationality for which we need to analyze the rationality not only of X but also of all the products $X \times \mathbb{P}^r$. For the Clemens-Griffiths method, this limitation is due to the fact that the rationality criterion they use works only in dimension 3. For the Iskovskikh-Manin method, the limitation is due to the fact that analyzing the birational automorphisms of $X \times \mathbb{P}^r$ seems to be very hard.

The third method due to Artin and Mumford not only works in any dimension, but also it rests on the introduction of invariants that have higher degree versions which are more and more subtle as the degree increases. A last crucial point is the fact that these invariants are stable birational invariants. They were the first to prove the following result:

Theorem 0.3. *[3] There exist unirational threefolds X which are not stably rational.*

The invariant used by Artin-Mumford is the torsion in Betti cohomology of degree 3 of a smooth projective model of X. We will describe this example in Section 1.1.1. The Artin-Mumford method has been further developed by Colliot-Thélène-Ojanguren [16] who used *higher degree unramified cohomology* groups with torsion coefficients as stable birational invariants in order to construct new examples of this phenomenon but having trivial Artin-Mumford invariants. We will introduce unramified cohomology in Section 1.2.2. We will also describe its main properties, and compute it in small degreee. The degree 2 case is in fact the Brauer group and it is immediately related to the Artin-Mumford invariant which is the topological version of it. The degree 3 case was shown in [19] to measure the defect of the Hodge conjecture in degree 4 with integral coefficients. These developments build on one hand on Bloch-Ogus theory [11] that we will survey in Section 1.2.1, and on the other hand on the Bloch-Kato conjecture proved by Voevodsky [58], that is the main recent new ingredient in the theory of unramified cohomology, together with Kerz' work [33].

We now explain our input to the subject. The theory of algebraic cycles of complex algebraic varieties received a great impulse from Bloch-Srinivas contribution [11] who gave an elegant proof and various generalizations of Mumford's theorem [44] saying that a smooth projective complex variety with trivial CH_0 group (in the sense that all points are rationally equivalent) has no nonzero algebraic differential form of degree > 0. Their approach used the "decomposition of the diagonal" which we will describe in Section 2.1. The decomposition of the diagonal is the beginning of a Künneth decomposition. It says that after removing the first term $X \times x$ of the diagonal Δ_X, the remaining cycle is supported on $D \times X$, where $D \subset X$ is a proper closed algebraic subset. We will show that for quantities with enough funtoriality under correspondences, such a decomposition allows to show that they are supported on D in a strong sense. The first instance of this phenomenon was of course the Bloch-Srinivas improvement of Mumford's theorem saying that if X has $\mathrm{CH}_0(X) = \mathbb{Z}$, the positive degree rational cohomology of X has coniveau ≥ 1: more precisely, it is supported on the divisor D appearing in the decomposition of the diagonal.

The Bloch-Srinivas decomposition of the diagonal is with $\mathbb{Q}$-coefficients, and as we will see, there are many further obstructions to get a decomposition of the diagonal (Chow-theoretic or cohomological) with *integral* coefficients. We will discuss many of them in Sections 2.3.2 and 2.3. Actually, in the cohomological setting, we have a complete understanding of this condition at least in dimension 3. The relevance of this study for rationality questions is the fact that *the existence of such a decomposition is a necessary criterion for stable rationality.* In the Chow setting, this property governs all the invariants of a Chow-theoretic/cohomological nature that we mentioned previously, including unramified cohomology (see Section 2.3.2), in the sense that they vanish if the variety has a Chow decomposition of the diagonal with integral coefficients.

What we realized in [62] is the fact that the existence of a Chow decomposition (and with some care, also of a cohomological decomposition) is stable under the following operation: degenerate (or specialize) a smooth general fiber X_t to a mildly singular special fiber X_0 and then desingularize X_0 to $\widetilde{X}_0$. (This statement is the degeneration theorem 3.4.) The paper [62] had considered only the simplest such mild singularities, namely nodal singularities in dimension at least 2. This already led us to the following conclusion:

Theorem 0.4. *There exist unirational threefolds which are not stably rational although all their unramified cohomology groups are trivial. The very general quartic double solids are such examples.*

Note that the only possibly nontrivial unramified cohomology groups for rationally connected threefolds are in fact the group $H^2_{nr}(X, \mathbb{Q}/\mathbb{Z})$, that is, the Artin-Mumford invariant. The gain over Theorem 0.3 is the fact that these varieties are very simple to construct and exhibit (in fact they are general hypersurfaces in a toric fourfold), while Fano threefolds with a nontrivial Artin-Mumford invariant as exhibited in [3] are hard to construct.

The quartic double solids appearing in Theorem 0.4 are Fano threefolds which specialize to Artin-Mumford double solids X_0, which are nodal. The situation is thus the following: The desingularized Artin-Mumford double solid $\widetilde{X}_0$ does not admit a decomposition of the diagonal because it has a nontrivial Artin-Mumford invariant. This implies that a general deformation X_t of X_0 neither admits a decomposition of the diagonal by the specialization theorem mentioned above. However for all deformations smoothifying a node, the Artin-Mumford invariant disappears. One can summarize the above argument in the following statement which does not involve explicitly the decomposition of the diagonal:

Proposition 0.5. *Let $\pi : X \to C$ be a flat projective morphism of relative dimension $n \geq 2$, where C is a smooth curve. Assume that the fiber X_t is smooth for $t \neq 0$, and has at worst* *ordinary quadratic singularities for $t = 0$. Then if* $\mathrm{Tors}\, H^3_B(\widetilde{X}_0, \mathbb{Z}) \neq 0$, *the general fiber X_t is not stably rational.*

The paper [18] describes the exact conditions on the singularities which make the specialization theorem (hence Proposition 0.5) work. Colliot-Thélène and Pirutka applied their method to the case of the general quartic hypersurface in $\mathbb{P}^4$ for which they proved the analogue of Theorem 0.4. Although the precise nature of the allowed singularities is not known because it is related to the rationality of the exceptional divisors, their criterion was explicit enough to allow many other applications that we will try to survey in Section 3.3. The most striking and important consequence is the following result obtained in [27]:

Theorem 0.6. *Stable rationality is not deformation invariant. There exist families of smooth projective varieties such that the fiber X_t is rational over a dense set (a countable union of algebraic subsets) of the base, but the very general fiber is not stably rational.*

We will also describe in that section the Totaro method [56], which combines the specialization method with the Kollár argument in [35] of reduction to nonzero characteristic and analysis of algebraic differential forms on the central fiber. Finally we will explain Schreieder's further improvement (see [52], [53]) of Proposition 0.5.

1 Birational invariants

We will say that a property or a quantity is birationally invariant, resp. stably birationally invariant, if it is constant on any birational equivalence class of varieties, resp. on any stable birational equivalence class of varieties.

1.1 Classical birational invariants and functoriality

The following obvious lemma allows to produce stable birational invariants:

Lemma 1.1. *Let $X \mapsto I(X)$ be a group defined for smooth varieties over a given field K. Assume that $I(X)$ is covariant for morphisms of K-varieties and has the property that:*
(i) $I(U) \to I(X)$ is surjective for $U \subset X$ a Zariski open set, and
(ii) it is an isomorphism if $\mathrm{codim}\,(X \setminus U \subset X) \geq 2$.
Then:
(a) $I(X)$ is a birational invariant for smooth projective varieties over K.
(b) If furthermore
(iii) $I(X) \cong I(X \times \mathbb{A}^1)$ (by push-forward) for any X,
then $I(X)$ is a stable birational invariant for smooth projective varieties X over K.

Proof. Let $\phi : X \dashrightarrow Y$ be a birational map between smooth and projective varieties over K. Then there is an open set $U \subset X$ such that $\mathrm{codim}\,(X \setminus U \subset X) \geq 2$ and $\phi_{|U}$ is a morphism. Then we have $I(X) \cong I(U)$ by (ii) and by covariant functoriality a morphism $\phi_{U*} : I(U) \to I(Y)$, hence a morphism $\phi_* : I(X) \to I(Y)$. It remains to see that ϕ_* is an isomorphism. Replacing ϕ by ϕ^{-1}, we get $\phi^{-1}_{V*} : I(V) \to I(X)$ for some Zariski open set V of Y such that $I(V) \cong I(Y)$. Let $U' \subset U$ be defined as $\phi^{-1}(V)$. Then $\phi^{-1} \circ \phi$ is the identity on U', hence $(\phi^{-1})_* \circ \phi_* : I(U') \to I(U')$ is the identity. As $I(U') \to I(X)$ is surjective by (i), we conclude that $(\phi^{-1})_* \circ \phi_* : I(X) \to I(X)$ is the identity which proves the first statement after exchanging ϕ and ϕ^{-1}. This proves (a).

For statement (b), we observe that (iii) implies that $I(X) \cong I(X \times \mathbb{A}^l)$ for any X and any l, and then that $I(X) \cong I(X \times \mathbb{P}^l)$ for any X and any l. Indeed we have

$$I(X) \cong I(X \times \mathbb{A}^l) \twoheadrightarrow I(X \times \mathbb{P}^l) \to I(X),$$

where the second arrow is surjective by (i) and the composite is the identity. Together with (a) (proved for smooth *projective* varieties), this implies (b). $\square$

The obvious application is the case of the fundamental group $\pi_1(X_{an})$ when $K = \mathbb{C}$, where X_{an} is $X(\mathbb{C})$ endowed with the Euclidean topology. It clearly satisfies properties (i) and (ii). Unfortunately, the birational invariant so constructed is trivial for rationally connected varieties by the following result which is due to Serre in the case of unirational varieties.

Theorem 1.2. *Let X be a smooth projective rationally connected variety over the complex numbers. Then $\pi_1(X_{an}) = \{e\}$.*

Proof. As X is rationally connected, there exist a smooth projective variety B, and a rational map

$$\phi : B \times \mathbb{P}^1 \dashrightarrow X \tag{1.1}$$

which has the property that (i) $\phi(B \times 0) = \{x\}$ for a fixed point $x \in X(\mathbb{C})$ and

(ii) $\phi_\infty := \phi_{|B\times\infty} : B \dashrightarrow X$ is dominant (say generically finite).

Using the same arguments as above, there is an induced morphism $\phi_* : \pi_1(B_{an} \times \mathbb{CP}^1) \to \pi_1(X_{an})$. This morphism is trivial by (i) and its image is of finite index by (ii). This implies that $\pi_1(X_{an})$ is finite. The end of the proof is an argument of Serre : Consider the universal cover $\widetilde{X}_{an} \to X_{an}$. Then $\widetilde{X}_{an}$ is the analytic space of an algebraic variety $\widetilde{X}$ which is rationally connected because all rational curves contained in X lift to $\widetilde{X}$. This implies that the degree of the covering map $\widetilde{X} \to X$ is 1 (and thus that X is in fact simply connected) by the following Euler-Poincaré characteristic argument: When X is rationally connected over a field of characteristic 0, one has $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$ (see Proposition 1.4 below). This implies that $\chi(X, \mathcal{O}_X) = 1$. This equality has to hold for both X and $\widetilde{X}$, giving

$$\chi(X, \mathcal{O}_X) = 1,\ \chi(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 1. \tag{1.2}$$

However, for a proper étale cover, one has

$$\chi(X, \mathcal{O}_X) = \deg(\widetilde{X}/X)\chi(\widetilde{X}, \mathcal{O}_{\widetilde{X}}).$$

Comparing with (1.2), we get that $\deg(\widetilde{X}/X) = 1$. □

The contravariant version of Lemma 1.1 is the following:

Lemma 1.3. *Let $X \mapsto I(X)$ be a group defined for smooth varieties over a given field k. Assume that $I(X)$ is contravariant and has the property that:*

(i) $I(X) \to I(U)$ is injective for $U \subset X$ a Zariski open set, and

(ii) it is an isomorphism if $\operatorname{codim}(X \setminus U \subset X) \geq 2$.

Then:

(a) $I(X)$ is a birational invariant for smooth projective varieties over K.

(b) If furthermore (iii) $I(X) = I(X \times \mathbb{A}^1)$ (by pull-back) for any X, then $I(X)$ is a stable birational invariant for smooth projective varieties over K.

This lemma applies to closed differential forms of fixed positive degree. In characteristic 0, algebraic differential forms on smooth projective varieties are closed, but of course this is not true on nonprojective varieties and in fact condition (b) is not satisfied for algebraic differential forms, while it is for closed differential forms. The stable birational invariant that we get is trivial for rationally connected varieties in characteristic 0 by the following proposition:

Proposition 1.4. *Let X be smooth projective rationally connected over a field of characteristic 0. Then $H^0(X, \Omega_X^{\otimes l}) = 0$ and $H^l(X, \mathcal{O}_X) = 0$ for any $l > 0$.*

Proof. The second statement is a consequence of the first by Hodge symmetry, which gives that over $\mathbb{C}$, $H^l(X, \mathcal{O}_X)$ is canonically isomorphic to the complex conjugate of $H^0(X, \Omega_X^l)$, which is naturally a subspace of $H^0(X, \Omega_X^{\otimes l})$. Consider a rational map $\phi : B \times \mathbb{P}^1 \dashrightarrow X$ as in (1.1). As X is projective, ϕ is well defined along a generic fiber $\mathbb{P}^1_b := b \times \mathbb{P}^1$. As $\phi_{B\times\infty}$ is dominant, we can assume that ϕ is a submersion at (b,∞). The differential ϕ_{b*} of ϕ along $\mathbb{P}^1_b$ gives a morphism

$$\phi_{b*} : T_{\mathbb{P}^1_b} \oplus T_{B,b} \otimes \mathcal{O}_{\mathbb{P}^1_b} \to \phi_b^* T_X$$

of vector bundles along $\mathbb{P}^1_b$. As $\phi(B \times 0)$ reduces to one point, ϕ_{b*} vanishes at 0. On the other hand, ϕ_{b*} is by assumption surjective at ∞. We thus conclude that the vector bundle $(\phi^* T_X)_{|\mathbb{P}^1_b}(-0)$ is generically generated by sections, hence that it is a direct sum of line bundles $\mathcal{O}_{\mathbb{P}^1_b}(a_i)$ with $a_i \geq 0$ on $\mathbb{P}^1_b$. Hence $(\phi^* T_X)_{|\mathbb{P}^1_b}$ is a direct sum of line bundles $\mathcal{O}_{\mathbb{P}^1_b}(b_i)$ with $b_i > 0$. It follows that for any $l > 0$, $H^0(\mathbb{P}^1_b, (\phi^*\Omega_X^{\otimes l})_{|\mathbb{P}^1_b}) = 0$. As $b \in B$ is generic, this implies that $H^0(X, \Omega_X^{\otimes l}) = 0$ for $l > 0$. □

Proposition 1.4 is not true in nonzero characteristic. The problem is that the dominant map ϕ could be nonseparable, hence nowhere submersive. Kollár [35] exhibited such a phenomenon for some mildly singular double covers of a hypersurfaces in projective space.

Theorem 1.5. *[35] Let $X \subset \mathbb{P}^n$ be a hypersurface of degree $2d$. Then X specializes to a double cover X_0 of a hypersurface of degree d branched along a hypersurface $Y_0 \subset X_0$ of degree $2d$. Assume $\operatorname{char} K = 2$ and $3d > n + 2$. Then X_0 has a desingularization $\widetilde{X}_0$ admitting a nonzero section of $\Omega^{n-2}_{\widetilde{X}_0} \otimes L^{-1}$, where the line bundle L is big and effective.*

We will see in Section 3.3 how Totaro uses this construction. Totaro only uses the effectivity of L, while Kollár needs the bigness of L, in order to apply the following result:

Proposition 1.6. *A separably uniruled (in particular, a ruled) variety Y does not admit a nonzero section of $\Omega^l_Y \otimes L^{-1}$ for some $l \leq \dim Y$, where the line bundle L is big.*

Proof. If there is a variety Z admitting a morphism $f : Z \to B$ with general fiber $\mathbb{P}^1$ and a separable dominant map $\phi : Z \dashrightarrow Y$ not mapping the fibers of f to points, then we may assume ϕ is generically finite separable and dominant. If there is a nonzero section of $\Omega^{n-2}_Y \otimes L^{-1}$, there is a nonzero section of $\Omega^l_Z \otimes \phi^* L^{-1}$, where the line bundle $\phi^* L$ is also big, and in particular has positive degree along the fibers of f. But this is clearly impossible as $\Omega_{Z|Z_b} = f^*\Omega_B \oplus \Omega_{Z_b}$, and the first term is a trivial vector bundle along the fiber Z_b while the second term is a negative line bundle along the fibers $Z_b \cong \mathbb{P}^1$. □

One major application obtained by Kollár is :

Theorem 1.7. *[35] If $X \subset \mathbb{P}^n_{\mathbb{C}}$ is a very general hypersurface of degree $d \geq 2^{\ulcorner \frac{n+3}{3} \urcorner}$, X is not ruled, hence not rational.*

Proof. (We give the argument only for even d.) It suffices to exhibit one hypersurface in the above range of degree and dimension which is not ruled. The crucial point is that ruledness is stable under specialization. This result is due to Matsusaka [41]. Consider a hypersurface X defined over $\mathbb{Z}$, which admits a reduction modulo 2 of the form described in Theorem 1.5. If X is ruled, so is the specialization X_0 or rather its desingularization $\widetilde{X}_0$. But Proposition 1.6 precisely says that $\widetilde{X}_0$ is not ruled. □

Remark 1.8. It is not true that (separable) rational connectedness is stable under specialization. Under specialization to nonzero characteristic, a family of rational curves sweeping-out X can specialize to a family of rational curves sweeping-out X_0 but nonseparably. This problem does not appear for ruledness because in this case the morphism from the family of curves to X has degree 1, hence also its specialization. Hence the specialized morphism is separable.

1.1.1 The Artin-Mumford invariant

Our last examples of classical birational invariants will need functoriality properties slightly different from what we used in Lemmas 1.1 and 1.3, namely functoriality under correspondences. Let $X \mapsto I(X)$ be an invariant of smooth projective varieties. Assume that any correspondence $\Gamma \subset X \times Y$ with $\dim \Gamma = \dim X = \dim Y$ induces $\Gamma^* : I(Y) \to I(X)$ and that this action is compatible with composition of correspondences. In particular a morphism $\phi : X \to Y$ between smooth projective varieties of the same dimensions induces $\phi^* : I(Y) \to I(X)$ and $\phi_* : I(X) \to I(Y)$. Assume also that the projection formula $\phi_* \circ \phi^* = (\deg \phi)\, Id : I(Y) \to I(Y)$ holds. Assume the characteristic is 0 or resolution of singularities holds in the following sense: for any rational map $\phi : X \dashrightarrow Y$, with Y projective, there exists a smooth variety $\tau : \widetilde{X} \to X$, obtained from X by a sequence of blow-ups along smooth centers, such that $\phi \circ \tau$ gives a morphism $\widetilde{X} \to Y$.

Lemma 1.9. *Let $X \mapsto I(X)$ be an invariant of smooth projective varieties satisfying the functoriality properties above. Then $I(X)$ is invariant under birational maps of smooth projective varieties if and only if it is invariant under blow-up.*

Proof. Let $\phi : X \dashrightarrow Y$ be a birational map. The graph $\Gamma_\phi \subset X \times Y$ induces a morphism $\Gamma_\phi^* : I(Y) \to I(X)$. If

$$\tilde{\phi} : \widetilde{X} \to Y,\ \tau : \widetilde{X} \to X$$

is a resolution of indeterminacies of ϕ (or singularities of Γ_ϕ), with τ a composition of blow-ups, one has

$$\Gamma_\phi^* = \tau_* \circ \phi^* \tag{1.3}$$

because $\Gamma_\phi = (\tau, Id_Y)(\Gamma_{\tilde{\phi}})$ or equivalently $\Gamma_\phi = {}^t\Gamma_\tau \circ \Gamma_{\tilde{\phi}}$. Invariance of I under blow-ups guarantees that $\tau_* : I(\widetilde{X}) \to I(X)$ is an isomorphism. But ϕ^* is injective on $I(Y)$ because $\phi_* \circ \phi^* = Id$ on $I(Y)$. Hence by (1.3), Γ_ϕ^* is injective. In order to prove surjectivity, we now use resolution of singularities for ϕ^{-1}. We thus have a diagram

$$\tilde{\phi}^{-1} : \widetilde{Y} \to X,\ \tau' : \widetilde{Y} \to Y$$

where τ' is a composition of blow-ups.

As before we have $\Gamma_\phi^* = \tilde{\phi}_*^{-1} \circ \tau'^*$, where now τ'^* is an isomorphism by assumption while $\tilde{\phi}_*^{-1}$ is surjective by the projection formula $\tilde{\phi}_*^{-1} \circ (\tilde{\phi}^{-1})^* = Id_{I(X)}$. Thus Γ_ϕ^* is surjective. □

Remark 1.10. The proposition above becomes a triviality if we use the weak factorization instead of resolutions of singularities.

Let us now introduce the Artin-Mumford invariant which was used in [3]. It will be generalized in the next section but the simplest version of it is the following: X is defined over the complex numbers and

$$I(X) = \mathrm{Tors}\, H_B^3(X, \mathbb{Z}), \tag{1.4}$$

where $H_B^i(X, A)$ denotes Betti cohomology group $H^i(X_{an}, A)$.

Proposition 1.11. *The Artin-Mumford invariant is a stable birational invariant of smooth projective varieties.*

Proof. As all the Betti cohomology groups with integral coefficients have functoriality under correspondences, the same holds for their torsion subgroups. Similarly for the projection formula. By Lemma 1.9, in order to show birational invariance of $\mathrm{Tors}\, H_B^3(X, \mathbb{Z})$, it thus suffices to show its invariance under blow-up. We now use the blow-up formula

$$H_B^i(\widetilde{X}, \mathbb{Z}) = H_B^i(X, \mathbb{Z}) \oplus H_B^{i-2}(Z, \mathbb{Z}) \oplus H_B^{i-4}(Z, \mathbb{Z}) \oplus \dots,$$

where $\tau : \widetilde{X} \to X$ is the blow-up of X along the smooth locus Z with exceptional divisor $\tau_E : E \to Z$ and the first map is τ^* while the other maps are $j_* \circ (e^s \cup) \circ \tau_E^*$, with $e = [E]_{|E} \in H^2_B(E, \mathbb{Z})$. The end of the proof follows from the observation that the blow-up formula remains true if we replace integral cohomology by its torsion and that $\mathrm{Tors}\, H^1_B(W, \mathbb{Z}) = 0$ for any topological space W. This last fact follows indeed from the cohomology long exact sequence associated with the short exact sequence of constant sheaves

$$0 \to \mathbb{Z} \stackrel{m}{\to} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$$

on W. The blow-up formula then gives

$$\mathrm{Tors}\, H^3_B(\widetilde{X}, \mathbb{Z}) = \mathrm{Tors}\, H^3_B(X, \mathbb{Z}).$$

In order to get stable birational invariance, it remains to see invariance under $X \mapsto X \times \mathbb{P}^r$. This follows from Künneth formula which gives $H^3_B(X \times \mathbb{P}^r, \mathbb{Z}) = H^3_B(X, \mathbb{Z}) \oplus H^1_B(X, \mathbb{Z})$, hence

$$\mathrm{Tors}\, H^3_B(X \times \mathbb{P}^r, \mathbb{Z}) = \mathrm{Tors}\, H^3_B(X, \mathbb{Z}) \oplus \mathrm{Tors}\, H^1_B(X, \mathbb{Z}) = \mathrm{Tors}\, H^3_B(X, \mathbb{Z}).$$

□

Remark 1.12. The same proof shows as well that $\mathrm{Tors}\, H^2_B(X, \mathbb{Z})$ is also a birational invariant. However, this invariant is trivial for rationally connected varieties, because they are simply connected by Theorem 1.2.

The Artin-Mumford invariant of X has an important interpretation as the topological part of the Brauer group of X, which detects Brauer-Severi varieties on X. These varieties are fibered over X into projective spaces, but are not projective bundles $\mathbb{P}(E)$ for some vector bundle E on X. Given such a fibration $\pi : Z \to X$ with fibers Z_x isomorphic to $\mathbb{P}^r$, $Z \cong \mathbb{P}(E)$ for some vector bundle of rank $r+1$ if and only if there exists a line bundle L on Z which restricts to $\mathcal{O}(1)$ on each fiber. The topological part of the obstruction to the existence of L is the obstruction to the existence of $\alpha \in H^2_B(Z, \mathbb{Z})$ which restricts to $c_1(\mathcal{O}_{Z_x}(1)) \in H^2_B(Z_x, \mathbb{Z})$. The relevant piece of the Leray spectral sequence of π gives the exact sequence

$$H^2(Z, \mathbb{Z}) \to H^0(X, R^2\pi_*\mathbb{Z}) \stackrel{d_2}{\to} H^3(X, R^0\pi_*\mathbb{Z}) = H^3(X, \mathbb{Z}),$$

where the second map is 0 with $\mathbb{Q}$-coefficients by the degeneration at E_2 of the Leray spectral sequence (or because there is a line bundle on Z whose restriction to the fibers is $\mathcal{O}_{Z_x}(-r-1)$, namely the canonical bundle K_Z). The image $d_2(h)$ is thus a torsion class in $H^3(X, \mathbb{Z})$, called the Brauer class. The same argument shows that the order of the Brauer class divides $r+1$.

The Artin-Mumford invariant was used by Artin and Mumford to exhibit unirational threefolds which are not stably rational. Let $S_f \subset \mathbb{P}^3$ be a quartic surface defined by a degree 4 homogeneous polynomial f. Let $X_f \to \mathbb{P}^3$ be the double cover of $\mathbb{P}^3$ ramified along S_f. It is defined as $\mathrm{Spec}\,(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-2))$, where the algebra structure $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ on $\mathcal{A} = \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-2))$ is natural on the summands $\mathcal{O}_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}$ and $\mathcal{O}_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(-2)$ and sends $\mathcal{O}_{\mathbb{P}^3}(-2) \otimes \mathcal{O}_{\mathbb{P}^3}(-2)$ to $\mathcal{O}_{\mathbb{P}^3}$ via the composition

$$\mathcal{O}_{\mathbb{P}^3}(-2) \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{O}_{\mathbb{P}^3}(-4) \stackrel{f}{\to} \mathcal{O}_{\mathbb{P}^3}.$$

The local equation for $X_f \subset Spec\,(\oplus_{l \geq 0} \mathcal{O}_{\mathbb{P}^3}(-2l))$ is thus $u^2 = f$, from which we conclude that X_f has ordinary quadratic singularities if S_f does. When S_f is smooth, X_f has trivial Artin-Mumford invariant. This follows from Lefschetz theorem on hyperplane sections as X_f can be seen as a hypersurface (not ample but positive) in the $\mathbb{P}^1$-bundle $\mathrm{Proj}\,(\mathrm{Sym}\,(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-2)))$ over $\mathbb{P}^3$. Assume now that S_f has ordinary quadratic singularities and let $\widetilde{X}_f$ be the desingularization of X_f by blow-up of the nodes. Note that $\widetilde{X}_f$ is unirational. This is true for all quartic double solids but becomes particularly easy once S_f has a node. Indeed, choose a node $O \in S_f$. The lines in $\mathbb{P}^3$ passing through O intersect S_f in the point O with multiplicity 2 and two other points. The inverse image of such a line Δ in X_f has a singular point at O (that we see now as a point of X_f), and its proper tranform C_Δ in $\widetilde{X}_f$ is the double cover of $\Delta \cong \mathbb{P}^1$ ramified over the two remaining intersection points of Δ and S_f. It follows that C_Δ is rational and we thus constructed a conic bundle structure $a : \widetilde{X}_f \to \mathbb{P}^2$ on $\widetilde{X}_f$. On the other hand, if we choose a generic plane P in $\mathbb{P}^3$, its inverse image $\Sigma_{f,P}$ in X_f is a del Pezzo surface, hence is rational, and via a, it is a double cover of $P \cong \mathbb{P}^2$. The double cover $\widetilde{X}_f \times_P \Sigma_{f,P}$ of X_f is then rational, being rational over the function field of $\Sigma_{f,P}$ since it is a conic bundle over $\Sigma_{f,P}$ which has a section. We thus constructed a degree 2 unirational parametrization of X_f:

$$\widetilde{X}_f \times_P \Sigma_{f,P} \overset{birat}{\cong} \mathbb{P}^3 \dashrightarrow X_f.$$

Artin and Mumford construct f in such a way that X_f is nodal and $\widetilde{X}_f$ has a nontrivial Artin-Mumford invariant. Their construction is as follows: Project S_f from one of its nodes O. Then this projection makes the blow-up $\widetilde{S}_f$ of S_f at O a double cover of $\mathbb{P}^2$ ramified along a sextic curve. This sextic curve D is not arbitrary: it has to be tangent to a conic $C \subset \mathbb{P}^2$ at any of their intersection points. This conic indeed corresponds to the exceptional curve of $\widetilde{S}_f$. Another way to see it is to write the equation f as $X_0^2 q + X_0 t + s$, where q, t, s are homogeneous of respective degrees 2, 3, 4 in three variables X_1, X_2, X_3. The ramification curve of the $2:1$ map $\widetilde{S}_f \to \mathbb{P}^2$ is defined by the discriminant of f seen as a quadratic polynomial in X_0, that is,

$$g = t^2 - 4qs. \tag{1.5}$$

The conic C is defined by $q = 0$ and (1.5) shows that $g_{|C}$ is a square, and g is otherwise arbitrary. Artin and Mumford choose g to be a product of two degree 3 polynomials, each of which satisfies the tangency condition along C. Note that S_f has then 9 extra nodes coming from the intersection of the two cubics.

Theorem 1.13. *[3] If the ramification curve D is the union of two smooth cubics E, F meeting transversally and tangent to C at each of their intersection points, the desingularized quartic double solid $\widetilde{X}_f$ has* $\mathrm{Tors}\, H_B^3(\widetilde{X}_f, \mathbb{Z}) \neq 0$.

Rather than giving the complete proof of this statement, we describe Beauville's construction [10] of the Brauer-Severi variety $Z \to \widetilde{X}_f$ providing a Brauer class which is a 2-torsion class in $H_B^3(\widetilde{X}_f, \mathbb{Z})$ as described previously. The Artin-Mumford condition implies that the polynomial f is the discriminant of a $(4,4)$-symmetric matrix M whose entries are linear forms in four variables (the quartic surface S_f is then called a quartic symmetroid). This defines a family of quadric surfaces $\mathcal{Q}$ over $\mathbb{P}^3$ if we see M as an equation of type $(2,1)$ on $\mathbb{P}_1^3 \times \mathbb{P}_2^3$, and the associated double cover of $\mathbb{P}_2^3$ parameterizes the choice of a ruling in the corresponding quadric $\mathcal{Q}_t \subset \mathbb{P}_1^3$. The family of lines in a given ruling on a given fiber is a curve $\Delta \cong \mathbb{P}^1$ but the natural embedding of Δ in $G(2,4)$ gives Δ as a conic. This way we get a family of rational curves over $\widetilde{X}_f$, smooth away from the surface S_f parameterizing singular quadrics. We refer to [10] and also to [37] for the local analysis which shows how to actually construct a $\mathbb{P}^1$-fibration on the whole of $\widetilde{X}_f$.

The last, less classical, birational invariant that we will mention is defined as follows. For a smooth complex variety X, one has the cycle class map

$$cl : \mathcal{Z}^2(X) \to H_B^4(X, \mathbb{Z})$$

and we will denote by $H_B^4(X, \mathbb{Z})_{alg} \subset H_B^4(X, \mathbb{Z})$ the image of cl. The group $H_B^4(X, \mathbb{Z})_{alg}$ is contained in the group $\mathrm{Hdg}^4(X, \mathbb{Z})$ of integral Hodge classes of degree 4 on X.

Lemma 1.14. *The groups* $\mathrm{Tors}\,(H^4_B(X,\mathbb{Z})/H^4_B(X,\mathbb{Z})_{alg})$ *and* $\mathrm{Hdg}^4(X,\mathbb{Z})/H^4_B(X,\mathbb{Z})_{alg}$ *are birational invariants of the smooth projective variety* X.

Proof. The two groups satisfy the functoriality conditions needed to apply Lemma 1.9, hence in order to show birational invariance, it suffices to show their invariance under blow-up. However, for the blow-up $\widetilde{X} \to X$ of $Z \subset X$, one has

$$H^4_B(\widetilde{X},\mathbb{Z}) = H^4_B(X,\mathbb{Z}) \oplus H^2_B(Z,\mathbb{Z}) \oplus H^0_B(Z,\mathbb{Z}),$$

where the last term appears only if $\mathrm{codim}\, Z \geq 3$. In this decomposition, all the maps are natural and induced by algebraic correspondences. In particular this is a decomposition into a direct sum of Hodge structures. This decomposition thus induces

$$H^4_B(\widetilde{X},\mathbb{Z})_{alg} = H^4_B(X,\mathbb{Z})_{alg} \oplus H^2_B(Z,\mathbb{Z})_{alg} \oplus H^0_B(Z,\mathbb{Z})_{alg},$$

and

$$\mathrm{Hdg}^4(\widetilde{X},\mathbb{Z}) = \mathrm{Hdg}^4(X,\mathbb{Z}) \oplus \mathrm{Hdg}^2(Z,\mathbb{Z}) \oplus \mathrm{Hdg}^0(Z,\mathbb{Z}).$$

Using the facts that $H^2_B(Z,\mathbb{Z})/H^2_B(Z,\mathbb{Z})_{alg}$ has no torsion and $\mathrm{Hdg}^2(Z,\mathbb{Z}) = H^2_B(Z,\mathbb{Z})_{alg}$, which both follow from the integral Hodge conjecture in degree 2 (or Lefschetz theorem on $(1,1)$-classes), we conclude that

$$\mathrm{Tors}\,(H^4_B(\widetilde{X},\mathbb{Z})/H^4_B(\widetilde{X},\mathbb{Z})_{alg}) = \mathrm{Tors}\,(H^4_B(X,\mathbb{Z})/H^4_B(X,\mathbb{Z})_{alg}),$$
$$\mathrm{Hdg}^4(\widetilde{X},\mathbb{Z})/H^4_B(\widetilde{X},\mathbb{Z})_{alg} = \mathrm{Hdg}^4(X,\mathbb{Z})/H^4_B(X,\mathbb{Z})_{alg},$$

which proves the desired result.

The invariance of these groups under $X \mapsto X \times \mathbb{P}^r$ is proved similarly. $\square$

Note that if the rational Hodge conjecture holds for degree 4 Hodge classes on X, these two groups are naturally isomorphic:

Lemma 1.15. *For any smooth projective variety* X,

$$\mathrm{Tors}\,(H^4_B(X,\mathbb{Z})/H^4_B(X,\mathbb{Z})_{alg})) = \mathrm{Tors}\,(\mathrm{Hdg}^4(X,\mathbb{Z})/H^4_B(X,\mathbb{Z})_{alg}).$$

Assume X *satisfies the rational Hodge conjecture in degree* 4, *the group* $\mathrm{Tors}\,(H^4_B(X,\mathbb{Z})/H^4_B(X,\mathbb{Z})_{alg}))$ *identifies with the group* $\mathrm{Hdg}^4(X,\mathbb{Z})/H^4_B(X,\mathbb{Z})_{alg}$ *which measures the defect of the Hodge conjecture for integral Hodge classes of degree* 4 *on* X.

Proof. Indeed, a torsion element in $H^4_B(X,\mathbb{Z})/H^4_B(X,\mathbb{Z})_{alg}$ is given by a class α on X such that $N\alpha$ is algebraic on X. Then α is an integral Hodge class on X, which proves the first statement. Finally, the rational Hodge conjecture in degree 4 for X is equivalent to the fact that the group $\mathrm{Hdg}^4(X,\mathbb{Z})/H^4_B(X,\mathbb{Z})_{alg}$ is of torsion, which proves the second statement. $\square$

1.2 Unramified cohomology

1.2.1 The Bloch-Ogus spectral sequence

Let X be an algebraic variety (in particular, it is irreducible and we can speak of its function field). If X is defined over $\mathbb{C}$, we can consider two topologies on $X(\mathbb{C})$, namely the Euclidean (or analytic) topology and the Zariski topology. We will denote X_{an}, resp. X_{Zar}, the topological space $X(\mathbb{C})$ equipped with the Euclidean topology, resp. the Zariski topology. As Zariski open sets are open for the Euclidean topology, the identity of $X(\mathbb{C})$ is a continuous map

$$f : X_{an} \to X_{Zar}.$$

Given any abelian group A, the Bloch-Ogus spectral sequence is the Leray spectral sequence of f, abutting to the cohomology $H^i_B(X,A) := H^i(X_{an}, A)$. It starts with

$$E_2^{p,q}(A) = H^p(X_{Zar}, \mathcal{H}^q(A)),$$

where $\mathcal{H}^q(A)$ is the sheaf on X_{Zar} associated with the presheaf $U \mapsto H^q_B(U, A)$. The Betti cohomology groups $H^n_B(X, A) = H^n(X_{an}, A)$ thus have a filtration, (which is in fact when X is smooth the coniveau filtration,) namely the Leray filtration for which $Gr^p_L H^{p+q}_B(X_{an}, A) = E^{p,q}_\infty$, the latter group being a subquotient of $E_2^{p,q}$.

A fundamental result of Bloch-Ogus [11] is the Gersten-Quillen resolution for the sheaves $\mathcal{H}^q(A)$. It is constructed as follows: For any variety Y, we denote by $H^i(\mathbb{C}(Y), A)$ the direct limit over all dense Zariski open sets $U \subset Y$ of the groups $H^i_B(U, A)$:

$$H^i(\mathbb{C}(Y), A) = \lim_{\substack{\rightarrow \\ \emptyset \neq U \subset Y, \mathrm{open}}} H^i_B(U, A). \tag{1.6}$$

Let now Z be a normal irreducible closed algebraic subset of X, and let Z' be an irreducible reduced divisor of Z. At the generic point of Z', both Z' and Z are smooth. There is thus a residue map $\partial : H^i(\mathbb{C}(Z), A) \to H^{i-1}(\mathbb{C}(Z'), A)$. It is defined as the limit over all pairs of dense Zariski open sets $V \subset Z_{reg}$, $U \subset Z'_{reg}$ such that $U \subset V \cap Z'_{reg}$, of the residue maps

$$Res_{Z,Z'} : H^i((V \setminus V \cap Z')_{an}, A) \to H^{i-1}(U_{an}, A).$$

If now $Z' \subset Z$ is a divisor, with Z not necessarily normal along Z', we can introduce the normalization $n : \widetilde{Z} \to Z$ with restriction $n' : Z'' \to Z'$, where $Z'' = n^{-1}(Z')$, and then define $\partial : H^i(\mathbb{C}(Z), A) \to H^{i-1}(\mathbb{C}(Z'), A)$ as the composite

$$H^i(\mathbb{C}(Z), A) \cong H^i(\mathbb{C}(\widetilde{Z}), A) \stackrel{\partial}{\to} H^{i-1}(\mathbb{C}(Z''), A) \stackrel{n'_*}{\to} H^{i-1}(\mathbb{C}(Z'), A). \tag{1.7}$$

In (1.7), the pushforward morphism

$$n'_* : H^{i-1}(\mathbb{C}(Z''), A) \to H^{i-1}(\mathbb{C}(Z'), A)$$

is defined by restricting to pairs of Zariski open sets $U \subset Z''_{reg}$, $V \subset Z'_{reg}$ such that n' restricts to a proper (in fact, finite) morphism $U \to V$. More precisely, as Z'' is not necessarily irreducible, we should in the above definition write $Z'' = \cup_j Z''_j$ as a union of irreducible components, and take the sum over j of the morphisms (1.7) defined for each Z''_j.

For each subvariety $j : Z \hookrightarrow X$, we consider the group $H^i(\mathbb{C}(Z), A)$ as a constant sheaf supported on Z and we get the corresponding sheaf $j_* H^i(\mathbb{C}(Z), A)$ on X_{Zar}. Finally, we observe that we have a natural sheaf morphism

$$\mathcal{H}^i(A) \to H^i(\mathbb{C}(X), A)$$

where we recall that the second object is a constant sheaf on X_{Zar}. This sheaf morphism is simply induced by the natural maps $H^i(U_{an}, A) \to H^i(\mathbb{C}(X), A)$ for any Zariski open set $U \subset X$, given by (1.6). The residue maps have the following property: Let D_1, $D_2 \subset Y$ be two smooth divisors in a smooth variety, let Z be a smooth reduced irreducible component of $D_1 \cap D_2$ and let $\alpha \in H^i_B(U, A)$, where $U : Y \setminus (D_1 \cup D_2)$. Then

$$\mathrm{Res}_Z(\mathrm{Res}_{D_1}(\alpha)) = -\mathrm{Res}_Z(\mathrm{Res}_{D_2}(\alpha)), \tag{1.8}$$

where on the left Z is seen as a divisor in D_1, and on the right it is seen as a divisor of D_2. Considering the case where $Y \subset X$ is the regular locus of any subvariety of codimension k of X, D, $D' \subset Y$ are of codimension $k+1$, and $Z \subset D \cap D' \subset Y$ is of codimension $k+2$ in X, we conclude from (1.8) that for any i, the two sheaf maps

$$\partial : \oplus_{\mathrm{codim}\, Y = k} H^i(\mathbb{C}(Y), A) \to \oplus_{\mathrm{codim}\, D = k+1} H^{i-1}(\mathbb{C}(D), A)$$

and

$$\partial : \oplus_{\mathrm{codim}\, D=k+1} H^{i-1}(\mathbb{C}(D), A) \to \oplus_{\mathrm{codim}\, Z=k+2} H^{i-2}(\mathbb{C}(Z), A)$$

satisfy $\partial \circ \partial = 0$.

Theorem 1.16. *(Bloch-Ogus, [11]) Let X be smooth. The complex*

$$0 \to \mathcal{H}^i(A) \to H^i(\mathbb{C}(X), A) \to \oplus_{\substack{D \text{ irred} \\ \mathrm{codim}\, D=1}} H^{i-1}(\mathbb{C}(D), A) \to \ldots \to \oplus_{\substack{Z \text{ irred} \\ \mathrm{codim}\, Z=i}} H^0(\mathbb{C}(Z), A) \to 0 \quad (1.9)$$

is an acyclic resolution of $\mathcal{H}^i(A)$.

It is clear that this resolution is acyclic. Indeed, all the sheaves appearing in the resolution are acyclic, being constant sheaves for the Zariski topology on algebraic subvarieties of X. Note that the codimension i subvarieties Z of X appearing above are all irreducible, so that $H^0(\mathbb{C}(Z), A) = A$ and the global sections of the last sheaf appearing in this resolution is the group $\mathcal{Z}^i(X) \otimes A$ of codimension i cycles with coefficients in A.

Theorem 1.16 says first that the sheaf map $\mathcal{H}^i(A) \to H^i(\mathbb{C}(X), A)$ is injective, which is by no means obvious. The meaning of this assertion is that if a class $\alpha \in H^i_B(U, A)$ vanishes on a dense Zariski open set $V \subset U$, then U can be covered by Zariski open sets V_i such that $\alpha_{|V_i} = 0$. This is a moving lemma for the support of cohomology.

We now come back to the Bloch-Ogus spectral sequence and describe the consequences of this theorem, following [11].

Theorem 1.17. *(i) For any two integers $p > q$, one has $E_2^{p,q}(A) = H^p(X_{Zar}, \mathcal{H}^q(A)) = 0$.*

(ii) For $p \leq q$, one has

$$H^p(X_{Zar}, \mathcal{H}^q(A)) = \frac{\mathrm{Ker}\,(\partial : \oplus_{\mathrm{codim}\, Z=p} H^{q-p}(\mathbb{C}(Z), A) \to \oplus_{\mathrm{codim}\, Z=p+1} H^{q-p-1}(\mathbb{C}(Z), A))}{\mathrm{Im}\,(\partial : \oplus_{\mathrm{codim}\, Z=p-1} H^{q-p+1}(\mathbb{C}(Z), A) \to \oplus_{\mathrm{codim}\, Z=p} H^{q-p}(\mathbb{C}(Z), A))} \quad (1.10)$$

(iii) The group $H^p(X, \mathcal{H}^p(\mathbb{Z}))$ is isomorphic to the group $\mathcal{Z}^p(X)/\mathrm{alg}$ of codimension p cycles of X modulo algebraic equivalence.

Proof. (i) Indeed, Theorem 1.16 says that $\mathcal{H}^q(A)$ has an acyclic resolution of length q.

(ii) As (1.9) is an acyclic resolution of $\mathcal{H}^q(A)$, the complex of global sections of (1.9) has degree p cohomology equal to $H^p(X_{Zar}, \mathcal{H}^q(A))$.This is exactly the contents of (1.10).

(iii) We use (ii), which gives in this case

$$H^p(X_{Zar}, \mathcal{H}^p(\mathbb{Z})) = \frac{\oplus_{\mathrm{codim}\, Z=p} H^0(\mathbb{C}(Z), \mathbb{Z})}{\mathrm{Im}\,(\partial : \oplus_{\mathrm{codim}\, Z=p-1} H^1(\mathbb{C}(Z), \mathbb{Z}) \to \oplus_{\mathrm{codim}\, Z=p} H^0(\mathbb{C}(Z), \mathbb{Z}))}.$$

We already mentioned that the numerator is the group $\mathcal{Z}^p(X)$. The proof is concluded by recalling the following two facts :

(1) A cycle Z of codimension p on X is algebraically equivalent to 0 if it belongs to the group generated by divisors homologous to 0 in the (desingularization of a) subvarieties of codimension $p-1$ of X.

(2) A divisor D in a smooth complex manifold is cohomologous to 0 if and only if there exists a degree 1 integral Betti cohomology class α on $X \setminus |D|$ such that $\mathrm{Res}\, \alpha = D$. Here we denote by $|D|$ the support of D. □

The vanishing (i) in Theorem 1.17 is very important. Let us give some applications taken from [11]. We will give further applications in Section 1.2.3. First of all, by the vanishing (i), we conclude that there is no nonzero Leray differential d_r, $r \geq 2$ starting from $E_2^{p,p}(\mathbb{Z}) = H^p(X_{Zar}, \mathcal{H}^p(\mathbb{Z}))$. It follows that $E_\infty^{p,p}(\mathbb{Z})$ is a quotient of the group $H^p(X_{Zar}, \mathcal{H}^p(\mathbb{Z}))$. Futhermore, by the same vanishing (i) above, the Bloch-Ogus filtration on $H^{2p}_B(X, \mathbb{Z})$ has $L^{p+1} = 0$, and thus $L^p H^{2p}_B(X, \mathbb{Z}) = Gr^p_L H^{2p}_B(X, \mathbb{Z}) = E_\infty^{p,p}(\mathbb{Z})$. We conclude that there is a natural composite map

$$H^p(X_{Zar}, \mathcal{H}^p(\mathbb{Z})) \to E_\infty^{p,p}(\mathbb{Z}) \hookrightarrow H^{2p}_B(X, \mathbb{Z}). \quad (1.11)$$

It is proved in [11] that, via the identification given by Theorem 1.17, (iii), this map is the cycle class map in Betti cohomology. Note that by definition, the kernel of the cycle class map $\mathcal{Z}^p(X)/\mathrm{alg} \to H^{2p}_B(X, \mathbb{Z})$ is the Griffiths group $\mathrm{Griff}^p(X)$. We finally have the following result for codimension 2 cycles which describes the kernel of the cycle class map.:

Theorem 1.18. *[11] Let X be a smooth variety over $\mathbb{C}$. There is a natural exact sequence*

$$H^3_B(X, \mathbb{Z}) \to H^0(X_{Zar}, \mathcal{H}^3(\mathbb{Z})) \to H^2(X_{Zar}, \mathcal{H}^2(\mathbb{Z})) \to H^4_B(X, \mathbb{Z}).$$

Proof. The maps are the natural ones. The first map is given by restriction to Zariski open sets. The second map is the differential d_2 of the Bloch-Ogus spectral sequence and the last map is the one appearing in (1.11) and just identified with the cycle class map. The proof of the exactness follows from inspection of the Bloch-Ogus spectral sequence. The kernel of the map $H^2(X_{Zar}, \mathcal{H}^2(\mathbb{Z})) = E^{2,2}_\infty \to H^4_B(X, \mathbb{Z})$ must be in the image of some d_r and obviously only $r = 2$ is possible. This shows exactness in the third term. Finally, by the vanishing of Theorem 1.17,(i), the only nonzero d_r starting from $H^0(X_{Zar}, \mathcal{H}^3(\mathbb{Z})$ is d_2. It follows that $\mathrm{Ker}\, d_2 = E^{0,3}_\infty$, and this is a quotient of $H^3_B(X, \mathbb{Z})$. This shows exactness in the second term. □

1.2.2 Unramified cohomology

The following definition was first introduced in [16] in the setting of étale cohomology.

Definition 1.19. *Let X be an algebraic variety over $\mathbb{C}$ and let A be an abelian group. Then $H^i_{nr}(X, A) = H^0(X_{Zar}, \mathcal{H}^i(A))$.*

This definition can be made in fact over other fields, with Betti cohomology replaced by étale cohomology. If A is finite, and X is over $\mathbb{C}$, étale and Betti cohomology compare naturally. The advantage of Betti cohomology is that we can consider integral coefficients, while étale cohomology needs coefficients like $\mathbb{Z}_\ell$ which are projective limits of $\mathbb{Z}/l^n\mathbb{Z}$. However a big advantage of étale cohomology is that it fits naturally with Galois cohomology. In fact, we have a natural isomorphism

$$\lim_{\substack{\rightarrow \\ U \subset X, \mathrm{open}}} H^i_{et}(U, A) = H^i_{\mathrm{Gal}}(\mathbb{C}(X), A), \tag{1.12}$$

where A is finite, and the direct limit is over the dense Zariski open sets of X. The term on the right is the cohomology of the Galois group of the field $\mathbb{C}(X)$ with coefficients in A,The term on the left is the analogue of what we defined to be $H^i(\mathbb{C}(X), A)$ in the Betti context. If A is finite, then

$$H^i_{et}(U, A) \cong H^i_B(U, A)$$

hence $H^i(\mathbb{C}(X), A) = H^i_{Gal}(\mathbb{C}(X), A)$.

One consequence of Theorem 1.16 is the following formula for unramified cohomology: this is actually cohomology without residues.

Proposition 1.20. *Assuming X smooth over $\mathbb{C}$, one has*

$$H^i_{nr}(X, A) = \mathrm{Ker}\,(H^i(\mathbb{C}(X), A) \stackrel{\partial}{\to} \oplus_{\mathrm{codim}\, Z=1} H^{i-1}(\mathbb{C}(Z), A). \tag{1.13}$$

In particular, the restriction map $H^i_{nr}(X, A) \to H^i_{nr}(U, A)$ is injective for any Zariski dense open set U of X.

Proof. Looking at Definition 1.19, this is a particular case of formula (1.10). □

We now get the following important consequence:

Theorem 1.21. *Unramified cohomology groups $H^i_{nr}(X, A)$ are birational invariants of smooth projective varieties.*

We should make precise here that we consider complex varieties over $\mathbb{C}$ if we want to work with Betti cohomology and any coefficients, and that for more general fields, we use étale cohomology and have to restrict coefficients as mentioned below.

Proof of Theorem 1.21. This is an immediate application of Proposition 1.20 and 1.3, because formula (1.13) shows that the natural restriction map $H^i_{nr}(X, A) \to H^i_{nr}(U, A)$ is injective when U is a dense Zariski open set of X, and that it is an isomorphism if $\mathrm{codim}\,(X \setminus U \subset X) \geq 2$. One uses of course the obvious contravariant functoriality of unramified cohomology. □

We refer to Section 2.3.2 for the proof that unramified cohomology is in fact a stable birational invariant. The following example shows that unramified cohomology generalizes Artin-Mumford invariant to higher degree.

Proposition 1.22. *Let X be a smooth projective complex variety. Then*

$$H^2_{nr}(X, \mathbb{Q}/\mathbb{Z}) = \mathrm{Tors}\, H^2(X_{an}, \mathcal{O}^*_{X_{an}}), \tag{1.14}$$

where $\mathcal{O}^*_{X_{an}}$ is the sheaf of invertible holomorphic functions on X_{an}, is the Brauer group of X. In particular, if X is rationally connected, $H^2_{nr}(X, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Tors}\, H^3_B(X, \mathbb{Z})$ is the Artin-Mumford group of X.

Proof. Let us show the following precise version of (1.14):

$$H^2_{nr}(X, \mathbb{Z}/n\mathbb{Z}) = n - \mathrm{Tors}\,(H^2(X_{an}, \mathcal{O}^*_{X_{an}})). \tag{1.15}$$

Consider the exact sequence

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathcal{O}^*_{X_{an}} \to \mathcal{O}^*_{X_{an}} \to 1,$$

where the second map is $x \mapsto x^n$ and $\mathbb{Z}/n\mathbb{Z}$ is identified with the group of n-th roots of unity. The associated long exact sequence shows that

$$n - \mathrm{Tors}\, H^2(X_{an}, \mathcal{O}^*_{X_{an}}) \cong H^2(X_{an}, \mathbb{Z}/n\mathbb{Z})/\mathrm{Im}\, cl_n,$$

where

$$cl_n : H^1(X_{an}, \mathcal{O}^*_{X_{an}}) = H^1(X, \mathcal{O}^*_X) = \mathrm{CH}^1(X) \to H^2(X_{an}, \mathbb{Z}/n\mathbb{Z})$$

is the cycle class modulo n. We consider the Bloch-Ogus exact sequence for the sheaf $\mathbb{Z}/n\mathbb{Z}$ on X_{an}. The $E_2^{p,q}$-terms in degree 2 are, by Theorem 1.17, (i)

$$E_2^{0,2} = H^0(X_{Zar}, \mathcal{H}^2(\mathbb{Z}/n\mathbb{Z})) = H^2_{nr}(X, \mathbb{Z}/n\mathbb{Z}),\ E_2^{1,1} = H^1(X_{Zar}, \mathcal{H}^1(\mathbb{Z}/n\mathbb{Z})).$$

The last term maps to $H^2(X_{an}, \mathbb{Z}/n\mathbb{Z})$ as all the higher d_r vanish on it, again by Theorem 1.17, (i). No d_r for $r \geq 2$ starts from or arrives to $E_2^{0,2}$, by Theorem 1.17, (i) again. Hence $E_2^{0,2} = E_\infty^{0,2}$ is the quotient of $H^2(X_{an}, \mathbb{Z}/n\mathbb{Z})$ by the image of $E_2^{1,1}$. One then proves that this image is $\mathrm{Im}\, cl_n$. □

1.2.3 Bloch-Kato conjecture and applications

Define the Milnor K-theory groups of a field K (or a ring R) as follows

$$K^M_i(K) = (K^*)^{\otimes i}/I,$$

where I is the ideal generated by $x \otimes (1-x)$ for $x \in K^*$, $1 - x \in K^*$. In particular, we have $K_1^M(K) = K^*$. Fix an integer n prime to the characteristic of K. The exact sequence of Galois modules

$$0 \to \mu_n \to \overline{K}^* \to \overline{K}^* \to 1,$$

where $\mu_n \subset \overline{K}^*$ is the group of n-th roots of unity, gives a map

$$\partial : K^*/n \to H^1(K, \mu_n) := H^1(G_K, \mu_n), \tag{1.16}$$

where $G_K = \mathrm{Gal}\,(\overline{K}/K)$, which is known by Hilbert's Theorem 90 to be an isomorphism (this is equivalent to the vanishing $H^1(G_K, \overline{K}^*) = 0$). More generally, one has a morphism (called the Galois symbol or norm residue map)

$$\partial_i : K_i^M(K)/n \to H^i(G_K, \mu_n^{\otimes i}) \tag{1.17}$$

which to $(x_1, \ldots, x_i)$ associates $\alpha(x_1) \cup \ldots \cup \alpha(x_i)$. The following fundamental result generalizing the isomorphism (1.16) is the Bloch-Kato conjecture solved by Voevodsky [58].

Theorem 1.23. *The map ∂_i is an isomorphism for any i and n prime to* $\mathrm{char}\, K$.

This result was known for $i = 2$ as the Merkur'ev-Suslin theorem [42]. The following result is proved in [19], and [6] (see also [7]) to be a consequence of the Bloch-Kato conjecture (now Voevodsky's theorem).

Theorem 1.24. *Let X be a smooth complex variety. Then the sheaves $\mathcal{H}^i(\mathbb{Z})$ on X_{Zar} have no torsion.*

In other words, if an integral Betti cohomology class α defined on a Zariski open set U of X is of n-torsion for some integer n, then U is covered by Zariski open sets V such that $\alpha_{|V} = 0$.

Proof of Theorem 1.24. The exact sequence of sheaves on X_{an}

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

provides an associated long exact sequence of sheaves on X_{Zar}

$$\ldots \mathcal{H}^i(\mathbb{Z}) \to \mathcal{H}^i(\mathbb{Z}/n\mathbb{Z}) \to \mathcal{H}^{i+1}(\mathbb{Z}) \xrightarrow{n} \mathcal{H}^{i+1}(\mathbb{Z}) \ldots$$

from which one concludes that the sheaves $\mathcal{H}^i(\mathbb{Z})$ have no n-torsion (for any n, i) if and only the natural sheaf maps

$$\mathcal{H}^i(\mathbb{Z}) \to \mathcal{H}^i(\mathbb{Z}/n\mathbb{Z}) \tag{1.18}$$

are surjective for all i, n. This is however implied by Voevodsky's theorem as follows: Voevodsky gives the isomorphisms

$$K_j^M(\mathbb{C}(D))/n \cong H^j_{Gal}(\mathbb{C}(D), \mathbb{Z}/n\mathbb{Z})$$

for all j, n and closed algebraic subsets D of X. If one combines these isomorphisms with the Bloch-Ogus resolution of $\mathcal{H}^i(\mathbb{Z}/n\mathbb{Z})$ on one hand and the Gersten-Quillen resolution of the sheaves $\mathcal{K}_i^M(\mathcal{O}_X)$ established by Kerz [33] on the other hand, one concludes that the natural maps

$$\mathcal{K}_i^M(\mathcal{O}_X) \to \mathcal{H}^i(\mathbb{Z}/n\mathbb{Z}) \tag{1.19}$$

are sheaf isomorphisms. On the other hand, we note that for a Zariski open set $U \subset X$, we have the inclusion

$$\Gamma(\mathcal{O}_U^*) \subset \Gamma(\mathcal{O}_{U_{an}}^*)$$

where on the right we consider the invertible *holomorphic* functions on U. There are natural maps given by the exponential exact sequence on U_{an}

$$K_1^M(\Gamma(\mathcal{O}_{U_{an}})) = \Gamma(\mathcal{O}_{U_{an}}^*) \to H^1(U_{an}, \mathbb{Z}),\ K_i^M(\Gamma(\mathcal{O}_{U_{an}})) \to H^i(U_{an}, \mathbb{Z})$$

and the maps $K_i^M(\Gamma(\mathcal{O}_U)) \to H^i(U_{an}, \mathbb{Z}/n\mathbb{Z})$ appearing in (1.19) fit in a commutative diagram

$$\begin{tikzcd} K_i^M(\Gamma(\mathcal{O}_U)) \arrow[r] \arrow[d, "f"] & K_i^M(\Gamma(\mathcal{O}_{U_{an}})) \arrow[r, "c"] \arrow[d, "f_{an}"] & H^i(U_{an}, \mathbb{Z}) \arrow[d, "g"] \\ K_i^M(\Gamma(\mathcal{O}_U))/n \arrow[r] & K_i^M(\Gamma(\mathcal{O}_{U_{an}}))/n \arrow[r, "c_n"] & H^i(U_{an}, \mathbb{Z}/n\mathbb{Z}) \end{tikzcd} \tag{1.20}$$

where the first vertical maps f and f_{an} given by reduction mod n are obviously surjective and the vertical map g is the map (1.18), or rather its global sections version over U. Voevodsky's theorem implies a fortiori the surjectivity of the bottom horizontal map c_n at the sheaf level, so by surjectivity of f_{an}, we conclude that $c_n \circ f_{an} = g \circ c$ is surjective at the sheaf level. A fortiori g is surjective at the sheaf level, that is, the sheaf maps (1.18) are surjective. □

Corollary 1.25. *The groups $H^i_{nr}(X, \mathbb{Z})$ have no torsion, for any smooth algebraic variety over $\mathbb{C}$.*

We will however also see that these groups are trivial for X unirational. (We refer to Theorem 2.22 in Section 2.3.2 for details of proof and for a more general statement.) The unirationality assumption guarantees by functoriality considerations that the groups $H^i_{nr}(X, \mathbb{Z})$ are torsion for $i > 0$. The torsion freeness statement then implies that they are trivial. It follows that we cannot use the unramified cohomology groups with integral coefficients to distinguish rational varieties from unirational ones. In fact, unramified cohomology with *torsion coefficients* are the right invariant to use, as it already appeared in Proposition 1.22. The following result proved in [19] uses Theorem 1.24 to describe the next group $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z})$ (or $H^3_{nr}(X, \mathbb{Z}/n\mathbb{Z})$). In fact we relate it to the birationally invariant group we introduced in Lemma 1.14.

Theorem 1.26. *[19] (i) For any smooth algebraic variety X over $\mathbb{C}$, there is an exact sequence*

$$0 \to H^3_{nr}(X, \mathbb{Z}) \otimes \mathbb{Z}/n\mathbb{Z} \to H^3_{nr}(X, \mathbb{Z}/n\mathbb{Z}) \to n - \mathrm{Tors}\,(H^4_B(X, \mathbb{Z})/H^4_B(X, \mathbb{Z})_{alg}) \to 0 \tag{1.21}$$

(ii) If X is rationally connected, then $H^4_{nr}(X, \mathbb{Z}/n\mathbb{Z}) \cong n - \mathrm{Tors}\,(H^4_B(X, \mathbb{Z})/H^4_B(X, \mathbb{Z})_{alg})$ and $H^4_{nr}(X, \mathbb{Q}/\mathbb{Z})$ measures the defect of the Hodge conjecture for degree 4 integral Hodge classes on X.

Proof. The second statement follows from the first by Theorem 2.22, (ii), using Lemma 1.15 and the fact that if X is rationally connected, the Hodge conjecture holds for rational Hodge classes of degree 4 on X (see [20], [12], and Section 2.3.1).

We now prove (i). The result is obtained by examining the Bloch-Ogus spectral sequence for degree 4 integral cohomology. Recall from Section 1.2.1 that we have $E_2^{p,q}$-terms $H^p(X_{Zar}, \mathcal{H}^q(\mathbb{Z}))$ with $p + q = 2$ converging to $H^4_B(X, \mathbb{Z})$. By Theorem 1.17, (i), only

$$H^0(X_{Zar}, \mathcal{H}^4(\mathbb{Z})),\ H^1(X_{Zar}, \mathcal{H}^3(\mathbb{Z})),\ H^2(X_{Zar}, \mathcal{H}^2(\mathbb{Z}))$$

appear. Furthermore, as we already saw, the group $H^2(X_{Zar}, \mathcal{H}^2(\mathbb{Z}))$ maps onto its image $E_\infty^{2,2}$ in $H^4_B(X, \mathbb{Z})$, which identifies with $H^4_B(X, \mathbb{Z})_{alg}$. We conclude that the Bloch-Ogus filtration on $H^4_B(X, \mathbb{Z})$ induces a filtration on $H^4_B(X, \mathbb{Z})/H^4_B(X, \mathbb{Z})_{alg}$ with two successive quotients, namely $E_\infty^{1,3}$ and $E_\infty^{0,4}$. The space $E_\infty^{0,4}$ is a subspace of $E_2^{0,4} = H^0(X_{Zar}, \mathcal{H}^4(\mathbb{Z}))$, hence it has no torsion by Theorem 1.24. It thus follows that

$$\mathrm{Tors}\,(H^4_B(X, \mathbb{Z})/H^4_B(X, \mathbb{Z})_{alg}) = \mathrm{Tors}\, E_\infty^{1,3}.$$

Finally, applying again Theorem 1.17, (i), we see that no d_r can start from $E_\infty^{1,3}$, so that $E_\infty^{1,3} = E_2^{1,3} = H^1(X_{Zar}, \mathcal{H}^3(\mathbb{Z}))$. Finally we have to compute the n-torsion of the last group and for this we use the short exact sequence of sheaves on X_{Zar} given by Theorem 1.24:

$$0 \to \mathcal{H}^3(\mathbb{Z}) \xrightarrow{n} \mathcal{H}^3(\mathbb{Z}) \to \mathcal{H}^3(\mathbb{Z}/n\mathbb{Z}) \to 0.$$

Taking the long exact sequence associated to it, we get

$$0 \to H^3_{nr}(X,\mathbb{Z}) \otimes \mathbb{Z}/n\mathbb{Z} \to H^3_{nr}(X,\mathbb{Z}/n\mathbb{Z}) \to n - \mathrm{Tors}\,(H^1(X_{Zar},\mathcal{H}^3(\mathbb{Z}))) \to 0,$$

that is, (1.21). □

It is well-known that the integral Hodge conjecture is not true in general in degree 4. This was first observed by Atiyah and Hirzebruch [4]; further examples were found by Kollár [34], and we refer to [50] for a development of Kollár's method. One question which remained open was whether there are such counterexamples to the integral Hodge conjecture for degree 4 Hodge classes on a rationally connected variety (such a variety is then stably irrational by Lemma 1.14). That such examples exist follows from Theorem 1.26 and earlier work of Colliot-Thélène-Ojanguren [16].

Theorem 1.27. *[16] There exist unirational* 6*-fold* X*, which satisfy* $H^3_{nr}(X,\mathbb{Z}/2\mathbb{Z}) \neq 0$*.*

The varieties X are constructed as quadric bundles over $\mathbb{P}^3$. No smooth model is provided in [16] but in fact, although the formulas we gave above need a smooth projective model, it is not actually needed to compute unramified cohomology, as this is a birational invariant, hence can be computed using only the function field, which is what the authors do in [16]. By Theorem 1.26 (ii), any smooth projective model of such a variety X has an integral Hodge class of degree 4 which is not algebraic, but it is not obvious to see it geometrically.

1.3 Further stable birational invariants

We work over the complex numbers. We describe in this section a number of interesting birational invariants constructed from the group of 1-cycles. It is not clear whether these invariants can be nontrivial for some rationally connected varieties.

1.3.1 Curve classes

Let X be a smooth projective rationally connected variety of dimension n over $\mathbb{C}$. As $H^2(X,\mathcal{O}_X) = 0$, the Hodge structure on $H^{2n-2}_B(X,\mathbb{Z}) \cong H_2^B(X,\mathbb{Z})$ is trivial, that is, purely of type $(n-1,n-1)$. For any smooth projective variety X as above, the cycle class map $\mathcal{Z}_1(X) \otimes \mathbb{Q} \to \mathrm{Hdg}^{2n-2}(X,\mathbb{Q}) := H^{2n-2}_B(X,\mathbb{Q}) \cap H^{n-1,n-1}(X)$ is surjective, as follows from the Lefschetz theorem on $(1,1)$-classes and the hard Lefschetz theorem in degree 2, which provides an isomorphism

$$l^{n-2} : \mathrm{Hdg}^2(X,\mathbb{Q}) \cong \mathrm{Hdg}^{2n-2}(X,\mathbb{Q}).$$

The following was observed in [50]:

Proposition 1.28. *The quotient group*

$$\mathrm{Hdg}^{2n-2}(X,\mathbb{Z})/H^{2n-2}(X,\mathbb{Z})_{alg}, \tag{1.22}$$

where $H^{2n-2}(X,\mathbb{Z})_{alg}$ *denotes the image of the cycle class map* $\mathcal{Z}_1(X) \to \mathrm{Hdg}^{2n-2}(X,\mathbb{Z})$*, is a stable birational invariant.*

Proof. Using Lemma 1.9, we only have to prove invariance under blow-up and under $X \mapsto X \times \mathbb{P}^r$. When we blow-up X along a smooth subvariety $Z \subset X$, the extra (Hodge) classes of degree $2n-2$ are generated by the classes of vertical lines of the exceptional divisor $E_Z \to Z$. They are all algebraic so that the quotient (1.22) remains unchanged. When taking the product with $\mathbb{P}^r$, the extra Hodge homology classes of degree 2 are generated by the class of a line in $\mathbb{P}^r$, hence are algebraic, so that the quotient (1.22) remains unchanged □

We conjectured in [64] that the group (1.22) is trivial for rationally connected varieties. This conjecture is proved in [65] in the case of threefolds. More generally, we prove the following:

Theorem 1.29. *[65] Let X be a smooth projective threefold which is uniruled or has trivial canonical bundle. Then the integral Hodge classes of degree 4 on X are algebraic.*

The conjecture is also proved in [29] for Fano fourfolds, building on the K-trivial case in Theorem 1.29. In the paper [65], we also proved that the conjecture would be a consequence of the Tate conjecture for divisor classes on surfaces defined over a finite field.

1.3.2 Griffiths group

Here is a more refined birational invariant that one can define using 1-cycles. Recall that the Griffiths group $\mathrm{Griff}_k(X)$ (see [25]) is defined as the group of k-cycles of X homologous to 0 modulo algebraic equivalence.

Proposition 1.30. *The group $\mathrm{Griff}_1(X)$ is a stable birational invariant of the smooth projective variety X.*

Proof. Using Lemma 1.9, we only have to prove invariance under blow-up and under $X \mapsto X \times \mathbb{P}^r$. When we blow-up X along a smooth subvariety $Z \subset X$, the blow-up formulas show that the extra elements in the group $\mathrm{Griff}_1(\widetilde{X})$ come from $\mathrm{Griff}_0(Z)$ which is 0 as 0-cycles homologous to 0 are algebraically equivalent to 0. When we take the product of X with $\mathbb{P}^r$, the extra 1-cycles in $X \times \mathbb{P}^r$ are coming from 0-cycles of X, and the extra 1-cycles homologous to 0 from 0-cycles homologous to 0 on X, which are algebraically equivalent to 0. ☐

It is not known if the group $\mathrm{Griff}_1(X)$ can be nonzero for a rationally connected variety. It is tempting to conjecture that it is always trivial for X rationally connected. This has been proved by Tian and Zong [55] for Fano complete intersections of index at least 2. For such a variety X, they prove that all rational curves deform to a union of lines.

Remark 1.31. If $\dim X = 3$, then $\mathrm{Griff}_1(X) = \mathrm{Griff}^2(X)$. If furthermore X is rationally connected, then $\mathrm{Griff}^2(X) = 0$ by Theorem 2.21.

1.3.3 Torsion 1-cycles with trivial Abel-Jacobi invariant

It is an important and classical result due to Roitman [48] that the kernel of the Albanese map

$$\mathrm{alb}_X : \mathrm{CH}_0(X)_{hom} \to \mathrm{Alb}(X)$$

has no torsion if X is a smooth projective variety over $\mathbb{C}$ (or any algebraically closed field of characteristic 0). Let $J_3(X) = J^{2n-3}(X)$ be the intermediate Jacobian built from the Hodge structure on $H_B^{2n-3}(X, \mathbb{Z}) \cong H_3^B(X, \mathbb{Z})$. If X is rationally connected, $H^{3,0}(X) = 0 = H^{n,n-3}(X)$ and $J_3(X)$ is an abelian variety which is the target of the Abel-Jacobi map

$$\phi_X^{n-1} : \mathrm{CH}_1(X)_{hom} \to J_3(X). \tag{1.23}$$

The Abel-Jacobi map for 3-dimensional varieties played an important role in the study of the rationality problem, thanks to Clemens-Griffiths criterion that we will revisit in Section 4.2. The following provides another birationally invariant group:

Proposition 1.32. *The group $\mathrm{Tors}\,(\mathrm{Ker}\,\phi_X^{n-1})$ is a stable birational invariant of the smooth projective variety X.*

Proof. Using Lemma 1.9, we only have to prove invariance under blow-up and under $X \mapsto X \times \mathbb{P}^r$. When we blow-up X along a smooth subvariety $Z \subset X$, the blow-up formulas for Chow groups and cohomology give

$$\mathrm{CH}_1(\widetilde{X}) = \mathrm{CH}_1(X) \oplus \mathrm{CH}_0(Z)$$

$$H_2^B(\widetilde{X}, \mathbb{Z}) = H_2^B(X, \mathbb{Z}) \oplus H_0(Z, \mathbb{Z}),$$

hence

$$\mathrm{CH}_1(\widetilde{X})_{hom} = \mathrm{CH}_1(X)_{hom} \oplus \mathrm{CH}_0(Z)_{hom}$$

and similarly $J_3(\widetilde{X}) = J_3(X) \oplus J_1(Z)$ where $J_1(Z) = \mathrm{Alb}(Z)$. The Abel-Jacobi map $\phi^{n-1}_{\widetilde{X}}$ is the direct sum of the Abel-Jacobi map ϕ^{n-1}_X and the Albanese map of Z. It follows that

$$\mathrm{Tors}\,(\mathrm{Ker}\,\phi^{n-1}_{\widetilde{X}}) = \mathrm{Tors}\,(\mathrm{Ker}\,\phi^{n-1}_X) \oplus \mathrm{Tors}\,(\mathrm{Ker}\,\mathrm{alb}_Z)$$

and the second group on the right is trivial by Roitman's theorem.

Similarly, for any $r \geq 1$, we have

$$\mathrm{CH}_1(X \times \mathbb{P}^r)_{hom} = \mathrm{CH}_1(X)_{hom} \oplus \mathrm{CH}_0(X)_{hom}$$

and $J_3(X \times \mathbb{P}^r) = J_3(X) \oplus J_1(X)$ where $J_1(X) = \mathrm{Alb}(X)$. The Abel-Jacobi map $\phi^{n-1+r}_{X\times\mathbb{P}^r}$ is the direct sum of the Abel-Jacobi map ϕ^{n-1}_X and the Albanese map of X. It follows that

$$\mathrm{Tors}\,(\mathrm{Ker}\,\phi^{n-1+r}_{X\times\mathbb{P}^r}) = \mathrm{Tors}\,(\mathrm{Ker}\,\phi^{n-1}_X) \oplus \mathrm{Tors}\,(\mathrm{Ker}\,\mathrm{alb}_X)$$

and the second group on the right is trivial by Roitman's theorem. □

It is again an open question whether a smooth projective rationally connected variety over $\mathbb{C}$ can have some nonzero torsion in $(\mathrm{Ker}\,\phi^{n-1}_X)$.

Remark 1.33. If $\dim X = 3$, then the 1-cycles are codimension 2 cycles and a difficult theorem of Bloch (see Theorem 2.19) thus applies and says that $\mathrm{Ker}\,\phi^{n-1}_X = \mathrm{Ker}\,\phi^2_X$ has no torsion in this case.

2 0-cycles

2.1 Bloch-Srinivas principle

The Bloch-Srinivas principle [12] says the following:

Theorem 2.1. *Let $Y \to B$ be a flat morphism of varieties defined over a field k, with B smooth, and let Z be a cycle on Y. Assume that $K \supseteq k$ is an algebraically closed field of infinite transcendence degree over k and that for any point $b \in B(K)$, the restricted cycle $Z_{|Y_b}$ is rationally equivalent to 0. Then there exist an integer $N > 0$ and a dense Zariski open set $U \subset B$ such that $NZ_{|Y_U} = 0$ in $\mathrm{CH}(Y_U)$, where $Y_U := \phi^{-1}(U) \subset Y$.*

The condition on K guarantees that it contains any finitely generated extension of k. The assumptions we imposed on B and ϕ are used to give a meaning to the restricted cycles $Z_{|Y_b}$. As the conclusion concerns only a dense Zariski open set of B, smoothness of B is not restrictive. The theorem is obtained by embedding $k(B)$ into K and by applying the assumption to the generic point η of B, which is defined over $k(B)$ but can be seen as defined over K via $k(B) \subset K$. As Z vanishes in $\mathrm{CH}(Y_{\eta_K})$, one easily concludes by a trace argument that it is torsion in $\mathrm{CH}(Y_\eta)$. Finally, as η is the generic point of B, the vanishing of NZ in $\mathrm{CH}(Y_\eta)$ implies the vanishing of NZ in $\mathrm{CH}(Y_U)$ for some dense Zariski open set U of B, which proves the theorem. Note that the same argument proves as well the following statement:

Proposition 2.2. *Under the same assumptions as in Theorem 2.1, there exist a dense Zariski open set $U \subset B_{reg}$ and a finite cover $U' \to U$ such that $Z_{U'} = 0$ in $\mathrm{CH}(Y_{U'})$, where $Y_{U'} := U' \times_U Y_U$ and $Z_{U'}$ is the pull-back of $Z_{|Y_U}$ to $Y_{U'}$.*

If X is a complex variety, then X is defined over a field k which has finite transcendence degree over $\mathbb{Q}$ and $\mathbb{C}$ satisfies the desired properties with respect to k. We then conclude:

Theorem 2.3. *Let $\phi : Y \to B$ be a morphism of complex varieties and let Z be a cycle on Y. Assume that for any complex point $b \in B(\mathbb{C})$, the restricted cycle $Z_{|Y_b}$ is rationally equivalent to 0. Then there exist an integer $N > 0$ and a dense Zariski open set $U \subset B$ such that $NZ_{|Y_U} = 0$ in $\mathrm{CH}(Y_U)$, where $Y_U := \phi^{-1}(U) \subset Y$.*

This theorem leads us to the "decomposition of the diagonal" first introduced by Bloch and Srinivas, some applications of which we will describe below:

Theorem 2.4. *[12] Let X be a variety of dimension n over $\mathbb{C}$ and assume that there exists a closed algebraic subset $W \subset X$ such that $\mathrm{CH}_0(W) \to \mathrm{CH}_0(X)$ is surjective. Then for some integer $N > 0$, one has a decomposition*

$$N\Delta_X = Z_W + Z \text{ in } \mathrm{CH}^n(X \times X), \tag{2.24}$$

where Δ_X is the diagonal of X, Z_W is supported on $X \times W$ and Z is supported on $D \times X$ for some proper closed algebraic subset $D \subset X$.

Proof. The assumption is equivalent, by the localization exact sequence, to the vanishing of $\mathrm{CH}_0(X \setminus W)$. We can then apply Theorem 2.3 and conclude that for some Zariski open set $U \subset X$, and for some integer $N > 0$,

$$N\Delta_{X\,|U\times(X\setminus W)} = 0 \text{ in } \mathrm{CH}^n(U \times (X \setminus W)).$$

By the localization exact sequence, letting $D := X \setminus U$, this is equivalent to the decomposition (2.24). ☐

In these notes, we are interested in rationally connected varieties X, which have "trivial" CH_0 group over an algebraically closed field, as all points of X are rationally equivalent in X. We are thus in the situation of Theorem 2.4, where we can take for W any point $x \in X$. One then gets:

Theorem 2.5. *Let X be a complex algebraic variety of dimension n, such that all points of X are rationally equivalent to any given point $x \in X$. Then there is a divisor $D \subset X$ and an integer N such that*

$$N\Delta_X = N(X \times x) + Z \text{ in } \mathrm{CH}^n(X \times X) \tag{2.25}$$

where Z is supported on $D \times X$.

The decomposition (2.25) is a Chow decomposition of the diagonal with rational coefficients, due to the presence of the coefficient N. It was used by Bloch and Srinivas to give a new proof and a generalization of Mumford's theorem [44]. Note conversely that, if X is smooth projective, and admits a decomposition as in (2.25), then $\mathrm{CH}_0(X) = \mathbb{Z}$. Indeed, for any $y \in X$, we get by letting act the correspondences appearing in (2.25) on any 0-cycle z :

$$Nz = N(\deg z)x \text{ in } \mathrm{CH}_0(X),$$

This shows that up to torsion, $\mathrm{CH}_0(X) = \mathbb{Z}$, and in particular that $\mathrm{Alb}\,X = 0$. On the other hand Roitman's theorem [48] says that the kernel of the Albanese map has no torsion, hence finally $\mathrm{CH}_0(X) = \mathbb{Z}$.

2.2 Universal Chow group of 0-cycles

The universal CH_0 group of X is not a group but a functor. If X is a variety defined over a field K, this functor, from the category of fields containing K to the category of abelian groups, associates to any field $L \supsetneqq K$ the group $\mathrm{CH}_0(X_L)$. The crucial point is that it provides much more information on X than the group $\mathrm{CH}_0(X)$, even if K is very big like $\mathbb{C}$, because the considered fields L are not algebraically closed. The interest of this notion for rationality questions comes from the following facts:

Lemma 2.6. *One has* $\mathrm{CH}_0(\mathbb{P}^n_K) = \mathbb{Z}$ *for any field* K. *One can take for generator the class of any* K*-point of* $\mathbb{P}^n_K$.

Proof. (See also [22, 1.9].) This follows indeed by induction from the localization exact sequence

$$\mathrm{CH}_0(\mathbb{P}^{n-1}_K) \to \mathrm{CH}_0(\mathbb{P}^n_K) \to \mathrm{CH}_0(\mathbb{A}^n_K) \to 0,$$

where $\mathbb{P}^{n-1}_K \subset \mathbb{P}^n_K$ is any hyperplane, and from $\mathrm{CH}_0(\mathbb{A}^n_K) = 0$ which is almost trivial: any effective 0-cycle of $\mathbb{A}^1_K$ is the divisor of a polynomial $P \in K[X]$. $\square$

Note that the same proof shows that $\mathrm{CH}_0(X \times \mathbb{P}^n_K) \cong \mathrm{CH}_0(X)$, see [22, 3.1].

The following definition appears in [5]:

Definition 2.7. *A variety* X *over* K *has universally trivial* CH_0*-group if* X *has a* 0*-cycle* z *of degree* 1 *and* $\mathrm{CH}_0(X_L) = \mathbb{Z}z$ *for any field* $L \supseteq K$.

We then have

Proposition 2.8. *If* X *and* Y *are smooth projective over* K *and are stably birational over* K, *then* X *has universally trivial* CH_0*-group if and only* Y *does.*

In particular, if X *is stably rational over* K, X *has universally trivially trivial* CH_0*-group.*

Let us first recall the following basic facts that will be used many times in the sequel. Let X be a smooth projective variety of dimension n. Any cycle $\Gamma \in \mathrm{CH}^n(X \times X)$ (also called a self-correspondence) acts on Chow groups of X in the following way: the upper-star action $\Gamma^* : \mathrm{CH}(X) \to \mathrm{CH}(X)$ is defined by

$$\Gamma^*(z) = pr_{1*}(pr_2^* z \cdot \Gamma), \tag{2.26}$$

where $pr_i : X \times X \to X$ are the two projections, and the lower-star action $\Gamma_* : \mathrm{CH}(X) \to \mathrm{CH}(X)$ is defined by

$$\Gamma_*(z) = pr_{2*}(pr_1^* z \cdot \Gamma). \tag{2.27}$$

Obviously $\Gamma_* = {}^t\Gamma^*$ where ${}^t\Gamma$ is the image of Γ under the involution of $X \times X$ exchanging the factors, but it is important for us in this section to use the two actions.

Proof of Proposition 2.8. The CH_0-group has the functoriality properties needed to apply Lemma 1.9. Hence assuming resolution of singularities, it suffices to show invariance under blow-up and invariance under $X \mapsto X \times \mathbb{P}^r$. The former follows more generally from the blow-up formulas for Chow groups, and the later was noted above. An alternative proof which does not use resolution of singularities is as follows: Let $\phi : X \dashrightarrow Y$ be a birational map. Then the graphs $\Gamma_\phi \subset X \times Y$ and $\Gamma_{\phi^{-1}} \subset Y \times X$ are correspondences which satisfy

$$\begin{aligned} \Gamma_{\phi^{-1}} \circ \Gamma_\phi &= \Delta_X + Z \text{ in } \mathrm{CH}^n(X \times X) \\ \Gamma_\phi \circ \Gamma_{\phi^{-1}} &= \Delta_Y + Z' \text{ in } \mathrm{CH}^n(Y \times Y) \end{aligned} \tag{2.28}$$

where the self-correspondences Z (resp. Z') have the property of being supported on $D \times X$ (resp. $D' \times Y$) for some proper closed algebraic subset D of X (resp. D' of Y). But a correspondence Z satisfying this property acts trivially on $\mathrm{CH}_0(X)$, and similarly for Z'. Thus we conclude that

$$(\Gamma_{\phi^{-1}})_* \circ (\Gamma_\phi)_* = Id_{\mathrm{CH}_0(X)},\ (\Gamma_\phi)_* \circ (\Gamma_{\phi^{-1}})_* = Id_{\mathrm{CH}_0(Y)}.$$

$\square$

We now consider the previous situation where X is a smooth complex projective variety. The precise relationship between CH_0-triviality and universal CH_0-triviality is described by the Bloch-Srinivas Theorem 2.5:

Proposition 2.9. *If* $\mathrm{CH}_0(X) = \mathbb{Z}$*, the universal* CH_0 *group of* X *is trivial modulo torsion; more precisely, there is an integer* $N > 0$ *such that* $N\mathrm{CH}_0(X_L)_0 = 0$ *for any field* $L \supseteq \mathbb{C}$.

Here $\mathrm{CH}_0(X_L)_0$ is the Chow group of 0-cycles of degree 0.

Proof. We use Bloch-Srinivas decomposition of the diagonal of $X/\mathbb{C}$

$$N\Delta_X = N(X \times x) + Z \text{ in } \mathrm{CH}^n(X \times X) \tag{2.29}$$

with Z supported on $D \times X$. It clearly remains true for X_L, which gives

$$N\Delta_{X_L} = N(X_L \times x) + Z_L \text{ in } \mathrm{CH}^n(X_L \times X_L), \tag{2.30}$$

with Z_L supported on $D_L \times X_L$. Both sides of this equality act on $\mathrm{CH}_0(X_L)_0$. The action of Z_{L*} and $N(X_L \times x)_*$ are clearly 0 on $\mathrm{CH}_0(X_L)_0$, while $(N\Delta_{X_L})_* = N\, Id_{\mathrm{CH}_0(X_L)_0}$. □

The next question (and the central subject of these notes) is whether one can get rid of the coefficient N, that is whether X has universally trivial CH_0 and we will see in the next sections that there are many obstructions to that, which all provide interesting obstuctions to stable rationality. We will shift to the language of decomposition of the diagonal, that was studied first in [61] in relation with rationality questions.

Definition 2.10. *A* n*-dimensional variety* X *over* K *admitting a* K*-point* x *(or a* 0*-cycle of degree* 1*) has a Chow decomposition of the diagonal if one can write*

$$\Delta_X = X \times x + Z \text{ in } \mathrm{CH}^n(X \times X), \tag{2.31}$$

where Z *is a cycle of* $X \times X$ *which is supported on* $D \times X$*, where* $D \subset X$ *is a proper closed algebraic subset.*

It is immediate to see that the definition is independent of the choice of x. The equivalence of the two definitions is contained in the following result proved in [5]:

Proposition 2.11. *A variety* X *over* K *admitting a* K*-point* x *(or a* 0*-cycle of degree* 1*) has a Chow decomposition of the diagonal if and only if* X *has universally trivial* CH_0 *group.*

Proof. If we look at the proof of Proposition 2.9, and put $N = 1$, we see that it proves the "iff" direction. Conversely, assume X has universally trivial CH_0 group and let $L = K(X)$. The diagonal of X provides a L-point η_X of X_L (namely the generic point). By assumption, we get that

$$\eta_X = x_L \text{ in } \mathrm{CH}_0(X_L). \tag{2.32}$$

Now we use the fact that

$$\mathrm{CH}_0(X_L) = \mathrm{CH}^n(X_L) = \lim_{\substack{\to \\ U \subset X}} \mathrm{CH}^n(U \times X),\ n = \dim X,$$

where the direct limit is over all the dense Zariski open sets of X. The points η_X and x_L are the limits of the cycles $(\Delta_X)_{|U \times X}$ and $U \times x$ respectively. Formula (2.32) thus says that there exists a Zariski open set $U \subset X$ such that

$$(\Delta_X)_{|U \times X} - U \times x = 0 \text{ in } \mathrm{CH}^n(U \times X),$$

which is equivalent to a decomposition of the diagonal (2.31) with $D = X \setminus U$ by the localization exact sequence. □

The study of the decomposition (2.31) and its consequences will allow us in next section to exhibit many obstructions, some topological, to the universal triviality of CH_0.

2.3 Decomposition of the diagonal : consequences

Here we will work over $\mathbb{C}$ and use integral Betti cohomology classes, but of course $\mathbb{Z}_\ell$-étale cohomology classes could be used in general. A cohomology class $\alpha \in H_B^{2n}(X \times X)$ acts on integral cohomology of X by the same formulas as (2.26) and (2.27). We will denote by α^* and α_* these actions. When α is an integral Hodge class, in particular when $\alpha = [\Gamma]$ is algebraic, the two maps $\alpha^* : H_B^l(X, \mathbb{Z}) \to H_B^l(X, \mathbb{Z})$ are morphisms of Hodge structures. In particular, when $l = 2j+1$ is odd, there are corresponding endomorphisms α^*, α_* of the associated Jacobian $J^l(X) = H_B^l(X, \mathbb{C})/(F^i H_B^l(X) \oplus H_B^l(X, \mathbb{Z})_{tf})$.

We will use freely the fact that the actions of Γ^*, resp. Γ_* on Chow groups are compatible via the cycle class map and Abel-Jacobi map with the action of $[\Gamma]^*$, resp. $[\Gamma]_*$, on cohomology and Jacobians, see [66, 9.2].

2.3.1 Consequences of a cohomological decomposition of the diagonal

Let X be smooth projective of dimension n over $\mathbb{C}$. We will say that X has a cohomological decomposition of the diagonal if one can write

$$[\Delta_X] = [X \times x] + [Z] \text{ in } H_B^{2n}(X \times X, \mathbb{Z}), \tag{2.33}$$

where Z is a cycle of $X \times X$ which is supported over $D \times X$, with $D \subset X$ a proper closed algebraic subset, that we can assume to be a divisor. Clearly, if X has a Chow decomposition of the diagonal as in (2.31), then it has a cohomological decomposition of the diagonal by taking cohomology classes. Note that (2.33) implies that $([\Delta_X] - [X \times x])_{|U \times X} = 0$ in $H_B^{2n}(U \times X, \mathbb{Z})$ but that this is a priori a stronger statement, because the latter is just saying that the homology class $[\Delta_X] - [X \times x]$ comes from an integral homology class β supported on $D \times X$ for some proper closed algebraic subset D, but it is not saying that this β can be taken algebraic on $D \times X$. In order to draw consequences of (2.33), we use it in the following form: We observe that we can choose D to be smooth generically along each component of $pr_1(\operatorname{Supp} Z)$. It then follows that the cycle Z lifts to a codimension $n-1$ cycle $\widetilde{Z}$ of $\widetilde{D} \times X$, where $\tilde{j} : \widetilde{D} \to X$ is a desingularization of $D \subset X$. Then (2.33) rewrites as

$$[\Delta_X] - [X \times x] = (\tilde{j}, Id_X)_*([\widetilde{Z}]) \text{ in } H_B^{2n}(X \times X, \mathbb{Z}). \tag{2.34}$$

We now get the following consequence:

Lemma 2.12. *If X has a cohomological decomposition of the diagonal as in (2.34), then for any $\alpha \in H_B^*(X, \mathbb{Z})$ of degree $* > 0$, one has*

$$\alpha = \tilde{j}_*([\widetilde{Z}]^*\alpha) \text{ in } H_B^*(X, \mathbb{Z}). \tag{2.35}$$

Similarly, for any $\alpha \in H^*(X, \mathbb{Z})$ of degree $* < 2n$, one has

$$\alpha = [\widetilde{Z}]_*(\tilde{j}^*\alpha) \text{ in } H_B^*(X, \mathbb{Z}). \tag{2.36}$$

Proof. For (2.35), we let both sides of (2.34) act on $H_B^*(X, \mathbb{Z})$ by the upper-star action. We observe that $[X \times x]^*\alpha = 0$ if $* = \deg \alpha > 0$ and $[\Delta_X]^*\alpha = \alpha$. Finally we have

$$((\tilde{j}, Id_X)_*([\widetilde{Z}]))^*(\alpha) = \tilde{j}_*([\widetilde{Z}]^*\alpha) \text{ in } H_B^*(X, \mathbb{Z}),$$

which proves (2.35).

For (2.36), we argue similarly but use the lower-star action. We observe that $[X \times x]_*\alpha = 0$ if $* = \deg \alpha < 2n$ and $[\Delta_X]_*\alpha = \alpha$. Finally we have

$$((\tilde{j}, Id_X)_*([\widetilde{Z}]))_*\alpha = [\widetilde{Z}]_*(\tilde{j}^*\alpha) \text{ in } H_B^*(X, \mathbb{Z}),$$

which proves (2.36). □

We now get the following:

Theorem 2.13. *If X has a cohomological decomposition of the diagonal, then the following hold:*

1. $H^{i,0}(X) = 0$ *(hence also $H^{0,i}(X) = 0$ for $i > 0$.*
2. $\mathrm{Tors}\, H^i_B(X, \mathbb{Z}) = 0$ *for $i \leq 3$. Dually* $\mathrm{Tors}\, H^i_B(X, \mathbb{Z}) = 0$ *for $i \geq 2n - 2$.*
3. *Integral Hodge classes of degree 4 on X are algebraic.*
4. *Integral cohomology classes of degree $2n - 2$ on X are algebraic.*

Remark 2.14. Statement 1 is due to Bloch and Srinivas [12] and uses only the cohomological decomposition of the diagonal with $\mathbb{Q}$-coefficients. Statement 3 is proved by Bloch and Srinivas in [12] with $\mathbb{Q}$-coefficients. Statements 2 and 3 appear in [19] and statement 4 in [61].

Remark 2.15. If X is rationally connected of dimension 3 over $\mathbb{C}$, the only property, among these four properties, which can be violated is the vanishing of $\mathrm{Tors}\, H^3_B(X, \mathbb{Z})$ and of $\mathrm{Tors}\, H^4_B(X, \mathbb{Z})$. Indeed, by Theorem 1.2, the other cohomology groups have no torsion. Furthermore, by Theorem 1.29, properties 3 and 4, which coincide in this case, are satisfied.

Proof of Theorem 2.13. We use formula (2.35). If $\alpha \in H^{i,0}(X)$ with $i > 0$, then $\tilde{j}_*([\widetilde{Z}]^*\alpha) = 0$ in $H^{i,0}(X)$ as this is a holomorphic form on X which vanishes on the dense Zariski open set $X \setminus D$. Thus (2.35) gives $\alpha = 0$, proving 1.

Remark 2.16. To make this argument totally rigorous, we should use the action of classes of correspondences on Dolbeault cohomology, rather than Betti cohomology (they coincide on X but not on U). We refer to the discussion starting the proof of Theorem 2.20 for more detail.

If α is torsion and of degree $* \leq 3$, then $[\widetilde{Z}]^*\alpha$ is torsion and of degree $* - 2 \leq 1$, hence vanishes in $H^{*-2}(\widetilde{D}, \mathbb{Z})$. Hence (2.35) gives $\alpha = \tilde{j}_*([\widetilde{Z}]^*\alpha) = 0$. The other statements are obtained by duality or can be obtained directly by using formula (2.36). This proves 2.

If α is an integral Hodge class of degree 4, then $[\widetilde{Z}]^*\alpha$ is an integral Hodge class of degree 2 on $\widetilde{D}$, hence is algebraic by the Lefschetz $(1,1)$-theorem. Thus $\alpha = \tilde{j}_*([\widetilde{Z}]^*\alpha)$ is algebraic and 3 holds.

For the remaining statement, we use (2.36). If α is an integral cohomology class of degree $2n-2$ on X, then $\tilde{j}^*\alpha \in H^{2n-2}(\widetilde{D}, \mathbb{Z})$ is algebraic on $\widetilde{D}$ which is smooth of dimension $n-1$. Thus $\alpha = [\widetilde{Z}]_*(\tilde{j}^*\alpha)$ is algebraic on X. □

2.3.2 Consequences of a Chow decomposition of the diagonal

We now describe consequences of a Chow decomposition of the diagonal that a priori cannot be obtained from a cohomological decomposition of the diagonal, for which we refer to Theorem 2.13 .

Theorem 2.17. *If X has a Chow decomposition of the diagonal, then*

1. *The Griffiths group* $\mathrm{Griff}_1(X)$ *is trivial.*
2. *The kernel of the Abel-Jacobi map* $\phi^{2n-3}_X : \mathrm{CH}^{n-1}(X)_{hom} \to J^{2n-3}(X)$ *has no torsion.*
3. *The kernel of the Abel-Jacobi map* $\phi^3_X : \mathrm{CH}^3(X)_{hom} \to J^5(X)$ *has no torsion.*

We start the proof by redoing in the Chow setting the analysis done previously in the cohomological setting. A Chow decomposition of the diagonal $\Delta_X = X \times x + Z$ in $\mathrm{CH}^n(X \times X)$ rewrites by desingularization in the form

$$\Delta_X - X \times x = (\tilde{j}, Id_X)_*(\widetilde{Z}) \text{ in } \mathrm{CH}^n(X \times X) \tag{2.37}$$

where $\widetilde{D}$ is smooth of dimension $n-1$ and maps to X via $\tilde{j}$. We get the following consequence by letting both sides of (2.37) act on $\mathrm{CH}(X)$, either by the lower star or by the upper star action:

Lemma 2.18. *If X has a Chow decomposition of the diagonal, then for any $z \in \mathrm{CH}^*(X)$ of codimension $* > 0$, one has*

$$z = \tilde{j}_*(\widetilde{Z}^* z) \text{ in } \mathrm{CH}^*(\mathrm{X}). \tag{2.38}$$

Similarly, for any $z \in \mathrm{CH}^*(X)$ of codimension $* < n$, one has

$$z = \widetilde{Z}_*(\tilde{j}^* z) \text{ in } \mathrm{CH}^*(X). \tag{2.39}$$

Proof of Theorem 2.17. By assumption, X has a Chow decomposition of the diagonal that we write as in (2.37). If $z \in \mathrm{CH}_1(X)$, we get $z = \tilde{j}_*(\widetilde{Z}^* z)$ in $\mathrm{CH}_1(X)$ by Lemma 2.18, and if z is homologous to 0, $\widetilde{Z}^* z$ is a 0-cycle homologous to 0 on $\widetilde{D}$. It is thus algebraically equivalent to 0 and so $z = \tilde{j}_*(\widetilde{Z}^* z)$ is also algebraically equivalent to 0. This proves 1.

Assume now that z is of torsion and annihilated by the Abel-Jacobi map. Then $\widetilde{Z}^* z$ is a torsion 0-cycle on $\widetilde{D}$ which is annihilated by the Albanese map and Roitman's theorem gives that $\widetilde{Z}^* z = 0$. Thus $z = 0$, which proves 2.

If $\dim X \geq 4$ and $z \in \mathrm{CH}^3(X)$, we have by Lemma 2.18, $z = \tilde{j}_*(\widetilde{Z}^* z)$ in $\mathrm{CH}^3(X)$, where $\widetilde{Z}^* z$ is a codimension 2 cycle on $\widetilde{D}$. If now z is of torsion and annihilated by the Abel-Jacobi map, $\widetilde{Z}^* z$ is of torsion and annihilated by the Abel-Jacobi map, hence it vanishes in $\mathrm{CH}^2(\widetilde{D})$ by the following result of Bloch:

Theorem 2.19. *(Bloch) The kernel of the Abel-Jacobi map for codimension 2 cycles homologous to zero on complex projective manifolds has no torsion.*

It then follows that $z = \tilde{j}_*(\widetilde{Z}^* z) = 0$, which proves 3. □

We conclude this section with an implication of cohomological type which is due to Totaro [56] and will be used in Section 3.3. The statement is due to Bloch and Srinivas when $\mathrm{char}\, K = 0$.

Theorem 2.20. *Let X be a smooth projective variety of dimension n defined over a field K of any characteristic. Assume X has a Chow decomposition of the diagonal. Then $H^0(X, \Omega_X^i) = 0$ for $i > 0$.*

Proof. We use the algebraic de Rham cycle class for cycles in any smooth variety Y over K. For any cycle $Z \in \mathrm{CH}^k(Y)$, we get a class $[Z] \in H^k(Y, \Omega_Y^k)$. Furthermore, if X is smooth projective of dimension n, a class $\alpha \in H^n(Y \times X, \Omega^n_{Y \times X})$, with Y smooth but not necessarily projective, induces a morphism

$$\alpha^* : H^p(X, \Omega_X^q) \to H^p(Y, \Omega_Y^q)$$

$$\alpha^*(a) = pr_{1*}(\alpha \cup pr_2^* a).$$

We now start from our Chow decomposition of the diagonal in the form

$$(\Delta_X)_{|U \times X} = U \times x \text{ in } \mathrm{CH}^n(U \times X) \tag{2.40}$$

for some Zariski dense open set of X. Taking de Rham cycle classes, we get

$$[\Delta_X]_{|U \times X} = [U \times x] \text{ in } H^n(U \times X, \Omega^n_{U \times X}). \tag{2.41}$$

We let both sides act on elements $a \in H^{i,0}(X)$ for $i > 0$. The right hand side acts by 0 and the left hand side acts by restriction of forms to U. We thus conclude that for any $a \in H^0(X, \Omega_X^i)$ with $i > 0$, $a_{|U} = 0$, hence $a = 0$ because $U \subset X$ is a dense Zariski open set. □

We finish this section with an important result due to Bloch and Srinivas [12] and uses in fact only the Chow decomposition of the diagonal with $\mathbb{Q}$-coefficients.

Theorem 2.21. *Let X be a smooth projective complex variety admitting a Chow decomposition of the diagonal with $\mathbb{Q}$-coefficients (equivalently, by Theorem 2.4, $\mathrm{CH}_0(X) = \mathbb{Z}$). Then the Griffiths group $\mathrm{Griff}^2(X)$ is trivial and the Abel-Jacobi map*

$$\phi_X^2 : \mathrm{CH}^2(X)_{hom} \to J^3(X)$$

is an isomorphism.

Proof. We write the decomposition of the diagonal as

$$N\Delta_X = N(X \times x) + (\tilde{j}, Id_X)_*(\widetilde{Z}) \text{ in } \mathrm{CH}^n(X \times X), \tag{2.42}$$

where $\tilde{j} : \widetilde{D} \to X$ s a morphism from a smooth variety of dimension $n-1$. This provides for any $z \in \mathrm{CH}^2(X)$ the equality

$$Nz = \tilde{j}_*(\widetilde{Z}^* z), \tag{2.43}$$

where $\widetilde{Z}^* z$ is a codimension 1 cycle on $\widetilde{D}$. If z is cohomologous to 0, so is $\widetilde{Z}^* z$, hence $\widetilde{Z}^* z$ is algebraically equivalent to 0 and Nz is algebraically equivalent to 0 by (2.43). We thus proved that $\mathrm{Griff}^2(X)$ is a torsion group. On the other hand, using the cohomological version of (2.42), we conclude that for any $\alpha \in H^3(X, \mathbb{Z})$, $N\alpha = \tilde{j}_*(\widetilde{Z}^* * \alpha)$, hence vanishes on $U = X \setminus D$. Using notation of Section 1.2, this implies that the map $H^3(X, \mathbb{Z}) \to H^0(X_{Zar}, \mathcal{H}^3(\mathbb{Z}))$ is of N-torsion, hence is trivial as the second group has no torsion by Theorem 1.24. Recalling the Bloch-Ogus exact sequence

$$H_B^3(X, \mathbb{Z}) \to H^0(X_{Zar}, \mathcal{H}^3(\mathbb{Z})) \to \mathrm{Griff}^2(X) \to 0$$

from Theorem 1.18, we conclude that in this case $\mathrm{Griff}^2(X) = H^0(X_{Zar}, \mathcal{H}^3(\mathbb{Z}))$ has no torsion. Hence it is in fact trivial, which proves the first statement.

The second statement is obtained as follows: we use again (2.43). If now z is homologous to 0 and annihilated by ϕ_X^2, then $\widetilde{Z}^* z$ is a codimension 1 cycle on $\widetilde{D}$ which is homologous to 0 and annihilated by $\phi_{\widetilde{D}}^1$, hence is trivial. Thus $Nz = 0$ in $\mathrm{CH}^2(X)$ by (2.43). We then conclude that $z = 0$ using Theorem 2.19. $\square$

We finally turn to unramified cohomology. The following result was proved in [19]:

Theorem 2.22. *Let X be a smooth projective complex variety. (i) If $N\Delta_X$ decomposes as in (2.25), $H_{nr}^i(X, A)$ is of N-torsion for any $i > 0$ and any coefficients A. In particular, if X has a Chow decomposition of the diagonal, the unramified cohomology groups $H_{nr}^i(X, A)$ vanish for any $i > 0$ and any coefficients A.*

(ii) If X satisfies $\mathrm{CH}_0(X) = \mathbb{Z}$, $H_{nr}^i(X, \mathbb{Z})$ vanishes for any $i > 0$.

Proof. Statement (ii) follows from (i), using Theorem 2.5, which guarantees the existence of a decomposition of $N\Delta_X$ assuming $\mathrm{CH}_0(X) = \mathbb{Z}$, and Corollary 1.25, which tells that $H_{nr}^i(X, \mathbb{Z})$ has no torsion.

The proof of (i) uses the fact that Chow correspondences $\Gamma \in \mathrm{CH}^i(X \times Y)$ with X, Y smooth and Y projective of dimension n act on unramified cohomology providing

$$\Gamma^* : H_{nr}^l(Y, A) \to H_{nr}^{l+i-n}(X, A). \tag{2.44}$$

We refer to the appendix of [19] for a precise construction of this action. It factors through the cycle class $[\Gamma]_{mot} \in H^i((X \times Y)_{Zar}, \mathcal{H}^i(\mathbb{Z})) = \mathcal{Z}^i(X \times Y)/\mathrm{alg}$ introduced in Theorem 1.17 (iii). The construction of the action then rests on the basic functoriality properties of unramified cohomology for pull-back, and push-forward under proper maps and the existence of a cup-product. Having this action, we simply let act on $H^i_{nr}(X,A)$ both sides of the decomposition

$$N[\Delta_X]_{mot} = N[X \times x]_{mot} + [Z]_{mot}$$

with Z supported on $D \times X$ for some proper closed algebraic subset $D \subset X$. The left hand side acts as NId. The term $N[X\times x]_{mot}$ acts trivially on $H^i_{nr}(X,A)$ for $i>0$. The fact that $[Z]^*_{mot} = 0$ on $H^i_{nr}(X,A)$ follows from the fact that, denoting $U := X \setminus D$ and $j_U : U \to X$ the inclusion, we clearly have

$$j_U^* \circ [Z]^*_{mot} = 0 : H^i_{nr}(X,A) \to H^i_{nr}(U,A)$$

for any i since Z is supported on $D \times X$. On the other hand, the restriction map j_U^* is injective on $H^i_{nr}(X,A)$ by Proposition 1.20. □

Corollary 2.23. *(i) The unramified cohomology of* $\mathbb{P}^n$ *with any coefficients vanishes in degree* >0.

(ii) Unramified cohomology with any coefficients is a stable birational invariant.

Proof. Clearly $\mathbb{P}^n$ admits a decomposition of the diagonal. This follows from the computation of $\mathrm{CH}(\mathbb{P}^n \times \mathbb{P}^n)$ as the free abelian group with basis $h_1^i \cdot h_2^j$, $0 \leq i,\, j \leq n$, where $h_1 = pr_1^* c_1(\mathcal{O}_{\mathbb{P}^n}(1))$, $h_2 = pr_2^* c_1(\mathcal{O}_{\mathbb{P}^n}(1)) \in \mathrm{CH}^1(\mathbb{P}^n \times \mathbb{P}^n)$. Thus (i) follows from Theorem 2.22, (i).

For the proof of (ii), as we already proved birational invariance of unramified cohomology in Theorem 1.21, it suffices to show invariance under $X \mapsto X \times \mathbb{P}^r$. We use for this the following partial or relative decomposition of the diagonal of $X \times \mathbb{P}^r$:

$$\Delta_{X\times\mathbb{P}^r} = \sum_{i=0}^{r} p_{13}^*\Delta_X \cdot p_{24}^*(h_1^i \cdot h_2^{r-i}), \tag{2.45}$$

where $p_{13} : X \times \mathbb{P}^r \times X \times \mathbb{P}^r \to X \times X$ and $p_{24} : X \times \mathbb{P}^r \times X \times \mathbb{P}^r \to \mathbb{P}^r \times \mathbb{P}^r$ are the obvious projections, and the h_1, h_2 are as above codimension 1 cycles on $\mathbb{P}^r \times \mathbb{P}^r$. We let act both sides of the decomposition above on $H^i_{nr}(X\times\mathbb{P}^r, A)$, say by the upper-star action. The left hand side acts trivially, and a term $p_{13}^*\Delta_X \cdot p_{24}^*(h_1^i \cdot h_2^{r-i})$ acts nontrivially on $H^i_{nr}(X\times\mathbb{P}^r, A)$ only if it dominates $X\times\mathbb{P}^r$ by the projection p_{12}, as follows from the argument already given above and using the injectivity of the restriction map to an open set. The only term which dominates $X \times \mathbb{P}^r$ by the projection p_{12} is

$$W := p_{13}^*\Delta_X \cdot p_4^*(h_2^r),$$

which acts on $H^i_{nr}(X\times\mathbb{P}^r, A)$ by the composite map:

$$H^i_{nr}(X\times\mathbb{P}^r, A) \stackrel{rest}{\to} H^i_{nr}(X \times pt, A) = H^i_{nr}(X,A) \stackrel{p_X^*}{\to} H^i_{nr}(X\times\mathbb{P}^r, A).$$

It follows from the above arguments that $W^* = Id$ on $H^i_{nr}(X\times\mathbb{P}^r, A)$, from which we conclude immediately that the pull-back map $p_X^* : H^i_{nr}(X,A) \to H^i_{nr}(X\times\mathbb{P}^r, A)$ is an isomorphism. □

2.3.3 Cohomological versus Chow decomposition

We explained above that the existence of a Chow decomposition of the diagonal has a priori stronger consequences than the existence of a cohomological decomposition of the diagonal. We are going to discuss here how the two properties relate.

Note first that, by Theorem 2.13, 1, a smooth complex projective variety admitting a cohomological decomposition of the diagonal has $h^{i,0}(X) = 0$ for $i>0$, hence Bloch's conjecture predicts that $\mathrm{CH}_0(X) = \mathbb{Z}$. The Bloch-Srinivas theorem 2.5 then shows that X admits a Chow decomposition of the diagonal with $\mathbb{Q}$-coefficients. In conclusion, when working with $\mathbb{Q}$-coefficients, having a cohomological and a Chow decomposition of the diagonal should be equivalent.

Turning to integral coefficients, the following result appears in [63].

Proposition 2.24. *A smooth projective variety defined over an algebraically closed field admits a Chow decomposition of the diagonal if and only if it admits a decomposition of the diagonal modulo algebraic equivalence.*

Proof. We use the following result of [57] and [59].

Theorem 2.25. *Let $\Gamma \in \mathrm{CH}^*(X \times X)$ be a self-correspondence which is algebraically equivalent to 0. Then Γ is nilpotent in the ring $\mathrm{CH}^*(X \times X)$ of self-correspondences of X.*

Starting from our decomposition

$$\Delta_X = X \times x + Z$$

modulo algebraic equivalence, with Z supported on $D \times X$, let $\Gamma = \Delta_X - X \times x - Z \in \mathrm{CH}^n(X \times X)$. Theorem 2.25 implies that $\Gamma^{\circ N} = 0$ in $\mathrm{CH}^n(X \times X)$ for some $N > 0$. We finally observe that $\Gamma^{\circ N} = \Delta_X - X \times x - Z'$ in $\mathrm{CH}^n(X \times X)$ for some Z' supported on $D' \times X$, for some proper closed algebraic subset $D' \subset X$. The equality

$$\Gamma^{\circ N} = 0 = \Delta_X - X \times x - Z' \text{ in } \mathrm{CH}^n(X \times X)$$

thus gives a Chow decomposition of the diagonal for X. □

In the surface case, we have the following result (proved in [63], and reproved in [32]).

Theorem 2.26. *Let X be a smooth projective surface with $\mathrm{CH}_0(X) = \mathbb{Z}$. Then the following are equivalent:*

1. *X admits a Chow decomposition of the diagonal.*
2. *X admits a cohomological decomposition of the diagonal.*
3. *$\mathrm{Tors}\, H^*_B(X, \mathbb{Z}) = 0$.*

Proof. The implications $1 \Rightarrow 2 \Rightarrow 3$ are clear (the second one is Theorem 2.13, 2). Let us prove $3 \Rightarrow 1$. The condition $\mathrm{Tors}\, H^*(X, \mathbb{Z}) = 0$ implies that X admits a Künneth decomposition with integral coefficients, so that we can write

$$[\Delta_X] = \sum_i \alpha_i \otimes \beta_i \text{ in } H^4_B(X \times X, \mathbb{Z}) \tag{2.46}$$

for some integral cohomology classes α_i, β_i. As $\mathrm{CH}_0(X) = \mathbb{Z}$ we have by Mumford's theorem [44] or Bloch-Srinivas that $H^{i,0}(X) = 0$ for $i > 0$, which in our case implies that the whole cohomology of X is algebraic. (In particular X has no odd degree cohomology.) In formula (2.46), the classes α_i, β_i are classes of algebraic cycles (points, curves, or X itself), which gives a cohomological decomposition of the diagonal that takes the form

$$[\Delta_X - X \times x - Z] = 0, \tag{2.47}$$

where Z is a cycle supported on $D \times X$ for some curve $D \subset X$. We then apply Theorem 2.21 to $Y = X \times X$ which has $\mathrm{CH}_0(Y) = \mathbb{Z}$. This theorem tells us that the group $\mathrm{Griff}^2(Y)$ is trivial. It follows that the cycle $\Delta_X - X \times x - Z$ homologous to 0 is algebraically equivalent to 0. We conclude that X has a decomposition of the diagonal modulo algebraic equivalence, hence admits a Chow decomposition of the diagonal by Proposition 2.24. □

The following question is open:

Question 2.27. *Do there exist smooth projective complex varieties which admit a cohomological decomposition of the diagonal, but no Chow decomposition of the diagonal?*

The answer might be affirmative in view of the discussion made in the previous sections concerning what is controled by the Chow, resp. cohomological decompositions of the diagonal. If we look at the proof of Proposition 2.24, we see that the key point is the nilpotence of self-correspondences algebraically equivalent to 0 (Theorem 2.25). A big conjecture in the theory of algebraic cycles is the following nilpotence conjecture:

Conjecture 2.28. *Self-correspondences $\Gamma \in \mathrm{CH}(X \times X)_{\mathbb{Q}}$ with $\mathbb{Q}$-coefficients and homologous to 0 are nilpotent, that is, $\Gamma^{\circ N} = 0$ in $\mathrm{CH}(X \times X)_{\mathbb{Q}}$, for any smooth projective variety X over $\mathbb{C}$.*

This conjecture is not formulated for self-correspondences $\Gamma \in \mathrm{CH}(X \times X)$, that is with $\mathbb{Z}$-coefficients, and is presumably false, although we are not aware of an explicit counterexample. In fact, there is a different and more general nilpotence conjecture by Voevodsky [57] which predicts the following:

Conjecture 2.29. *For any smooth projective variety Y, any cycle $Z \in \mathrm{CH}(Y)_{\mathbb{Q}}$ with $\mathbb{Q}$-coefficients and homologous to 0 is smash-nilpotent, namely $Z^N = 0$ in $\mathrm{CH}(X^N)_{\mathbb{Q}}$ for some $N > 0$.*

This conjecture implies Conjecture 2.28 by putting $Y = X \times X$ and realizing that $\Gamma^{\circ N}$ is obtained from $\Gamma^N \in \mathrm{CH}((X \times X)^N)_{\mathbb{Q}}$ by a natural correspondence. However, Conjecture 2.29 is shown not to be true with integral coefficients in [50, Theorem 5].

3 The degeneration method

3.1 A specialization result

First of all, let us explain a version of Fulton's specialization map [22, 20.3].

Proposition 3.1. *Let $\pi : Y \to C$ be a flat morphism to a smooth curve over $\mathbb{C}$. Let Z be a cycle on Y such that for the very general complex point $t \in C$, $Z_{|Y_t}$ is rationally equivalent to 0, where $Y_t := \pi^{-1}(t) \subset Y$. Then for any $t \in C$, $Z_{|Y_t}$ is rationally equivalent to 0.*

Remark 3.2. There is no smoothness assumption in this statement, neither for Y, nor for the morphism π. Indeed, by flatness of π and smoothness of C, the fibers Y_t are Cartier divisors, so the restricted cycle $Z_{|Y_t}$ is well-defined.

Proof. We apply Proposition 2.2. It says that there exist a base change $C' \to C$, where we obviously can assume that C' is smooth, and a Zariski open set $U' \subset C'$, such that $Z_{U'} = 0$ in $\mathrm{CH}(Y_{U'})$. The cycle $Z_{C'} \in \mathrm{CH}(Y_{C'})$ thus vanishes on the Zariski open set $Y_{U'} \subset Y_{C'}$, and it follows from the localization exact sequence that there are finitely many fibers $Y_{t'_i} \subset Y_{C'}$ such that $Z_{C'}$ is supported on the union of the $Y_{t'_i}$'s. Clearly the restriction $Z_{|Y_{t'}}$ vanishes for any $t \neq t'_i$ for all i, but in fact this is also true for $t' = t'_i$. Indeed, let $j_i : Y_{t'_i} \to Y_{C'}$ be the inclusion. Then $Y_{t'_i}$ is a Cartier divisor, which furthermore has the property that $\mathcal{O}_{Y_{t'_i}}(Y_{t'_i})$ is trivial. It follows that $j_i^* \circ j_{i*} : \mathrm{CH}(Y_{t'_i}) \to \mathrm{CH}(Y_{t'_i})$ is 0. This proves the result also for the special points t'_i since we know that $Z_{C'} = \sum_i j_{i*}(Z_i)$. (One uses here the fact that the fibers of $Y_{C'} \to C'$ and $Y \to C$ are the same.) □

Corollary 3.3. *Let $X \to C$ be a flat morphism over $\mathbb{C}$ where C is a smooth curve and X is irreducible. Assume that for a very general point $t \in C$, the fiber X_t has a Chow decomposition of the diagonal. Then any fiber X_t has a Chow decomposition of the diagonal.*

Proof. Consider the flat morphism $Y := X \times_C X \to C$. By assumption, for a very general point $t \in C$, there exist a divisor $D_t \subset X_t$ and a point $x_t \in X_t$ such that

$$\Delta_{X_t} = X_t \times x_t + Z_t \text{ in } \mathrm{CH}(X_t \times X_t),$$

where the cycle Z_t is supported on $D_t \times X_t$. The data such as D_t, Z_t or x_t are parameterized by a countable union of Chow varieties which are proper over C', and we conclude that, after base change $C' \to C$, there exists a divisor $D \subset X_{C'}$ which does not contain any fiber, there exist a section $\sigma : C' \to X_{C'}$ and a cycle Z supported on $D \times_{C'} X_{C'}$ such that the cycle

$$\Gamma := \Delta_{X'/C'} - X' \times_{C'} \sigma(C') - Z \in \mathrm{CH}(Y_{C'}) \tag{3.48}$$

has the property that for a very general point $t \in C'$, $\Gamma_{|Y_t} = 0$. Note that we can assume that D is the Zariski closure of its generic fiber over C', as the only constraint it has to satisfy, namely (3.48), concerns its generic fiber. By Proposition 3.1, this remains true for any $t \in C'$. As X is irreducible and flat over C, the fibers of $X \to C$ are equidimensional of dimension n, hence we can assume that D does not contain any component of any fiber of $X_{C'} \to C'$ (such a fiber would form an irreducible component of D that does not dominate C', hence would not be in the Zariski closure of the generic fiber of D). Thus $D \cap X_t$ is a proper divisor for any point $t \in C'$, and the condition $\Gamma_{|Y_t} = 0$ for any t thus says that X_t has a Chow decomposition of the diagonal. □

The following result is proved in [62].

Theorem 3.4. *Let $\pi : X \to C$ be a flat projective morphism of relative dimension $n \geq 2$, where C is a smooth curve. Assume that the fiber X_t is smooth for $t \neq 0$, and has at worst isolated ordinary quadratic singularities for $t = 0$. Then*

(i) If for general $t \in B$, X_t admits a Chow theoretic decomposition of the diagonal (equivalently, $CH_0(X_t)$ is universally trivial), the same is true for any smooth projective model $\widetilde{X}_0$ of X_0.

(ii) If for general $t \in B$, X_t admits a cohomological decomposition of the diagonal, and the even degree integral homology of a smooth projective model $\widetilde{X}_0$ of X_0 is algebraic (i.e. generated over $\mathbb{Z}$ by classes of subvarieties), $\widetilde{X}_0$ also admits a cohomological decomposition of the diagonal.

In order to prove (ii), we will need an intermediate step involving the notion of a homological decomposition of the diagonal for singular projective varieties: to make sense of this, we just need to know that cycles Z have a homology class $[Z]_{hom}$ in Betti integral *homology*, which is standard. Then a homological decomposition of the diagonal of a singular but projective X of pure dimension n is an equality

$$[\Delta_X]_{hom} = [X \times x]_{hom} + [Z]_{hom} \text{ in } H_{2n}(X \times X, \mathbb{Z}),$$

where as usual Z is a cycle supported on $D \times X$ for some nowhere dense closed algebraic subset D of X.

Proof of Theorem 3.4 . By Corollary 3.3 and under the assumptions made on the general fibers in (i), the central fiber admits a Chow decomposition of the diagonal. This step does not need any assumption on the singularities of the fibers. Similarly, under the assumptions made on the general fibers in (ii), the central fiber admits a homological decomposition of the diagonal. The proof here uses the fact that as we are over $\mathbb{C}$, for any proper flat analytic morphism $X' \to \Delta$, after shrinking Δ if necessary, there is a continuous retraction $X' \to X_0$. Passing to $X' \times_\Delta X'$, this retraction maps the diagonal $\Delta_{X'_t}$ to the diagonal $\Delta_{X'_0}$. This implies that a homological relation $[\Gamma_t] = 0$ in $H_{2n}(X'_t \times X'_t, \mathbb{Z})$, where Γ_t is as in (3.48) implies a homological relation $[\Gamma_0] = 0$ in $H_{2n}(X'_0 \times X'_0, \mathbb{Z})$, which provides a homological decomposition of the diagonal for $X'_0 = X_0$.

The second step is passing from X_0 to $\widetilde{X}_0$ and this is here that we use the assumption on the singularities. Let us first concentrate on (i). From the decomposition

$$\Delta_{X_0} = X_0 \times x + Z \text{ in } \mathrm{CH}_n(X_0 \times X_0),$$

where Z is supported on $D \times X_0$, we deduce by restriction to

$$U \times U, \quad U := X_0 \setminus \operatorname{Sing} X_0 = \widetilde{X}_0 \setminus E,$$

where E is the exceptional divisor of the resolution of singularities of X_0 obtained by blowing-up the singular points:

$$\Delta_U = U \times x + Z_{|U \times U} \text{ in } \mathrm{CH}_n(U \times U).$$

By the localization exact sequence, we get a decomposition on $\widetilde{X}_0$ which takes the following form:

$$\Delta_{\widetilde{X}_0} = \widetilde{X}_0 \times x_0 + \widetilde{Z} + \Gamma_1 + \Gamma_2 \text{ in } \mathrm{CH}_n(\widetilde{X}_0 \times \widetilde{X}_0),$$

where $\widetilde{Z}$ is supported on $D' \times \widetilde{X}_0$ for some $D' \subsetneq \widetilde{X}_0$, and Γ_1 is supported on $E \times \widetilde{X}_0$, Γ_2 is supported on $\widetilde{X}_0 \times E$. Of course the cycle Γ_1 does not dominate $\widetilde{X}_0$ by the first projection, so we need only to understand Γ_2. But E is a disjoint union of smooth quadrics Q_i of dimension ≥ 1, and for each of them, n-dimensional cycles in $\widetilde{X}_0 \times Q_i$ decompose as $n_i \widetilde{X}_0 \times x_i + Z_i$, where Z_i does not dominate $\widetilde{X}_0$ by the first projection, n_i is an integer, and x_i is any point of Q_i. At this point, we obtained a decomposition of the form

$$\Delta_{\widetilde{X}_0} = \widetilde{X}_0 \times x_0 + \sum_i n_i \widetilde{X}_0 \times x_i + Z \text{ in } \mathrm{CH}(\widetilde{X}_0 \times \widetilde{X}_0). \tag{3.49}$$

In order to conclude, we have to use the assumption $n \geq 2$. It implies that $\widetilde{X}_0$ is irreducible or equivalently, connected. Indeed the general fiber X_t is connected as this is a consequence of the existence of a decomposition of the diagonal for X_t. Formula (3.49) tells us by letting both sides act on $\mathrm{CH}_0(\widetilde{X}_0)$ that $\mathrm{CH}_0(\widetilde{X}_0)$ is generated over $\mathbb{Z}$ by x_0 and the x_i. By Roitman's theorem [48], this implies that $\mathrm{CH}_0(\widetilde{X}_0) = \mathbb{Z}$, so that all the x_i's are rationally equivalent to x_0 in $\widetilde{X}_0$. Then (3.49) gives a Chow decomposition of the diagonal for $\widetilde{X}_0$.

The proof of (ii) is quite similar although the tools are slightly different. It is important here to realize that homology and algebraic cycles do not work completely in the same way. For example, we do not have in homology the localization exact sequence.

We know that the central fiber has a homological decomposition of the diagonal in $H_*(X_0 \times X_0, \mathbb{Z})$. A fortiori it has a homological decomposition in the relative homology $H_*(X_0 \times X_0, B, \mathbb{Z})$ where $B = \operatorname{Sing}(X_0) \times X_0 \cup X_0 \times \operatorname{Sing}(X_0)$ As $\widetilde{X}_0 \setminus E \cong X_0 \setminus \operatorname{Sing} X_0$, it follows that we get for $\widetilde{X}_0$ a homological decomposition of the diagonal modulo $E \times \widetilde{X}_0 \cup \widetilde{X}_0 \times E$. This shows that we have a relation

$$[\Delta_{\widetilde{X}_0}]_{hom} = [\widetilde{X}_0 \times x_0] + [\widetilde{Z}] + \alpha \text{ in } H_{2n}(\widetilde{X}_0 \times \widetilde{X}_0, \mathbb{Z}), \tag{3.50}$$

where $\alpha \in H_{2n}(E \times \widetilde{X}_0 \cup \widetilde{X}_0 \times E, \mathbb{Z})$. We use now the fact that $E = \sqcup Q_i$ so that the union above is the union of the $Q_i \times \widetilde{X}_0$ and $\widetilde{X}_0 \times Q_j$ intersecting along the union of the $Q_i \times Q_j$. As $Q_i \times Q_j$ has trivial odd degree cohomology, it follows that $H_{2n}(E \times \widetilde{X}_0 \cup \widetilde{X}_0 \times E, \mathbb{Z})$ is generated by the subgroups $H_{2n}(Q_i \times \widetilde{X}_0, \mathbb{Z})$ and $H_{2n}(\widetilde{X}_0 \times Q_i, \mathbb{Z})$. Hence $\alpha = \sum_i \alpha_i + \beta_i$ with $\alpha_i \in H_{2n}(Q_i \times \widetilde{X}_0, \mathbb{Z})$, $\beta_i \in H_{2n}(\widetilde{X}_0 \times Q_i, \mathbb{Z})$.

We assume for simplicity that $H^{2*}(\widetilde{X}_0, \mathbb{Z})$ is algebraic (we only assumed that this assumption holds for some variety birationally equivalent to $\widetilde{X}_0$). We then get using the Künneth decomposition of the even degree cohomology (or homology) of $\widetilde{X}_0 \times Q_i$ and $Q_i \times \widetilde{X}_0$ that each α_i is algebraic and each β_i is algebraically decomposable, that is of the form $\sum_l [Z_l \times Z'_l]$ for some algebraic cycles on each summand. Clearly α_i is then the class of a cycle z_i in $\widetilde{X}_0 \times \widetilde{X}_0$ which does not dominate $\widetilde{X}_0$ by the first projection. For the $\beta_i = \sum_l [Z_l \times Z'_l]$, if $\dim Z'_l > 0$, then $\dim Z_l < n$ and Z_l does not dominate $\widetilde{X}_0$ by the first projection. Finally, if $\dim Z'_l = 0$, then one gets a contribution $[\widetilde{X}_0 \times x_i]$. Putting this decomposition in (3.50) and using the fact that $[x_i] = [x_0]$ in $H^{2n}(\widetilde{X}_0, \mathbb{Z})$, this clearly provides a homological (or equivalently cohomological as $\widetilde{X}_0$ is smooth) decomposition of the diagonal of $\widetilde{X}_0$. We used in the last step the fact that $n \geq 2$ to guarantee that $H_B^{2n}(\widetilde{X}_0, \mathbb{Z}) = \mathbb{Z}$ is generated by the class of the point x_0. $\square$

Remark 3.5. The assumptions on the singularities in Theorem 3.4 are too strong and this will be discussed in next section, but some assumptions on the singularities are necessary. Consider the case of the cubic surface degenerating to a cone over an elliptic curve. The general fiber is rational hence has a Chow decomposition of the diagonal, but the desingularization $\widetilde{S}_0$ of the central fiber has nonzero holomorphic forms, so it does not admit a decomposition of the diagonal by Theorem 2.13.

As a first application, let us prove Proposition 0.5 stated in the introduction:

Proof of Proposition 0.5. Indeed, if X_t was stably rational, it would admit a Chow decomposition of the diagonal. Then $\widetilde{X}_0$ would also admit a Chow decompositon of the diagonal by Theorem 3.4, because clearly the fiber dimension has to be ≥ 2. By Theorem 2.13, this contradicts the fact that $\operatorname{Tors} H^3(\widetilde{X}_0, \mathbb{Z}) \neq 0$. $\square$

3.1.1 The very general quartic double solid is not stably rational

Recall that a quartic double solid is a hypersurface X in $\mathbb{L} := \operatorname{Spec}(\operatorname{Sym} \mathcal{O}_{\mathbb{P}^3}(-2)) \xrightarrow{\pi} \mathbb{P}^3$ defined by the equation $u^2 = p^* f$, where u is the canonical extra section of $\pi^* \mathcal{O}_{\mathbb{P}^3}(2)$ on $\mathbb{L}$ and $f \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4))$. Thus quartic double solids are parameterized by $\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4)))$. We described in Section 1.1.1 the Artin-Mumford double solid X_0 which is nodal, with the property that $\widetilde{X}_0$ has a nontrivial Artin-Mumford invariant.

Theorem 3.6. *The very general quartic double solid X does not admit a cohomological (hence a fortiori Chow-theoretic) decomposition of the diagonal. Similarly, the desingularization of the very general quartic double solid X with $k \leq 7$ nodes in general position does not admit a cohomological decomposition of the diagonal.*

Here we observe that given $k \leq 7$ general points in $\mathbb{P}^3$, there is a linear space of dimension $34 - 4k > 0$ of quartic homogeneous polynomials f having multiplicity ≥ 2 at these k points. There is thus an irreducible variety parameterizing quartic double solids with k nodes in general position. As usual, "very general" in Theorem 3.6 means that the statement is true for a parameter f in the complement of a countable union of proper closed algebraic subsets of this variety.

Theorem 3.6 immediately follows from Theorem 3.4 by degeneration to the Artin-Mumford double solid. Indeed, if X_0 is the Artin-Mumford double solid, $\widetilde{X}_0$ does not admit a cohomological decomposition of the diagonal by Theorem 2.13, because the Artin-Mumford invariant of $\widetilde{X}_0$ is not trivial. Furthermore, the even degree integral Betti cohomology of $\widetilde{X}_0$ is algebraic by Theorem 1.29 because $\widetilde{X}_0$ is a rationally connected threefold. For the nodal case, one needs to check that the Artin-Mumford double solid smoothifies partially to the k-nodal quartic double solid with k nodes in general position, for $k \leq 7$.

A a consequence of Theorem 3.6, one gets the following

Corollary 3.7. *The desingularization of the very general quartic double solid with $k \leq 7$ nodes in general position is not stably rational.*

Note that by Endrass [21], if $\widetilde{X}$ is as in Theorem 3.7, $\widetilde{X}$ has trivial Artin-Mumford invariant. In fact Endrass proves that the desingularization of a quartic double solid with less than 10 points has no torsion in its third Betti cohomology. To our knowledge, the only criterion for stable irrationality of rationally connected threefolds used previously was the Artin-Mumford invariant.

3.2 Colliot-Thélène-Pirutka and Schreieder's work

It was noticed in [62] that the assumptions on the singularities in theorem 3.4 were too strong, even if, according to Remark 3.5, some assumptions are necessary. The paper by Colliot-Thélène and Pirutka [18], written in the equivalent language of universally CH_0-trivial varieties (see Section 2.2), provides a similar specialization result under weaker assumptions. They prove the following theorem that we in turn reformulate below in the language of decomposition of the diagonal. We will state the result over any algebraically closed field k. The only difference when working over $\mathbb{C}$ is the fact that, $\mathbb{C}$ being a large field, we can use the assumption on the very general fiber of a morphism as being equivalent to the similar assumption on the geometric generic fiber (see the discussion in Section 2.1). Note however that the setting of Colliot-Thélène-Pirutka's work is that of a scheme over a DVR, which includes specialization from varieties defined over a number field to varieties defined over a finite field. This is very important in Totaro's work (Theorem 3.15) that we will review later on.

The Colliot-Thélène and Pirutka's condition in [18] asks that the resolution map $\tau : \widetilde{X}_0 \to X_0$ is universally Chow-trivial, which means that for any field L containing the base field k, the morphism $\tau_* : \mathrm{CH}_0(\widetilde{X}_{0,L}) \to \mathrm{CH}_0(X_{0,L})$ is an isomorphism. This condition is rather strong and needs to be carefully checked geometrically. It says that for each subvariety $M \subset X_0$, the generic fiber $\widetilde{X}_{0,M}$ of the induced morphism $\tau^{-1}(M) \to M$, which is a variety over $k(M)$, has CH_0 universally trivial.

Let us first consider the following condition (*) that is slightly stronger than the Colliot-Thélène-Pirutka condition but is explicit geometrically:

(*) *For any irreducible subvariety $Y \subset X_0$, the map $\tau_Y : E_Y := \tau^{-1}(Y) \to Y$ has a rational section (or a 0-cycle of degree 1) and its generic fiber is smooth geometrically irreducible and has a decomposition of the diagonal over $k(Y)$.*

Remark 3.8. In practice, condition (*) is proved by checking that each generic fiber $E_{Y,\eta}$ is smooth rational over $k(Y)$.

Here the decomposition of the diagonal for $E_{Y,\eta}$ is supposed to hold with respect to the given point or 0-cycle $y_\eta \in E_{Y,\eta}(k(Y))$. Note that we use here the Chow decomposition of the diagonal for any variety defined over any field, in particular not algebraically closed.

Theorem 3.9. *Let $X \to C$ be a flat morphism, where C is a smooth curve over $\mathbb{C}$. Assume the very general fiber X_t is smooth and has a Chow decomposition of the diagonal. Then if the central fiber has a desingularization $\tau : \widetilde{X}_0 \to X_0$ satisfying (*), $\widetilde{X}_0$ has a Chow decomposition of the diagonal.*

Remark 3.10. We recover the case of nodal singularities by considering the standard resolution by blow-up. The condition that the fibers have dimension at least 2 is hidden in condition (*), because in dimension 2, the exceptional fiber of the resolution over a singular point consists of two points, which does not satisfy (*).

Proof of Theorem 3.9. The proof starts as the proof of Theorem 3.4: we thus conclude that X_0 has a Chow decomposition of the diagonal and we want to deduce that $\widetilde{X}_0$ also has one, so that we get by lifting the cycles to $\widetilde{X}_0 \times \widetilde{X}_0$:

$$\Delta_{\widetilde{X}_0} = \widetilde{X}_0 \times x_0 + \widetilde{Z} + \Gamma \text{ in } \mathrm{CH}_n(\widetilde{X}_0 \times \widetilde{X}_0), \tag{3.51}$$

where $\widetilde{Z}$ is supported on $D' \times \widetilde{X}_0$ for some proper closed algebraic subset $D' \subset \widetilde{X}_0$, and Γ is supported on $E \times \widetilde{X}_0 \cup \widetilde{X}_0 \times E$. Here E is the exceptional locus of the considered desingularization of X_0. A key point observed by Colliot-Thélène and Pirutka is the fact that for some dense Zariski open set U of X_0, the cycle Γ satisfies

$$(\tau,\tau)_*\Gamma_{|U\times X_0} = 0 \text{ in } \mathrm{CH}_n(U \times X_0).$$

The end of the proof then rests on the following statement:

Lemma 3.11. *Let $\phi : W \to V$ be a proper dominant morphism with irreducible smooth generic fiber. Assume there is a generic relative decomposition of the diagonal for ϕ, namely there exist a rational section ψ of ϕ with image $S \subset W$, a proper closed algebraic subset $W_1 \subset W$ and a cycle $T \subset W \times_V W \to V$ which is supported over $W_1 \subset W$, such that*

$$\Delta_W = W \times_V S + T \text{ in } \mathrm{CH}(W \times_V W). \tag{3.52}$$

Then for any smooth projective variety Y of dimension n and any cycle $\Gamma_1 \in \mathrm{CH}_n(Y \times W)$ such that $(Id_Y, \phi)_*\Gamma_1$ vanishes in $\mathrm{CH}(U \times V)$ for some dense Zariski open set U of Y, there exist a proper closed algebraic subset $V' \subset V$ and a cycle $\Gamma'_1 \in \mathrm{CH}_n(Y \times W')$, where $W' := \phi^{-1}(V')$, such that $\phi'_*\Gamma'_1 = 0$ in $\mathrm{CH}_n(U \times V')$ and $\Gamma'_1 = \Gamma_1$ in $\mathrm{CH}_n(U \times W)$ for some dense Zariski open set U of Y.

Proof. As $(Id_Y, \phi)_*\Gamma_1$ vanishes in $\mathrm{CH}(U \times V)$ for some dense Zariski open set U of Y, and there is a rational section of ϕ, we can assume that $(Id_Y, \phi)_*\Gamma_1$ actually vanishes as a n-cycle of $U \times V$ by replacing the cycle $\Gamma_1 \in \mathcal{Z}_n(Y \times W)$ by $\Gamma_1 - (Id, \psi)_*(Id, \phi)_*(\Gamma_1)$ which is rationally equivalent to it. Moving cycles, we can assume that the support $\mathrm{Supp}\,\Gamma_1$ of Γ_1 does not have its image in W contained in W_1. As $\dim \mathrm{Supp}\,\Gamma_1 = n = \dim Y$, there exists a dense Zariski open set of Y (that we can assume to be U), such that, over U, $\mathrm{Supp}\,\Gamma_1$ and $pr_W^{-1}(W_1)$ do not intersect. Let $m = \dim W$ and let V^0 be a dense Zariski open set over which $\phi : W \to V$ is smooth and let $W^0 := \phi^{-1}(V^0)$. The group $\mathrm{CH}_m(W^0 \times_{V^0} W^0)$ acts on $\mathrm{CH}_n(Y \times W^0)$ by composition over V^0. The diagonal Δ_{W^0} acts as the identity and $W^0 \times_{V^0} S$ acts as $(Id, \psi)_* \circ (Id, \phi)_*$. It thus follows from (3.52) and from the vanishing of $(Id, \phi)_*\Gamma_1$ in $\mathrm{CH}_n(U \times V)$, that

$$\Gamma_{1|U \times W^0} = T \circ \Gamma_1 \text{ in } \mathrm{CH}_n(U \times W^0). \tag{3.53}$$

As T is supported over W_1 and $\mathrm{Supp}\,\Gamma_1$ does not meet $pr_W^{-1}(W_1)$ over U, the cycle $T \circ \Gamma_1$ vanishes over the Zariski open set $U \times W^0$ of $Y \times W$. By (3.53) and the localization exact sequence, $\Gamma_{1|U \times W}$ is rationally equivalent to a cycle Γ'_1 supported over a proper closed algebraic subset $V' \subset V$. Denoting by $\phi' : W' := \phi^{-1}(V') \to V$, it remains to see that $(Id, \phi')_*(\Gamma'_1) = 0$ in $\mathrm{CH}(U \times V')$ if U is small enough. This is because, taking the limit over the Zariski open sets U of Y, Γ_1 can be seen as a 0-cycle of W_K, with $K = k(Y)$, which vanishes in $\mathcal{Z}_0(V_K)$. When we apply the map T_*, to it, the resulting cycle also vanishes as a 0-cycle of V_K, and at the same time it is supported on V'_K. Hence it vanishes in $\mathcal{Z}_0(V'_K)$. ◻

We apply Lemma 3.11 in an iterated way, with $Y = \widetilde{X}_0$, starting from the situation where $W = \widetilde{X}_0$, $V = X_0$, $\phi = \tau$ and Γ_1 is the component of the cycle Γ appearing in (3.51) which is supported on $\widetilde{X}_0 \times E$. (We do not care about the component supported on $E \times \widetilde{X}_0$ as it does not dominate $Y = \widetilde{X}_0$ by the first projection.) We then conclude using the condition (*) and Lemma 3.11 that we can decrease step by step the dimension of $\tau(\mathrm{Supp}\,\Gamma_{k(Y)})$ until finally we conclude that the cycle Γ_1 vanishes in $U \times \widetilde{X}_0$, for a small enough dense Zariski open set $U \subset \widetilde{X}_0$, and equivalently, Γ_1 is supported on $D \times \widetilde{X}_0$ for some proper closed algebraic subset D of $\widetilde{X}_0$. Formula (3.51) then provides a Chow decomposition of the diagonal for $\widetilde{X}_0$. The proof of the theorem is thus finished. ◻

Combining Theorem 3.9, Theorem 2.22 and Remark 3.8, one gets the following improvement of Proposition 0.5:

Proposition 3.12. *Let $\phi : X \to C$ be a flat morphism, where C is a smooth curve over an algebraically closed field k. Then if the central fiber has a desingularization $\tau : \widetilde{X}_0 \to X_0$ satisfying assumption (*) (for example, if τ has rational generic fibers E_{Y_η} over $k(Y)$ for any $Y \subset \widetilde{X}_0$), and $\widetilde{X}_0$ has a nontrivial Brauer group, the geometric generic fiber $X_{\overline{\eta_C}}$ of ϕ is not stably rational. If $k = \mathbb{C}$, the very general fiber X_t is not stably rational.*

We now come to Schreieder's improvement of Propositions 0.5 and 3.12. This is a very simple observation but it is very useful in practice because it does not need any control of the singularities of the special fiber X_0. The statement is as follows:

Theorem 3.13. *(See [52, Proposition 26], [53, Proposition 3.1].) Let $\phi : X \to C$ be a flat morphism where C is a smooth curve over an algebraically closed field k. Assume that the central fiber has a desingularization $\tau : \widetilde{X}_0 \to X_0$ satisfying the following property: There exists a nontrivial unramified cohomology class η of positive degree on $\widetilde{X}_0$ such that any component E_i of the exceptional divisor is smooth and satisfies $\eta_{|E_i} = 0$. Then the geometric generic fiber $X_{\overline{\eta_C}}$ of ϕ is not stably rational. If $k = \mathbb{C}$, the very general fiber X_t is not stably rational.*

Proof. By the first step in the proofs of Theorems 3.4 and 3.9, it suffices to show that the central fiber X_0 itself does not admit a Chow decomposition of the diagonal. Lifting such a decomposition to $\widetilde{X}_0$ would provide as before an equality

$$\Delta_{\widetilde{X}_0} = \widetilde{X}_0 \times x_0 + Z + \Gamma \text{ in } \mathrm{CH}_n(\widetilde{X}_0 \times \widetilde{X}_0), \tag{3.54}$$

where Z is supported on $D \times \widetilde{X}_0$ for some $D \subsetneq \widetilde{X}_0$, and Γ is supported on $E \times \widetilde{X}_0 \cup \widetilde{X}_0 \times E$. Here $E = \cup_i E_i$ is the exceptional locus of the considered desingularization of X_0.

Now we write $\Gamma = \sum_i \Gamma_i + \sum_i \Gamma'_i$ where Γ_i is supported on $E_i \times \widetilde{X}_0 \cup \widetilde{X}_0 \times E_i$, and we let both sides of the equality (3.54) act on η by the upper-star action. We observe here that $\Gamma_i^* \eta = 0$ by Proposition 1.20, because this is a unramified cohomology class which vanishes on the dense Zariski open set $\widetilde{X}_0 \setminus \cup_i E_i$. Next $(\Gamma'_i)^* \eta = 0$, because $\eta_{|E_i} = 0$. As $\deg \eta > 0$, $\widetilde{X}_0 \times x_0$ acts trivially on η, and thus (3.54) provides

$$\eta = Z^* \eta$$

where the right hand side is 0 again by Proposition 1.20 because $Z^*\eta$ vanishes on $\widetilde{X}_0 \setminus D$. ☐

Remark 3.14. We used here unramified cohomology but other invariants as discussed in Section 1 can be used as well, for example the differential forms in nonzero characteristic.

3.3 Further developments and consequences

In this section, we will describe further variants of Proposition 3.12, and some applications. In the paper [56], Totaro uses a version of the specialization theorem where the geometric degeneration is replaced by specialization mod p of a variety defined over a number field. This generalization is already present in Colliot-Thélène-Pirutka's paper. The second important ingredient is the fact that he uses as an obstruction to the Chow decomposition of the diagonal (or universal triviality of CH_0) for the desingularized central fiber $\widetilde{X}_0$ the space of algebraic differential forms of positive degree as discussed in Section 2.3.2 (Theorem 2.20).

Finally, the specialization he uses is the same as in [35], although the degree range in the final statement is slightly different. Kollár's specialization produces in characteristic 2 limits of hypersurfaces in $\mathbb{P}^{n+1}$, of any even degree $\geq 2\ulcorner(\frac{n+2}{3})\urcorner$ which admit nonzero algebraic differential forms of degree $n-1$. The important point to be discussed here is the nature of the singularities : not only the forms have to extend on the desingularization, a point which is discussed in Kollár's paper, but the singularities have to satisfy the condition (*) of Colliot-Thélène and Pirutka. This is done in [56].

Combining all these ingredients, Totaro finally proves the following theorem, where the ground field is assumed to be uncountable of characteristic 0 or 2:

Theorem 3.15. *[56] A very general hypersurface of degree $\geq 2\ulcorner(\frac{n+2}{3})\urcorner$ in $\mathbb{P}^{n+1}$, $n \geq 3$, is not stably rational.*

The method of the proof actually shows that such hypersurfaces defined over a number field exist, and not only they are not stably rational, but they in fact do not have universally trivial CH_0 group.

Let us finally state the following spectacular asymptotic improvement of Totaro's Theorem. This result by Schreieder [53] uses in an essential way Theorem 3.13.

Theorem 3.16. *(Schreieder [53]) A very general complex projective hypersurface of dimension n and degree at least $\log_2 n + 2$, $n \geq 3$, is not stably rational.*

We conclude this section with some hints on the following theorem solving a longstanding question:

Theorem 3.17. *[27] Let $Y \subset \mathbb{P}^2 \times \mathbb{P}^3$ be a very general hypersurface of bidegree $(2, 2)$. Then Y is not stably rational.*

On the other hand, there is a dense set of points b in the parameter space for these Y's, such that Y_b is rational. In particular, rationality and stable rationality are not invariant under deformation.

The proof uses the specialization method described above. Let us give a complete proof for the density statement which is not hard but useful. We will use the following fact from [17, Section 3] (see also [19, Section 8]):

Proposition 3.18. *Let Y be a smooth fourfold fibered in 2-dimensional quadrics over a surface. Then integral Hodge classes of degree 4 on Y are algebraic.*

We also have the following standard lemma due to Springer [54] (it is in fact true in any dimension and over any field).

Lemma 3.19. *Let Q be a smooth quadric surface over a field k of characteristic 0. Then Q has a k-point, hence is rational over k, if and only if Q has a 0-cycle z of odd degree.*

Proof. Indeed, let C be the family of lines in Q. The curve $C_{\overline{k}}$ is the disjoint union of two copies of $\mathbb{P}^1_{\overline{k}}$. Let $k \subset k'$ be the degree 2 (or 1) extension on which the two geometric components of $C_{\overline{k}}$ are defined. Then $C_{k'}$ is the disjoint union of two curves C_1, C_2 which become isomorphic to $\mathbb{P}^1_{\overline{k'}}$ over $\overline{k'}$. But each of these curves C_i has a divisor of odd degree defined over k', namely the incidence divisor $P^*z \in \mathrm{CH}^1(C_i)$, where $P \subset C_i \times Q$ is the universal correspondence. It follows that each component C_i is isomorphic to $\mathbb{P}^1_{k'}$, and has a k'-point l, providing a line $l \subset Q$ defined over k'. Let i be the Galois involution acting on $C(k')$. Then if $i(l) = l$, (so that in fact $k = k'$ and $i = Id$), l is defined over k and Q has a k-point. Otherwise we get two different conjugate lines l and $i(l)$ in Q which belong to different rulings of Q, and their intersection point is defined over k. $\square$

Corollary 3.20. *Let Y be a fourfold as in Theorem 3.17. Then Y is rational if Y has an integral Hodge class α of degree 4 which has odd intersection number with the fibers Q_s of the morphism $pr_1 : Y \to \mathbb{P}^2$.*

Proof. Indeed, Y is fibered via pr_1 into quadric surfaces over $\mathbb{P}^2$. Proposition 3.18 thus applies to Y and α is the class of a codimension algebraic cycle Z on Y. Restricting Z to the generic fiber Y_η of pr_1, we get a 0-cycle of odd degree on Y_η defined over the function field $\mathbb{C}(\eta)$ of $\mathbb{P}^2$ and Lemma 3.19 then tells that Y_η is rational over $\mathbb{C}(\eta)$. A fortiori, Y is rational. $\square$

Corollary 3.20 reduces the proof of the density statement to the following proposition:

Proposition 3.21. *Let B be the family of all smooth fourfolds Y described in Theorem 3.17. Then the set of points $b \in B$ such that Y_b has an integral Hodge class α of degree 4 which has odd intersection number with the fibers of pr_1 is dense in B for the usual topology.*

Proof. This will follow by applying the following infinitesimal criterion (Proposition 3.22) below : Consider our family of fourfolds $\mathcal{Y} \to B$. We have an associated infinitesimal variation of Hodge structures (see [66, 5.1.2]) at any point $t \in B$

$$H^{2,2}(\mathcal{Y}_t) \to \mathrm{Hom}\,(T_{B,t}, H^{1,3}(\mathcal{Y}_t)),$$

$$\alpha \mapsto \overline{\nabla}(\alpha) : T_{B,t} \to H^{1,3}(\mathcal{Y}_t).$$

Using the fact that the Hodge structure on $H^4(\mathcal{Y}_t, \mathbb{Q})$ is of Hodge niveau 2, that is, $H^{4,0}(\mathcal{Y}_t) = 0$, we have (see [66, 5.3.4]):

Proposition 3.22. *If there exist $t_0 \in B$ and $\alpha \in H^{2,2}(\mathcal{Y}_{t_0})$ such that $\overline{\nabla}(\alpha) : T_{B,t_0} \to H^{1,3}(\mathcal{Y}_{t_0})$ is surjective, then for any Euclidean open set $U \subset B$ containing t_0, the image of the natural map*

$$T_{t_0} : \mathcal{H}^{2,2}_{\mathcal{Y}_U,\mathbb{R}} \to H^4(\mathcal{Y}_{t_0}, \mathbb{R})$$

defined by composing the inclusion $\mathcal{H}^{2,2}_{\mathcal{Y},\mathbb{R}} \to \mathcal{H}^4_{\mathcal{Y},\mathbb{R}}$ with a local flat trivialization over U of $\mathcal{H}^4_{\mathcal{Y},\mathbb{R}}$, contains an open subset V_U of $H^4(\mathcal{Y}_{t_0}, \mathbb{R})$.

Here $\mathcal{H}^4_{\mathcal{Y},\mathbb{R}}$ is the flat real vector bundle with fiber $H^4(\mathcal{Y}_b, \mathbb{R})$ over any $b \in B$ and $\mathcal{H}^{2,2}_{\mathcal{Y},\mathbb{R}}$ is the real vector bundle over B with fiber over $t \in B$ the space $H^{2,2}(\mathcal{Y}_t)_\mathbb{R}$ of real cohomology classes of type $(2,2)$ on $\mathcal{Y}_t$. Note that the image of T_{t_0} is by definition the set of real degree 4 cohomology classes on $\mathcal{Y}_t$ which are of type $(2,2)$ at some point $t' \in U$.

Corollary 3.23. *Under the same assumption, for any $t \in B$, and any Euclidean open set $U \subset B$ containing t, there exists $t' \in U$ and $\alpha_{t'} \in H^{2,2}(\mathcal{Y}_{t'}) \cap H^4(\mathcal{Y}_{t'}, \mathbb{Z})$ such that the degree of α on the fibers Q_s of $pr_1 : \mathcal{Y}_{t'} \dashrightarrow \mathbb{P}^2$ is odd.*

Proof. We observe first that the condition on t_0 in Proposition 3.22 is Zariski open, hence is satisfied on a dense open set. We also note that the open subset V_U of $H^4(\mathcal{Y}_{t_0}, \mathbb{R})$ appearing in Proposition 3.22 is in fact a subcone. It is then immediate to prove that a non-empty open subcone of $H^4(\mathcal{Y}_{t_0}, \mathbb{R}) = H^4(\mathcal{Y}_{t_0}, \mathbb{Z}) \otimes \mathbb{R}$ has to contain an integral class which has odd degree on the fibers Q_s. □

What remains to be done is to check the infinitesimal criterion, which is quite well-understood thanks to the Carlson-Griffiths theory of variation of Hodge structures of hypersurfaces (see [66, 5.3.4]). □

4 Cohomological decomposition of the diagonal and the Abel-Jacobi map

4.1 Intermediate Jacobians, Abel-Jacobi map and universal cycle

We already encountered in the previous sections the Abel-Jacobi map

$$\phi^2_X : \mathrm{CH}^2(X)_{hom} \to J^3(X) \tag{4.55}$$

which is an isomorphism by Theorem 2.21 when X is a smooth projective complex manifold with $\mathrm{CH}_0(X) = \mathbb{Z}$. The right hand side is an abelian variety but the left hand side is not an algebraic variety, even if it is more than an abstract group. Namely, we can use the families of codimension 2 algebraic cycles on X given by codimension 2 cycles $\mathcal{Z} \in \mathrm{CH}^2(B \times X)$ parameterized by smooth connected varieties B and the associated maps $\mathcal{Z}_* : B \to \mathrm{CH}^2(X)_{alg}$, $b \mapsto \mathcal{Z}_b - \mathcal{Z}_{b_0}$, where $b_0 \in B$ is a fixed reference point, to say that ϕ^2_X is a "regular homomorphism". This notion was introduced by Murre [45] and it says that for any cycle $\mathcal{Z}$ as above, the map

$$\phi_\mathcal{Z} := \phi^2_X \circ \mathcal{Z}_* : B \to J^3(X)$$

is a morphism of algebraic varieties.

The question left open concerning the isomorphism (4.55) is the existence of a *universal codimension* 2 *cycle*, which was first asked in [61]:

Definition 4.1. *A universal codimension* 2 *cycle for X is a codimension* 2 *cycle* $\mathcal{Z} \in \mathrm{CH}^2(J^3(X) \times X)$ *such that* $\mathcal{Z}_0 = 0$ *and the associated map*

$$\phi_{\mathcal{Z}} : J^3(X) \to J^3(X)$$

is the identity.

For codimension 1 cycles, the universal cycle exists and is called the Poincaré divisor. Its existence in this case can be proved using the fact that the complete family of sufficiently ample divisors of given cohomology class on X is via ϕ_X^1 a honest projective bundle on $J^1(X)$. Indeed, the fiber over a divisor class L is the projective space $|L|$ and a point $x \in X$ determines for any L the hyperplane $|L|_x \subset |L|$ of divisors in $|L|$ passing through x.

We will see in next section that, as a consequence of the degeneration method, there are Fano threefolds which do not admit a universal codimension 2 cycle. Note that, once one knows that the Abel-Jacobi map ϕ_X^2 is surjective, there exists a codimension 2 cycle $\mathcal{Z} \in \mathrm{CH}^2(J^3(X) \times X)$ such that $\mathcal{Z}_0 = 0$ and the associated map

$$\phi_{\mathcal{Z}} : J^3(X) \to J^3(X)$$

is N times the identity for a certain integer $N > 0$. Indeed, we use for this the fact that ϕ_X^2 is regular. There are countably many complete families of codimension 2 algebraic cycles on X, so the surjectivity of the Abel-Jacobi map implies that there exist a smooth projective variety B and a cycle $\mathcal{Z} \in \mathrm{CH}_2(B \times X)$, such that the morphism

$$\phi_{\mathcal{Z}} : B \to J^3(X)$$

is surjective. We can replace B by a subvariety B' containing the reference point b_0 such that the restriction ϕ' of $\phi_{\mathcal{Z}}$ to B' is a generically finite map. Then we consider the cycle

$$\mathcal{Z}_J = (\phi', Id_X)_*(\mathcal{Z}') \in \mathrm{CH}^2(J^3(X) \times X),$$

where $\mathcal{Z}' := \mathcal{Z}'_{|B' \times X}$. The integer N obtained by this construction is $\deg \phi'$.

The existence of a universal cycle for codimension 1 cycles allows us to prove the following result:

Proposition 4.2. *[61] If X has a cohomological decomposition of the diagonal, X has a universal codimension* 2 *cycle.*

Proof. We write the decomposition in the form

$$[\Delta_X] = [X \times x] + (\tilde{j}, Id_X)_*[\widetilde{Z}] \text{ in } H_B^{2n}(X \times X, \mathbb{Z}), \tag{4.56}$$

where $\tilde{j} : \widetilde{D} \to X$ is a morphism from a smooth projective variety of dimension $n-1$. As we used several times, this implies that for any $\alpha \in H_B^3(X, \mathbb{Z})$

$$\alpha = \tilde{j}_*([\widetilde{Z}]^* \alpha). \tag{4.57}$$

The considered morphisms are morphisms of Hodge structures of odd weight and they induce as well morphisms between the associated intermediate Jacobians. (4.57) then says that

$$\tilde{j}_* \circ [\widetilde{Z}]^* = Id_{J^3(X)} : J^3(X) \to J^3(X). \tag{4.58}$$

Let now $\mathcal{D} \in \mathrm{CH}^1(J^1(\widetilde{D}) \times \widetilde{D})$ be a universal codimension 1-cycle. By pull-back to $J^3(X)$ it provides a codimension 1-cycle on $J^3(X) \times \widetilde{D}$ and by push-forward to X, we get finally a codimension 2 cycle on $J^3(X) \times X$ defined by the formula

$$\mathcal{Z} = (Id_{J^3(X)}, \tilde{j})_*(([\widetilde{Z}]^*, Id_{\widetilde{D}})^* \mathcal{D}).$$

The map $\phi_{\mathcal{Z}} : J^3(X) \to J^3(X)$ equals by construction $\tilde{j}_* \circ [\widetilde{Z}]^*$, hence it is the identity of $J^3(X)$ by (4.58). $\square$

4.2 Extending Clemens-Griffiths criterion

The discussion in this section is specific to dimension 3, although it concerns stable rationality for them. The stable rationality of X says that $X \times \mathbb{P}^r$ is rational for some r, hence it involves birational geometry of higher dimensional varieties. Because of this, the Clemens-Griffiths criterion that we now describe concerns only rationality of threefolds and not stable rationality.

Let X be a smooth complex projective threefold. Let us assume that $H^1_B(X,\mathbb{Z}) = 0$ and $H^{3,0}(X) = 0$, which will be the case if X is rationally connected. Consider the intermediate Jacobian $J^3(X) = H^3_B(X,\mathbb{C})/(F^2H^3_B(X,\mathbb{C}) \oplus H^3_B(X,\mathbb{Z})_{tf})$, which in this case equals $H^{1,2}(X)/H^3_B(X,\mathbb{Z})_{tf}$. Here $H^3_B(X,\mathbb{Z})_{tf}$ denotes the abelian group $H^3_B(X,\mathbb{Z})$ modulo torsion. By definition, one has a canonical isomorphism

$$H_1^B(J^3(X),\mathbb{Z}) \cong H^3_B(X,\mathbb{Z})_{tf}. \tag{4.59}$$

The unimodular intersection pairing $\langle\, ,\, \rangle_X$ on $H^3_B(X,\mathbb{Z})_{tf}$ provides, thanks to the Hodge-Riemann relations, a principal polarization on $J^3(X)$ of class $\theta_X \in H^2_B(J^3(X),\mathbb{Z})$. If $g = \dim J^3(X)$, the integral degree $2g-2$ cohomology class (or degree 2 homology class) $\frac{\theta_X^{g-1}}{(g-1)!}$ on $J^3(X)$ is called the minimal class. It is not known if it is algebraic for a general principally polarized abelian variety (A,θ_A), although it is when $(A,\theta_A) = (J^1(C),\theta_C)$ is the Jacobian of a smooth projective curve, or a product of them. The celebrated Clemens-Griffiths criterion [15] says the following:

Theorem 4.3. *If a smooth projective threefold X is rational, then $(J^3(X),\theta_X)$ is the direct product of Jacobians $(J^1(C_i),\theta_{C_i})$ of curves.*

This theorem follows indeed from the fact that a principally polarized abelian variety splits uniquely into a direct sum of simple principally polarized abelian varieties. Furthermore the Jacobian of a smooth projective curve is indecomposable as a ppav, by Riemann's theorem which implies that its Theta divisor is irreducible. This decomposition changes under blow-up of a curve $C \subset X$ by the addition of an orthogonal direct summand which is the Jacobian of C. We then conclude that the Griffiths component of X, namely the sum in the decomposition above of all summands not isomorphic as ppav's to Jacobians of curves, does not change under blow-up and thus is a birational invariant (one also uses the fact that if $\phi : X \to Y$ is a morphism which is birational, that is, of degree 1, the morphism $\phi^* : H^3_B(Y,\mathbb{Z}) \to H^3_B(X,\mathbb{Z})$ is compatible with polarizations and thus makes $H^3(Y,\mathbb{Z})$ an orthogonal direct summand of $H^3_B(X,\mathbb{Z})$).

In the Jacobian $J^1(C)$ of a curve C, the image of C by the Albanese map gives an effective 1-cycle Z whose class is the minimal class. The Matsusaka criterion [40] says the following:

Theorem 4.4. *A principally polarized abelian variety (A,θ_A) is a product of Jacobians of curves if and only if it carries an* effective *1-cycle $Z = \sum_i n_iC_i$, $n_i > 0$ whose class $[Z] \in H_2^B(A,\mathbb{Z})$ is the minimal class.*

The following result proved in [63] is thus a version of Clemens-Griffiths theorem for stable rationality.

Theorem 4.5. *Let X be a smooth projective threefold. If X has a cohomological decomposition of the diagonal, the minimal class $\frac{\theta_X^{g-1}}{(g-1)!}$ of $J^3(X)$ is algebraic. In particular, if X is stably rational, the minimal class $\frac{\theta_X^{g-1}}{(g-1)!}$ of $J^3(X)$ is algebraic.*

This condition says that there is a 1-cycle $Z = \sum_i n_iC_i$ whose class $[Z] \in H_2^B(J^3(X),\mathbb{Z})$ is the minimal class. The difference with Clemens-Griffiths criterion is that we do not ask it to be effective.

Proof of Theorem 4.5. Recalling the isomorphism (4.59), or rather its dual

$$i : H^1_B(J^3(X),\mathbb{Z}) \cong H^3_B(X,\mathbb{Z})_{tf}, \tag{4.60}$$

the minimal class $\gamma \in H_2^B(J^3(X),\mathbb{Z})$ is characterized by the fact that

$$\int_\gamma \alpha \cup \beta = \langle i(\alpha), i(\beta)\rangle_X \tag{4.61}$$

for any α, $\beta \in H^1_B(J^3(X),\mathbb{Z})$. We now assume that X has a cohomological decomposition of the diagonal

$$[\Delta_X] = \widetilde{j}_*([\widetilde{Z}]) + [X \times x] \text{ in } H^6_B(X \times X, \mathbb{Z}) \tag{4.62}$$

for some cycle $\widetilde{Z} \in \mathrm{CH}^2(\widetilde{D} \times X)$. Recalling that $\widetilde{D}$ is the desingularization of a divisor in X, we can assume after blowing-up X that $\widetilde{D} = \sqcup D_i$, where each $j_i = \widetilde{j}_{|D_i}$ is an embedding and that $\widetilde{j}(\widetilde{D})$ has normal crossings. We denote by Z_i the restriction of $\widetilde{Z}$ to $D_i \times X$, and by W_{il} the curve which is the intersection $j_i(D_i) \cap j_l(D_l)$, that we can see as a divisor in either surface D_i or D_l. Formula (4.62) gives for any $\alpha \in H^3_B(X,\mathbb{Z})_{tf}$

$$\alpha = \sum_i j_{i*}([Z_i]^*\alpha).$$

It follows that, for any $\alpha \in H^3_B(X,\mathbb{Z})_{tf}$,

$$\begin{aligned}
\langle \alpha, \beta\rangle_X &= \sum_{il} \langle j_{i*}([Z_i]^*\alpha), j_{l*}([Z_l]^*\beta)\rangle_X \qquad (4.63)\\
&= \sum_{i\neq l} \int_{W_{il}} [Z_i]^*\alpha \cup [Z_l]^*\beta + \sum_i \langle j_{i*}([Z_i]^*\alpha), j_{i*}([Z_i]^*\beta)\rangle_X \\
&= \sum_{i<l} \int_{W_{il}} ([Z_i]^*\alpha \cup [Z_l]^*\beta + [Z_l]^*\alpha \cup [Z_i]^*\beta) + \sum_i \int_{D_i} j_i^*[D_i] \cup [Z_i]^*\alpha \cup [Z_i]^*\beta \\
&= \sum_{i<l} \int_{W_{il}} ([Z_i]+[Z_l])^*\alpha \cup ([Z_i]+[Z_l])^*\beta \\
&\quad - \sum_{i<l} \int_{W_{il}} ([Z_i]^*\alpha \cup [Z_i]^*\beta + [Z_l]^*\alpha \cup [Z_l]^*\beta) + \sum_i \int_{W_i} [Z_i]^*\alpha \cup [Z_i]^*\beta,
\end{aligned}$$

where in the last term W_i is the 1-cycle $j_i^* D_i$ of D_i. The conclusion of (4.63) is that we found smooth projective curves C_s (namely the W_{il}'s and the supports of the W_i's), integers n_s (namely the coefficients of the 1-cycle W_i) and codimension 2 cycles $Z'_s \in \mathrm{CH}^2(C_s \times X)$, such that

$$\langle \alpha, \beta\rangle_X = \sum_s n_s \langle [Z'_s]^*\alpha), [Z'_s]^*\beta\rangle_{C_s}. \tag{4.64}$$

Let $\phi_{Z'_s} : C_s \to J^3(X)$ be the associated Abel-Jacobi map. For any class $\eta \in H^1_B(J^3(X),\mathbb{Z})$ the class $\alpha = i(\eta)$ satisfies by definition of the isomorphism i:

$$\phi^*_{Z'_s}\eta = [Z'_s]^*\alpha \text{ in } H^1_B(C_s,\mathbb{Z}). \tag{4.65}$$

Thus (4.64) rewrites as

$$\langle \alpha, \alpha'\rangle_X = \sum_s n_s \int_{C_s} \phi^*_{Z'_s}\eta \cup \phi^*_{Z'_s}\eta' \tag{4.66}$$

for any η, $\eta' \in H^1_B(J^3(X),\mathbb{Z})$. Comparing with (4.61), we conclude that $\sum_s n_s[\phi_{Z'_s}(C_s)] = \gamma$ is the minimal class. □

We now give a necessary and sufficient set of conditions for a smooth projective threefold to admit a cohomological decomposition of the diagonal. Part of these results were obtained in [61], and they were finally completed in [63]. We assume that X has $H^{3,0}(X) = 0$ and $H^1_B(X, \mathbb{Z}) = 0$ because this is necessary for X to have a cohomological decomposition with rational coefficients. These vanishing conditions allow us to speak of the ppav $(J^3(X), \theta_X)$.

Theorem 4.6. *A smooth complex projective threefold with $H^{3,0}(X) = 0$ and $H^1_B(X, \mathbb{Z}) = 0$ admits a cohomological decomposition of the diagonal if and only if the following conditions are satisfied:*

1. *$H^*_B(X, \mathbb{Z})$ has no torsion.*
2. *$H^4_B(X, \mathbb{Z})$ is algebraic.*
3. *X admits a universal codimension 2 cycle.*
4. *The minimal class of $(J^3(X), \theta_X)$ is algebraic.*

Proof. We already proved that these conditions are necessary: 1 and 2 were proved to be necessary in Theorem 2.13. 3 is necessary by Proposition 4.2 and 4 is necessary by Proposition 4.5.

We now prove that these conditions are sufficient. If X satisfies these conditions, then by 1, X has a Künneth decomposition of the diagonal

$$[\Delta_X] = \delta_{6,0} + \delta_{5,1} + \delta_{4,2} + \delta_{3,3} + \delta_{2,4} + \delta_{0,6}$$

where $\delta_{i,j} \in H^i_B(X, \mathbb{Z}) \otimes H^j_B(X, \mathbb{Z})$ and acts as the projector on $H^j_B(X, \mathbb{Z})$.

As we assumed $H^1_B(X, \mathbb{Z}) = 0$, $\delta_{1,5}$ and $\delta_{5,1}$ are zero. Assuming 2, the even degree cohomology of X is algebraic, since this implies $H^{2,0}(X) = 0$, so that $H^2_B(X, \mathbb{Z})$ is also algebraic by Lefschetz. It follows that the terms $\delta_{6,0}$, $\delta_{4,2}$, $\delta_{2,4}$ can be written as $\sum_i n_i[Z_i \times Z'_i]$, where $\operatorname{codim} Z_i > 0$. They are thus contained in $D \times X$ for some closed proper algebraic subset D of X. Finally the term $\delta_{0,6}$ is the class of $X \times x$.

It thus remains to show that the class $\delta_{3,3}$ is the class of a cycle supported on $D \times X$ for some closed proper algebraic subset D of X, and by the previous analysis of the other Künneth terms, it suffices in fact that there is a cycle supported on $D \times X$ for some closed proper algebraic subset D of X whose Künneth component of type $(3, 3)$ is $\delta_{3,3}$.

Let $\Gamma = \sum_i n_i C_i$ be a 1-cycle of $J^3(X)$ representing the minimal class. Let $Z \in \mathrm{CH}^2(J^3(X) \times X)$ be a universal codimension 2 cycle and let $Z_i \in \mathrm{CH}^2(\widetilde{C}_i \times X)$ be its pull-back to $\widetilde{C}_i \times X$, where $\widetilde{C}_i$ is the normalization of C_i. We consider the following cycle

$$T := \sum_i n_i (Z_i, Z_i)_* \Delta_{\widetilde{C}_i} \text{ in } \mathrm{CH}^3(X \times X), \tag{4.67}$$

where as usual $\Delta_{\widetilde{C}_i}$ is the diagonal of $\widetilde{C}_i$ and

$$(Z_i, Z_i) := pr_{13}^* Z_i \cdot pr_{24}^* Z_i \in \mathrm{CH}^4(\widetilde{C}_i \times \widetilde{C}_i \times X \times X).$$

Observe that each Z_i is of dimension 2, so that the support of Z_i does not dominate X by the second projection. It follows that T is supported on $D \times X$ for some closed proper algebraic subset D of X. We now have:

Lemma 4.7. *The $(3, 3)$-Künneth component of T is equal to $\delta_{3,3}$.*

Proof. We have to show that for any α, $\beta \in H^3_B(X, \mathbb{Z})$

$$\langle [T]^* \alpha, \beta \rangle_X = \langle \alpha, \beta \rangle_X. \tag{4.68}$$

We claim that for any curve C and any codimension 2 cycle Z in $C \times X$, one has, denoting $Z' := (Z,Z)_*\Delta_C$,

$$\langle [Z']^*\alpha, \beta\rangle_X = \langle [Z]^*\alpha, [Z]^*\beta\rangle_C. \tag{4.69}$$

Assuming this equality, we get

$$\langle [T]^*\alpha, \beta\rangle_X = \sum_i n_i \langle [Z_i]^*\alpha, [Z_i]^*\beta\rangle_{\widetilde{C}_i}$$

$$= \sum_i n_i \langle j_i^*([Z]^*\alpha), j_i^*([Z]^*\beta)\rangle_{\widetilde{C}_i},$$

where $j_i : \widetilde{C}_i \to J^3(X)$ is the natural map. As Z is a universal cycle, one has $[Z]^* = i^{-1}$ and thus, as $\sum_i n_i j_{i*}([\widetilde{C}_i])$ is the minimal class, the last term is $\langle \alpha, \beta\rangle_X$ by (4.61).

It remains to prove (4.69). This follows from the fact that

$$[Z,Z]^*(\alpha\otimes\beta) = [Z]^*\alpha \otimes [Z]^*\beta \text{ in } H^1_B(C\times C, \mathbb{Z}), \tag{4.70}$$

where $\alpha\otimes\beta := pr_1^*\alpha \cup pr_2^*\beta$ for both X and C.

It follows from (4.70) that

$$\langle [\Delta_C], [Z]^*\alpha\otimes[Z]^*\beta\rangle_{C\times C} = \langle [\Delta_C], [Z,Z]^*(\alpha\otimes\beta)\rangle_{C\times C}$$

$$= \langle [Z,Z]_*([\Delta_C]), \alpha\otimes\beta\rangle_{X\times X} = \langle [Z'], \alpha\otimes\beta\rangle_{X\times X}.$$

The last term is easily seen to be $\langle [Z']^*\alpha, \beta\rangle_X$. ☐

The proof of Theorem 4.6 is finished. ☐

In the case of rationally connected threefolds, we know that $H^2_B(X,\mathbb{Z})$ and $H^5_B(X,\mathbb{Z})$ have no torsion by Theorem 1.2. We also know that $H^4_B(X,\mathbb{Z})$ is algebraic by Theorem 1.29. We thus get in this case

Theorem 4.8. *A smooth complex projective rationally connected threefold admits a cohomological decomposition of the diagonal if and only if the following conditions are satisfied:*

1. *$H^3_B(X,\mathbb{Z})$ has no torsion.*
2. *X admits a universal codimension 2 cycle.*
3. *The minimal class of $(J^3(X), \theta_X)$ is algebraic.*

It is interesting to note that 1 is the Artin-Mumford invariant, while 3 is our generalization of Clemens-Griffiths criterion that works for stable rationality. The condition 2 has no obvious classical analogue but in [62], we compute it as "universal degree 3 unramified cohomology" of X. In fact this condition is related to the integral Hodge conjecture for $X \times J^3(X)$ or rather its $(3,1)$-Künneth component.

We now deduce the following consequence:

Corollary 4.9. *There are rationally connected threefolds not admitting a universal codimension 2 cycle.*

Proof. The example is the desingularization of a very general quartic double solid with 7 nodes. It does not admit a cohomological decomposition of the diagonal by Theorem 3.6. On the other hand, its intermediate Jacobian has dimension 3, so it is a Jacobian and the minimal class is algebraic. Finally it has trivial Artin-Mumford invariant by work of Endrass [21]. The condition that fails in Theorem 4.8 must thus be Condition 2. ☐

4.3 The case of cubic hypersurfaces

The rationality or stable rationality of cubic hypersurfaces is an almost completely open problem. The results available are:

• *A smooth plane cubic is not rational* as it has $H^{1,0} \neq 0$.

• *A smooth cubic surface X over an algebraically closed field is rational*: This is a particular case of Castelnuovo theorem but it can be proved explicitly in this case: take any two not intersecting lines Δ, Δ' in X. Then for $x \in \Delta$, $x' \in \Delta'$, the line $\langle x, x' \rangle$ in $\mathbb{P}^3$ meets X in a third point $\phi(x, x')$. This defines a birational map

$$\phi : \Delta \times \Delta' \dashrightarrow X.$$

The inverse map is constructed as follows: start from a general point $y \in X$, and let $Q_y := \langle y, \Delta \rangle$, $Q'_y := \langle y, \Delta' \rangle$. Then $Q_y \cap \Delta' = \{x'\}$, $Q'_y \cap \Delta = \{x\}$ and $\phi(x, x') = y$. This construction shows more generally:

• *Any smooth cubic hypersurface of dimension $2m$ containing two m-planes P, P' which do not meet is rational.*

• *A smooth cubic threefold is not rational.* This is the celebrated Clemens-Griffiths theorem, proved in [15] for cubics defined over $\mathbb{C}$, and by Murre (see [46]) in any nonzero characteristic different from 2.

This is essentially all that we know about cubic hypersurfaces. Let us state a few open questions:

Question 4.10. *Does there exist a smooth cubic hypersurface of odd dimension which is rational or stably rational?*

Question 4.11. *Is a smooth cubic threefold stably irrational? Does there exist a stably rational smooth cubic threefold?*

Question 4.12. *Is a very general smooth cubic hypersurface of even dimension ≥ 4 irrational?*

In this section, we are going to study the weaker question whether a smooth cubic hypersurface has a decomposition of the diagonal. We will see that even for cubic threefolds, this already rises serious difficulties.

4.3.1 General cubic hypersurfaces

The following construction is certainly classical. In dimension 1, it allows to construct the group structure on a plane cubic curve. Let X be a smooth cubic hypersurface in $\mathbb{P}^n$. The variety $F(X)$ of lines in X is smooth of dimension $2n-6$. If $x \in X$ and l is a line in $\mathbb{P}^n$ passing through x, then $l \cap X$ contains x and two residual points y, $z \in X$. Conversely, starting from two points y, z in X, the line $l_{y,z} := \langle y, z \rangle$ intersects X in a third point $x \in X$. This shows that there is a birational map

$$\Phi : X^{[2]} \dashrightarrow Q_X,$$

where $Q_X \to X$ is the projective bundle with fiber over x the $\mathbb{P}^{n-1}$ parameterizing lines in $\mathbb{P}^n$ passing through x. The following is proved in [63], see also [23]:

Proposition 4.13. *The map Φ induces an isomorphism between the blow-up of $X^{[2]}$ along $C(X)$ and the blow-up of Q_X along Q_{XX}.*

Here the loci $C(X)$ and Q_{XX} are defined as follows:

- $C(X) \subset X^{[2]}$ is the locus of length 2 subschemes of X that are contained in a line contained in X. Thus $C(X)$ is a $\mathbb{P}^2$-bundle over $F(X)$.

- The locus $Q_{XX} \subset Q_X$ is the set of pairs $(x, [l])$, such that $x \in l$ and the line l is contained in X. It is thus naturally isomorphic to the universal $\mathbb{P}^1$-bundle over $F(X)$.

We now explain two consequences of this proposition. The first one is due to Galkin and Shinder [23] and gives a beautiful evidence for the link between rationality of cubic fourfolds and $K3$ surfaces which has been proposed and studied in [26] and more recently explicitly conjectured and studied in [36], [2] , [1]. Let $K_0(\mathrm{Var}_K)$ be the Grothendieck ring whose generators are isomorphism classes of algebraic varieties defined over K, with relation

$$[U] + [Z] = [X] \tag{4.71}$$

whenever $X = U \sqcup Z$, with Z closed, U open. The ring structure is given by product. Denote by $\mathbb{L} \in K_0(\mathrm{Var}_K)$ the class of the affine line and $\langle \mathbb{L} \rangle$ the ideal of $K_0(\mathrm{Var}_K)$ generated by $\mathbb{L}$. The following result is proved in [38].

Theorem 4.14. *[38] Let K be a field of characteristic zero. The quotient-ring $K_0(\mathrm{Var}_K)/\langle \mathbb{L} \rangle$ is naturally isomorphic to the free abelian group generated by stable birational equivalence classes of smooth projective connected varieties over K together with its natural ring structure. In particular, if X and $Y_1, \ldots, Y_m$ are smooth projective connected varieties and*

$$[X] = \sum_i n_i [Y_i] \text{ in } K_0(\mathrm{Var}_K)/\langle \mathbb{L} \rangle$$

for some $n_i \in \mathbb{Z}$, then X is stably birationally equivalent to one of the Y_i's.

The class of $\mathbb{P}^n$ is equal to $\sum_{i=0}^n \mathbb{L}^i$, as one argue by induction using (4.71) and

$$\mathbb{P}^n \setminus \mathbb{P}^{n-1} = \mathbb{A}^n = (\mathbb{A}^1)^n.$$

Similarly, one gets that the class of a projective bundle $\mathbb{P}(E) \to X$ with $\operatorname{rank} E = r$ is given by

$$[\mathbb{P}(E)] = [\mathbb{P}^{r-1}][X] = (\sum_{i=0}^{r-1} \mathbb{L}^i)[X]. \tag{4.72}$$

For the blow-up $\widetilde{X}$ of a smooth variety X along a smooth subvariety Z of codimension r, the isomorphism $\widetilde{X} \setminus E \cong X \setminus Z$ gives by (4.71)

$$[\widetilde{X}] - (\sum_{i=0}^{r-1} \mathbb{L}^i)[Z] = [X] - [Z]$$

or equivalently

$$[\widetilde{X}] = [X] + (\sum_{i=1}^{r-1} \mathbb{L}^i)[Z]. \tag{4.73}$$

The following result is due to Galkin and Shinder [23].

Theorem 4.15. *Let X be a smooth cubic hypersurface in $\mathbb{P}^n_K$. Then the following equality holds in $K_0(\mathrm{Var}_K)$:*

$$[X^{(2)}] - \mathbb{L}^2[F(X)] = [X](1 + \mathbb{L}^{n-1}). \tag{4.74}$$

If rmchar $K =$, and X is rational of dimension 4, then either there is an explicit nonzero element in $K_0(\mathrm{Var}_K)$ annihilated by $\mathbb{L}^2$, or the Fano variety of lines is birational to $S^{[2]}$, where S is a $K3$ surface.

Proof. Applying Proposition 4.13 and the projective bundle and blow-up formulas (4.72), (4.73), we get

$$[X^{[2]}] + \mathbb{L}(1+\mathbb{L}+\mathbb{L}^2)[F(X)] = [X](\sum_{i=0}^{n-1}\mathbb{L}^i) + (\mathbb{L}+\mathbb{L}^2)(1+\mathbb{L})[F(X)]. \tag{4.75}$$

Now we notice that the difference $X^{[2]}\setminus E_X$ is isomorphic to $X^{(2)}\setminus X$, where $X\subset X^{(2)}$ is the diagonal and $E_X\to X$, $E_X\subset X^{[2]}$, is the exceptional divisor over the diagonal. Plugging again (4.71) and the projective bundle formula for E_X in formula (4.75), we get (4.74).

We now turn to the proof of the second statement. We observe that the symmetric product operation $s^{(2)}:[Y]\mapsto[Y^{(2)}]$ satisfies the following property

$$s^{(2)}([Y]+[Y']) = s^{(2)}([Y]) + s^{(2)}([Y']) + [Y]\cdot[Y'], \tag{4.76}$$

as this is the case for disjoint unions. We also have $s^{(2)}(\mathbb{L}[Y]) = \mathbb{L}^2 s^{(2)}([Y])$ as follows from the fact that $\mathbb{A}^{(2)} = \mathbb{A}^2$ in $K_0(\mathrm{Var}_K)$.

Suppose now that a smooth cubic fourfold X is rational. Then by a sequence of smooth blow-ups starting from X, one gets something isomorphic to a variety Y which is also obtained from $\mathbb{P}^4$ by a sequence of smooth blow-ups. Let us assume for simplicity that we blew-up only surfaces S_i on the X side and T_j on the $\mathbb{P}^4$ side. Then in $K_0(\mathrm{Var}_K)$, we get using (4.73)

$$[X] + \sum_i \mathbb{L}[S_i] = [\mathbb{P}^4] + \sum_j \mathbb{L}[T_j]. \tag{4.77}$$

Taking symmetric products and applying (4.76), we get

$$\begin{aligned}
&[X^{(2)}] + \sum_i \mathbb{L}^2[S_i^{(2)}] + \sum_{i\neq i'}\mathbb{L}^2[S_i][S_{i'}] + \sum_i \mathbb{L}[X][S_i] \\
&= [(\mathbb{P}^4)^{(2)}] + \sum_j \mathbb{L}^2[T_j^{(2)}] + \sum_{j\neq j'}\mathbb{L}^2[T_j][T_{j'}] + \sum_j \mathbb{L}[\mathbb{P}^4][T_j].
\end{aligned}$$

We now replace in this formula $[X^{(2)}]$ by its expression given in formula (4.74) and get

$$\begin{aligned}
&\mathbb{L}^2[F(X)] + [X](1+\mathbb{L}^4) + \sum_i \mathbb{L}^2[S_i^{(2)}] + \sum_{i\neq i'}\mathbb{L}^2[S_i][S_{i'}] + \sum_i \mathbb{L}[X][S_i] \\
&= [(\mathbb{P}^4)^{(2)}] + \sum_j \mathbb{L}^2[T_j^{(2)}] + \sum_{j\neq j'}\mathbb{L}^2[T_j][T_{j'}] + \sum_j \mathbb{L}[\mathbb{P}^4][T_j].
\end{aligned}$$

Using again (4.77) in the form

$$[X] = 1 + \mathbb{L} + \sum_i \mathbb{L}[S_i] - \sum_j \mathbb{L}[T_j] \text{ modulo } \mathbb{L}^2,$$

and the relation

$$[(\mathbb{P}^4)^{(2)}] = [\mathbb{P}^4] + \mathbb{L}^2 \text{ modulo } \mathbb{L}^3,$$

we get after simplification

$$\begin{aligned}
\mathbb{L}^2([F(X)] - \sum_{i<i'}[S_i][S_{i'}] - \sum_j [T_j^{(2)}] - \sum_{j\neq j'}[T_j][T_{j'}] - 1 \\
- \sum_i [S_i] + \sum_{i,j}[S_i][T_j] - \sum_j [T_j] + \mathbb{L}\alpha) = 0
\end{aligned} \tag{4.78}$$

for some $\alpha \in K_0(\mathrm{Var}_K)$. We thus conclude that either the class $[F(X)] - \sum_{i<i'}[S_i][S_{i'}] - \sum_j [T_j^{(2)}] - \sum_{j\neq j'}[T_j][T_{j'}] - 1 - \sum_i [S_i] + \sum_{i,j}[S_i][T_j] - \sum_j [T_j] + \mathbb{L}\alpha$ is nonzero in $K_0(Var)$ but annihilated by $\mathbb{L}^2$, or the following relation holds in $K_0(\mathrm{Var}_K)/\langle\mathbb{L}\rangle$:

$$[F(X)] - \sum_{i<i'}[S_i][S_{i'}] - \sum_j [T_j^{(2)}] - \sum_{j\neq j'}[T_j][T_{j'}] - 1 \qquad (4.79)$$
$$- \sum_i [S_i] + \sum_{i,j}[S_i][T_j] - \sum_j [T_j] = 0.$$

In the latter case, as this provides an equality

$$[F(X)] = \sum_{i<i'}[S_i][S_{i'}] + \sum_j [T_j^{(2)}] + \sum_{j\neq j'}[T_j][T_{j'}] + [\mathbb{P}^4] + \sum_i [S_i \times \mathbb{P}^2] - \sum_{i,j}[S_i][T_j] + \sum_j [T_j \times \mathbb{P}^2]$$

in $K_0(Var)/\langle\mathbb{L}\rangle$ of combinations of classes of four-dimensional varieties, and that $F(X)$ is irreducible, we conclude by Theorem 4.14 that $F(X)$ is stably birational to one of the terms appearing on the right. Using the fact that $F(X)$ has a unique nondegenerate holomorphic 2-form (see [9]), hence is not rationally connected, we easily conclude that $F(X)$ is in fact birational to $[T_j^{(2)}]$ for some smooth surface T_j with $h^{2,0}(T_j) = 1$. Finally one concludes by surface classification that T_j is birational to a $K3$ surface. □

Theorem 4.15 does not allow to conclude that a very general smooth cubic fourfold is not rational because multiplication by $\mathbb{L}$ is not injective as shown by Borisov [13]. Still it gives a beautiful evidence for the relationship between rationality and the existence of an associated $K3$ surface.

We now turn to another application of Proposition 4.13, which can be found in [63].

Theorem 4.16. *Let X be a smooth cubic hypersurface. Assume that X satisfies the Hodge conjecture for integral Hodge classes modulo 2. Then X has a Chow decomposition of the diagonal if and only if it has a cohomological decomposition of the diagonal.*

Note that the assumption is satisfied by odd dimensional cubic hypersurfaces, because three times their even degree cohomology comes from projective space by Lefschetz theorem on hyperplane sections, hence is algebraic. It is also true for cubic fourfolds by [60] (the later result has been reproved recently by Mongardi and Ottem in [43]).

Sketch of proof of Theorem 4.16. First of all we show the following result, which uses our assumption on integral Hodge classes and also the fact that the integral cohomology of X has no torsion. We will denote by $\Gamma \subset X \times X \times X^{[2]}$ the graph of the natural rational map $X^2 \dashrightarrow X^{[2]}$.

Proposition 4.17. *If a smooth cubic hypersurface X has a cohomological decomposition of the diagonal, there exists a cycle W cohomologous to 0 in $X^{[2]}$ such that*

$$\Delta_X - X \times x - Z = \Gamma^* W \text{ in } \mathrm{CH}(X \times X), \qquad (4.80)$$

where as usual Z is supported on $D \times X$, with $D \subset X$ proper closed algebraic.

The difficulty to achieve (4.80) is the following: our assumption is that $\Delta_X - X \times x - Z$ is cohomologous to 0, and we can also arrange to make this cycle symmetric, hence coming from a cycle on $X^{[2]}$. The point of (4.80) is that we want it to come from a cycle which is also cohomologous to 0 on $X^{[2]}$.

Having Proposition 4.17, we now use Proposition 4.13 which allows us to analyze the cycle W. In the case of the cubic threefold, the proof is very short and as follows: We know that after blow-up, $X^{[2]}$ becomes isomorphic to the blow-up of a projective bundle over X along a subvariety which is a projective bundle over the surface $F(X)$. Both X and $F(X)$

have trivial Griffiths groups in all dimensions, hence it follows from the blow-up formulas that $X^{[2]}$ also has trivial Griffiths groups. Hence the cycle W which is given by Proposition 4.80 is algebraically equivalent to 0 on $X^{[2]}$. This means that the equality $\Delta_X - X\times x - Z = 0$ holds modulo algebraic equivalence. We can then apply Proposition 2.24 and conclude that X has a Chow decomposition of the diagonal. $\square$

Let us mention the following application of Theorem 4.16 (see [63] for the proof).

Proposition 4.18. *A cubic fourfold X such that $\mathrm{Hdg}^4(X,\mathbb{Z})$ has rank 2 and discriminant not divisible by 4 has universally trivial CH_0-group.*

Here the discriminant is the discriminant of the the restricted intersection pairing $\langle\,,\,\rangle_X$ on the rank 2 lattice $\mathrm{Hdg}^4(X,\mathbb{Z})$. Such cubics are said special and were studied first by Hassett [26].

4.3.2 The case of the cubic threefold

Recall from section 4.2 that a smooth projective threefold X with $h^{1,0}(X) = h^{3,0}(X) = 0$ has an associated principally polarized abelian variety $(J^3(X), \theta_X)$ of dimension $g = b_3(X)/2$. The minimal class $\theta_X^{g-1}/(g-1)! \in \mathrm{Hdg}^{2g-2}(J^3(X),\mathbb{Z})$ is an integral Hodge class of degree $2g-2$ (or homology class of degree 2. For $g \geq 4$, it is not known to be algebraic. Note that Mongardi and Ottem made recent progress [43] on the similar problem for hyper-Kähler manifolds, which in some sense are close to abelian varieties.

In the case where $g = 4$, 5, it is known that the generic principally polarized abelian variety (A, θ_A) is a Prym variety $P(\widetilde{C}/C)$, and there is then a copy of the curve $\widetilde{C}$ in A, whose class is twice the minimal class. Many interesting rationally connected threefolds appear as conic bundles over a rational surface, for which the intermediate Jacobian is a Prym variety (see [8]). This applies particularly to cubic threefolds, whose intermediate Jacobian is well-known to be a Prym variety, thanks to the representation of X as a conic bundle.

Theorem 4.19. *[63] A smooth cubic threefold X has a decomposition of the diagonal (or has universally trivial CH_0 group) if and only if the minimal class of $J^3(X)$ is algebraic.*

Proof. By Theorem 4.16, it suffices to prove the result for "cohomological decomposition" instead of "Chow decomposition". We now use Theorem 4.6. It says in particular that the algebraicity of the minimal class is a necessary condition for the existence of a cohomological decomposition of the diagonal. It remains to show that it is also sufficient. We know that $H^*(X,\mathbb{Z})$ has no torsion and that $H^4(X,\mathbb{Z})$ is algebraic, being generated by the class of a line, so the only condition to check is the existence of a universal codimension 2 cycle on X. This follows from the following statement which is taken from [61]:

Proposition 4.20. *Let X be a smooth projective threefold with $h^{1,0}(X) = h^{2,0}(X) = 0$ and such that the minimal class of $J^3(X)$ is algebraic. Then if furthermore there exist a smooth projective variety B and a codimension 2 cycle $Z \in \mathrm{CH}^2(B\times X)$ such that*

$$\phi_Z : B \to J^3(X)$$

is surjective with rationally connected fibers, X has a universal codimension 2 cycle.

Proof. Let $\Gamma = \sum_i n_i C_i$ be a 1-cycle in the minimal class, where $C_i \subset J^3(X)$ are curves, that we can even assume to be smooth. By the Graber-Harris-Starr theorem [24], the map ϕ_Z with rationally connected fibers has sections over each C_i (or a general translate of it). This provides lift $s_i : C_i \to B$, and thus for each C_i, we get a codimension 2 cycle $Z_i \in \mathrm{CH}^2(C_i \times X)$ with the property that

$$\phi_{Z_i} : C_i \to J^3(X) \tag{4.81}$$

is the natural inclusion of C_i into $J^3(X)$.

Let $g := \dim J^3(X)$. As a consequence of the fact that the class of $\Gamma = \sum_i n_i C_i$ is the minimal class, we have

$$\Gamma^{*g} = g! J^3(X). \tag{4.82}$$

Here $*$ is the Pontryagin product on cycles of $J^3(X)$, which is defined by

$$\gamma * \gamma' = \mu_*(\gamma \times \gamma'),$$

where $\mu : J^3(X) \times J^3(X) \to J^3(X)$ is the sum map.

The meaning of equation (4.82) is clear if we assume that $\Gamma = C_i$ is a single curve : it says then that the sum map induces a birational map

$$j_g : C_i^{(g)} \to J^3(X)$$

is birational. In this case, constructing a universal codimension 2 cycle on $J^3(X)$ is easy: namely, starting from Z_i, we construct a codimension 2-cycle on $C_i^g \times X$ defined as $\sum_{i=1}^g ((p_i, p_X)^* Z_i$, which is clearly symmetric, hence descend to a codimension 2 cycle $Z_i^{(g)}$ on $C_i^{(g)} \times X$. One has

$$\phi_{Z_i^{(g)}} = j_g$$

which is birational by assumption, so that via (j_g, Id_X), $Z_i^{(g)}$ descends to a universal codimension 2 cycle on $J^3(X) \times X$.

The general case works similarly, using the curve $C = \sqcup_i C_i$, the cycle Z which is Z_i on the component C_i, and viewing Γ as a 1-cycle on C. $\square$

The cubic threefold satisfies the assumptions of the proposition by work of Markushevich-Tikhomirov [39]. Indeed they show that the Abel-Jacobi map on the family of elliptic curves of degree 5 in X has rationally connected fibers. What they prove is that a general such elliptic curve $E \subset X$ determines a rank 2 vector bundle $\mathcal{E}$ on X with 6 sections. The fiber of the Abel-Jacobi map passing through $[E]$ identifies to $\mathbb{P}(H^0(X, \mathcal{E}))$. Proposition 4.20 thus applies and the theorem is proved. $\square$

Let us mention the following consequence: We already mentioned that twice the minimal class is algebraic on $J^3(X)$. So if $(J^3(X), \theta_X)$ has an odd degree isogeny to $(J(C), \theta_C)$ for some genus 5 curve C, an odd multiple of the minimal class of $J^3(X)$ is algebraic, hence the minimal class itself is algebraic. This condition happens along a sublocus of codimension ≤ 3 in the moduli space of cubic threefolds, because the locus of Jacobians in $\mathcal{A}_5$ has codimension 3. Playing on this observation, we get the following:

Theorem 4.21. *There is a non-empty codimension ≤ 3 locus in the moduli space of cubic threefolds parameterizing cubic threefolds with universally trivial* CH_0*-group.*

The following questions remain open:

Question 4.22. *Does a general cubic threefold admit a universal codimension* 2*-cycle?*

This problem has been rephrased in [62] as computing universal unramified degree 3 cohomology of X with torsion coefficients.

Remark 4.23. The Markushevich-Tikhomirov parameterization of $J^3(X)$ with generic fiber isomorphic to $\mathbb{P}(H^0(X, \mathcal{E}))$ does not solve this problem because the fibration into projective spaces over a Zariski open set of $J^3(X)$ so constructed is a nontrivial Brauer-Severi variety over $\mathbb{C}(J^3(X))$. It does not admit a rational section.

Question 4.24. *Is the minimal class of the intermediate Jacobian $J^3(X)$ algebraic for a general cubic threefold X? Is the minimal class of a general principally polarized abelian variety of dimension* 5 *algebraic?*

Note that the question whether a smooth cubic of large dimension has universally trivial Chow group is also completely open.

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