The Hodge conjecture

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Abstract This is an introduction to the Hodge conjecture, which, although intended to a general mathematical audience, assumes some knowledge of topology and complex geometry. The emphasis will be put on the importance of the notion of Hodge structure in complex algebraic geometry.

1 Introduction

The Hodge conjecture stands between algebraic geometry and complex geometry. It relates data coming from topology (a Betti cohomology class), complex geometry (the Hodge decomposition or filtration) and algebraic geometry (the algebraic subvarieties of a complex algebraic variety). We can state it very quickly by saying that it provides a conjectural characterization of *algebraic classes*, that is cohomology classes generated over $\mathbb{Q}$ by classes of algebraic subvarieties of a given dimension of a complex projective manifold X, as *Hodge classes*, that is those rational cohomology classes of degree $2k$ which admit de Rham representatives which are closed forms of type (k,k) for the complex structure on X. The geometry behind this condition is the fact that the integration current defined by a complex submanifold of dimension $n-k$ annihilates forms of type (p,q) with $(p,q) \neq (n-k,n-k)$.

Not much is known about the Hodge conjecture, apart from the Lefschetz theorem on $(1,1)$-classes (Theorem 2) and a beautiful evidence (Theorem 6) provided by Cattani, Deligne and Kaplan, which says roughly that Hodge classes behave in family as if they were algebraic, that is, satisfied the Hodge conjecture. What we plan to do is to explain the basic notions in Hodge theory (Hodge structure, coniveau) giving a strong motivation for the Hodge conjecture (and still more for its generalization, the generalized Hodge conjecture, see Conjecture 3). The Hodge structures

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on rational cohomology groups are very rich objects associated to a smooth projective complex variety, and the belief is that they carry a lot of qualitative information on the variety: Torelli theorems state that under some assumptions, the isomorphism class of these Hodge structures determine the variety itself. In another direction, the Hodge conjecture is part of a general picture predicting that these Hodge structures determine the "motive" or at least the Chow groups of the variety. We also wish to present some of the most important facts from Hodge theory allowing to prove some conditional results or implications between various subconjectures. Some very important cases of the Hodge conjecture are summarized under the name of standard conjectures (see [21]), the main one being the Lefschetz standard conjecture (Conjecture 2). These instances of the Hodge conjecture concern Hodge classes of a very special type, which satisfy extra arithmetic conditions (being absolute Hodge, see Definition 5) satisfied by algebraic classes but not known to be satisfied by all Hodge classes (see Conjecture 7). An example of such conditional statement underlining the importance of the Lefschetz standard conjecture is Theorem 7 concerning the variational form of the Hodge conjecture which asks whether, starting from a variety X with a Hodge class α which is algebraic, and deforming X in a family $(X_t)_{t\in B}$ in such a way that the class α remains Hodge along the deformation, the class α_t also remains algebraic on X_t.

This paper is organized as follows: in Section 2, we will define Hodge structures, polarizations on them and Hodge classes. In Section 3, we will present the Hodge conjecture, its generalized version, and the few cases in which it is known. We will also discuss the standard conjectures. Finally we will turn in Section 4 to variational aspects of the Hodge conjecture. Sections 3 and 4 use in an essential way the theory of mixed Hodge structures which is summarized in Section 3.3.

This quick presentation of the Hodge conjecture does not contain many examples. It is an invitation to read the book [23] where many specific known cases of the Hodge conjecture are presented.

2 Hodge structures, Hodge classes

2.1 Hodge decomposition

Let X be a complex manifold. The complex structure on X allows to decompose the vector bundle of complex differential 1-forms on X as

$$\Omega_{X,\mathbb{C}} = \Omega_X^{1,0} \oplus \Omega_X^{0,1}, \tag{1}$$

where $\Omega_X^{1,0}$ is the vector bundle of 1-forms which are $\mathbb{C}$-linear for the complex structure on T_X, locally generated by dz_i, the z_i's being local holomorphic coordinates, and $\Omega_X^{0,1} = \overline{\Omega_X^{1,0}}$ is its complex conjugate, locally generated by $d\overline{z_i}$. From (1), we deduce a decomposition of the sheaf of C^∞ complex differential forms of degree k:

$$\mathscr{A}^k_{X,\mathbb{C}} = \oplus_{p+q=k}\mathscr{A}^{p,q}_X, \tag{2}$$

where $\mathscr{A}^{p,q}_X$ is the sheaf of differential forms of type (p,q), which can be written in local holomorphic coordinates z_i as

$$\alpha = \sum_{|I|=p,|J|=q} \alpha_{I,J} dz_I \wedge d\overline{z_J}. \tag{3}$$

It is clear from (3) that the exterior differential $d : \mathscr{A}^k_{X,\mathbb{C}} \to \mathscr{A}^{k+1}_{X,\mathbb{C}}$ satisfies $d\mathscr{A}^{p,q}_X \subset \mathscr{A}^{p+1,q}_X \oplus \mathscr{A}^{p,q+1}_X$. There is thus no reason that the decomposition (2) induces a decomposition on the level of de Rham cohomology, that is on the space

$$H^k(X,\mathbb{C}) = \frac{\mathrm{Ker}\,(d : A^k(X) \to A^{k+1}(X))}{\mathrm{Im}\,(d : A^{k-1}(X) \to A^k(X))}.$$

Here $A^k(X) := \Gamma(X,\mathscr{A}^k_{X,\mathbb{C}})$ is the space of C^∞ complex differential k-forms on X. However, when X is compact Kähler (and a fortiori projective), the Hodge decomposition theorem says the following:

Theorem 1 *(Hodge [17]) If X is a compact Kähler manifold, one has a canonical decomposition*

$$H^k(X,\mathbb{C}) = \oplus_{p+q=k} H^{p,q}(X), \tag{4}$$

where $H^{p,q}(X)$ is the set of de Rham cohomology classes of closed differential forms on X which are of type (p,q).

The simplest consequence of this statement is the following restriction on the topology of compact Kähler manifolds:

Corollary 1 *If k is odd, and X is a compact Kähler manifold, $b_k(X)$ is even.*

Indeed, the definition we gave of $H^{p,q}(X)$ clearly shows that the Hodge decomposition (4) satisfies the *Hodge symmetry* property:

$$\overline{H^{p,q}(X)} = H^{q,p}(X), \tag{5}$$

where complex conjugation acts naturally on $H^k(X,\mathbb{C}) = H^k(X,\mathbb{R}) \otimes \mathbb{C}$. The conclusion of Corollary 1 is not satisfied by the simplest example of non-Kähler compact complex surface, namely the Hopf surface S, which is the quotient of $\mathbb{C}^2 \setminus \{0\}$ by the action of $\mathbb{Z}$ given by multiplication by $\lambda \neq 0$, where $|\lambda| \neq 1$. Indeed, $\pi_1(T) = \mathbb{Z}$ hence $b_1(T) = 1$.

Note on the other hand that by the change of coefficients theorem, we have

$$H^k(X,\mathbb{C}) = H^k(X,\mathbb{Q}) \otimes \mathbb{C}.$$

This leads us to introduce the basic definition of a *Hodge structure of weight k*:

Definition 1 *A rational Hodge structure of weight k is the data of a finite rank $\mathbb{Q}$-vector space L, together with a decomposition*

$$L_{\mathbb{C}} := L \otimes \mathbb{C} = \oplus_{p+q=k} L^{p,q}, \tag{6}$$

where the $L^{p,q} \subset L_{\mathbb{C}}$ *are complex vector subspaces satisfying the Hodge symmetry condition* $\overline{L^{p,q}} = L^{q,p}$.

The data of the Hodge decomposition (6) is equivalent to that of the Hodge filtration (which is a decreasing filtration on $L_{\mathbb{C}}$)

$$F^r L_{\mathbb{C}} := \oplus_{p+q=k, p \geq r} L^{p,q}, \tag{7}$$

since $L^{p,q} = F^p L_{\mathbb{C}} \cap \overline{F^q L_{\mathbb{C}}}$.

Hodge structures coming from geometry are "effective", meaning that $L^{p,q} = 0$ for $p < 0$ or $q < 0$. However it is natural to introduce the dual $(L^*, (L^{p,q})^*)$ of a Hodge structure $(L, L^{p,q})$ of weight k and to give it weight $-k$, so the effectivity condition should not be part of the definition.

Morphisms of Hodge structures $(L, L^{p,q})$ of weight k and $(L', L'^{p,q})$ of weight k' are defined only when $k' = k + 2r$, as the set of morphisms $\phi : L \to L'$ of $\mathbb{Q}$-vector spaces satisfying

$$\phi_{\mathbb{C}}(L^{p,q}) \subset L'^{p+r,q+r}.$$

The Tate twist $L(r)$ of a Hodge structure of weight k is the Hodge structure L' of weight $k - 2r$ which has the same underlying vector space $L' = L$ and Hodge decomposition $L'^{p,q} = L^{p+r,q+r}$. If X is a compact Kähler manifold, Poincaré duality provides an isomorphism of weight $-k$ Hodge structures

$$H^k(X, \mathbb{Q})^* \cong H^{2n-k}(X, \mathbb{Q})(n).$$

If X and Y are compact Kähler manifolds and $\phi : X \to Y$ is a holomorphic map,

$$\phi^* : H^k(Y, \mathbb{Q}) \to H^k(X, \mathbb{Q})$$

is a morphism of Hodge structures since the pull-back by ϕ of a closed form of type (p,q) on X is a closed form of type (p,q) on Y; by Poincaré duality, the Gysin morphism

$$\phi_* : H^k(X, \mathbb{Q}) \to H^{k+2r}(Y, \mathbb{Q}),\ r = \dim Y - \dim X$$

is also a morphism of Hodge structures.

2.2 Hodge structures and polarizations

Given a morphism of Hodge structures $\phi : L \to L'$, it is obvious how to define a Hodge structure on $\mathrm{Ker}\,\phi$ and on $\mathrm{Im}\,\phi$ since morphisms of Hodge structures are those which are bigraded after tensoring by $\mathbb{C}$. Hence rational Hodge structures form an abelian category. However this category is not semi-simple. This phenomenon already appears for weight 1 Hodge structures. An effective weight 1 Hodge structure on L is determined by the choice of vector subspace $L^{1,0} \subset L_{\mathbb{C}}$ which has to be in direct sum with its complex conjugate. Suppose now that $(L, L^{1,0})$ contains a Hodge substructure $L' \subset L$, $L'^{1,0} = L'_{\mathbb{C}} \cap L^{1,0}$. The only condition on the space $L^{1,0}$ determining the Hodge structure on L is that its intersection with $L'_{\mathbb{C}}$ has dimension $\frac{1}{2}\dim L'$. We claim that for a general pair (L', L) of Hodge structures as above, there is no splitting $L = L' \oplus L''$ as Hodge structures. Indeed, there are countably many choices of such splitting over $\mathbb{Q}$, and for a given splitting, the condition that $L'' \subset L$ is also a Hodge structure means that $L^{1,0} \cap L''_{\mathbb{C}}$ has dimension $\frac{1}{2}\dim L''$. The complex dimension of the algebraic subset of the Grassmannian $\mathrm{Grass}(k, 2k)$ parameterizing the Hodge structures on L for which $L' \subset L$ is a Hodge substructure is thus equal to $k'^2 + (k-k')k$ while the algebraic subset of the Grassmannian $\mathrm{Grass}(k, 2k)$ parameterizing the Hodge structures on L for which L is a direct sum $L' \oplus L''$ of Hodge structures is a countable union of algebraic subsets of dimension $k'^2 + (k-k')^2$. As $k'^2 + (k-k')k > k'^2 + (k-k')^2$, the claim is proved. The phenomenon described above does not appear in algebraic geometry where the Hodge structures we get are polarized.

Definition 2 *A polarization on a rational Hodge structure L of weight k is a nondegenerate intersection form* $(\,,\,)$ *on L which is symmetric if k is even, skew-symmetric if k is odd and satisfies the Hodge-Riemann bilinear relations:*
(1) $(\alpha, \overline{\beta}) = 0$ *for* $\alpha \in L^{p,q}$, $\beta \in L^{p',q'}$ *and* $(p,q) \neq (p',q')$.
(2) $\iota^k(-1)^p(\alpha, \overline{\alpha}) > 0$ *for* $\alpha \in L^{p,q}$, $\alpha \neq 0$.

Admittedly, the sign rules in (2) are complicated, but they are imposed on us by geometry. The importance of the notion comes from the following:

Lemma 1 *Let* $L' \subset L$ *a Hodge substructure of a polarized rational Hodge structure. Then there exists a Hodge substructure* $L'' \subset L$ *such that L is isomorphic to* $L' \oplus L''$ *as Hodge structure.*

Proof. Indeed, let q be the intersection form giving the polarization on L. It suffices to prove that the restricted form $q_{|L'}$ is nondegenerate since then the orthogonal complement $L'' := L'^{\perp_q}$ is defined over $\mathbb{Q}$, is a Hodge substructure of L by property (1) above and satisfies $L' \oplus L'' = L$. Let $h(u,v) = \iota^k q(u, \overline{v})$ be the Hermitian bilinear form on $L_{\mathbb{C}}$ associated to q. It suffices to show that $h_{|L'_{\mathbb{C}}}$ is nondegenerate. But the Hodge decomposition of $L'_{\mathbb{C}}$ is orthogonal for h by (1) above and each $h_{|L'^{p,q}}$ is nondegenerate by (2) above. Hence $h_{|L'_{\mathbb{C}}}$ is nondegenerate.

The construction of a polarization on the Hodge structure on $H^k(X, \mathbb{Q})$ when X is a smooth complex projective manifold goes as follows: Let $l \in H^2(X, \mathbb{Q})$ be the chern class of an ample line bundle on X. Then the hard Lefschetz theorem [30, 6.2.3] gives an isomorphism of Hodge structures

$$l^{n-k} \smile : H^k(X, \mathbb{Q}) \to H^{2n-k}(X, \mathbb{Q}),\ n = \dim X.$$

We can thus assume $k \leq n$. We then consider the nondegenerate intersection pairing

$$(\alpha, \beta)_l := \int_X l^{n-k} \smile \alpha \smile \beta,\ \alpha,\ \beta \in H^k(X, \mathbb{Q}).$$

It is nondegenerate by the hard Lefschetz theorem, does not satisfy property (2) above, but satisfies property (1) above. We finally modify it as follows: the Hodge structure $H^k(X,\mathbb{Q})$ admits the Lefschetz decomposition as a direct sum of Hodge substructures

$$H^k(X,\mathbb{Q}) = \oplus_{2r\leq k} l^r \smile H^{k-2r}(X,\mathbb{Q})_{prim}, \tag{8}$$

where the primitive cohomology is defined by $H^{k-2r}(X,\mathbb{Q})_{prim} := \mathrm{Ker}\,(l^{n-k+2r+1}\smile : H^{k-2r}(X,\mathbb{Q}) \to H^{2n-k+2r+2}(X,\mathbb{Q}))$. This decomposition is orthogonal for $(\,,\,)_l$. The polarization $(\,,\,)$ on $H^k(X,\mathbb{Q})$ is the unique intersection pairing for which the Lefschetz decomposition is orthogonal, and which is equal to $(-1)^r(\,,\,)_l$ on $l^r \smile H^{k-2r}(X,\mathbb{Q})_{prim}$. The fact that this polarizes (up to a sign) the Hodge structure on $H^k(X,\mathbb{Q})$ is exactly the contents of the Hodge-Riemann bilinear relations (see [30, 6.3.2]).

2.3 Hodge classes and cycle classes

2.3.1 Hodge classes

Let H be a Hodge structure of even weight $2k$, with Hodge decomposition $H_{\mathbb{C}} = \oplus_{p+q=2k} H^{p,q}$.

Definition 3 *The Hodge classes in H are the classes in H (hence rational) which, via the inclusion $H \subset H_{\mathbb{C}}$, belong to $H^{k,k}$.*

We will denote $\mathrm{Hdg}^{2k}(H)$ the space $H \cap H^{k,k}$ of Hodge classes. Note that this space can be reduced to 0 and will be 0 for a general Hodge structure with given Hodge numbers $h^{p,q} = \dim H^{p,q}$ unless $h^{p,q} = 0$ for $p \neq q$, since the space $H^{k,k} \subset H_{\mathbb{C}}$ needs not be defined over $\mathbb{Q}$, but only over $\mathbb{R}$ (as implied by the Hodge symmetry property, that is condition (5)). If X is a smooth projective variety, we will denote $\mathrm{Hdg}^{2k}(X)$ the space $\mathrm{Hdg}^{2k}(H^{2k}(X,\mathbb{Q}))$.

2.3.2 Cycle classes

Let X be a smooth complex projective or compact Kähler variety of (complex) dimension n, and let $Z \overset{j}{\hookrightarrow} X$ be a closed analytic subset (which in the projective case is the same thing according to Chow as a closed algebraic subset) of codimension k. If Z is smooth, then Z is a codimension $2k$ real submanifold endowed with the complex orientation, so it has a fundamental homology class $[Z]_{fund} \in H_{2n-2k}(Z,\mathbb{Z})$ which gives a homology class

$$j_*[Z]_{fund} \in H_{2n-2k}(X,\mathbb{Z}) \cong H^{2k}(X,\mathbb{Z}),$$

where the last isomorphism is the Poincaré duality isomorphism. If Z is not smooth, then according to Hironaka, one can construct a smooth projective variety $\widetilde{Z}$ with a morphism $\tau : \widetilde{Z} \to Z$ of degree 1. Letting $\tilde{j} := j \circ \tau : \widetilde{Z} \to X$, we can define the class $[Z]$ of Z by

$$[Z] = \tilde{j}_*[\widetilde{Z}]_{fund} \in H^{2k}(X, \mathbb{Z}).$$

Lemma 2 *The class of a closed analytic subset Z in a compact Kähler manifold X is a Hodge class.*

Proof. Let n be the dimension of X. Then we have Poincaré duality

$$H^{2k}(X, \mathbb{C}) = H^{2n-2k}(X, \mathbb{C})^*$$

identifying the space $H^{k,k}(X)$ with the subspace of $H^{2k}(X, \mathbb{C})$ which is orthogonal to $\oplus_{p+q=2n-2k,(p,q)\neq(n-k,n-k)} H^{p,q}(X)$. It thus suffices to show that for $\beta \in H^{p,q}(X)$, $p+q = 2n-2k, (p,q) \neq (n-k, n-k)$, one has $\langle [Z], \beta \rangle_X = 0$. Recall from Section 2.1 that $H^{p,q}(X)$ consists of classes of closed forms of type (p,q). The class β is thus represented by a closed form $\tilde{\beta}$ which is closed of type (p,q) and introducing a desingularization $\tilde{j} : \widetilde{Z} \to X$ of Z, we have, by definition of the Gysin morphism,

$$\langle [Z], \beta \rangle_X = \langle \tilde{j}_*[\widetilde{Z}]_{fund}, \tilde{\beta} \rangle_X = \langle [\widetilde{Z}]_{fund}, \tilde{j}^* \tilde{\beta} \rangle_{\widetilde{Z}} = \int_{\widetilde{Z}} \tilde{j}^* \tilde{\beta}.$$

The last expression vanishes since the form $\tilde{j}^* \tilde{\beta}$ vanishes on $\widetilde{Z}$ for type reasons.

Important examples of Hodge classes are provided by the following lemma 3.

Lemma 3 *Let H, H' be two Hodge classes of weights k, $k' = k + 2r$. Then the Hodge classes of the weight $2r$ Hodge structure* $\mathrm{Hom}\,(H, H')$ *are exactly the morphisms of Hodge structures $H \to H'$.*

Here the Hodge structure on H^* has been introduced previously, and the Hodge structure on the tensor product $H^* \otimes H' = \mathrm{Hom}\,(H, H')$ is given by

$$(H^* \otimes H')^{p,q} = \oplus_{t+t'=p, s+s'=q} (H^*)^{t,s} \otimes (H')^{t',s'}. \tag{9}$$

Proof. Indeed a morphism $\phi \in \mathrm{Hom}\,(H, H') = H^* \otimes H'$ is of type (r,r) for the tensor product Hodge structure if and only if it satisfies $\phi_{\mathbb{C}} \in \oplus_{(t,s)} (H^*)^{t,s} \otimes (H')^{r-t,r-s}$. As we have $(H^*)^{t,s} = (H^{-t,-s})^*$, this is equivalent to

$$\phi_{\mathbb{C}} \in \oplus_{(t,s)} (H^{t,s})^* \otimes (H')^{r+t,r+s} = \oplus_{(t,s)} \mathrm{Hom}\,(H^{t,s}, (H')^{r+t,r+s}),$$

that is, to the fact that $\phi_{\mathbb{C}}$ shifts the Hodge decomposition by (r,r).

3 The Hodge and generalized Hodge conjectures

3.1 The Hodge conjecture

Conjecture 1 *(Hodge 1951) Let X be a projective complex manifold. Then for any k, the space $\mathrm{Hdg}^{2k}(X)$ is generated over $\mathbb{Q}$ by classes $[Z]$ of codimension k closed algebraic subsets of X.*

A codimension k cycle on X is a formal combination $Z = \sum_i \alpha_i Z_i$, $\alpha_i \in \mathbb{Q}$. We will call cycle classes $[Z] := \sum_i \alpha_i [Z_i]$ *algebraic classes*, and will use the notation $H^{2k}(X,\mathbb{Q})_{alg}$ for the space of algebraic classes. We have $H^{2k}(X,\mathbb{Q})_{alg} \subset \mathrm{Hdg}^{2k}(X)$ and the Hodge conjecture states that $H^{2k}(X,\mathbb{Q})_{alg} = \mathrm{Hdg}^{2k}(X)$.

3.1.1 Why is the conjecture important?

There are very few morphisms in algebraic geometry so it is important to consider multivalued morphisms which are given by their graphs $\Gamma \subset X \times Y$. This leads to consider the group $\mathscr{Z}^m(X \times Y)$ of codimension m cycles in $X \times Y$, or better cycles modulo an adequate equivalence relation $\sim$, like rational equivalence, which provides Chow groups, or homological equivalence. When X and Y are smooth and projective, cycles in $X \times Y$ act on many objects, like Chow groups or cohomology. Given an adequate equivalence relation $\sim$ on cycles, the action of $\Gamma \in \mathscr{Z}^m(X \times Y)$ takes the general form

$$\Gamma^*(\alpha) = pr_{1*}(\Gamma \cdot pr_2^* \alpha) \in \mathscr{Z}^{k+m-\dim Y}(X)/\sim, \forall \alpha \in \mathscr{Z}^k(X)/\sim,$$

where pr_{1*} is pushforward by the first projection, pr_2^* is pull-back by the second projection and "$\cdot$" is the intersection product. When the equivalence relation is homological equivalence, cycles Z mod. $\sim$ are cohomology classes and the push-forward map is the Gysin map, the intersection product is the cup-product.

The importance of the Hodge conjecture in this context is that, combined with Lemma 3, it predicts exactly which morphisms $Z^* : H^*(Y,\mathbb{Q}) \to H^*(X,\mathbb{Q})$ can be constructed from cycle classes in $X \times Y$. Namely, one should get exactly the morphisms of Hodge structures. The geometric importance of this prediction is obvious: we mentioned in the introduction Torelli type questions, asking whether a variety is determined by its Hodge structures. The Hodge conjecture predicts that if two smooth projective varieties X, Y have isomorphic Hodge structures, they are related by algebraic cycles in $X \times Y$ inducing isomorphisms in cohomology. In a more motivic direction, the Hodge conjecture can thus pedantically rephrased by saying that the category of polarizable Hodge structures contains the category of cohomological motives as a *full* subcategory, so that structure results for the category of polarizable Hodge structures (like semisimplicity, see Lemma 1) also should hold for the category of cohomological motives. This adequation of Hodge theory and algebraic geometry fits also very well with conjectures of Bloch and Beilinson (see [6], [19],

[32]) predicting that to a large extent, Hodge structures control Chow groups. In our mind however, the generalized Hodge conjecture which will be explained in Section 3.3 is much more important than the Hodge conjecture itself as it says much more, qualitatively, on the relationship between Hodge structures and algebraic cycles than the Hodge conjecture does.

A more technical but important justification of the interest of the Hodge conjecture concerns the Hodge classes which appear in the standard conjecture. Roughly speaking, these Hodge classes are those which can be produced by linear algebra starting from classes of algebraic cycles. The classes so obtained, which will be described in Section 3.2, are still Hodge classes for linear algebra reasons, but it is not known if they are algebraic. The importance of these classes also comes from the consideration of the theory of motives.

3.1.2 Positive evidences

The only instances of the Hodge conjecture which are known for any smooth complex projective n-fold X are first of all the two trivial cases $H^0(X,\mathbb{Q}) = \mathrm{Hdg}^0(X,\mathbb{Q}) = \mathbb{Q}[X]_{fund}$, (where X is assumed to be connected), and $H^{2n}(X,\mathbb{Q}) = \mathrm{Hdg}^{2n}(X,\mathbb{Q}) = \mathbb{Q}[\text{point}]$, and secondly the Lefschetz theorem on $(1,1)$-classes (Theorem 2) which concerns divisor (that is degree 2) classes and its corollary which concerns curve (that is degree $2n-2$) classes.

Theorem 2 *(Degree 2) Let X be a complex projective manifold and let $\alpha \in \mathrm{Hdg}^2(X,\mathbb{Z})$ be an integral Hodge class. Then α is a combination with integral coefficients of classes $[D] \in H^2(X,\mathbb{Z})$ of hypersurfaces $D \subset X$.*

Corollary 2 *(Degree $2n-2$) Let X be a complex projective n-fold and let $\alpha \in \mathrm{Hdg}^{2n-2}(X)$ be a Hodge class. Then α is a combination with rational coefficients of classes $[C] \in H^{2n-2}(X,\mathbb{Z})$ of curves $C \subset X$.*

Remark 1 The first three cases mentioned above (degrees 0, 2 or $2n$) are the only cases where the Hodge conjecture is true for *integral* Hodge classes, that is integral cohomology classes whose image in rational cohomology is a Hodge class. This follows from Atiyah-Hirzebruch and Kollár counterexamples [3], [20] for integral Hodge classes.

Proof (Proof of Theorem 2). There is a beautiful description in [13] of the original Lefschetz proof. It relies on the notion of normal functions associated to a Hodge class. Given a Hodge class $\alpha \in \mathrm{Hdg}^2(X,\mathbb{Z})$, we choose a pencil of hyperplane sections $(X_t)_{t\in\mathbb{P}^1}$ of X and assume that $\alpha_{|X_t} = 0$. The Hodge class α lifts to a class $\tilde{\alpha}$ in the Deligne cohomology group $H^2_{\mathcal{D}}(X,\mathbb{Z}(1))$ (see [30, 12.3.1]). Then $\tilde{\alpha}_{|X_t}$ belongs to

$$\mathrm{Ker}\,(H^2_{\mathcal{D}}(X_t,\mathbb{Z}(1)) \to H^2(X_t,\mathbb{Z})) = J^1(X_t) = \mathrm{Pic}^0(X_t).$$

Associated to α we thus found a family of divisors $t \mapsto \tilde{\alpha}_{|X_t} \in \mathrm{Pic}^0(X_t)$. A large part of this argument works as well for any Hodge class on a smooth projective variety X

vanishing on the fibers X_t of a pencil on X. Indeed, the Deligne cohomology group $H^{2k}_{\mathscr{D}}(X,\mathbb{Z}(k))$ fits in the exact sequence

$$0 \to J^k(X) \to H^{2k}_{\mathscr{D}}(X,\mathbb{Z}(k)) \to \mathrm{Hdg}^{2k}(X,\mathbb{Z}) \to 0$$

and similarly for X_t. We can thus lift a Hodge class on X to a Deligne cohomology class and restrict it to the fibers X_t. The problem is that the normal function one shall get this way will be a holomorphic section of the family of intermediate Jacobians $J^k(X_t)_{t\in\mathbb{P}^1}$, and one does not know for $k \geq 2$ what is the image of the Abel-Jacobi map $\mathscr{Z}^k(X_t)_{hom} \to J^k(X_t)$.

The modern proof of Theorem 2 uses the exponential exact sequence and goes as follows:

1) The Picard group of holomorphic line bundles of an analytic space X identifies to $H^1(X,\mathscr{O}_X^*)$, where $\mathscr{O}_X^*$ is the sheaf of invertible holomorphic functions. The exponential exact sequence

$$0 \to \mathbb{Z} \stackrel{2\iota\pi}{\to} \mathscr{O}_X \stackrel{\exp}{\to} \mathscr{O}_X^* \to 1$$

provides the associated cohomology long exact sequence

$$\ldots H^1(X,\mathscr{O}_X^*) \stackrel{c_1}{\to} H^2(X,\mathbb{Z}) \to H^2(X,\mathscr{O}_X)\ldots$$

defining c_1.

2) If X is compact Kähler, the kernel of the natural map $H^2(X,\mathbb{Z}) \to H^2(X,\mathscr{O}_X)$ appearing above is exactly the set of integral Hodge classes. This follows from the fact that this map identifies using Hodge theory with the composite

$$H^2(X,\mathbb{Z}) \stackrel{2\iota\pi}{\to} H^2(X,\mathbb{C}) \to H^{0,2}(X) \cong H^2(X,\mathscr{O}_X),$$

where all maps are natural and the map $H^2(X,\mathbb{C}) \to H^{0,2}(X)$ is the projection given by Hodge decomposition. It thus follows that a class $\alpha \in H^2(X,\mathbb{Z})$ which maps to 0 in $H^2(X,\mathscr{O}_X)$ has $\alpha^{0,2} = 0$ in the Hodge decomposition. But then it also has $\alpha^{2,0} = 0$ since it is real, and thus it is of type $(1,1)$ hence a Hodge class.

3) At this point we proved that if X is compact Kähler, the set of Hodge classes of degree 2 is equal to the set $c_1(L)$ where L runs through the set of holomorphic line bundles on X. assume now that X is projective. By Serre GAGA principle [26], holomorphic line bundles and algebraic line bundles are the same objects on X : equivalently, any holomorphic line bundle has a nonzero meromorphic section. Choosing a nonzero meromorphic section σ of L, we introduce its divisor D_σ which is a codimension 1 cycle on X and the final step is Lelong's formula [30, Theorem 11.33] which says that the class $[D_\sigma]$ is equal to $c_1(L)$.

Proof (Proof of Corollary 2). We use for this the Lefschetz isomorphism

$$l^{n-2} \smile : H^2(X,\mathbb{Q}) \to H^{2n-2}(X,\mathbb{Q})$$

given by the choice of a very ample line bundle $\mathscr{L}$ on X with first Chern class l, which is obviously an isomorphism of Hodge structures. A Hodge class β of degree $2n-2$ can thus be written as $\beta = l^{n-2} \smile \alpha$, where α is a Hodge class of degree 2. The class α is the class of a divisor $D = \sum_i \alpha_i D_i$, where the D_i's are hypersurfaces in X, and thus $\beta = \sum_i \alpha_i [C_i]$ where the curve C_i is the intersection of D_i with a surface $L_1 \cap \ldots \cap L_{n-2}$ complete intersection of hypersurfaces L_i in the linear system $|\mathscr{L}|$ (hence of class l) in general position.

Apart from these four known cases, the best positive evidence in favour of the Hodge conjecture is the fact that Hodge classes behave geometrically as if they were algebraic as predicted by the Hodge conjecture. The precise statement will be explained in Section 4.2.

3.1.3 Negative evidences

Many complex geometry results have been proved in the past by analytic methods working as well in the compact Kähler setting, for example the Hodge decomposition itself, or the study of positivity of divisors by curvature and currents methods [12], or the proof of the existence of Hermite-Einstein metrics on stable vector bundles [28]. In the case of the Hodge conjecture, it has been known for a long time (see [35]) that in the compact Kähler setting, there are not enough closed analytic cycles to generate the Hodge classes: the example, due to Mumford, is a very general complex torus of dimension at least 2 admitting a holomorphic line bundle $\mathscr{L}$ with nontrivial Chern class which is neither positive not negative: such a torus does not contain any hypersurface, while $c_1(\mathscr{L})$ is a nontrivial Hodge class. However, in this example, one can argue that the problem is a lack of effectivity (or positivity), and that we still have a complex geometric object which is a good substitute for the hypersurfaces, namely the line bundle itself (in the projective case, by the existence of rational sections of line bundles, Chern classes of line bundles are combinations of classes of hypersurfaces).

In the paper [29], I constructed examples of Hodge classes on complex tori T, which do not belong to the $\mathbb{Q}$-vector space generated by Chern classes of coherent sheaves on T. It seems that in these cases, there is no way of extending the Hodge conjecture: there is no holomorphic object on T explaining the presence of a Hodge class on T.

The second point which makes not very plausible a solution of the Hodge conjecture by analytic methods is the lack of uniform solutions to the Hodge conjecture, assuming they exist, that is the lack of bound on the cycles (supposed minimal in some way) representing a given Hodge class. This follows from the analysis of some of the known counterexamples to the integral Hodge conjecture. In the case of Kollár counterexamples [20], which are just hypersurfaces X of degree d in projective space $\mathbb{P}^{n+1}$ with the generator α of $H^{2n-2}(X,\mathbb{Z})$ not being algebraic while $d\alpha$ is algebraic, it was observed in [27] that the following phenomenon holds: Let U be the Zariski open set in the space of homogeneous polynomials of degree d such that the corresponding hypersurface is smooth. Then the (locally constant) class

$\alpha_t \in H^{2n-2}(X_t,\mathbb{Z})$ is Hodge on X_t for any $t \in U$, the set of points $t \in U$ such that the class α_t is algebraic on X_t is dense in U for the usual topology, while Kollár proves that this set is not the whole of U. This means that for a very general point $0 \in U$, there is a sequence of points $t_n \in U$ converging to 0 and for which the class α_{t_n} is the class of an algebraic cycle Z_n on X_{t_n}. Thus the cycle Z_n is of the form $Z_n^+ - Z_n^-$, but the degrees of the positive part Z_n^+ and the negative part Z_n^- of Z_n cannot be bounded, although the difference Z_n has class α_{t_n} which is locally constant hence bounded. Indeed, if these degrees were bounded, we could use compactness results to make the cycles Z_n^+ and Z_n^- converge respectively to cycles Z^+ and Z^- on X_0 with $[Z^+]-[Z^-]=\alpha$, which is not true.

3.2 The standard conjectures

The main source of construction of Hodge classes is Lemma 3. Let X be a complex projective n-fold, and consider $X \times X$. For any integer k, we have

$$\mathrm{End}\, H^k(X,\mathbb{Q}) \cong H^{2n-k}(X,\mathbb{Q}) \otimes H^k(X,\mathbb{Q}) \subset H^{2n}(X\times X,\mathbb{Q})$$

and Lemma 3 tells us that a morphism $\phi \in \mathrm{End}\, H^k(X,\mathbb{Q})$ provides a Hodge class on $X \times X$ by the composite map above if and only if ϕ is a morphism of Hodge structure. In particular, the identity of $H^k(X,\mathbb{Q})$ is a morphism of Hodge structures, hence provides a Hodge class $\delta_k \in \mathrm{Hdg}^{2n}(X\times X,\mathbb{Q})$. The sum $\sum_k \delta_k$ is the identity of $H^*(X,\mathbb{Q})$, hence is the class of the diagonal $\Delta_X \subset X \times X$. Hence $\sum_k \delta_k$ is algebraic but it is not known if individually each class δ_k is algebraic, that is, satisfies the Hodge conjecture. The classes δ_k are called the Künneth components of the diagonal of X. The varieties for which it is known that the Künneth components of the diagonal are algebraic include the abelian varieties (that is, projective complex tori) and smooth complete intersections in projective space, for which the non-algebraic cohomology is concentrated in degree n. If A is an abelian variety (or complex torus), A is an abelian group, hence we have for each l the multiplication map

$$\mu_l : A \to A,\ a \mapsto la.$$

We have $\mu_l^* = l^k Id$ on $H^k(A,\mathbb{Q})$ and it easily follows that we can write the Künneth components of A as linear combinations of the classes of the graph Γ_l of μ_l for various l (note that $\mu_l^* = [\Gamma_l]^* : H^*(A,\mathbb{Q}) \to H^*(A,\mathbb{Q})$).

A more subtle construction involves the properties of the Lefschetz operator. Recall from Section 2.2 that if l is the first Chern class of an ample line bundle $\mathscr{L}$ on X, the cup-product map

$$l^{n-k}\smile : H^k(X,\mathbb{Q}) \to H^{2n-k}(X,\mathbb{Q}),\ n = \dim X \qquad (10)$$

is an isomorphism for any k. It is clear that $l^{n-k}\smile$ acting on $H^*(X,\mathbb{Q})$ is the action of the following cycle on $X \times X$: let $L_1,\ldots,L_{n-k}$ be general hypersurfaces in the linear system $|\mathscr{L}|$ (we may assume $\mathscr{L}$ very ample), and let $Z = L_1 \cap \ldots \cap L_{n-k}$. Then $[Z] = l^{n-k}$ by Lelong's theorem, and $[i_{\Delta *}Z] \in H^{4n-2k}(X \times X, \mathbb{Q})$ acts on $H^*(X, \mathbb{Q})$ by $l^{n-k} \smile$, where $i_{\Delta *}Z$ is the cycle Z supported on the diagonal $\Delta_X \cong X \subset X \times X$. Next we can consider the inverse $\lambda_{n-k} : H^{2n-k}(X, \mathbb{Q}) \to H^k(X, \mathbb{Q})$ of the Lefschetz isomorphism (10). This is a morphism of Hodge structures, hence this provides a Hodge class on $X \times X$.

Conjecture 2 *(Lefschetz standard conjecture) There exists a codimension k cycle Z on $X \times X$ such that $[Z]^* : H^{2n-k}(X, \mathbb{Q}) \to H^k(X, \mathbb{Q})$ is equals to λ_{n-k}.*

Again the answer is positive in the case of an abelian variety A, and this is due to the existence of an interesting line bundle $\mathscr{P}$ on $A \times A$, defined as $\mu^*\mathscr{L}$ where $\mu : A \times A \to A$ is the sum map. The line bundle $\mathscr{P}$ is called the Poincaré divisor and its class $p := c_1(\mathscr{P}) \in \mathrm{Hdg}^2(A \times A)$ and its powers $p^k \in \mathrm{Hdg}^{2k}(A \times A)$ are algebraic classes on $A \times A$ which allow to solve the Lefschetz conjecture in this case.

The Lefschetz standard conjecture is very important in the theory of motives (see [1]), because of the semisimplicity Lemma 1. This lemma uses the polarization to construct, given a polarized Hodge structure L and a Hodge substructure $L' \subset L$, a decomposition

$$L = L' \oplus L''. \tag{11}$$

The construction of these polarizations when $L = H^k(X, \mathbb{Q})$ for some smooth projective variety X is quite involved, as it uses the Lefschetz decomposition in order to modify the natural pairing into one which satisfies the polarization axioms. If now $L = H^k(X, \mathbb{Q})$ and $L' \subset L$ is defined as the image of a morphism $[Z]^*$ for some algebraic cycle Z on $X \times X$, the Lefschetz standard conjecture is exactly what would be needed in order to construct the orthogonal complement L'' via the action of an algebraic cycle on $X \times X$.

The most concrete consequence of the Lefschetz standard conjecture is the following (cf. [21]):

Lemma 4 *Let X be a smooth complex projective variety of dimension n. Assume the Lefschetz standard conjecture holds for X and some ample class $l \in \mathrm{Hdg}^2(X)$ in all even degrees $2k$. Then for any k, the intersection pairing between $H^{2k}(X, \mathbb{Q})_{alg}$ and $H^{2n-2k}(X, \mathbb{Q})_{alg}$ is nondegenerate.*

Proof. Indeed, if the Lefschetz conjecture holds for X in any even degree, then the Lefschetz isomorphism (10) induces an isomorphism $l^{n-2k} \smile : H^{2k}(X, \mathbb{Q})_{alg} \cong H^{2n-2k}(X, \mathbb{Q})_{alg}$ for all $k \leq n/2$, because the inverse λ_{n-k} preserves algebraic classes. It follows that the space $H^{2k}(X, \mathbb{Q})_{alg}$ is stable under the Lefschetz decomposition (8). It remains only to prove that for $k \leq n/2$ the pairing $(\,,\,)_l$ on $H^{2k}(X, \mathbb{Q})$ defined by $(\alpha, \beta)_l = \langle l^{n-2k} \smile \alpha, \beta \rangle_X$, is nondegenerate on $H^{2k}(X, \mathbb{Q})_{alg} \subset H^{2k}(X, \mathbb{Q})$. By the Hodge-Riemann bilinear relations, the Lefschetz decomposition is orthogonal for this pairing and on each piece $l^r \smile H^{2k-2r}(X, \mathbb{R})_{prim}$, the pairing $(\,,\,)_l$ restricted to the subspace $H^{k-r,k-r}(X)_{\mathbb{R},prim} \subset H^{2k-2r}(X, \mathbb{Q})_{prim}$ of real classes of Hodge type $(k-r, k-r)$ is definite of a sign which depends only on $k-r$. As

$l^r \smile H^{2k-2r}(X,\mathbb{Q})_{alg,prim}$ is contained in $H^{k-r,k-r}(X)_{\mathbb{R},prim}$, it follows that the pairing $(\,,\,)_l$ restricted to $H^{2k-2r}(X,\mathbb{Q})_{alg,prim}$ remains definite, and in particular nondegenerate. Hence $(\,,\,)_l$ is nondegenerate on $H^{2k-2r}(X,\mathbb{Q})_{alg}$.

Let us give two corollaries:

Corollary 3 *(i) Let $j: Y \to X$ be a morphism, where X, Y are smooth complex projective varieties. Assume X and Y satisfy the Lefschetz standard conjecture. Then if Z is an algebraic cycle on Y whose class $[Z] \in H^{2k}(Y,\mathbb{Q})$ is equal to $j^*\beta$ for some class $\beta \in H^{2k}(X,\mathbb{Q})$, there exists a codimension k cycle Z' on X such that*

$$j^*[Z'] = [Z] \text{ in } H^{2k}(Y,\mathbb{Q}). \tag{12}$$

(ii) If Z is an algebraic cycle on X such that the class $[Z] \in H^{2k}(X,\mathbb{Q})$ is equal to $j_*\beta$ for some class $\beta \in H^{2k-2r}(Y,\mathbb{Q})$, $r = \dim X - \dim Y$, there exists a codimension $k-r$ cycle Z' on Y such that $j_*[Z'] = [Z]$ in $H^{2k}(X,\mathbb{Q})$.

Proof. (i) The class β gives by the Poincaré pairing on X a linear form on $H^{2n-2k}(X,\mathbb{Q})_{alg}$, $n = \dim X$, which by Lemma 4 applied to X is of the form $\langle [Z'],\ \rangle_X$ for some codimension k cycle Z' on X. We now prove that the class $[Z']$ satisfies (12). By Lemma 4, it suffices to show that for any cycle W on Y,

$$\langle [Z']_{|Y}, [W]\rangle_Y = \langle [Z], [W]\rangle_Y. \tag{13}$$

The left hand side is equal to $\langle [Z'], j_*[W]\rangle_X$ where j is the inclusion morphism of Y in X, and by definition of $[Z']$, this is equal to $\langle \beta, j_*[W]\rangle_X$. Finally, by definition of the Gysin morphism j_*, we have $\langle \beta, j_*[W]\rangle_X = \langle j^*\beta, [W]\rangle_Y = \langle [Z], [W]\rangle_Y$.

(ii) is proved exactly in the same way.

The following corollary appears in [32] where it is proved that the conclusion (for all X and Y) is essentially equivalent to the Lefschetz conjecture:

Corollary 4 *(see [32]) Assume the Lefschetz conjecture. Let X be a smooth projective variety and let $Y \subset X$ be a closed algebraic subset. Let Z be a codimension k cycle on X whose cohomology class $[Z]$ vanishes in $H^{2k}(X \setminus Y,\mathbb{Q})$. Then there exists an algebraic cycle Z' supported on Y such that $[Z] = [Z']$ in $H^{2k}(X,\mathbb{Q})$.*

Proof. Our assumption is that there is a homology class $\beta \in H_{2n-2k}(Y,\mathbb{Q})$ such that the image of $j_*\beta \in H_{2n-2k}(X,\mathbb{Q}) \cong H^{2k}(X,\mathbb{Q})$ is equal to $[Z]$. We now apply Lemma 6, which says that if $\tilde{j}: \widetilde{Y} \to X$ is a desingularization of Y, there exists a class $\beta' \in H^{2k-2r}(\widetilde{Y},\mathbb{Q})$ such that $\tilde{j}_*\beta' = [Z]$, where $r = \dim X - \dim Y$. We then conclude with Corollary 3, (ii).

3.3 Mixed Hodge structures and the generalized Hodge conjecture

In [8], Deligne discovered a very important generalization of Hodge structures, namely mixed Hodge structures. The definition is as follows:

Definition 4 *A mixed Hodge structure is the data of a finite dimensional $\mathbb{Q}$-vector space L equipped with an increasing exhaustive filtration W (the weight filtration), together with a decreasing exhaustive filtration F on $L_{\mathbb{C}}$ with the property that the induced filtration on Gr_W^i, defined by $F^pGr_W^i = F^p \cap W_iL_{\mathbb{C}}/F^p \cap W_{i+1}L_{\mathbb{C}}$, comes from a Hodge structure (see (7)) of weight i on Gr_W^i.*

Morphisms of mixed Hodge structures are morphisms of $\mathbb{Q}$-vector spaces preserving both filtrations. The following result is crucial for geometric and topological applications of this notion.

Lemma 5 *(Deligne [8]) Morphisms of mixed Hodge structures are strict for both filtrations.*

Denoting by $\phi : L \to M$ such a morphism, this means that

$$(\mathrm{Im}\,\phi_{\mathbb{C}}) \cap F^pM_{\mathbb{C}} = \phi_{\mathbb{C}}(F^pL_{\mathbb{C}}),\ (\mathrm{Im}\,\phi) \cap W_iM_{\mathbb{C}} = \phi(W_iL).$$

We will call the pure Hodge substructure of a mixed Hodge structure the smallest nonzero piece $W_iL \subset L$ and the pure quotient the quotient L/W_iL where i is maximal such that $W_iL \neq L$. they both carry a Hodge structure.

Deligne proves the following result:

Theorem 3 *For any quasiprojective variety X, its homology groups and cohomology groups carry mixed Hodge structures, which are functorial under pull-back on cohomology and functorial under pushforward on homology.*

If X is smooth, the pure Hodge substructure on $H^k(X,\mathbb{Q})$ has weight k (so all weights are $\geq k$) and is equal to $\mathrm{Im}\,(H^k(\overline{X},\mathbb{Q}) \to H^k(X,\mathbb{Q}))$ for any smooth projective compactification $\overline{X}$ of X.

If X is projective, the pure quotient Hodge structure of $H^k(X,\mathbb{Q})$ has weight k (so all weights are $\leq k$) and is equal to $\mathrm{Im}\,(H^k(X,\mathbb{Q}) \to H^k(\widetilde{X},\mathbb{Q}))$ for any smooth projective desingularization $\widetilde{X}$ of X. The dual statement is that the pure Hodge substructure of $H_k(X,\mathbb{Q})$ is the image $\mathrm{Im}\,(H_k(\widetilde{X},\mathbb{Q}) \to H_k(X,\mathbb{Q}))$ for any smooth projective desingularization $\widetilde{X}$ of X.

Let now X be a smooth projective variety, and $Y \subset X$ be a closed algebraic subset of X. Assume for simplicity that all the irreducible components of Y are of codimension r.

Theorem 4 *Let $U := X \setminus Y$. Then the kernel*

$$\mathrm{Ker}\,(H^k(X,\mathbb{Q}) \to H^k(U,\mathbb{Q}))$$

is a Hodge substructure L_Y of $H^k(X,\mathbb{Q})$ which is of Hodge coniveau $\geq r$, meaning that $L_Y^{p,q} = 0$ for $p < r$ or $q < r$.

Proof. We will use the following consequence of Theorem 3 and Lemma 5 which is of independent interest:

Lemma 6 *In the situation of Theorem 4, the kernel $\mathrm{Ker}\,(H^k(X,\mathbb{Q}) \to H^k(U,\mathbb{Q}))$ is equal to the image of the composite map*

$$\widetilde{j}_* : H_{2n-k}(\widetilde{Y}, \mathbb{Q}) \to H_{2n-k}(X, \mathbb{Q}) \overset{PD}{\cong} H^k(X, \mathbb{Q}), \tag{14}$$

where $\tilde{j} : \widetilde{Y} \to X$ *is a desingularization of* Y.

Proof. This kernel is the image of the composite map

$$H_{2n-k}(Y, \mathbb{Q}) \to H_{2n-k}(X, \mathbb{Q}) \overset{PD}{\cong} H^k(X, \mathbb{Q}).$$

This map is a morphism of mixed Hodge structures, the right hand side being a pure Hodge structure of weight k. Comparing weights and applying Lemma 5 and Theorem 3, the image of this map is the same as the image of the pure Hodge substructure of $H_{2n-k}(Y, \mathbb{Q})$, that is $\mathrm{Im}\,(H_{2n-k}(\widetilde{Y}, \mathbb{Q}) \to H_{2n-k}(Y, \mathbb{Q}))$, which concludes the proof.

Of course, as $\widetilde{Y}$ is smooth and projective, the composite in (14) is the same as the Gysin morphism $\widetilde{j}_* : H^{k-2r}(\widetilde{Y}, \mathbb{Q}) \to H^k(X, \mathbb{Q})$. As $\widetilde{j}_*$ is a morphism of Hodge structures of bidegree (r, r), its image is a substructure of $H^k(X, \mathbb{Q})$ which is of Hodge coniveau $\geq r$.

The generalized Hodge conjecture due to Grothendieck [15] states the following:

Conjecture 3 *Let* X *be a smooth complex projective variety and let* $L \subset H^k(X, \mathbb{Q})$ *be a Hodge substructure of Hodge coniveau* $\geq r$. *Then there exists a closed algebraic subset* $Y \subset X$ *of codimension* $\geq r$ *such that* $L \subset \mathrm{Ker}\,(H^k(X, \mathbb{Q}) \to H^k(U, \mathbb{Q}))$, $U := X \setminus Y$.

The Hodge conjecture 1 is the particular case of Conjecture 3 where $k = 2r$. Indeed, a Hodge substructure of $H^{2r}(X, \mathbb{Q})$ which is of Hodge coniveau $\geq r$ is made of Hodge classes. Conjecture 3 predicts in this case that L vanishes away from a closed algebraic subset $Y \subset X$ of codimension r, which is the same as saying that L is generated by classes of irreducible components of Y (see [30, 11.1.2]). Conjecture 3 corrects an overoptimistic formulation of the Hodge conjecture (see [18]), where any rational cohomology class α of degree k with Hodge decomposition

$$\alpha_{\mathbb{C}} = \alpha^{k-r,r} + \ldots + \alpha^{r,k-r},$$

that is, satisfying $\alpha^{p,q} = 0$ for $p < r$ or $q < r$, is conjectured to be supported on a codimension r closed algebraic subset. This is wrong by Theorem 4 which says that if α is supported on a codimension r closed algebraic subset, then the minimal Hodge substructure $L \subset H^k(X, \mathbb{Q})$ containing α also satisfies $L^{p,q} = 0$ for $p < r$ or $q < r$ (see [15], [31, Exercise 1 p 184]).

The generalized Hodge conjecture 3 cannot be deduced from the Hodge conjecture, unless the following conjecture is answered affirmatively:

Conjecture 4 *Let* X *be a smooth projective complex variety and let* $L \subset H^k(X, \mathbb{Q})$ *be a Hodge substructure of Hodge coniveau* $\geq r$ *(thus* $L(r)$ *is effective of weight* $k - 2r$*). Then there exists a smooth projective variety* Y*, such that* $L(r)$ *is isomorphic to a Hodge substructure of* $H^{k-2r}(Y, \mathbb{Q})$.

We now have:

Proposition 1 *Conjecture 4 combined with the Hodge conjecture implies Conjecture 3.*

Proof. Note that by the hard Lefschetz theorem, it suffices to prove Conjecture 3 for $L \subset H^k(X,\mathbb{Q})$ with $k \leq n$. Next assume Conjecture 4. Then since $k \leq n$ we can assume by the Lefschetz theorem on hyperplane section that $\dim Y = n - r$. Now $L(r)$ is a direct summand of $H^{k-2r}(Y,\mathbb{Q})$ and the Hodge structure isomorphism $L(r) \cong L \subset H^k(X,\mathbb{Q})$ provides by Lemma 3 a Hodge class α of degree $2n$ on $Y \times X$. Assuming the Hodge conjecture, α is algebraic, which provides a cycle $Z = \sum_i \alpha_i Z_i$, $Z_i \subset Y \times X$ $\dim Z_i = n - r$, such that $L = \mathrm{Im}\,([Z]_* : H^{k-2r}(Y,\mathbb{Q}) \to H^k(X,\mathbb{Q}))$. But then L vanishes away from the codimension $\geq r$ closed algebraic subset $Y' := \cup_i pr_2(Z_i)$ of X.

4 Variational Hodge conjecture

4.1 The global invariant cycles theorem

The following result is due to Deligne [8]. Let $\phi : X \to B$ be a holomorphic map from a smooth projective variety X to a complex manifold, and let $\phi^0 : X^0 \to B^0$ be the restriction of ϕ over the open subset B^0 of B of regular values of ϕ. By definition, $\phi^0 : X^0 \to B^0$ is proper with smooth fibers, hence is a topological fibration. There is thus a monodromy representation $\rho : \pi_1(B^0, b) \to \mathrm{Aut}\, H^k(X_b,\mathbb{Q})$, where $b \in B^0$ is a regular value.

Theorem 5 *The image of the restriction map $H^k(X,\mathbb{Q}) \to H^k(X_b,\mathbb{Q})$ is equal to the subspace $H^k(X_b,\mathbb{Q})^\rho$ of monodromy invariant cohomology classes.*

Proof (Sketch of proof). The proof of this theorem splits into two parts. First of all, Deligne proves in [11] that the Leray spectral sequence for ϕ^0 degenerates at E_2, a result which was also known to Blanchard [4]. This implies that the space $H^k(X_b,\mathbb{Q})^\rho$, which is also the image of $H^0(B^0, R^k\phi^0_*\mathbb{Q})$ in $H^k(X_b,\mathbb{Q})$, is equal to the image of the restriction map

$$H^k(X^0,\mathbb{Q}) \to H^k(X_b,\mathbb{Q}). \tag{15}$$

The second step uses the full strength of Theorem 3. The morphism (15) is a morphism of mixed Hodge structures, the Hodge structure on the right being pure, that is, equal to its minimal Hodge substructure. The mixed Hodge structure on the left has for minimal Hodge substructure (or pure part) the image of the restriction map $H^k(X,\mathbb{Q}) \to H^k(X^0,\mathbb{Q})$. Comparing weights, it then follows from Lemma 5 that the two restriction maps $H^k(X^0,\mathbb{Q}) \to H^k(X_b,\mathbb{Q})$ and $H^k(X,\mathbb{Q}) \to H^k(X_b,\mathbb{Q})$ have the same image.

4.2 The algebraicity theorem and application to the variational Hodge conjecture

The following theorem proved in [7] is the best known evidence for the Hodge conjecture. It says that Hodge classes behave geometrically as if they were algebraic. Let $\phi : \mathscr{X} \to B$ be a projective everywhere submersive morphism, with $\mathscr{X}, B$ smooth quasi-projective. For any $b \in B$, denote by $\mathscr{X}_b$ the fiber $\phi^{-1}(b)$. Let $\alpha \in \mathrm{Hdg}^{2k}(\mathscr{X}_b)$ be a Hodge class. The Hodge locus of α is defined as the set of points $t \in B$, such that for some path $\gamma : [0,1] \to B$ with $\gamma(0) = b$, $\gamma(1) = t$, the class $\alpha_s \in H^{2k}(\mathscr{X}_{\gamma(s)}, \mathbb{Q})$ remains a Hodge class for any $s \in [0,1]$. Here α_s is the class α transported to $\mathscr{X}_{\gamma(s)}$ using the natural isomorphism $H^{2k}(\mathscr{X}_b, \mathbb{Q}) \cong H^{2k}(\mathscr{X}_{\gamma(s)}, \mathbb{Q})$ given by topological trivialization of the pulled-back family $\mathscr{X}_\gamma \to [0,1]$.

Theorem 6 *(Cattani, Deligne, Kaplan 1995) The Hodge locus of α is a countable union of closed algebraic subsets of B.*

Note that the local structure of this locus, say in an open ball $B' \subset B$, as a countable union of closed analytic subsets of B' was understood since the developments of the theory of variations of Hodge structures due to Griffiths [14]. The difficulty here lies in the comparison between the analytic and the algebraic category (the basis B is almost never projective in the above theorem).

That this is indeed the structure predicted by the Hodge conjecture for the Hodge locus of α follows from the existence of relative Hilbert schemes (or Chow varieties) which are projective over B and parameterize subschemes (or effective cycles) $Z_t \subset X_t$ of a given cohomology class. Using these relative Hilbert schemes M_i, we can construct a countable union of varieties M_{ij} projective over B, defined by $M_{ij} = M_i \times_B M_j$ parameterizing cycles $Z_t = Z_t^+ - Z_t^-$ in the fibers $\mathscr{X}_t$. For any point $t \in B$, if the class α_t on $\mathscr{X}_t$ is algebraic, α_t is the class of a cycle $Z_t^+ - Z_t^-$ parameterized by a point in the fiber of at least one of these varieties M_{ij}. Hence the Hodge locus is the union of the images of M_{ij} in B over the pairs (i, j) such that the cycles parameterized by M_{ij} are of class α.

Let us explain the importance of this theorem in the context of the "variational Hodge conjecture". Here the situation is the following: $\mathscr{X}$ is a complex manifold, Δ is a complex ball centered at 0, $\mathscr{X} \to \Delta$ is a proper submersive holomorphic map with projective fibers $\mathscr{X}_t$, $t \in \Delta$, and $\alpha \in H^{2k}(\mathscr{X}, \mathbb{Q})$ is a cohomology class which has the property that $\alpha_t := \alpha_{|\mathscr{X}_t}$ is a degree $2k$ Hodge class on $\mathscr{X}_t$ for any $t \in B$.

Conjecture 5 *(Variational Hodge conjecture) Assume that α_0 satisfies the Hodge conjecture, that is, is algebraic on $\mathscr{X}_0$. Does it follow that α_t is also algebraic?*

Theorem 7 *The variational Hodge conjecture is implied by the Lefschetz conjecture.*

Proof. The family of projective varieties $(\mathscr{X}_b)_{b \in \Delta}$ is the pullback of an algebraic family $\mathscr{X}^{alg} \to B$ via a holomorphic map $f : \Delta \to B$. Our assumption is that $f(\Delta)$ is contained in the Hodge locus B_α of the Hodge class α_0 on $\mathscr{X}_0$. By Theorem 6, this Hodge locus is algebraic, and we can thus replace Δ by an irreducible component B'_α of B_α passing through 0 and containing $f(\Delta)$. We can assume that B'_α is smooth by desingularization. By definition of B_α, the class α_t deduced by parallel transport from the class α_0 is Hodge on all fibers $\mathscr{X}_t$ of the family $\mathscr{X}_\alpha^{alg} \to B'_\alpha$. The monodromy has finite orbits on the set of cohomology classes in fibers which are Hodge everywhere (see [34, Theorem 4.1]). Replacing B'_α by a finite étale cover, we can thus assume that the class α_0 is monodromy invariant on B'_α. Let us introduce a smooth projective completion $\overline{\mathscr{X}_\alpha^{alg}}$ of $\mathscr{X}_\alpha^{alg}$. By Theorem 5, there exists a class $\beta \in H^{2k}(\overline{\mathscr{X}_\alpha^{alg}},\mathbb{Q})$ such that $\beta_{|\mathscr{X}_0} = \alpha_0$. We now apply Corollary 3 (i) to $X = \overline{\mathscr{X}_\alpha^{alg}}$, $Y = \mathscr{X}_0$. As the class $\alpha_0 = \beta_{|\mathscr{X}_0}$ is algebraic, there exists assuming the Lefschetz standard conjecture a cycle Z on $\overline{\mathscr{X}_\alpha^{alg}}$ such that $[Z]_{|\mathscr{X}_0} = \alpha_0$, hence $[Z]_{|\mathscr{X}_t} = \alpha_t$, $\forall t \in \Delta \subset B'_\alpha$, and thus α_t is also algebraic.

4.3 Algebraic de Rham cohomology and absolute Hodge classes

The following arithmetic counterpart of Theorem 6 is completely open (see however [33], [25] for some partial results) :

Conjecture 6 *In the situation of Theorem 6, assume the family $\mathscr{X} \to B$ is defined over a field K (in fact, we can always assume K to be a number field). Then the Hodge locus of α is a countable union of closed algebraic subsets of B which are defined over a finite extension of K.*

Using the global invariant cycle theorem, this conjecture would allow to reduce the Hodge conjecture to the case of varieties X defined over a number field (see [33]). It would be disproved by the existence of a variety X not defined over a number field, with a Hodge class α such that the pair (X,α) is rigid (meaning that under a nontrivial deformation of X, the class α does not remain Hodge).

We next introduce the notion of *absolute Hodge class*. Let X be a smooth projective variety defined over $\mathbb{C}$. In the following, we will write X^{an} for the complex manifold associated with X and cohomology on X will be coherent cohomology with respect to the Zariski topology on X. We have a chain of isomorphisms whose combination gives the Grothendieck comparison isomorphism [16]:

$$\mathbb{H}^k(X,\Omega^\bullet_{X/\mathbb{C}}) \cong \mathbb{H}^k(X^{an},\Omega^\bullet_{X^{an}}) \cong H^k(X^{an},\mathbb{C}).$$

The first term is algebraic de Rham cohomology of X over $\mathbb{C}$. The second term is holomorphic de Rham cohomology of X^{an} and the first isomorphism comes from Serre's GAGA theorem [26]. The second isomorphism comes from the fact that the holomorphic de Rham complex is a resolution of the constant sheaf $\mathbb{C}$ on X^{an}. Note that the Grothendieck isomorphism gives an algebraic definition of the Hodge filtration, namely, it induces for any p an isomorphism

$$\mathbb{H}^k(X,\Omega^{\bullet\geq p}_{X/\mathbb{C}}) \cong F^pH^k(X^{an},\mathbb{C}). \tag{16}$$

Let now $\tau : \mathbb{C} \to \mathbb{C}$ be a field automorphism. Clearly τ induces an isomorphism (which is not $\mathbb{C}$-linear)

$$\tau_* : \mathbb{H}^k(X, \Omega^\bullet_{X/\mathbb{C}}) \cong \mathbb{H}^k(X_\tau, \Omega^\bullet_{X_\tau/\mathbb{C}}), \tag{17}$$

where X_τ is the complex algebraic variety whose equations are obtained by applying τ to the coefficients of the defining equations of X. Composing this automorphism with the Grothendieck isomorphisms

$$\mathbb{H}^k(X, \Omega^\bullet_{X/\mathbb{C}}) \cong H^k(X^{an}, \mathbb{C}) \tag{18}$$

for X and X_τ, we get an isomorphism $H^k(X^{an}, \mathbb{C}) \cong H^k(X_\tau^{an}, \mathbb{C})$, $\alpha \mapsto \alpha_\tau$. This isomorphism is compatible with the Hodge filtrations by (16).

Definition 5 *Let α be a degree $2k$ Hodge class on X. We say that α is an absolute Hodge class if the class $(2\imath\pi)^k\alpha =: \alpha'$ has the property that for any field automorphism τ of $\mathbb{C}$, α'_τ belongs to $(2\imath\pi)^k H^{2k}(X_\tau^{an}, \mathbb{Q})$.*

Remark 2 The class α'_τ is then $(2\imath\pi)^k$ times a Hodge class on X_τ, as it belongs to $F^k H^{2k}(X_\tau^{an}, \mathbb{C})$ since α' belongs to $F^k H^{2k}(X^{an}, \mathbb{C})$.

We now use the existence of an algebraic cycle class $Z \mapsto [Z_{dR}]$ with value in algebraic de Rham cohomology (see [5] for an explicit construction). It is clear that if τ is a field automorphism of $\mathbb{C}$, and Z is a codimension k algebraic cycle on X,

$$\tau_*[Z]_{dR} = [Z_\tau]_{dR} \text{ in } \mathbb{H}^{2k}(X_\tau, \Omega^\bullet_{X_\tau/\mathbb{C}}),$$

where Z_τ is the cycle of X_τ obtained by applying τ to the defining equations of the components Z_i of Z. Finally we use the comparison formula saying that, via the Grothendieck isomorphism (18), $[Z]_{dR} = (2\imath\pi)^k[Z]$. We then get:

Proposition 2 *Cycle classes on smooth projective varieties are absolute Hodge.*

Conjecture 6 is a weak form (see [33]) of the following conjecture 7 (which by Proposition 2 is part of the Hodge conjecture).

Conjecture 7 *Hodge classes are absolute Hodge.*

Deligne [9] proves Conjecture 7 for abelian varieties. It follows from the compatibility properties of the Kuga-Satake construction [22] (see [10]) that it is true as well for (powers of) hyper-Kähler varieties.

In general, one can say from the above discussion that the Hodge conjecture has two independent parts, each of which might be true or wrong, namely Conjecture 7 and the conjecture that absolute Hodge classes are algebraic, which is in the same spirit as the Lefschetz standard conjecture 2 but also concerns more mysterious classes, like Weil classes on abelian varieties with complex multiplication.

Let us conclude with an example of an absolute Hodge class which is not known to be "motivated" in the sense of André [2]. André defines the set of motivated classes as the smallest set of classes on smooth projective algebraic varieties containing algebraic classes, and stable under the operators λ_{n-k} inverse of the Lefschetz operators and under any other algebraic correspondence. Motivated classes include classes $\alpha_t \in \mathrm{Hdg}^{2k}(X_t)$, for some Hodge class α on a smooth projective variety $X \to B$ (where B is connected), such that for some regular value $0 \in B$, $\alpha_0 \in \mathrm{Hdg}^{2k}(X_0)$ is algebraic.

Example 1. Let X be smooth complex projective, and let $b_{2k} := \dim H^{2k}(X,\mathbb{Q})$. Then the space

$$\bigwedge^{b_{2k}} H^{2k}(X,\mathbb{Q}) \subset H^{2k}(X,\mathbb{Q})^{\otimes b_{2k}} \subset H^{2kb_{2k}}(X^{b_{2k}},\mathbb{Q})$$

is clearly a Hodge substructure which is of rank 1, hence generated by a Hodge class on $X^{b_{2k}}$. This class is clearly an absolute Hodge class. Note that one can make the same construction with odd degree cohomology, but in this case the existence of a polarization easily implies that the classes one gets are algebraic, or at least motivated. For this reason, by specializing to Fermat hypersurfaces, the class constructed above is motivated for all smooth hypersurfaces.

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