Remarks and questions on coisotropic subvarieties and 0-cycles of hyper-Kähler varieties

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0 Introduction

This paper proposes a conjectural picture for the structure of the Chow ring $\mathrm{CH}^*(X)$ of a (projective) hyper-Kähler variety X, that seems to emerge from the recent papers [9], [24], [25], [26], with emphasis on the Chow group $\mathrm{CH}_0(X)$ of 0-cycles (in this paper, Chow groups will be taken with $\mathbb{Q}$-coefficients). Our motivation is Beauville's conjecture (see [5]) that for such an X, the Bloch-Beilinson filtration has a natural, multiplicative, splitting. This statement is hard to make precise since the Bloch-Beilinson filtration is not known to exist, but for 0-cycles, this means that

$$\mathrm{CH}_0(X) = \oplus \mathrm{CH}_0(X)_i,$$

where the decomposition is given by the action of self-correspondences Γ_i of X, and where the group $\mathrm{CH}_0(X)_i$ depends only on $(i,0)$-forms on X (the correspondence Γ_i should act as 0 on $H^{j,0}$ for $j \neq i$, and Id for $i = j$). We refer to the paragraph 0.1 at the end of this introduction for the axioms of the Bloch-Beilinson filtration and we will refer to it in Section 3 when providing some evidence for our conjectures. Note that a hyper-Kähler variety X has $H^{i,0}(X) = 0$ for odd i, so the Bloch-Beilinson filtration F_{BB} has to satisfy $F_{BB}^i \mathrm{CH}_0(X) = F_{BB}^{i+1} \mathrm{CH}_0(X)$ when i is odd. Hence we are only interested in the F_{BB}^{2i}-levels, which we denote by F'^i_{BB}. Note also that there are concrete consequences of the Beauville conjecture that can be attacked directly, namely, the 0-th piece $\mathrm{CH}(X)_0$ should map isomorphically via the cycle class map to its image in $H^*(X, \mathbb{Q})$ which should be the subalgebra $H^*(X, \mathbb{Q})_{alg}$ of algebraic cycle classes, since the Bloch-Beilinson filtration is conjectured to have $F_{BB}^1 \mathrm{CH}^*(X) = \mathrm{CH}^*(X)_{hom}$. Hence there should be a subalgebra of $\mathrm{CH}^*(X, \mathbb{Q})$ which is isomorphic to the subalgebra $H^{2*}(X, \mathbb{Q})_{alg} \subset H^{2*}(X, \mathbb{Q})$ of algebraic classes. Furthermore, this subalgebra has to contain $\mathrm{NS}(X) = \mathrm{Pic}(X)$. Thus a concrete subconjecture is the following prediction (cf. [5]):

Conjecture 0.1. *(Beauville) Let X be a projective hyper-Kähler manifold. Then the cycle class map is injective on the subalgebra of $\mathrm{CH}^*(X)$ generated by divisors.*

This conjecture has been enlarged in [29] to include the Chow-theoretic Chern classes of X, $c_i(X) := c_i(T_X)$ which should thus be thought as being contained in the 0-th piece of the conjectural Beauville decomposition. Our purpose in this paper is to introduce a new set of classes which should also be put in this 0-th piece, for example, the constant cycles subvarieties of maximal dimension (namely n, with $\dim X = 2n$, because they have to be isotropic, see Section 1) and their higher dimensional generalization, which are algebraically coisotropic. Let us explain the motivation for this, starting from the study of 0-cycles.

Based on the case of $S^{[n]}$ where we have the results of [4], [23], [26] that concern the CH_0 group of a $K3$ surface but will be reinterpreted in a slightly different form in Section 2, we introduce the following decreasing filtration $S_\bullet$ on $\mathrm{CH}_0(X)$ for any hyper-Kähler manifold

(the definition can in fact be introduced for any algebraic variety but we think it is interesting only in the hyper-Kähler case). First of all, recall that the orbit O_x of a point $x \in X$ under rational equivalence, defined as

$$O_x = \{y \in X,\, y \equiv_{rat} x \text{ in } X\},$$

is a countable union of closed algebraic subsets of X. It thus has a dimension, which is the supremum of the dimensions of the closed algebraic subsets of X appearing in this union. If $\dim O_x = i$, there exists a subvariety $Z \subset X$ of dimension i, such that all points of Z are rationally equivalent to x in X. The variety Z is by definition a "constant cycle subvariety" in the sense of Huybrechts [14].

Definition 0.2. *We define $S_i X \subset X$ to be the set of points in X whose orbit under rational equivalence has dimension $\geq i$.*

The filtration $S_\bullet$ is then defined by letting $S_i \mathrm{CH}_0(X)$ be the subgroup of $\mathrm{CH}_0(X)$ generated by classes of points $x \in S_i X$.

The set $S_i X \subset X$ is a countable union of closed algebraic subsets of X and by definition $S_i \mathrm{CH}_0(X)$ is the $\mathbb{Q}$-vector subspace of $\mathrm{CH}_0(X)$ generated by the points in $S_i X$.

Let us explain the relationship between this definition and the one introduced by O'Grady in [23]. O'Grady introduces a decreasing filtration S_{OG} on the CH_0 group of a $K3$ surface S, defined (up to a shift of indices) by

$$S_{OG,i}\mathrm{CH}_0(S)_d = \{z \in \mathrm{CH}_0(S)_d,\, z \equiv_S z' + i\,o_S,\, z' \in S^{(d-i)}\}. \tag{1}$$

Here, $\mathrm{CH}_0(S)_d$ is the set of 0-cycles of degree d modulo rational equivalence on S, and $o_S \in \mathrm{CH}_0(S)_1$ is the "Beauville-Voisin" canonical 0-cycle of S introduced in [4]. The symbol $\equiv_S$ means "rationally equivalent in S". A variant of this definition where we replace rational equivalence in S by rational equivalence in $S^{[n]}$ (or equivalently $S^{(n)}$) provides a filtration $N_\bullet$ on $\mathrm{CH}_0(S^{[n]})$, namely

$$N_i \mathrm{CH}_0(S^{[n]}) = \mathrm{Im}\left((i o_S)_* : \mathrm{CH}_0(S^{[n-i]}) \to \mathrm{CH}_0(S^{[n]})\right). \tag{2}$$

This filtration exists for any surface S equipped with a base-point. It depends however on the choice of the point, or at least of the rational equivalence class of the point. Here, the only specificity of $K3$ surfaces is thus the fact that there is a canonically defined rational equivalence class of a point, namely the canonical zero-cycle o_S. It is proved in [26, Theorem 1.4] that an equivalent definition of O'Grady's filtration (1) on $\mathrm{CH}_0(S)$ can be given as follows (here we are assuming $i \geq 0$ and the assumption "dim ≥ 0" means in particular "non-empty"):

$$S_{OG,i}\mathrm{CH}_0(S)_d = \{z \in \mathrm{CH}_0(S)_d,\, \dim O_z^S \geq i\}. \tag{3}$$

Here $O_z^S \subset S^{(d)}$ is the orbit of z for rational equivalence in S, that is

$$O_z^S = \{z' \in S^{(d)},\, z' \equiv_S z\}.$$

Note that this is different from the orbit O_z of z as a point of $S^{(n)}$ or $S^{[n]}$ for *rational equivalence in $S^{[n]}$*. One has however the obvious inclusion $O_z \subset O_z^S$ which will be exploited in this paper. As we will prove in Section 2, the main result in [26] also implies

Theorem 0.3. *(Cf. Theorem 2.5) The filtration $S_\bullet$ introduced in Definition 0.2 coincides with the filtration $N_\bullet$ of (2) on 0-cycles on $S^{[n]}$, when S is a $K3$ surface and o_S is a point representing the canonical 0-cycle of S.*

Moreover, for any regular surface S and choice of point o_S, the filtration $N_\bullet$ provides a splitting of the Bloch-Beilinson filtration on $\mathrm{CH}_0(S^{[n]})$ (see Proposition 2.2 in Section 2). Hence, when S is a $K3$ surface and o_S is the canonical 0-cycle, the filtration $S_\bullet$ provides a splitting of the Bloch-Beilinson filtration on $\mathrm{CH}_0(S^{[n]})$. Analogous results have been obtained by Lin [17] in the case of genralized Kummer varieties. Our hope and guiding idea in this paper is that this is the general situation for hyper-Kähler manifolds. A first concrete conjecture in this direction is the following:

Conjecture 0.4. *Let X be projective hyper-Kähler manifold of dimension $2n$. Then for any nonnegative integer $i \leq n$, the set*

$$S_i X := \{x \in X, \dim O_x \geq i\}$$

has dimension $2n - i$.

The case $i = n$, that is Lagrangian constant cycles subvarieties, was first asked by Pacienza (oral communication). This conjecture and the axioms of Bloch-Beilinson filtration would imply in particular that the natural map

$$S_i \mathrm{CH}_0(X) \to \mathrm{CH}_0(X)/F'^{n-i+1}_{BB}\mathrm{CH}_0(X)$$

is surjective (see Lemma 3.9). We conjecture in fact that this map is an isomorphism (cf. Conjecture 0.8).

A good evidence for Conjecture 0.4 is provided by the results of Charles and Pacienza [9], which deal with the deformations of $S^{[n]}$ (case $i = 1$), and the deformations of $S^{[2]}$ (case $i = 2$), and Lin [16] who constructs constant cycles Lagrangian subvarieties in hyper-Kähler manifolds admitting a Lagrangian fibration. Another evidence is given by the complete family of hyper-Kähler 8-folds constructed by Lehn-Lehn-Sorger-van Straten in [18] that we will study in Section 4 (see Corollary 4.9). We will prove there that they satisfy Conjecture 0.4. In fact, we describe a parametrization of them which should make accessible for them a number of conjectures made in this paper, [5], or [29], by reduction to the case of the variety of lines of a cubic fourfold.

We will explain in Section 1 that Conjecture 0.4 contains as a by-product an existence conjecture for algebraically coisotropic (possibly singular) subvarieties of X of codimension i. By this we mean the following:

Definition 0.5. *A subvariety $Z \subset X$ is coisotropic if for any $z \in Z_{reg}$, $T^{\perp \sigma}_{Z,z} \subset T_{Z,z}$.*

Here σ is the $(2,0)$-form on X. Given a coisotropic subvariety $Z \subset X$, the open set Z_{reg} has an integrable distribution (a foliation) given by the vector subbundle $T^{\perp \sigma}_Z$, with fiber $T^{\perp \sigma}_{Z,z} \subset T_{Z,z}$, or equivalently, the kernel of the restricted form $\sigma_{|Z}$ which has by assumption the constant minimal rank.

Definition 0.6. *A subvariety $Z \subset X$ of codimension i is algebraically coisotropic if the distribution above is algebraically integrable, by which we mean that there exists a rational map $\phi : Z \dashrightarrow B$ onto a variety B of dimension $2n - 2i$ such that $\sigma_{|Z}$ is the pull-back to Z of a $(2,0)$-form on B, $\sigma_{|Z} = \phi^* \sigma_B$.*

Any divisor in a hyper-Kähler variety is coisotropic. However, only few of them are algebraically coisotropic: In fact, Amerik and Campana prove in [2] that if $n \geq 2$, a smooth divisor is algebraically coisotropic if and only if it is uniruled. The regularity assumption here is of course crucial.

The link between Conjecture 0.4 and the existence of algebraically coisotropic subvarieties is provided by Mumford's theorem [22] on pull-backs of holomorphic forms and rational equivalence. The following result will be proved in Section 1, where we will also describe the restrictions satisfied by the cohomology classes of coisotropic subvarieties.

Theorem 0.7. *(Cf. Theorem 1.3) Let Z be a codimension i subvariety of a hyper-Kähler manifold X. Assume that any point of Z has an orbit of dimension $\geq i$ under rational equivalence in X (that is $Z \subset S_i X$). Then Z is algebraically coisotropic and the fibers of the isotropic fibration are i-dimensional orbits of X for rational equivalence.*

In Section 3, starting from the case of $S^{[n]}$, where things work very well thanks to Theorem 0.3, we will then discuss the following "conjecture":

Conjecture 0.8. *Let X be a projective hyper-Kähler manifold of dimension $2n$. Then the filtration $S_\bullet$ is opposite to the filtration F'_{BB} and thus provides a splitting of it.*

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Concretely, this means that

$$S_i \mathrm{CH}_0(X) \cong \mathrm{CH}_0(X)/F'^{n-i+1}_{BB}\mathrm{CH}_0(X) \tag{4}$$

for any $i \geq 0$. Assuming (4) holds, we have a natural decomposition of CH_0 into a direct sum

$$\mathrm{CH}_0(X) = \oplus_j \mathrm{CH}_0(X)_{2j},$$

where $\mathrm{CH}_0(X)_{2j} := S_{n-j}\mathrm{CH}_0(X) \cap F^{2j}_{BB}\mathrm{CH}_0(X)$, and this gives a splitting of the Bloch-Beilinson filtration.

We will explain in Section 3 how this conjecture would fit with the expected multiplicativity property of the Beauville filtration, and in particular with the following expectation:

Conjecture 0.9. *The classes of codimension i subvarieties of X contained in $S_i X$ belong to the 0-th piece of the Beauville decomposition and their cohomology classes generate the space of coisotropic classes.*

Again, this has concrete consequences that can be investigated for themselves and independently of the existence of a Bloch-Beilinson filtration, namely the fact that the cycle class map is injective on the subring of $\mathrm{CH}(X)$ generated by these classes and divisor classes.

The paper is organized as follows: In section 1, we will describe the link between families of constant cycles subvarieties and algebraically coisotropic subvarieties. In section 2, we will compare the filtrations $N_\bullet$ and $S_\cdot$ for $X = \mathrm{Hilb}(K3)$. Section 3 is devoted to stating conjectures needed to construct a Beauville decomposition starting from the filtration $S_\bullet$. Finally, Section 4 will provide a number of geometric constructions and various evidences for these conjectures, in three cases: Hilbert schemes of $K3$ surfaces, Fano varieties of lines of cubic fourfolds, and finally the more recent 8-folds constructed in [18] starting from the Hilbert scheme of cubic rational curves on cubic fourfolds.

Thanks. I thank F. Charles, L. Fu, K. O'Grady, G. Pacienza, G. Saccà and Ch. Vial for interesting discussions related to this paper.

0.1 Bloch-Beilinson filtration

The Bloch-Beilinson filtration $F^\bullet_{BB}$ on the Chow groups with $\mathbb{Q}$-coefficients of smooth projective varieties is a decreasing filtration which is subject to the following axioms:

1. It is preserved by correspondences: If $\Gamma \in \mathrm{CH}(X \times Y)$, then for any i, $\Gamma_*(F^i\mathrm{CH}(X)) \subset F^i\mathrm{CH}(Y)$.

2. $F^0\mathrm{CH}(X) = \mathrm{CH}(X)$, $F^1\mathrm{CH}(X) = \mathrm{CH}(X)_{hom}$.

3. It is multiplicative: $F^i\mathrm{CH}(X) \cdot F^j\mathrm{CH}(X) \subset F^{i+j}\mathrm{CH}(X)$, where $\cdot$ is the intersection product.

4. $F^{i+1}\mathrm{CH}^i(X) = 0$ for all i and X.

Note that items 2 and 3 imply that a correspondence $\Gamma \in \mathrm{CH}(X \times Y)$ which is cohomologous to 0 shifts the Bloch-Beilinson filtration:

$$\Gamma_*(F^i\mathrm{CH}(X)) \subset F^{i+1}\mathrm{CH}(Y). \tag{5}$$

One can also be more precise at this point, namely asking that $Gr^i_F\mathrm{CH}^l(X)$ is governed only by $H^{2l-i}(X)$ and its Hodge structure. Then (5) becomes:

$$\Gamma_*(F^i\mathrm{CH}^l(X)) \subset F^{i+1}\mathrm{CH}^{l+r}(Y), \tag{6}$$

if $[\Gamma]_* : H^{2l-i}(X, \mathbb{Q}) \to H^{2l+2r-i}(Y, \mathbb{Q})$ vanishes (so we are considering only one Künneth component of $[\Gamma] \in H^{2r+2n}(X \times Y, \mathbb{Q})$, where $n = \dim X$, $\Gamma \in \mathrm{CH}^{r+n}(X \times Y)$). The

reason why, assuming a Bloch-Beilinson filtration exists, the graded pieces of it for 0-cycles depend only on holomorphic forms is the fact that according to (5), $Gr_F^i \mathrm{CH}_0(X)$ should be governed by the cohomology group $H^{2n-i}(X, \mathbb{Q})$ or dually $H^i(X, \mathbb{Q})$. On the other hand, $\mathrm{CH}_0(X)$ is not sensitive to the cohomology of X which is of geometric coniveau ≥ 1, that is, supported on a divisor, hence assuming the generalized Hodge conjecture holds, it should be sensitive only to the group $H^i(X, \mathbb{Q})/N^1 H^i(X, \mathbb{Q})$, where $N^1 H^i(X, \mathbb{Q})$ is the maximal Hodge substructure of $H^i(X, \mathbb{Q})$ which is of Hodge coniveau ≥ 1, that is, contained in $F^1 H^i(X, \mathbb{C})$. But clearly $H^i(X, \mathbb{Q})/N^1 H^i(X, \mathbb{Q}) = 0$ if and only if $H^{i,0}(X) = 0$.

We will refer to the set of axioms above in the form "if a Bloch-Beilinson filtration exists", with the meaning that it exists for all X (this is necessary as the axiom 1 is essential).

Note that there exist many varieties for which we have natural candidates for the Bloch-Beilinson filtration (for example surfaces, complete intersections in projective space) but apart from curves and more generally varieties with representable Chow groups, none are known to satisfy the full set of axioms above.

1 Constant cycles subvarieties and coisotropic classes

Let X be a smooth projective variety over $\mathbb{C}$ and let $f : Z \to X$ be a morphism from a smooth projective variety admitting a surjective morphism $p : Z \to B$, where B is smooth, with the following property:

(*) *The fibers of p map via f to "constant cycle" subvarieties*,

that is, all points in a given fiber of p map via f to rationally equivalent points in X. Mumford's theorem [22] then implies:

Lemma 1.1. *Under assumption (*), for any holomorphic form α on X, there is a holomorphic form α_B on B such that $f^*\alpha = p^*\alpha_B$.*

Proof. Let $B' \subset Z$ be a closed subvariety such that $p_{|B'} =: p'$ is generically finite and let $N := \deg B'/B$, $f' := f_{|B'}$. We have two correspondences between Z and X, namely Γ_f, which is given by the graph of f, and Γ', which is defined as the composition with p of the correspondence

$$\Gamma'' := \frac{1}{N}(p', Id)_*(\Gamma_{f'})$$

between B and X. Assumption (*) says that

$$(\Gamma_f)_* = \Gamma'_* : \mathrm{CH}_0(Z) \to \mathrm{CH}_0(X).$$

Mumford's theorem then says that for any holomorphic form α on X, one has

$$f^*\alpha = \Gamma_f^*\alpha = {\Gamma'}^*\alpha = (\Gamma'' \circ p)^*\alpha = p^*({\Gamma''}^*\alpha),$$

which proves the result with $\alpha_B = {\Gamma''}^*\alpha$. $\square$

We now consider the case where X is a projective hyper-Kähler manifold of dimension $2n$ with holomorphic 2-form σ. A particular case of Lemma 1.1 is the case where B is a point, which gives the following statement:

Corollary 1.2. *Let $\Gamma \subset X$ be a constant cycle subvariety. Then Γ is an isotropic subvariety, that is*

$$\sigma_{|\Gamma_{reg}} = 0.$$

In particular, $\dim \Gamma \leq n$, and in the case of equality, Γ is a Lagrangian (possibly singular) subvariety.

In this paper, we will discuss coisotropic subvarieties whose study started only recently (see [2], [3]). Such subvarieties can be constructed applying the following result:

Theorem 1.3. *Assume $S_i X$ (see Definition 0.2) contains a closed algebraic subvariety Z of codimension $\leq i$. Then:*

(i) The codimension of Z is equal to i and (the smooth locus of) Z is algebraically coisotropic (see Definition 0.6).

(ii) Furthermore, the general fibers of the coisotropic fibration $Z \dashrightarrow B$ are constant cycles subvarieties of X of dimension i.

Proof. By assumption, for any $z \in Z$, there is a subvariety $K_z \subset X$ which is contained in O_z and has dimension i. Using the countability of Hilbert schemes, there exists a generically finite cover $\alpha : Z' \to Z \subset X$, and a family $p : K \to Z'$ of varieties of dimension i over Z', together with a morphism $f : K \to X$ satisfying the following properties:

1. For any $k \in K$, $f(k)$ is rationally equivalent to $\alpha \circ p(k)$ in X.

2. The morphism f restricted to the general fiber of p is generically finite over (or even birational to) its image in X.

Property 1 and Lemma 1.1 imply that

$$f^* \sigma = p^*(\alpha^* \sigma) \text{ in } H^0(K, \Omega_K^2). \tag{7}$$

Formula (7) tells us that the 2-form $f^* \sigma$ has the property that for the general point $k \in K$, the tangent space $T_{K_k, k}$ at the point k to the fiber K_k of p passing through k is contained in the kernel of $f^* \sigma_{|k} \in \Omega_{K,k}^2$. Equivalently, the vector space $f_*(T_{K_k,k}) \subset T_{X,f(k)}$ is contained in the kernel of the form $\sigma_{|f(K)}$ at the point $f(k)$. From now on, let us denote $Z'' := f(K) \subset X$. By assumption 2, $f_*(T_{K_k})$ has dimension i for general k, and because σ is nondegenerate, this implies that the rank of the map f at k is at most $2n - i$. By Sard's theorem, it follows that Z'' has dimension $\leq 2n - i$ and that the generic rank of the 2-form $\sigma_{|Z''}$ is at most $2n - 2i$. Hence the rank of the 2-form $f^* \sigma = p^*(\alpha^* \sigma)$ on K is at most $2n - 2i$, and as p is dominating and α is generically finite, this implies that the rank of $\sigma_{|Z_{reg}}$ is $\leq 2n - 2i$. Hence Z satisfies codim $Z \geq i$, hence in fact codim $Z = i$, and furthermore the rank of $\sigma_{|Z_{reg}}$ is equal to $2n - 2i$, so that Z is a coisotropic subvariety of X.

Let us now prove that Z is algebraically coisotropic. We proved above that the varieties $f(K_k) \subset Z''$ have their tangent space contained in the kernel of the form $\sigma_{|Z''}$. Let now $\Gamma \subset K$ be a subvariety which is generically finite over Z' and dominates Z''. Such a Γ exists since we proved that $\dim Z'' \leq 2n - i = \dim Z'$. Denote by

$$f_\Gamma : \Gamma \to Z'', \ q_\Gamma : \Gamma \to Z$$

the restrictions to Γ of f and $\alpha \circ p$ respectively. Restricting (7) to Γ gives

$$f_\Gamma^*(\sigma_{|Z''}) = q_\Gamma^*(\sigma_{|Z}) \text{ in } H^0(\Gamma, \Omega_\Gamma^2). \tag{8}$$

The varieties $f_\Gamma^{-1}(f(K_k))$ are tangent to the kernel of the form $f_\Gamma^*(\sigma_{|Z''})$, hence of the form $q_\Gamma^*(\sigma_{|Z})$, which means that their images

$$q_\Gamma(f_\Gamma^{-1}(f(K_k))) \subset Z$$

are tangent to the kernel of the form $\sigma_{|Z}$. As they are (for general k) of dimension $\geq i$ because q_Γ is generically finite, and as $\dim Z = 2n - i$, one concludes that they are in fact of dimension i, and are thus algebraic integral leaves of the distribution on Z_{reg} given by $\mathrm{Ker}\, \sigma_{|Z}$. One still needs to explain why this is enough to imply that Z is algebraically coisotropic. We already proved that Z is swept-out by algebraic varieties Z_t which are i-dimensional and tangent to the distribution on Z_{reg} given by $\mathrm{Ker}\, \sigma_{|Z}$. We just have to construct a dominant rational map $Z \dashrightarrow B$ which admits the Z_t's as fibers. However, we observe that if B is the Hilbert scheme parameterizing i-dimensional subvarieties of Z tangent to this distribution along Z_{reg}, and $M \to B$ is the universal family of such subvarieties, the morphism $M \to X$ is

birational since there is a unique leaf of the distribution at any point of Z_{reg}. This provides us with the desired fibration. This proves (i).

(ii) We have to prove that the fibers of the coisotropic fibration of Z are constant cycle subvarieties of X of dimension i. By construction, they are the varieties $q_\Gamma(f_\Gamma^{-1}(f(K_k)))$, $k \in K$. But $f(K_k)$ is by definition a constant cycle subvariety of dimension i of X, all of whose points are rationally equivalent to $\alpha \circ p(k)$, by condition 1. It follows from condition 1 again that all points in $q_\Gamma(f_\Gamma^{-1}(f(K_k)))$ are rationally equivalent in X to $f(k)$.

$\square$

1.1 Classes of coisotropic subvarieties

This subsection is devoted to the description of the restrictions on the cohomology classes of coisotropic subvarieties of a hyper-Kähler manifold X of dimension $2n \geq 4$. More precisely, we will only study those classes which can be written as a polynomial in divisor classes and the class $c \in S^2 H^2(X, \mathbb{Q}) \subset H^4(X, \mathbb{Q})$ defined as follows: The Beauville-Bogomolov form q on $H^2(X, \mathbb{R})$, which is characterized up to a coefficient by the condition that for any $\lambda \in H^2(X, \mathbb{Q})$,

$$\int_X \lambda^{2n} = \mu_X q(\lambda)^n \tag{9}$$

for some nonzero constant μ_X, is nondegenerate. The form q provides an invertible symmetric map

$$H^2(X, \mathbb{Q}) \to H^2(X, \mathbb{Q})^*$$

with inverse

$$c \in \mathrm{Hom}_{sym}(H^2(X, \mathbb{Q})^*, H^2(X, \mathbb{Q})) = S^2 H^2(X, \mathbb{Q}) \subset H^4(X, \mathbb{Q}).$$

It is also easy to check that for every nonnegative integer $i \leq n$, there exists a nonzero constant $\mu_{i,X}$ such that for any $\lambda \in H^2(X, \mathbb{Q})$

$$\int_X c^i \lambda^{2n-2i} = \mu_{i,X} q(\lambda)^{n-i}. \tag{10}$$

Our goal in this section is to compute the "coisotropic classes" which can be written as polynomials $P(c, l_j)$, where $l_j \in \mathrm{NS}^2(X)$. The following lemma justifies Definition 1.5 of a coisotropic class:

Lemma 1.4. *Let $Z \subset X$ be a codimension i subvariety and let $[Z] \in H^{2i}(X, \mathbb{Q})$ its cohomology class. Then Z is coisotropic if and only*

$$[\sigma]^{n-i+1} \cup [Z] = 0 \text{ in } H^{2n+2}(X, \mathbb{C}), \tag{11}$$

where $[\sigma] \in H^2(X, \mathbb{C})$ is the cohomology class of σ.

Proof. Indeed, Z is coisotropic if and only if the restricted form $\sigma_{|Z}$ has rank $2n - 2i$ on Z_{reg}, which is equivalent to the vanishing

$$\sigma_{|Z_{reg}}^{n-i+1} = 0 \text{ in } H^0(Z_{reg}, \Omega_{Z_{reg}}^{2n-2i+2}). \tag{12}$$

We claim that this vanishing in turn is equivalent to the condition

$$[\sigma]^{n-i+1} \cup [Z] = 0 \text{ in } H^{2n+2}(X, \mathbb{C}). \tag{13}$$

Indeed, (12) clearly implies (13). In the other direction, we can assume $i \geq 2$ since for $i = 1$ there is nothing to prove (all divisor classes are coisotropic). The vanishing of $[\sigma]^{n-i+1} \cup [Z]$ in $H^{2n+2}(X, \mathbb{C})$ then implies that the cup-product $l^{i-2} \cup [\sigma]^{n-i+1} \cup [\bar{\sigma}]^{n-i+1} \cup [Z]$ vanishes in $H^{4n}(X, \mathbb{C})$, where l is the first Chern class of a very ample line bundle L on X, so that

(12) implies the vanishing of the integral $\int_{Z'} \sigma^{n-i+1} \wedge \overline{\sigma}^{n-i+1}$, where $Z' \subset Z$ is the complete intersection of $i - 2$ general members in $|L|$ and the integral has to be understood as an integral on a desingularization of Z'. But the form $\sigma^{n-i+1}_{|Z'} \wedge \overline{\sigma}^{n-i+1}_{|Z'}$ can be written (along Z'_{reg}) as $\pm f\,\nu$ where ν is a volume form on Z'_{reg}, and the continuous function f is real nonnegative.

It follows that the vanishing of the integral implies that $\sigma^{n-i+1}_{|Z'} = 0$ and as the tangent space of Z' at a given point can be chosen arbitrarily (assuming L ample enough), this implies that $\sigma^{n-i+1}_{Z_{reg}} = 0$. (Of course, if Z is smooth, we can directly apply the second Hodge-Riemann relations to Z.)

$\square$

We thus make the following definition

Definition 1.5. *A coisotropic class on X of degree $2i$ is a Hodge class z of degree $2i$ which satisfies the condition*

$$[\sigma]^{n-i+1} \cup z = 0 \text{ in } H^{2n+2}(X, \mathbb{C}). \tag{14}$$

The contents of Lemma 1.4 is thus that an *effective* class is the class of a coisotropic subvariety if and only if it is coisotropic.

Remark 1.6. It is not known if the class c is algebraic. However it is known to be algebraic for those X which are deformations of $\mathrm{Hilb}^n(K3)$ (see [19]). In general however, the class $c_2(X)$ is of course always algebraic and its projection to $S^2 H^2(X, \mathbb{Q})$ is a nonzero multiple of c. (This projection is well-defined, using the canonical decomposition $H^4(X, \mathbb{Q}) \cong S^2 H^2(X, \mathbb{Q}) \oplus S^{2n-2} H^2(X, \mathbb{Q})^\perp$.)

Formula (14) in Definition 1.5 may give the feeling that the space of coisotropic classes depends on the period point $[\sigma] \in H^2(X, \mathbb{C})$. This is not true, as we are going to show, at least if we restrict to classes which can be written as polynomials $P(c, l_j)$ involving only divisor classes and the class c, and X is very general in moduli.

To state the next theorem, we need to introduce some notation. Let $H^2(X, \mathbb{Q})_{tr} \subset H^2(X, \mathbb{Q})$ be the orthogonal complement of $\mathrm{NS}(X)_{\mathbb{Q}} = \mathrm{Hdg}^2(X, \mathbb{Q})$ with respect to the Beauville-Bogomolov form q, and let $QH^2(X, \mathbb{C})_{tr} \subset H^2(X, \mathbb{Q})_{tr}$ be the quadric defined by q. The $\mathbb{Q}$-vector space $H^2(X, \mathbb{Q})$ splits as the orthogonal direct sum

$$H^2(X, \mathbb{Q})_{tr} \oplus^\perp \mathrm{NS}(X)_{\mathbb{Q}},$$

and it is immediate to check that $c \in S^2 H^2(X, \mathbb{Q})$ decomposes as

$$c = c_{tr} + c_{alg},$$

where $c_{tr} \in S^2 H^2(X, \mathbb{Q})_{tr}$ and $c_{alg} \in S^2 \mathrm{NS}(X)_{\mathbb{Q}}$.

If X is a projective hyper-Kähler manifold, let $\rho := \rho(X)$. We will say that X is ρ-very general if X corresponds to a very general point in the family of deformations of X preserving $\mathrm{NS}(X)$, that is, with Picard number at least ρ.

Theorem 1.7. *(i) A Hodge class $z \in \mathrm{Hdg}^{2i}(X, \mathbb{Q})$ is coisotropic if*

$$\alpha^{n-i+1} \cup z = 0 \text{ in } H^{2n+2}(X, \mathbb{C}) \tag{15}$$

for any $\alpha \in QH^2(X, \mathbb{C})_{tr}$. If X is ρ-very general, the two conditions (being coisotropic and satisfying (15)) are equivalent.

(ii) If $z = P(c, l_j)$ is a polynomial as above, z is coisotropic if, for any $\alpha \in QH^2(X, \mathbb{C})_{tr}$, for any $\beta \in H^2(X, \mathbb{Q})$,

$$\alpha^{n-i+1} \cup \beta^{n-1} \cup z = 0 \text{ in } H^{4n}(X, \mathbb{C}) \stackrel{\int_X}{\cong} \mathbb{C}. \tag{16}$$

If X is ρ-very general, the two conditions (being coisotropic and satisfying (16)) are equivalent.

(iii) If $\rho(X) = 1$, the space of coisotropic classes which are polynomials $P(c, l)$, $l \in \mathrm{NS}(X)$, is of dimension ≥ 1, for any $0 \leq i \leq n$. If $\rho(X) = 2$, it has dimension $\geq i + 1$.

(iv) Assumptions being as in (iii), if l is an ample class on X, a nonzero coisotropic class

$$z = \lambda_0 l^i + \lambda_1 l^{i-2} c_{tr} + \ldots + \lambda_j l^{i-2j} c_{tr}^j, \ j := \lfloor i/2 \rfloor$$

has $\lambda_0 \neq 0$. In particular, the space of such classes has dimension exactly 1.

Proof. (i) The first statement in (i) follows from Definition 1.5 since $\sigma \in QH^2(X, \mathbb{C})_{tr}$. For the (partial) converse, as X is assumed ρ-very general, its period point $[\sigma]$ is a very general point of $QH^2(X, \mathbb{C})_{tr}$ and it follows that the Mumford-Tate group $MT(X)$ of the Hodge structure on $H^2(X, \mathbb{Q})_{tr}$ is equal to the orthogonal group $SO(H^2(X, \mathbb{Q})_{tr}, q)$. Denote by $< QH^2(X, \mathbb{C})_{tr}^{n-i+1} >$ the complex vector subspace of $S^{n-i+1} H^2(X, \mathbb{C})_{tr}$ generated by x^{n-i+1} for all x satisfying $q(x) = 0$. This vector space is defined over $\mathbb{Q}$, that is, it is the complexification of a $\mathbb{Q}$-vector subspace

$$< QH^2(X, \mathbb{Q})_{tr}^{n-i+1} > \subset S^{n-i+1} H^2(X, \mathbb{Q})_{tr},$$

and it is in fact a sub-Hodge structure of $S^{n-i+1} H^2(X, \mathbb{Q})_{tr}$. This Hodge structure is simple under our assumption. Indeed, the Mumford-Tate group $MT(X)$ is the full orthogonal group $SO(H^2(X, \mathbb{Q})_{tr}, q)$ and $< QH^2(X, \mathbb{C})_{tr}^{n-i+1} >$ is an irreducible representation of $SO(H^2(X, \mathbb{C})_{tr}, q)$. This implies a fortiori that $< QH^2(X, \mathbb{Q})_{tr}^{n-i+1} >$ is an irreducible representation of $SO(H^2(X, \mathbb{Q})_{tr}, q)$, and the simplicity of the Hodge structure follows since by definition of the Mumford-Tate group $MT(X)$, sub-Hodge structures correspond to sub-representations of $MT(X)$. The proof of (i) is now immediate. First of all, by (14), a Hodge class z of degree $2i$ on X is a coisotropic class if and only if

$$[\sigma]^{n-i+1} \cup z = 0 \text{ in } H^{2n+2}(X, \mathbb{C}). \tag{17}$$

An equivalent way of stating this property is to say that the class

$$[\sigma]^{n-i+1} \in < QH^2(X, \mathbb{C})_{tr}^{n-i+1} >$$

is annihilated by the morphism of Hodge structures

$$\cup z : < QH^2(X, \mathbb{Q})_{tr}^{n-i+1} > \to H^{2n+2}(X, \mathbb{Q}). \tag{18}$$

As the Hodge structure on the left is simple, this morphism is either injective or identically 0, so the vanishing (17) is equivalent to the vanishing of $\cup z$ in (18).

(ii) Using (i), it suffices to prove that the two conditions (15) and (16) are equivalent for $z = P(c, l_i)$. This is proved as follows. As the class c belongs to $S^2 H^2(X, \mathbb{Q})$, the class $z = P(c, l_j)$, $l_j \in \mathrm{NS}(X)$, belongs to the image of $S^i H^2(X, \mathbb{Q})$ in $H^{2i}(X, \mathbb{Q})$. We now use the results of [8] which say that the subalgebra of $H^*(X, \mathbb{Q})$ generated by $H^2(X, \mathbb{Q})$ is Gorenstein, that is self-dual with respect to Poincaré duality. Hence a class $\alpha^{n-i+1} \cup z$, with $z = P(c, j_j)$, vanishes if and only if its cup-product with any class in $S^{n-1} H^2(X, \mathbb{Q}) \subset H^{2n-2}(X, \mathbb{Q})$ vanishes.

(iii) It suffices to prove the statement for classes satisfying (15) since we know by (i) that they are coisotropic. As shown by Fujiki [11], it follows from formula (10) that for any $l, e \in H^2(X, \mathbb{Q})$, and any polynomial $P(c, l, e)$ of weighted degree i, there exists a polynomial R depending only on P, in the variables $q(\alpha, l), q(\alpha, e), q(\alpha), q(\beta, l), q(\beta, e), q(\beta), q(\alpha, \beta)$, such that for any $\alpha, \beta \in H^2(X, \mathbb{Q})$,

$$\int_X \alpha^{n-i+1} \cup \beta^{n-1} \cup P(c, l, e) = R(q(\alpha, l), q(\alpha, e), q(\alpha), q(\beta, l), q(\beta, e), q(\beta), q(\alpha, \beta)). \tag{19}$$

Assume now that $\alpha \in QH^2(X, \mathbb{Q})_{tr}$ and l, e form a basis of $\mathrm{NS}(X)$ (so $\rho(X) = 2$). Then

$$q(\alpha) = 0, \ q(\alpha, e) = 0, \ q(\alpha, l) = 0$$

and thus (19) becomes

$$\int_X \alpha^{n-i+1} \cup \beta^{n-1} \cup P(c, l, e) = R_0(q(\beta, l), q(\beta, e), q(\beta), q(\alpha, \beta)), \tag{20}$$

where R_0 is the restriction of R to the subspace where the first three coordinates vanish. The left hand side is homogeneous of degree $n - i + 1$ in α and homogeneous of degree $n - 1$ in β, so we conclude that the right hand side has to be of the form

$$q(\alpha, \beta)^{n-i+1} R_1(q(\beta, l), q(\beta, e), q(\beta)),$$

where the polynomial R_1 has to be homogeneous of degree $i - 2$ in β, hence R_1 has to be of weighted degree $i - 2$ in the three variables $q(\beta, l)$, $q(\beta, e)$ of degree 1 and $q(\beta)$ of degree 2. In conclusion, the space of coisotropic classes z which can be written as polynomials in c, l and e is the kernel of the linear map $P \mapsto R_1$ which sends the space $W^i_{2,1,1}$ of polynomials of weighted degree i in the variables c, l, e to the space of polynomials $W^{i-2}_{2,1,1}$ of weighted degree $i - 2$ in $q(\beta)$, $q(\beta, l)$, $q(\beta, e)$. Thus its kernel has dimension $\geq \dim W^i_{2,1,1} - \dim W^{i-2}_{2,1,1} = i + 1$.

The argument when $\rho(X) = 1$ is exactly the same, except that there is only one variable l instead of the two variables l, e. We conclude as before that the space of coisotropic classes z which can be written as polynomials in c, l has dimension $\geq \dim W^i_{2,1} - \dim W^{i-2}_{2,1} = 1$.

(iv) Writing z as a polynomial in c_{tr} and l, the non-vanishing of the l^i coefficient is equivalent to the fact that z is not of the form $c_{tr} \cup z'$, for some Hodge class z' of degree $2i - 4$. So assume that $z = c_{tr} \cup z'$ is coisotropic. One then has

$$\sigma^{n-i+1} \cup c_{tr} \cup z' = 0 \text{ in } H^{2n+2}(X, \mathbb{C}). \tag{21}$$

We now have the following lemma

Lemma 1.8. *The cup-product map*

$$\cup c_{tr} : H^{2n-2}(X, \mathbb{C}) \to H^{2n+2}(X, \mathbb{C})$$

is injective on the subspace $S^{n-1} H^2(X, \mathbb{C}) \subset H^{2n-2}(X, \mathbb{C})$.

Proof. We first claim that the cup-product map

$$\cup c : H^{2n-2}(X, \mathbb{C}) \to H^{2n+2}(X, \mathbb{C})$$

is injective on the subspace $S^{n-1} H^2(X, \mathbb{C}) \subset H^{2n-2}(X, \mathbb{C})$. (Note that this is equivalent to saying that it induces an isomorphism between $S^{n-1} H^2(X, \mathbb{C}) \subset H^{2n-2}(X, \mathbb{C})$ and the subspace of $H^{2n+2}(X, \mathbb{C})$ which is the image of $S^{n+1} H^2(X, \mathbb{C})$.) The claim follows from the fact that the kernel of the map

$$S^{n+1} H^2(X, \mathbb{C}) \to H^{2n+2}(X, \mathbb{C})$$

is according to [8] equal to $< QH^2(X, \mathbb{C}) >_{n+1}$. On the other hand, representation theory of the orthogonal group shows that the natural map

$$\oplus_k < QH^2(X, \mathbb{C}) >_{n+1-2k} \to S^{n+1} H^2(X, \mathbb{C})$$

which is the multiplication by c^k on $< QH^2(X, \mathbb{C}) >_{n+1-2k}$, is an isomorphism. Writing a similar decomposition for $S^{n-1} H^2(X, \mathbb{C})$ makes clear that the image of the multiplication by c from $S^{n-1} H^2(X, \mathbb{C})$ to $S^{n+1} H^2(X, \mathbb{C})$ and the subspace $< QH^2(X, \mathbb{C}) >_{n+1}$ have their intersection reduced to 0, This proves the claim.

To conclude the proof, we observe that the intersection pairing q restricted to $H^2(X,\mathbb{C})_{tr}$ is nondegenerate, and that $c_{tr} \in S^2 H^2(X,\mathbb{C})_{tr}$ is the analogue of the class c for $(H^2(X,\mathbb{C})_{tr}, q)$. So Lemma 1.8 applies to the multiplication maps

$$c_{tr} : S^{k-1} H^2(X,\mathbb{C})_{tr} \to S^{k+1} H^2(X,\mathbb{C})_{tr}/ < QH^2(X,\mathbb{C})_{tr} >_{k+1}, \qquad (22)$$

showing they are all injective.

To conclude, we observe that as $H^2(X,\mathbb{C}) = H^2(X,\mathbb{C})_{tr} \oplus \mathbb{C}l$, the space $S^{n-1} H^2(X,\mathbb{C})$ decomposes as

$$S^{n-1} H^2(X,\mathbb{C}) = \oplus_{k=0}^{n-1} l^k S^{n-1-k} H^2(X,\mathbb{C})_{tr}, \qquad (23)$$

and similarly

$$S^{n+1} H^2(X,\mathbb{C}) = \oplus_{k=0}^{n+1} l^k S^{n+1-k} H^2(X,\mathbb{C})_{tr}. \qquad (24)$$

The three spaces $S^{n-1} H^2(X,\mathbb{C})$, $S^{n+1} H^2(X,\mathbb{C})$ and $< QH^2(X,\mathbb{C}) >_{n+1} \subset S^{n+1} H^2(X,\mathbb{C})$ are filtered by the respective subspaces

$$l^k S^{n-1-k} H^2(X,\mathbb{C}), \ l^k S^{n+1-k} H^2(X,\mathbb{C}), \ l^k S^{n+1-k} H^2(X,\mathbb{C}) \cap < QH^2(X,\mathbb{C}) >_{n+1},$$

and denoting by $L^\bullet$ these filtrations, it is easy to check that

$$Gr_L^k S^{n-1} H^2(X,\mathbb{C}) \cong S^{n-1-k} H^2(X,\mathbb{C})_{tr}, \qquad (25)$$
$$Gr_L^k S^{n+1} H^2(X,\mathbb{C}) \cong S^{n+1-k} H^2(X,\mathbb{C})_{tr},$$
$$Gr_L^k < QH^2(X,\mathbb{C}) >_{n+1} \cong < QH^2(X,\mathbb{C})_{tr} >_{n+1-k} .$$

The multiplication (or cup-product) map by c_{tr} is compatible with the filtrations $L^\bullet$ (where we also use the induced filtration on the quotient $S^{n+1} H^2(X,\mathbb{C})/ < QH^2(X,\mathbb{C}) >_{n+1} \subset H^{2n+2}(X,\mathbb{C})$) and looking at the identifications (25), we see that it induces between the graded pieces $Gr_L^k S^{n-1} H^2(X,\mathbb{C})$, $Gr_L^k S^{n+1} H^2(X,\mathbb{C})/ < QH^2(X,\mathbb{C}) >_{n+1}$ the isomorphisms

$$c_{tr} : S^{n-1-k} H^2(X,\mathbb{C})_{tr} \cong S^{n+1-k} H^2(X,\mathbb{C})_{tr}/ < QH^2(X,\mathbb{C}) >_{n+1-k}$$

of (22).

As the multiplication by c_{tr} induces an isomorphisms on each graded piece, it is an isomorphism.

$\square$

Using Lemma 1.8, (21) implies that

$$\sigma^{n-i+1} \cup z' = 0 \text{ in } H^{2n-2}(X,\mathbb{C}). \qquad (26)$$

On the other hand, we know by [8] that the natural map $S^* H^2(X,\mathbb{C}) \to H^*(X,\mathbb{C})$ is injective in degree $* \leq n$, so that (26) implies $z' = 0$, hence $z = 0$. So (iv) is proved.

$\square$

2 The case of $\mathrm{Hilb}^n(K3)$

Let S be a smooth surface and let $x_1, \ldots, x_i \in S$ be i different points. We then get rational maps

$$S^{[n-i]} \dashrightarrow S^{[n]},$$

$$z \mapsto \{x_1, \ldots, x_i\} \cup z,$$

which is well-defined at the points $z \in S^{[n-i]}$ parameterizing subschemes of S disjoint from the x_l's. These maps induce morphisms $\mathrm{CH}_0(S^{[n-i]}) \to \mathrm{CH}_0(S^{[n]})$.

Remark 2.1. If we work with the symmetric products $S^{(k)}$ instead of the Hilbert schemes $S^{[k]}$, the indeterminacies of these maps do not appear anymore. Furthermore, as the Hilbert-Chow map has rationally connected fibers, $\mathrm{CH}_0(S^{[k]}) = \mathrm{CH}_0(S^{(k)})$. Finally, the fact that $S^{(k)}$ is a quotient allows to work with correspondences and Chow groups of $S^{(k)}$ despite their singularities (see [10]). Because of this, for the computations below, we will work with the symmetric products, and this allows us to take $x_1 = \ldots = x_i$.

According to the remark above, we choose now a point $o \in S$ and do $x_1 = \ldots = x_i = o$. We denote by $(io) : S^{(n-i)} \to S^{(n)}$ the map $z \mapsto io + z$. The elementary computations leading to the following result already appear in [7], [27], [20], [24]. In the following statement, we denote by $\Sigma_{k-1,k} \subset S^{(k-1)} \times S^{(k)}$ the correspondence

$$\Sigma_{k-1,k} = \{(z, z') \in S^{(k-1)} \times S^{(k)}, \ z \leq z'\}.$$

These correspondences induce morphisms

$$\Sigma_{k-1,k}^* : \mathrm{CH}_0(S^{(k)}) \to \mathrm{CH}_0(S^{(k-1)})$$

where it is prudent here to take Chow groups with $\mathbb{Q}$-coefficients, due to the singularities of $S^{(k)}$.

Proposition 2.2. *(i) The natural (but depending on o) decreasing filtration N_i defined on $\mathrm{CH}_0(S^{[n]}) = \mathrm{CH}_0(S^{(n)})$ by*

$$N_i\mathrm{CH}_0(S^{(n)}) := \mathrm{Im}\left((io)_* : \mathrm{CH}_0(S^{(n-i)}) \to \mathrm{CH}_0(S^{(n)})\right) \tag{27}$$

induces a splitting

$$\mathrm{CH}_0(S^{(n)}) = \oplus_{0 \leq i \leq n}(io)_*(\mathrm{CH}_0(S^{(n-i)})^0), \tag{28}$$

where

$$\mathrm{CH}_0(S^{(n-i)})^0 := \mathrm{Ker}\left(\Sigma_{n-i-1,n-i}^* : \mathrm{CH}_0(S^{(n-i)}) \to \mathrm{CH}_0(S^{(n-i-1)})\right).$$

(ii) This splitting also induces a decomposition of each N_k:

$$N_k\mathrm{CH}_0(S^{(n)}) = \oplus_{i \geq k}(io)_*(\mathrm{CH}_0(S^{(n-i)})^0), \tag{29}$$

(iii) If $b_1(S) = 0$, this decomposition gives a decomposition of the Bloch-Beilinson filtration, in the sense that it is given by projectors P_i acting on $\mathrm{CH}_0(S^{(n)})$, the corresponding action on holomorphic forms being given by

$$P_i^* = Id \ \text{on} \ H^{2n-2i,0}(S^{(n)}), \ P_i^* = 0 \ \text{on} \ H^{j,0}(S^{(n)}), \ j \neq 2n - 2i.$$

Proof. (i) The proof is based on the following easy formula (see [27]):

$$\Sigma_{n-1,n}^* \circ (o)_* = Id + (o)_* \circ \Sigma_{n-2,n-1}^* : \mathrm{CH}_0(S^{(n-1)}) \to \mathrm{CH}_0(S^{(n-1)}). \tag{30}$$

Note that in this formula, the first $(o)_*$ belongs to $\mathrm{Hom}\left(\mathrm{CH}_0(S^{(n-1)}), \mathrm{CH}_0(S^{(n)})\right)$ and the second one belongs to $\mathrm{Hom}\left(\mathrm{CH}_0(S^{(n-2)}), \mathrm{CH}_0(S^{(n-1)})\right)$. We deduce from this formula that if $z = (o)_*(z') \in \mathrm{Ker}\,\Sigma_{n-1,n}^* \cap \mathrm{Im}\,(o)_*$, then

$$z' = -(o)_* \circ \Sigma_{n-2,n-1}^* z'. \tag{31}$$

Thus $z' \in \mathrm{Im}\,(o)_*$. Applying $\Sigma_{n-2,n-1}^*$ to both sides of equality (31) and formula (30), one gets

$$\Sigma_{n-2,n-1}^* z' = -\Sigma_{n-2,n-1}^* z' - (o)_*(\Sigma_{n-3,n-2}^* \circ \Sigma_{n-2,n-1}^* z'),$$

and applying $(o)_*$ again, one gets

$$z' = -(o)_* \circ \Sigma_{n-2,n-1}^* z' = \frac{1}{2}(2o)_*(\Sigma_{n-3,n-2}^* \circ \Sigma_{n-2,n-1}^* z'),$$

so that in fact $z' \in \text{Im}\,(2o)_*$. Iterating this argument, we finally conclude that $z = 0$.

On the other hand, any cycle in $S^{(n)}$ is the image of a 0-cycle in S^n and each point of S^n can be written (as a 0-cycle of S^n) as

$$(x_1, \ldots, x_n) = pr_1^*(x_1 - o) \cdot \ldots \cdot pr_n^*(x_n - o) + z' \tag{32}$$

where $z' = \sum_i n_i z_i'$ is a cycle of S^n supported on points $z_i' = (z_{i,1}', \ldots, z_{i,n}')$ having at least one factor equal to o. Projecting to $S^{(n)}$, we conclude that every 0-cycle in $S^{(n)}$ is the sum of a 0-cycle in $\text{Im}\,(o)_*$, and of a 0-cycle $(x_1 - o) * \ldots * (x_n - o)$, where the $*$-product used here is the external product appearing in (32) followed by the projection to $S^{(n)}$. It is immediate to check that $(x_1 - o) * \ldots * (x_n - o)$ is annihilated by $\Sigma_{n-1,n}^*$, and thus we get the existence of a decomposition

$$\text{CH}_0(S^{(n)}) = \text{Im}\,(o)_* + \text{Ker}\,\Sigma_{n-1,n}^*, \tag{33}$$

this decomposition being in fact a direct sum decomposition by the previous argument. Using the decomposition (33), (i) is proved by induction.

(ii) This follows directly from the definition of N_k and the decomposition (28) applied to $S^{(n-k)}$.

(iii) We work by induction on n. We observe that the proof of (i) shows that

$$\text{Ker}\,(\Sigma_{n-1,n}^* : \text{CH}_0(S^{(n)}) \to \text{CH}_0(S^{(n-1)}))$$

identifies for $n > 0$ to the image of the map

$$*_n : \text{Sym}^n(\text{CH}_0(S)_0) \to \text{CH}_0(S^{(n)})$$

induced by the $*$-product, where $\text{CH}_0(S)_0$ denotes the group of 0-cycles of degree 0. Furthermore, we have a projector P_n from $\text{CH}_0(S^{(n)})$ to $\text{Im}\,*_n$, which to $x_1 + \ldots + x_n$ associates the cycle $(x_1 - o) * \ldots * (x_n - o)$. This projector annihilates $\text{Im}\,(o_* : \text{CH}_0(S^{(n-1)}) \to \text{CH}_0(S^{(n)}))$ and it is thus the projector on the summand $\text{Ker}\,(\Sigma_{n-1,n}^*)$ in the decomposition (33). Finally, we note that P_n acts as the identity on the space $H^{2n,0}(S^{[n]}) = \text{Sym}^n H^{2,0}(S)$ and as 0 on the spaces of holomorphic forms of even degree $< 2n$. $\qquad\square$

Remark 2.3. Under the assumption made in (iii), $S^{(n)}$ (or rather $S^{[n]}$) has no nonzero odd degree holomorphic form. For this reason, the Bloch-Beilinson filtration on $\text{CH}_0(S^{[n]})$ jumps only in even degree, that is $F_{BB}^{2j}\text{CH}_0(S^{[n]}) = F_{BB}^{2j-1}\text{CH}_0(S^{[n]})$. It is thus more natural in this case and also in the case of hyper-Kähler varieties to work with the filtration $F'^i_{BB}\text{CH}_0(S^{[n]}) := F_{BB}^{2i}\text{CH}_0(S^{[n]})$, whose graded pieces are governed by the $(2i, 0)$-forms.

Remark 2.4. One may prefer to use Proposition 2.2, (ii) as giving a construction of the Bloch-Beilinson filtration F'^i_{BB} on $\text{CH}_0(S^{[n]})$, where S is any surface with $q = 0$. One remark is that, putting

$$F'^i_{BB}\text{CH}_0(S^{[n]}) = F'^i_{BB}\text{CH}_0(S^{(n)}) := \oplus_{k \leq n-i}(ko)_*(\text{CH}_0(S^{(n-k)})^0)$$

the filtration F'_{BB} does not depend on the choice of the point o.

With this notation, Proposition 2.2, (ii) implies that the fitrations N and F'_{BB} are opposite, in the sense that the natural composite map

$$N_i\text{CH}_0(S^{[n]}) \to \text{CH}_0(S^{[n]}) \to \text{CH}_0(S^{[n]})/F'^{n-i+1}_{BB}\text{CH}_0(S^{[n]})$$

is an isomorphism. Let now S be a projective $K3$ surface and let $o_S \in \text{CH}_0(S)$ be the canonical 0-cycle of degree 1 on S which is introduced in [4]. We choose for o any point representing the cycle o_S and conclude that the filtration $N_\bullet$ appearing in Proposition 2.2 is canonically defined on $\text{CH}_0(S^{[n]})$.

The following is obtained by reinterpreting Theorem 2.1 of [26]:

Theorem 2.5. *Let $X = S^{[n]}$. Then the filtration $N_\bullet$ introduced above and the filtration $S_\bullet$ introduced in Definition 0.2 coincide on $\mathrm{CH}_0(X)$.*

Proof. Let $x \in S_i X$. This means by definition that there exists a subvariety $W_x \subset X$ which is of dimension at least i, such that all points in W_x are rationally equivalent to x in X. A fortiori, the degree n effective 0-cycles of S parameterized by W_x are constant in $\mathrm{CH}_0(S)$, since they are obtained by applying the universal subvariety $\Sigma_n \subset S^{[n]} \times S$, seen as a correspondence between $S^{[n]} = X$ and S, to the points of W_x. We now apply the following result which can be proved by a slight modification of the proofs of [26, Theorem 2.1 and Variant 2.4].

Theorem 2.6. *Let S be a projective K3 surface and let $Z \subset S^{[n]}$ be a subvariety of dimension i, such that all the degree i effective 0-cycles of S parameterized by Z are rationally equivalent in S. Then for some constant cycle curve $C \subset S$, the image of Z in $S^{(n)}$ intersects $C^{(i)} + S^{(n-i)} \subset S^{(n)}$.*

We apply this result to W_x and conclude that the image $\overline{W}_x$ of W_x in $S^{(n)}$ intersects $C^{(i)} + S^{(n-i)} \subset S^{(n)}$ in a point z. As all points in W_x are rationally equivalent to x in X, the image $\overline{x}$ of x in $S^{(n)}$ is rationally equivalent in $S^{(n)}$ to $z \in C^{(i)} + S^{(n-i)}$. But for any such point, which is of the form $z = z_1 + z_2$ with z_1 effective of degree i supported in C, $z_2 \in S^{(n-i)}$, we have $z_1 = \sum_j x_j$, $x_j \in C \subset S$, hence x_j is rationally equivalent to o_S in S, and thus $\sum_j x_j + z_2$ is rationally equivalent in $S^{(n)}$ to the point $io_S + z_2$. We conclude that z, hence also $\overline{x}$, is rationally equivalent in $S^{(n)}$ to a point in the image of the map io_S, so that its class in $\mathrm{CH}_0(S^{(n)}) = \mathrm{CH}_0(S^{[n]}) = \mathrm{CH}_0(X)$ belongs to $N_i \mathrm{CH}_0(X)$, proving the inclusion $S_i \mathrm{CH}_0(X) \subset N_i \mathrm{CH}_0(X)$.

The reverse inclusion is obvious since for any constant cycle curve $C \subset S$, a point $z = io_S + z'$, $z' \in S^{(n-i)}$ in $\mathrm{Im}\,(io_S : S^{(n-i)} \to S^{(n)})$ contains in its orbit the i-dimensional subvariety $C^{(i)} + z'$ which lifts to an i-dimensional subvariety of X, so that any lift of z to X belongs to $S_i X$.

$\square$

This result suggests that the filtration $S_\bullet$, which is defined for any variety, could be in the case of general hyper-Kähler manifolds the natural substitute for the filtration $N_\bullet$ (which was defined only for Hilbert schemes of surfaces). We will formulate this more explicitly in Section 3.

3 Conjectures on the Chow groups of hyper-Kähler varieties

Let X be a projective hyper-Kähler manifold. We have the notion of coisotropic class introduced in Definition 1.5. We proved in Section 1 that the class of a codimension i subvariety Z of X contained in $S_i X$ is coisotropic (see Theorem 1.3 which proves a stronger statement). The computations made in Section 1.1 (see Theorem 1.7) show that nonzero coisotropic Hodge classes always exist (and even nonzero coisotropic algebraic classes, assuming the algebraicity of the class c, see Remark 1.6). Let us state the following more precise version of Conjecture 0.4:

Conjecture 3.1. *For any X as above and any $i \leq n$, the space of coisotropic classes of degree $2i$ is generated over $\mathbb{Q}$ by classes of codimension i subvarieties Z of X contained in $S_i X$.*

Remark 3.2. It might even be true, as suggested by the work of Charles and Pacienza [9], that the space of coisotropic classes of degree $2i$ is generated over $\mathbb{Q}$ by classes of codimension i subvarieties Z of X which are swept-out by i-dimensional rationally connected varieties.

Remark 3.3. There are three different problems hidden in Conjecture 3.1:

1) The Hodge conjecture: one has to prove that coisotropic Hodge classes are algebraic.

2) The existence problem for coisotropic subvarieties: one has to prove that the classes of coisotropic subvarieties generate the space of coisotropic classes. As the previous one, this problem does not appear for divisors, as all divisors are coisotropic.

3) The existence problem for algebraically coisotropic subvarieties, and even algebraically coisotropic subvarieties obtained as codimension i components of $S_i X$. The last problem is unsolved even for divisors, but there are progresses in this case (for example the problem is solved by Charles-Pacienza [9] if X is a deformation of $\mathrm{Hilb}(K3)$).

Let us prove the following conditional result.

Theorem 3.4. *Assume X satisfies Conjecture 3.1. Then*

$$S_i \mathrm{CH}_0(X) = \mathrm{Im}\,(z : \mathrm{CH}_i(X) \to \mathrm{CH}_0(X)),$$

for any adequate combination $z = \sum_j n_j z_j \in \mathrm{CH}^i(X)$ of classes of subvarieties $Z_j \subset S_i X$ of codimension i in X, such that the class $[z]$ is a nonzero polynomial in c and an ample divisor class $l \in \mathrm{NS}(X)$.

Proof. Note first of all that such a z exists if Conjecture 3.1 holds, since Theorem 1.7, (iii) shows the existence of a nonzero coisotropic class which is a polynomial in c and any given ample class l.

Next, for any such z, the inclusion $\mathrm{Im}\,(z : \mathrm{CH}_i(X) \to \mathrm{CH}_0(X)) \subset S_i \mathrm{CH}_0(X)$ is obvious, since $\mathrm{Supp}\, z \subset S_i X$ and by definition $S_i \mathrm{CH}_0(X)$ is generated by the classes of the points in $S_i X$.

Let us prove the inclusion $S_i \mathrm{CH}_0(X) \subset \mathrm{Im}\,(z : \mathrm{CH}_i(X) \to \mathrm{CH}_0(X))$. Let $x \in S_i X$. By assumption, there exists an i-dimensional subvariety $W_x \subset X$ all of whose points are rationally equivalent to x in X. We claim that $\deg(W_x \cdot z) \neq 0$. Assuming the claim, $W_x \cdot z \in \mathrm{CH}_0(X)$ is a 0-cycle of degree different from 0 supported on W_x, hence a nonzero multiple of $x \in \mathrm{CH}_0(X)$. This gives us the desired inclusion

since by definition $S_i \mathrm{CH}_0(X)$ is generated by the classes of such points x.

Let us prove the claim. We use the fact (see Corollary 1.2) that a constant cycle subvariety W is isotropic, that is $\sigma_{|W_{reg}} = 0$ as a form. Equivalently, the pull-back of σ to a desingularization $\widetilde{W}$ of W has a vanishing class in $H^2(\widetilde{W}, \mathbb{C})$. The Hodge structure on $H^2(X, \mathbb{Q})_{tr}$ being simple because $h^{2,0}(X) = 1$, it follows that the restriction map $H^2(X, \mathbb{Q})_{tr} \to H^2(\widetilde{W}, \mathbb{Q})$ vanishes identically. This implies that the class c_{tr}, which by construction belongs to $S^2 H^2(X, \mathbb{Q})_{tr} \subset H^4(X, \mathbb{Q})$, vanishes in $H^4(\widetilde{W}, \mathbb{Q})$. As we know by Theorem 1.7, (iv) that $[z] = \lambda l^i + c_{tr} l^{i-2} + ...$, with $\lambda \neq 0$, we get that $\deg(W \cdot z) = \lambda \deg(W \cdot l^i) \neq 0$. $\qquad \square$

This result suggests that Beauville's conjectural splitting of the Bloch-Beilinson filtration could be obtained by considering the action of the classes in $\mathrm{CH}(X)$ of codimension i subvarieties of X contained in $S_i X$. More precisely, we would like to impose the following rule: Let $C^i(X) \subset \mathrm{CH}^i(X)$ be the $\mathbb{Q}$-vector space generated by codimension i components of $S_i X$.

(*) *The subspace $C^i(X) \subset \mathrm{CH}^i(X)$ is contained in the 0-th piece $\mathrm{CH}^i(X)_0$ of the Beauville conjectural splitting.*

For this to be compatible with multiplicativity (and the axiom that $F^1_{BB}\mathrm{CH} = \mathrm{CH}_{hom}$), one needs to prove the following concrete conjecture:

Conjecture 3.5. *Let X be hyper-Kähler of dimension $2n$. Then the cycle class map is injective on the subalgebra of $\mathrm{CH}^*(X)$ generated by $\oplus_i C^i(X)$.*

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Restricting to the $\mathbb{Q}$-vector subspace $\oplus_i C^i(X)$ itself, this conjecture predicts the following:

Conjecture 3.6. *For any i such that $0 \leq i \leq n$, the cycle class map is injective on the $\mathbb{Q}$-vector subspace $C^i(X)$ of $\mathrm{CH}^i(X)$.*

In particular, for $i = n$, $2n = \dim X$, the cycle class map is injective on the subspace $C^n(X)$ generated by classes in $\mathrm{CH}(X)$ of constant cycle Lagrangian subvarieties of X.

An evidence for this conjecture is provided by Proposition 4.7, which establishes it for the Fano variety of lines of a cubic fourfold.

A last conjecture suggested by the results in Section 2 concerns the possibility of constructing the conjectural Beauville decomposition from the filtration $S_\bullet$ studied in the previous section, at least on some part of $\mathrm{CH}(X)$. Here we assume of course the existence of the Bloch-Beilinson filtration. First of all, let us consider the case of CH_0.

Conjecture 3.7. *Let X be hyper-Kähler of dimension $2n$. For any integer i such that $0 \leq i \leq n$, the filtration $S_\bullet$ is opposite to the filtration F'_{BB} in the sense that*

$$S_i\mathrm{CH}_0(X) \cong \mathrm{CH}_0(X)/F'^{n-i+1}_{BB}\mathrm{CH}_0(X) = 0.$$

The main evidences for this conjecture are the cases of $S^{[n]}$, where S is a $K3$ surface (see Sections 2 and 4) and of the Fano variety of lines of a cubic fourfold (see Proposition 4.5), for which we already have candidates for the Bloch-Beilinson filtrations.

Remark 3.8. In the case $i = n$, Conjecture 3.7 takes a more concrete form which does not assume the existence of the Bloch-Beilinson filtration. Indeed, we then have $F'^{n-i+1}_{BB}\mathrm{CH}_0(X) = F^2_{BB}\mathrm{CH}_0(X) = \mathrm{CH}_0(X)_{hom}$ and we are considering 0-cycles supported on constant cycles Lagrangian subvarieties. The conjecture is that they are rationally equivalent to 0 if and only if they are of degree 0.

The following observation illustrates the importance of Conjecture 0.4 for our constructions:

Lemma 3.9. *Assuming Conjecture 0.4, the map*

$$S_i\mathrm{CH}_0(X) \to \mathrm{CH}_0(X)/F'^{n-i+1}_{BB}\mathrm{CH}_0(X) \tag{34}$$

is surjective.

Proof. First we claim that for any component Z of codimension i of $S_i X$, the pull-back map

$$H^0(X, \Omega^l_X) \to H^0(\widetilde{Z}, \Omega^l_{\widetilde{Z}})$$

is injective for $l \leq 2(n-i)$, where $\widetilde{Z}$ is a desingularization of Z. Indeed, the space $H^0(X, \Omega^l_X)$ is equal to 0 for odd l and is generated by the form $\sigma^{l'}$ for $l = 2l'$. The form σ being everywhere nondegenerate, the rank of $\sigma_{|Z_{reg}}$ is at least $2n - 2i$, which implies that σ^{n-i} does not vanish in $H^0(\widetilde{Z}, \Omega^{n-i}_{\widetilde{Z}})$.

By the general axioms concerning the Bloch-Beilinson filtration, the claim above guarantees the surjectivity of the map (34). $\qquad\square$

Conjecture 3.7 thus concerns the injectivity of this map. One case would be also an easy consequence of Conjecture 3.1:

Lemma 3.10. *(i) (Charles-Pacienza [9]) Conjecture 3.7 holds for constant Lagrangian surfaces in very general algebraic deformations of $\mathrm{Hilb}^2(S)$.*

(ii) More generally, Conjecture 3.7 holds for $i = n$ if X contains a constant Lagrangian subvariety which is connected and of class $\lambda_n l^n + \lambda_{n-2} c_{tr} l^{n-2} + \ldots$ for some ample class $l \in \mathrm{NS}(X)$.

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Proof. By Remark 3.8, what we have to prove when $i = n$ is the equality

$$S_n \mathrm{CH}_0(X) \cap \mathrm{CH}_0(X)_{hom} = 0.$$

In both cases (i) and (ii), we have the existence of a *connected* Lagrangian constant cycle subvariety $W \subset X$ of class $w = \lambda_n l^n + \lambda_{n-2} c_{tr} l^{n-2} + \ldots$, where l is an ample divisor class on X and $\lambda_n \neq 0$. (In the case (i), this is because all Lagrangian surfaces have their class proportional to $\lambda l^2 + c_{tr}$, and in case (ii) this follows from our assumptions, using Theorem 1.7 (iv)). The same argument as in the proof of Theorem 3.4 then shows that for any Lagrangian constant cycle subvariety $\Gamma \subset X$, one has $\deg(\Gamma \cdot W) \neq 0$. It follows that $\Gamma \cap W \neq \emptyset$, and thus that any point of Γ is rationally equivalent in X to any point of W. Thus $S_n \mathrm{CH}_0(X) = \mathbb{Q}$, and $S_n \mathrm{CH}_0(X) \cap \mathrm{CH}_0(X)_{hom} = 0$.

$\square$

Another small evidence for Conjecture 3.7 is provided by the following result: If z is the class of any subvariety of X contained in $S_i X$ and of codimension i, and $\Gamma \in \mathrm{CH}^{2n-i}(X)$ is any cycle, $z \cdot \Gamma$ belongs to $S_i \mathrm{CH}_0(X)$. Hence if $\Gamma \in F_{BB}^{2n-2i+1} \mathrm{CH}^{2n-i}(X)$ we have $z \cdot \Gamma \in F_{BB}^{2n-2i+1} \mathrm{CH}^{2n-i}(X) \cap S_i \mathrm{CH}_0(X)$ and Conjecture 3.7 then predicts that

$$z \cdot \Gamma = 0 \text{ in } \mathrm{CH}_0(X).$$

This is in fact true, as shows the following result (which assumes the existence of the Bloch-Beilinson filtration):

Proposition 3.11. *Let $Z \subset X$ be a codimension i subvariety contained in $S_i X$. Then the intersection product*

$$Z : F_{BB}^{2n-2i+1} \mathrm{CH}^{2n-i}(X) \to \mathrm{CH}_0(X)$$

vanishes identically.

Proof. We use Theorem 1.3, (ii) which says that a desingularization $\widetilde{Z} \xrightarrow{\tilde{i}} X$ of Z admits a fibration

$$p : \widetilde{Z} \to B$$

where $\dim B = 2n - 2i$, and the i-dimensional fibers of p map via $\tilde{i}$ to constant cycle subvarieties of X. It follows that the morphism

$$\tilde{i}_* : \mathrm{CH}_0(\widetilde{Z}) \to \mathrm{CH}_0(X)$$

factors through $\mathrm{CH}_0(B)$. Indeed, let $B' \subset \widetilde{Z}$ be generically finite of degree N over B, and let $p' : B' \to B$ be the restriction of p. Then for any point $z \in \widetilde{Z}$, one has

$$\tilde{i}_*(z) = \frac{1}{N} \tilde{i}_*(p'^*(p_* z)) \text{ in } \mathrm{CH}_0(X),$$

which provides the desired factorization.

Now, for $\Gamma \in F^{2n-2i+1} \mathrm{CH}^{2n-i}(X)$ one has $Z \cdot \Gamma = \tilde{i}_*(\tilde{i}^* \Gamma)$. As $\tilde{i}^* \Gamma \in F^{2n-2i+1} \mathrm{CH}_0(\widetilde{Z})$, one has $p_*(\tilde{i}^* \Gamma) \in F^{2n-2i+1} \mathrm{CH}_0(B)$ where the last space is equal to $\{0\}$ because $\dim B = 2n - 2i$. By the factorization above, this implies that $\tilde{i}_*(\tilde{i}^* \Gamma) = 0$ in $\mathrm{CH}_0(X)$.

$\square$

Conjecture 3.7 would allow to construct the Beauville decomposition on $\mathrm{CH}_0(X)$ as

$$\mathrm{CH}_0(X)_{2k} = S_{n-k} \mathrm{CH}_0(X) \cap F'^k_{BB} \mathrm{CH}_0(X), \ \mathrm{CH}_0(X)_{2k+1} = 0,$$

and we would have equivalently

$$S_i \mathrm{CH}_0(X) = \oplus_{j \leq n-i} \mathrm{CH}_0(X)_{2j},$$

$$F'^i_{BB}\mathrm{CH}_0(X) = \oplus_{j \geq i}\mathrm{CH}_0(X)_{2j}.$$

If we try to extend this construction to other cycles, having in mind that the Beauville decomposition is supposed to be multiplicative and have the divisor classes in its 0-th piece, the following proposal seems to be compatible with Theorem 3.4 and the previous assignments:

From a decomposition

$$\mathrm{CH}^k(X) = \oplus_{0 \leq j \leq k}\mathrm{CH}^k(X)_j \tag{35}$$

with

$$F^i_{BB}\mathrm{CH}^k(X) = \oplus_{i \leq j \leq k}\mathrm{CH}^k(X)_j, \tag{36}$$

one can construct another decreasing filtration T defined by

$$T^i\mathrm{CH}^k(X) = \oplus_{0 \leq j \leq k-i}\mathrm{CH}^k(X)_j. \tag{37}$$

One clearly has

$$\mathrm{CH}^k(X)_j = T^{k-j}\mathrm{CH}^k(X) \cap F^j_{BB}\mathrm{CH}^k(X).$$

Conversely, a decreasing filtration T which is opposite to the filtration F_{BB} in the sense that

$$T^l\mathrm{CH}^k \cong \mathrm{CH}^k/F^{k+1-l}_{BB}\mathrm{CH}^k$$

leads to a decomposition (35) satisfying (36) and (37).

For 0-cycles, we put $T^{2i}\mathrm{CH}_0(X) = S_i\mathrm{CH}_0(X)$ and assuming Conjecture 3.7, we have the desired decomposition.

We now propose the following assignment for the filtration T:

(**) Let $\Gamma \subset X$ be an i-dimensional constant cycle subvariety. Then its class $\gamma \in \mathrm{CH}^{2n-i}(X)$ belongs to $T^i\mathrm{CH}(X)$.

For a filtration T satisfying the assignment above to be opposite to the Bloch-Beilinson filtration, one needs the following:

Conjecture 3.12. *Let $C_i(X) \subset \mathrm{CH}_i(X) = \mathrm{CH}^{2n-i}(X)$ be the $\mathbb{Q}$-vector space generated by constant cycle subvarieties of dimension i. Then the Bloch-Beilinson filtration F_{BB} satisfies*

$$F^{2n-2i+1}_{BB}C_i(X) = 0. \tag{38}$$

Note that the general finiteness condition for the Bloch-Beilinson filtration (see Section 0.1) is

$$F^{2n-i+1}\mathrm{CH}^{2n-i} = 0,$$

which is weaker than (38). Note also that Conjecture 3.12 generalizes the case $i = n$ of Conjecture 3.6 (i.e. the case of Lagrangian constant cycle subvarieties). Indeed, for $i = n$, one has $2n - 2i + 1 = 1$, hence $F^{2n-2i+1}_{BB}\mathrm{CH}^{2n-i}(X) = \mathrm{CH}^{2n-i}(X)_{hom}$, and Conjecture 3.12 says that the cycle class map is injective on the subspace $C_n(X) = C^n(X)$ of $\mathrm{CH}^n(X)$ generated by classes of Lagrangian constant cycle subvarieties.

Remark 3.13. The three conjectures 0.8, 3.5 and 3.12 can be unified as follows: Consider the inclusion $S_iX \subset X$, and let $S'_iX \subset S_iX$ be the union of the components of S_iX which are of codimension i in X. According to Theorem 1.3, each component Z of S'_iX (or rather a birational model of it) admits a fibration $Z \to B$ into i-dimensional constant cycles subvarieties, with $\dim B = 2n - 2i$. For each such Z, we thus have a correspondence between B and X given by the two maps

$$\tilde{i} : Z \to X, \; p : Z \to B,$$

and we thus have three maps, namely :

1) The morphism $\overline{\tilde{i}_*} : \mathrm{CH}_0(B) \to \mathrm{CH}_0(X)$ factoring through $p_* : \mathrm{CH}_0(Z) \to \mathrm{CH}_0(B)$ the natural morphism $\tilde{i}_* : \mathrm{CH}_0(Z) \to \mathrm{CH}_0(X)$, using the fact that the fibers of p are mapped by $\tilde{i}$ to constant cycles subvarieties of X;

2) The morphism $Z_* = \tilde{i}_* \circ p^* : \mathrm{CH}_0(B) \to \mathrm{CH}_i(X)$, whose image is contained in the subgroup $C_i(X)$ generated by classes of constant cycles subvarieties of dimension i;

3) The morphism $Z_* = \tilde{i}_* \circ p^* : \mathrm{CH}^0(B) \to \mathrm{CH}^i(X)$, whose image belongs to the subgroup $C_{2n-i}(X)$.

Observing that the Bloch-Beilinson filtration on $\mathrm{CH}(B)$ satisfies

$$F_{BB}^{2n-2i+1} \mathrm{CH}_0(B) = 0, \ F^1 \mathrm{CH}^0(B) = 0,$$

for all Z, B's as above and taking the disjoint union of all components $Z_j \subset X$ of $S_i'X$, and of the corresponding B_j's, our conjectures can be unified and even strengthened saying that the three maps above are strict for the Bloch-Beilinson filtrations.

Remark 3.14. There are remarkable relations between these three maps, namely assuming B is connected, there are coefficients μ, ν_l with $\nu_l \neq 0$ depending on an ample class l, such that

$$Z_*(\alpha) \cdot Z_*(\gamma) = \mu \, \overline{\tilde{i}_*}(\alpha \cdot \gamma) \ \text{in} \ \mathrm{CH}_0(X), \tag{39}$$

for any $\alpha \in \mathrm{CH}^0(B)$, $\gamma \in \mathrm{CH}_0(B)$, and

$$l^i \cdot Z_*(\gamma) = \nu_l \overline{\tilde{i}_*}(\gamma) \ \text{in} \ \mathrm{CH}_0(X), \tag{40}$$

for any $\gamma \in \mathrm{CH}_0(B)$.

Both relations follow immediately from the fact that the fibers of p are constant cycles subvarieties of X, and they just say that that for any such fiber Z_t, the intersection $Z \cdot Z_t$, resp. $l^i \cdot Z_t$ is proportional to $\overline{\tilde{i}_*}(t)$ in $\mathrm{CH}_0(X)$.

The relation (40) provides a close link between Conjectures 3.7 and Conjecture 3.12 (we assume here the existence of a Bloch-Beilinson filtration).

Lemma 3.15. *(i) One has* $\mathrm{Ker}\, Z_* \subset \mathrm{Ker}\, \overline{\tilde{i}_*} \subset \mathrm{CH}_0(B)$.

(ii) Assuming Conjecture 3.7, for any codimension i component $Z \subset X$ of S_iX, and any 0-cycle $\gamma \in \mathrm{CH}_0(B)$, one has the implications

$$Z_*\gamma \in F^{2n-2i+1}\mathrm{CH}_i(X) \Rightarrow \overline{\tilde{i}_*}(\gamma) = 0 \ \text{in} \ \mathrm{CH}_0(X).$$

In particular, if furthermore one has equality in (i), then

$$Z_*\gamma \in F^{2n-2i+1}\mathrm{CH}_i(X) \Rightarrow Z_*\gamma = 0 \ \text{in} \ \mathrm{CH}_i(X),$$

which is essentially Conjecture 3.12.

Proof. Indeed, (i) is an obvious consequence of (40). As for (ii), if $Z_*\gamma \in F^{2n-2i+1}\mathrm{CH}_i(X)$, then $\overline{\tilde{i}_*}(\gamma) \in F^{2n-2i+1}\mathrm{CH}_0(X)$ by (40). Conjecture 3.7 says now that $F^{2n-2i+1}\mathrm{CH}_0(X) \cap S_i\mathrm{CH}_0(X) = 0$, hence that $\overline{\tilde{i}_*}(\mathrm{CH}_0(B)) \cap F^{2n-2i+1}\mathrm{CH}_0(X) = 0$. This implies that $\overline{\tilde{i}_*}(\gamma) = 0$. $\qquad\square$

From a slightly different point of view, let us explain how Conjecture 3.12 would lead to multiplicativity statements for the associated decomposition. First of all, let us observe the following results along the same lines as Theorem 3.4:

Lemma 3.16. *(i) Let $\Gamma \subset X$ be a constant cycle subvariety of dimension i. Then for any $l \in \mathrm{NS}(X) = \mathrm{Pic}(X)$, $l^i\Gamma \in S_i\mathrm{CH}_0(X) = T^{2i}\mathrm{CH}_0(X)$.*

(ii) Assuming Conjecture 3.1, $S_i\mathrm{CH}_0(X) = T^{2i}\mathrm{CH}_0(X)$ is generated by products $Z \cdot \Gamma$, where Z is a codimension i subvariety of X contained in S_iX, and Γ a constant cycle subvariety of X of dimension i.

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Proof. (i) Indeed, as all points of Γ belong to $S_i X$ by definition, so does the 0-cycle $l^i \Gamma$ which is supported on Γ.

(ii) Let $x \in S_i X$ and let Γ_x be a constant cycle subvariety of dimension i. Then for any cycle $z \in CH^i(X)$ such that $\deg(z \cdot \Gamma_x) \neq 0$, $z \cdot \Gamma_x \in CH_0(X)$ is a nonzero multiple of the class of any point of X supported on Γ_x, hence of x. Assuming Conjecture 3.1, the same argument as in the proof of Theorem 3.4 shows that there is a combination $z \in C_i(X)$ of classes of codimension i subvarieties of X contained in $S_i X$, such that $\deg(z \cdot \Gamma_x) \neq 0$ and thus the class of x is a multiple in $CH_0(X)$ of $z \cdot \Gamma$, which shows that $S_i CH_0(X)$ is generated by products $Z \cdot \Gamma$, where $Z \in C_{2n-i}(X)$, and $\Gamma \in C_i(X)$, since by definition $S_i CH_0(X)$ is generated by the classes of points in $S_i X$. The other inclusion is obvious since any cycle $Z \cdot \Gamma$ with $Z \subset S_i X$ of codimension i in X is supported on Z, hence belongs to $S_i CH_0(X)$. $\square$

Let now Γ be a constant cycle subvariety of dimension i. Then according to (**), the cycle $\Gamma \in CH^{2n-i}(X)$ should belong to $T^i CH^{2n-i}(X) = \oplus_{j \leq 2n-2i} CH^{2n-i}(X)_j$. According to (*), the class z of any subvariety of X contained in $S_i X$ and of codimension i should be in $CH^i(X)_0$. By multiplicativity of the conjectural Beauville decomposition, one should have

$$z \cdot \Gamma \in \oplus_{j \leq 2n-2i} CH_0(X)_j, \tag{41}$$

the right hand side being equal to $T^{2i} CH_0(X) = S_i CH_0(X)$. Equation (41) is in fact satisfied by the easy inclusion in Lemma 3.16, (ii), thus providing some evidence for the multiplicativity of the decomposition we started to construct. Similarly, if we take for z a degree i polynomial in divisor classes on X, then z should belong to $CH^i(X)_0$ and thus we should have according to (**) and multiplicativity

$$z \cdot \Gamma \in \oplus_{j \leq 2n-2i} CH_0(X)_j = S_i CH_0(X)$$

which is proved in Lemma 3.16, (i).

4 Examples

The purpose of this section is to collect some examples providing evidences for the conjectures proposed in this paper.

4.1 The case of $\mathrm{Hilb}(K3)$

We first examine Conjecture 3.1 concerning the existence of many algebraic coisotropic subvarieties obtained as codimension i components of $S_i X$.

Let us consider a very general algebraic $K3$ surface S, so that $NS(S)$ has rank 1 and is generated by the class of $L \in \mathrm{Pic}\, S$, and let $X := S^{[n]}$. The Néron-Severi group $NS(X)$ has then rank 2, and is generated over $\mathbb{Q}$ by the class e of the exceptional divisor and the class l of the pull-back to X, via the Hilbert-Chow morphism

$$s : X = S^{[n]} \to S^{(n)}$$

of the divisor $C + S^{(n-1)} \subset S^{(n)}$ where $C \in |L|$.

1) Obvious examples of constant cycles subvarieties of X are provided by the fibers of s. It is indeed known that these fibers are rationally connected, so that they are constant cycles subvarieties. We know that for each stratum $S_0^{(\mu)} \subset S^{(n)}$ determined by multiplicities $\mu_1, \ldots, \mu_l$ such that $\sum_i \mu_i = n$, the inverse image $s^{-1}(S_0^{(\mu)})$ is of codimension i in $S^{[n]}$, and fibered into constant cycles subvarieties of dimension i, namely the fibers of s over points

$z \in S_0^{(\mu)}$. Here the notation is as follows: The number i is equal to $n - l$, and $S_0^{(\mu)}$ is the locally closed stratum determined by μ, namely

$$S_0^{(\mu)} := \{\sum_j \mu_j x_j, \ x_j \in S, \ x_j \neq x_k, \ j \neq k\}.$$

The Zariski closure E_μ of $s^{-1}(S_0^{(\mu)})$ in $S^{[n]}$ is thus an example of a codimension i algebraically coisotropic subvariety of $S^{[n]}$ fibered into i-dimensional constant cycles subvarieties, as studied in Section 1.

1bis) With the same notation as above, let $W \subset S_0^{(\mu)} \subset S^{(n)}$ be a codimension j subvariety which is fibered by j-dimensional subvarieties Z_t of $S^{(\mu)}$ which are constant cycles for $S^{(n)}$ in the sense that for each t, the natural map $Z_t \to \mathrm{CH}_0(S^{(n)})$ is constant. Then the locally closed varieties $s^{-1}(Z_t) \subset S^{[n]}$ are of dimension $j + n - l(\mu)$ and they sweep-out the locally closed subvariety $s^{-1}(W)$ which has codimension $n - l(\mu) + j$. Hence its Zariski closure in X is algebraically coisotropic, fibered into constant cycles subvarieties.

2) Starting from a constant cycle curve $C \subset S$, for example a rational curve, we can also get codimension i subvarieties of $S_i X$, by taking the image of the rational map

$$C^{(i)} \times S^{[n-i]} \dashrightarrow S^{[n]},$$

$$(z, z') \mapsto z \cup z'.$$

3) A more subtle example (which however does not work for all possible pairs (i, g), where $L^2 = 2g - 2$) is given by applying the Lazarsfeld construction [15] in any possible range for the Brill-Noether theory of smooth curves in $|L|$. Let us describe this construction in more detail. Let $C \in |L|$ be smooth, and let $|D| \in G_n^1(C)$ be a base-point free pencil. By the main result of [15], such a D exists if and only if $2(g - n + 1) \leq g$, that is $g \leq 2n - 2$. Let F be the rank 2 vector bundle on S which is defined as the kernel of the (surjective) evaluation map

$$H^0(D) \otimes \mathcal{O}_S \to \mathcal{O}_C(D)$$

and let $E := F^*$. Then E fits in an exact sequence

$$0 \to H^0(D)^* \otimes \mathcal{O}_S \to E \to K_C(-D) \to 0.$$

It follows that

$$h^0(S, E) = 2 + g - n + 1, \ \deg c_2(E) = n, \tag{42}$$

so that 0-sets of sections of E provide constant cycles subvarieties of $S^{[n]}$ of dimension $g + 2 - n$ (we observe here that E satisfies $h^0(S, E(-L)) = 0$, hence any nonzero section of E has a 0-dimensional vanishing locus, so that the morphism $\mathbb{P}(H^0(S, E)) \to S^{[n]}$ is well-defined, obviously non-constant and thus finite to one onto its image). Let us now compute the dimension of the subvariety $Z \subset S^{[n]}$ we get by letting (C, D) deform in the space of pairs consisting of a curve $C \in |L|$ and an effective divisor D which is a g_n^1 on C. The curve C moves in the g-dimensional linear system $|L|$ and $\mathcal{O}_C(D)$ moves in the codimension $2(g - n + 1)$ subvariety of the relative Picard variety $\mathrm{Pic}(\mathcal{C}/|L|)$ which has dimension $2g$ (here $\mathcal{C} \to |L|$ is the universal curve; we work in fact over the open set parameterizing smooth curves, and we use Lazarsfeld's theorem [15] saying that the dimensions are the expected ones). The choice of $D \in C^{(n)}$ instead of the line bundle $\mathcal{O}_C(D)$ provides one more parameter. This gives us a subvariety W of $\mathcal{C}^{(n/B)}$ of dimension $2g + 1 - (2(g - n + 1)) = 2n - 1$. Finally, the fiber over a point $z \in S^{[n]}$ of the surjective map $W \to Z$ obtained by restricting to W the natural morphism

$$\mathcal{C}^{(n/B)} \to S^{[n]}$$

identifies to the set of curves $C \in |L|$ containing the 0-dimensional subscheme $z \subset S$, and this is a projective subspace of $|L| \cong \mathbb{P}^g$ which is of dimension $g - n + 1$ since these z's impose exactly $n - 1$ conditions to $|L|$. We conclude that $\dim Z = 2n - 1 - (g - n + 1) = 3n - g - 2$, and $\text{codim } Z = g + 2 - n$. Hence $Z \subset S_{g+2-n} X$ and has codimension $g + 2 - n$.

Remark 4.1. In this example, the base of the coisotropic fibration of Z is birationally equivalent to a moduli space of rank 2 vector bundles on S, hence to a possibly singular hyper-Kähler variety. This needs not to be the case in general. For example, there is a uniruled divisor in the variety of lines F of a cubic fourfold which is uniruled over a surface of general type, namely the indeterminacy locus of the rational self-map $\phi : F \dashrightarrow F$ constructed in [28] (see also Subsection 4.2).

4) For $n = 2$, S admits a covering by a 1-parameter family of elliptic curves. Each such curve E carries the "Beauville-Voisin" 0-cycle o_S, that is contains a point x that is rationally equivalent to o_S in S, and $2x$ moves in a pencil in E. This way we get a 2-dimensional orbit Σ of o_S in $S^{[2]}$, which is a Lagrangian surface.

4bis) We can combine construction 4) and the sum map

$$\mu : S^{[2]} \times S^{[n-2]} \dashrightarrow S^{[n]}$$

to construct codimension 2 subvarieties of $S^{[n]}$ contained in $S_2 S^{[n]}$: Namely, let Σ be as above, then for each $z \in S^{[n-2]}$, the image $\mu(\Sigma \times z)$ is a surface in $S^{[n]}$ all of whose points are rationally equivalent in $S^{[n]}$ hence $\mu(\Sigma \times S^{[n-2]})$ is contained in $S_2 S^{[n]}$.

Let us now prove the following result, which is Conjecture 3.1 for degree 4 coisotropic classes on $S^{[n]}$, S a very general algebraic $K3$ surface. Note that the cases $n = 2, 3, n \geq 4$ differ from the viewpoint of computing cohomology of degree 4. In the case $n = 2$, the degree 4 cohomology (resp. the space of degree 4 Hodge classes) is equal to $S^2 H^2(S^{[2]}, \mathbb{Q})$ (resp. is generated by c, l, e), while for $n \geq 3$, by the results of de Cataldo and Migliorini [10], the cohomology of degree 4 of $S^{[n]}$ is generated by $S^2 H^2(S, \mathbb{Q})$ (coming from the cohomology of $S^{(n)}$), two copies of $H^2(S, \mathbb{Q})$, (coming via the exceptional divisor from the cohomology of the codimension 2 stratum, which has for normalization $S^{(n-2)} \times S$) and by the classes of the codimension 2 subvarieties E_μ over strata $S^{(\mu)}$ of S, with $l(\mu) = n - 2$. If $n = 3$ there is only one such stratum and E_μ in this case is the set of schemes of length 3 whose support consists of a single point. For $n \geq 4$, there are 2 codimension 4 strata in $S^{(n)}$ corresponding to the partitions $\{1, \ldots, 1, 2, 2\}$ and $\{1, \ldots, 1, 3\}$, and we thus get two codimension 2 subvarieties E_{μ_1}, E_{μ_2} in $S^{[n]}$, $n \geq 4$.

Proposition 4.2. *Let S be a very general projective $K3$ surface with Picard number 1. Then for any integer n, the space of coisotropic classes of degree 4 on $S^{[n]}$ is generated by classes of codimension 2 subvarieties contained in $S_2 S^{[n]}$.*

Proof. First of all, it is immediate to see looking at the proof of Theorem 1.7, (iii) that for any ρ-very general X hyper-Kähler manifold with $\rho(X) = 2$, there is exactly a 3-dimensional space of isotropic classes which can be written as polynomials of weighted degree 2 in c, l, e (that is, the inequality given in Theorem 1.7, (iii) is an equality).

Now we first do the case $n = 2$. In this case, the algebraic cohomology of $S^{[2]}$ for S very general is given by polynomials of weighted degree 2 in c, l, e so it suffices to exhibit three surfaces contained in $S_2 S^{[2]}$ (that is, constant cycles surfaces) with independent classes. Construction 1bis) gives us such a surface, starting from a constant cycle curve $C \subset S = \Delta_S \subset S^{(2)}$, and taking $\Sigma_1 = s^{-1}(S) \subset E \subset S^{[2]}$. The surface Σ_1 is of class $l \cdot e$. Construction 2 gives us the surface $\Sigma_2 = C^{(2)} \subset S^{[2]}$ and clearly the class of Σ_2 is not proportional to the class of Σ_1 because the latter is annihilated by $s_* : H_4(S^{[2]}, \mathbb{Q}) \to H_4(S^{(2)}, \mathbb{Q})$ while the former is not. Finally construction 4 gives us a constant cycle surface $\Sigma_3 \subset S^{[2]}$. The class of Σ_3 is not contained in the space generated by the classes of Σ_1 and Σ_2 because the latter are annihilated by the morphism

$$p_{1*} \circ p_2^* : H^4(S^{[2]}, \mathbb{Q}) \to H^0(S, \mathbb{Q}),$$

where $\Sigma \subset S \times S^{[2]}$ is the incidence subvariety and p_1, p_2 are the restrictions to Σ of the two projections, while the former is not.

The general case follows by analyzing the coisotropic classes on $S^{[n]}$ which are not polynomials in c, l, e. Indeed, for $n \geq 4$ (the case $n = 3$ is slightly different but can be analyzed similarly), we observed that there are two extra degree 4 algebraic classes which are the classes of the varieties $s^{-1}(S^{(\mu_1, \ldots, 1, 3)})$ and $s^{-1}(S^{(\mu_1, \ldots, 1, 2, 2)})$. These two codimension 2 subvarieties are coisotropic subvarieties fibered into constant cycle surfaces in $S^{[n]}$, hence contained in $S_2 S^{[n]}$ (see construction 1)), so we can work modulo their classes. Next, modulo these two classes, the algebraic cohomology of $S^{[n]}$ supported on the exceptional divisor E is generated as follows: The normalization $\widetilde{E}$ of E admits a morphism f to $S \times S^{(n-2)}$ and a morphism j to $S^{[n]}$. Then we have the two classes

$$j_*(f^*(pr_1^* c_1(L))), \ j_*(f^*(pr_2^* c_1(L_{n-2})))$$

which are both classes of subvarieties of codimension 2 of $S^{[n]}$ contained in $S_2 S^{[n]}$ because f has generic fibers isomorphic to $\mathbb{P}^1$ and choosing a constant cycle curve $C \subset S$ which is a member of $|L|$, $pr_1^* c_1(L) = C \times S^{(n-2)}$ is contained in $S_1(C \times S^{(n-2)})$ and similarly for $pr_2^* c_1(L_{n-2})$. We are thus reduced to study degree 2 algebraic isotropic classes on $S^{[n]}$ modulo those which are supported on the exceptional divisor; it is immediate to see that they are generated by polynomials in c, l and e, and we then prove they are generated by classes of subvarieties of codimension 2 contained in $S^{[2]}$ starting from the case $n = 2$ and applying the sum construction.

$\square$

We now turn to the Chow-theoretic conjectures made in Section 3. In this case, there exists a natural splitting of the Bloch-Beilinson filtration which is given by the de Cataldo-Migliorini decomposition [10] and the decompositions of the motives of the ordinary self-products or symmetric products of S given by the choice of the class $o_S \in \mathrm{CH}_0(S)$ as in Section 2. This decomposition is multiplicative, as proved by Vial [25], hence is the obvious candidate for the Beauville decomposition in this case.

We observe first that by definition, the induced decomposition on CH_0 is the one already described in Section 2 (see Proposition 2.2), with $o = o_S$, for which Conjecture 0.8 has been proved to hold.

Concerning the other conjectures, we deduce from the definition of the de Cataldo-Migliorini decomposition and from the construction 1), above a reduction of Conjectures 3.5 and 3.12 to the case of ordinary self-products or symmetric products of S.

Finally, we also have the following evidence for Conjecture 3.5, which immediately follows from the definition of the de Cataldo-Migliorini decomposition:

Lemma 4.3. *For any partition μ of n, the codimension i subvarieties*

$$s^{-1}(S^{(\mu)}) \subset S_i S^{[n]}$$

appearing in construction 1), with $i = n - l(\mu)$, have their class in $\mathrm{CH}^i(S^{[n]})_0$.

4.2 The Fano variety of lines in a cubic fourfold

It is well-known since [6] that the variety F of lines in a smooth cubic fourfold $W \subset \mathbb{P}^5$ is a smooth hyper-Kähler fourfold which is a deformation of $S^{[2]}$ for a K3 surface S of genus 14 (and an adequate polarization on $S^{[2]}$). The Chow ring of such varieties F has been studied in [29], confirming in particular Conjecture 0.1 for them, and even its variant involving the Chern classes of F. This variety satisfies Conjecture 3.1. This is implied by the following result summarizing observations made in [28], [29] (or can be obtained as an application of [9]).

Proposition 4.4. *Let W be a general smooth cubic fourfold, then*

(a) There are rational surfaces in F which can be obtained by considering the surface of lines in a hyperplane section of X with 5 nodes.

(b) There is a uniruled divisor in X obtained as follows: F admits a rational self-map $\phi_F : F \dashrightarrow F$ which is of degree 16. The exceptional divisor of a desingularization $\tilde{\phi}$ of ϕ maps to a uniruled divisor in F.

Note that for such an F, the degree 4 cohomology $H^4(F, \mathbb{Q})$ is equal to $S^2 H^2(F, \mathbb{Q})$ and when W is very general, Hodge classes of degree 4 are linear combinations of l^2, $c_2(F)$, where l is the Plücker polarization. This case is quite interesting because Shen and Vial constructed in [24] a "Beauville decomposition" of $\mathrm{CH}(F)$. We now have the following proposition showing that our proposal to construct a decomposition on $\mathrm{CH}_0(X)$, for X a general hyper-Kähler manifold fits well with their results:

Proposition 4.5. *For the Shen-Vial decomposition, one has*

$$S_1 \mathrm{CH}_0(F) = \mathrm{CH}_0(F)_0 \oplus \mathrm{CH}_0(F)_2,$$

$$S_2 \mathrm{CH}_0(F) = \mathrm{CH}_0(F)_0.$$

In particular, F satisfies Conjecture 3.7.

Proof. Indeed, [24, Proposition 19.5] says the following: Let $\Sigma_2 \subset F$ be the surface of lines $L \subset W$ such that there exists a $\mathbb{P}^3$ everywhere tangent to W along L. The surface Σ_2 is clearly the indeterminacy locus of the rational map $\phi : F \dashrightarrow F$ introduced above.

Proposition 4.6. *(Shen-Vial [24]) One has*

$$\mathrm{CH}_0(F)_0 \oplus \mathrm{CH}_0(F)_2 = \mathrm{Im}\,(\mathrm{CH}_0(\Sigma_2) \to \mathrm{CH}_0(F)). \tag{43}$$

On the other hand, the Shen-Vial decomposition is preserved by the map ϕ_* and thus, if D is the uniruled divisor mentioned in Proposition 4.4, (b), that is, the image under the desingularized map $\tilde{\phi} : \widetilde{F} \to F$ of the exceptional divisor over Σ_2, one has

$$\mathrm{Im}\,(\mathrm{CH}_0(\Sigma_2) \to \mathrm{CH}_0(F)) = \mathrm{Im}\,(\mathrm{CH}_0(D) \to \mathrm{CH}_0(F)) \subset S_1 \mathrm{CH}_0(F). \tag{44}$$

Finally, if $x \in F$ belongs to $S_1 F$, the orbit O_x contains a curve, which has to intersect D, as D is ample. Hence x is rationally equivalent in F to a point of D. Thus we conclude that $S_1 \mathrm{CH}_0(F) \subset \mathrm{Im}\,(\mathrm{CH}_0(D) \to \mathrm{CH}_0(F))$ so finally

$$\mathrm{Im}\,(\mathrm{CH}_0(D) \to \mathrm{CH}_0(F)) \subset S_1 \mathrm{CH}_0(F). \tag{45}$$

Combining (45), (43) and (44), we get

$$S_1 \mathrm{CH}_0(F) = \mathrm{CH}_0(F)_0 \oplus \mathrm{CH}_0(F)_2$$

as desired.

The second statement follows from the fact that the group $\mathrm{CH}_0(F)_0$ of the Shen-Vial decomposition is generated by the canonical 0-cycle of F, which can be constructed (using the results of [29]) by taking any 0-cycle o_F of degree different from 0, which can be expressed as a weighted degree 4 polynomial in $c_2(\mathcal{E})$ and l, where $\mathcal{E}$ is the restriction to $F \subset G(1,5)$ of the universal rank 2 bundle on the Grassmannian $G(1,5)$. However, the rational surfaces described in Proposition 4.4, a) represent the class $c_2(\mathcal{E})$ in $\mathrm{CH}^2(F)$. Thus o_F, being supported on a rational surface, belongs to $S_2 \mathrm{CH}_0(F)$. This gives the inclusion $\mathrm{CH}_0(F)_2 \subset S_2 \mathrm{CH}_0(F)$ and the fact that this is an equality follows from Lemma 3.10, (ii) which implies that the right hand side is isomorphic to $\mathbb{Q}$. $\qquad\square$

Let us finish this section with the following conditional result concerning Conjecture 3.6:

Proposition 4.7. *Assume the Shen-Vial decomposition on $\mathrm{CH}^2(F)$ is a splitting of the Bloch-Beilinson filtration on $\mathrm{CH}^2(F)$. Then the cycle class map is injective on the subgroup of $\mathrm{CH}^2(F)$ generated by constant cycles surfaces.*

Proof. It follows from [24, Proposition 21.10 and Section 20] that the group $CH^2(F)$ splits as

$$CH^2(F) = CH^2(F)_0 \oplus CH^2(F)_2,$$

where $CH^2(F)_2 \subset CH^2(F)_{hom}$. Our assumption is that the summand $CH^2(F)_0$ maps isomorphically onto $CH^2(F)/F_{BB}^1 CH^2(F)$ or equivalently that the cycle class map is injective on $CH^2(F)_0$ or that $CH^2(F)_2 = CH^2(F)_{hom}$. The space $CH^2(F)_2$ is constructed in [24, proof of Proposition 21.10] as the image of the map

$$I_* : CH_0(F)_{hom} \to CH^2(F)_{hom}$$

induced by the codimension 2 incidence correspondence $I \subset F \times F$ and it is proved more precisely in *loc. cit.* that

$$I_*(g^2\sigma) = -6\sigma \text{ in } CH^2(F) \tag{46}$$

for $\sigma \in CH^2(F)_2$. Suppose now that $\sigma \in CH^2(F)$ is a combination of classes of constant cycle surfaces which is homologous to 0. Then by our assumption, $\sigma \in CH^2(F)_2$ and on the other hand $g^2\sigma = 0$ in $CH_0(F)$ because F satisfies Conjecture 3.7. It then follows from (46) that $\sigma = 0$.

$\square$

4.3 The case of the LLSS 8-folds

Let again W be a cubic 4-fold. As mentioned in the previous section, the variety $F := F_1(W)$ of lines in W is a smooth hyper-Kähler fourfold. It is a deformation of $S^{[2]}$ for some $K3$ surfaces with adequate polarization, but for very general W, it has $\rho(F) = 1$. Much more recently, Lehn-Lehn-Sorger-van Straten proved in [18] that starting from the variety $F_3(W)$ of cubic rational curves in W, one can construct a hyper-Kähler 8-fold Z, which has Picard number 1 for very general W, and which has been proved in [1] to be a deformation of a hyper-Kähler manifold birationally equivalent to $S^{[4]}$. The variety Z is constructed by observing first that each cubic rational curve $C \subset W$ moves in a 2-dimensional linear system in the cubic surface $S_C = <C> \cap W$, where $<C>$ is the $\mathbb{P}^3$ generated by C. Finally there is a boundary divisor which can be contracted in the base of this $\mathbb{P}^2$-fibration on $F_3(W)$, and this produces the variety Z. Thus there is a morphism $q : F_3(W) \to Z$ which is birationally a $\mathbb{P}^2$-bundle.

Let us prove the following result:

Proposition 4.8. *There is a degree* 6 *dominant rational map*

$$\psi : F \times F \dashrightarrow Z$$

such that

$$\psi^*\sigma_Z = pr_1^*\sigma_F - pr_2^*\sigma_F. \tag{47}$$

Here σ_Z, resp. σ_F denotes the holomorphic 2-form of Z, resp. F.

Proof. Let L, L' be two lines in W and denote by l, resp l' the corresponding points in F. Assume l, l' are general points of F; then L and L' are in general position in W and they generate a $\mathbb{P}_{L,L'}^3 :=< L, L' >$. The surface $S_{L,L'} := \mathbb{P}_{L,L'}^3 \cap W$ is a smooth cubic surface containing both L and L' and we claim that the linear system $|\mathcal{O}_S(L - L')(1)|$ is a 2-dimensional linear system of rational cubics on S. This can be verified by computing its self-intersection and intersection with $K_{S_{L,L'}}$ but it is even easier by observing that for any choice of point $x \in L$, the plane $< x, L' >$ intersects L into one point, and intersects W along the union of the line L' and a residual conic C'. Thus we get a member of this linear system which is the union of L and of C' meeting in one point: this is a rational cubic curve. Note that for each pair (L, L') we get a $\mathbb{P}^1 \cong L$ of such curves.

To compute the degree of the rational map ψ so constructed, we start from a cubic surface $S \subset W$ with a 2-dimensional linear system of rational cubic curves. This system provides a birational map $\tau : S \to \mathbb{P}^2$ contracting 6 exceptional curves, which are lines in S. The curves in this linear system are the pull-backs of lines in $\mathbb{P}^2$, and they become reducible when the line passes through one of the 6 points blown-up by τ. We thus get 6 $\mathbb{P}^1$'s of such curves which correspond to 6 possible choices of pairs (L, L'), each one giving rise to a $\mathbb{P}^1 \cong L$ of reducible curves $L \cup_x C$.

Let us finally prove formula (47). In fact, we observe that according to the constructions of [6] and [18], the 2-forms σ_F and $p^* \sigma_Z$ are deduced from the choice of a generator of the 1-dimensional vector space $H^{3,1}(W)$ by applying the correspondences $P \subset F \times W$, $\mathcal{C}_3 \subset F_3(X) \times W$ given by the universal families of curves. Here $p : F_3(W) \to Z$ is the forgetting morphism whose description has been sketched above.

Next we observe that the rational map ψ has a lift

$$\psi_1 : P_1 \dashrightarrow F_3(W)$$

where $p_1 : P \times F = P_1 \to F \times F$ is the pull-back by the first projection $F \times F \to F$ of the universal $\mathbb{P}^1$-bundle $P \to F$, and ψ_1 associates to a general triple (l, x, l') with $x \in L$, the cubic curve

$$L \cup_x C,$$

where $C \subset W$ is the residual conic of L' contained in the plane $< x, L' >$. The equality of forms stated in (47) is then a consequence of Mumford's theorem [22] and the fact that if we restrict the universal family $\mathcal{C}_3$ to the divisor D in $F_3(W)$ parameterizing reducible rational curves $C_3 = L \cup C$, where C is a conic in X with residual line L', then for any $(l, x, l') \in P \times F$, the curve C_3 parametrized by $\psi_1(l, x, l')$ is rationally equivalent in W to $L - L'$ up to a constant. It follows that we have the equality of forms pulled-back from W via the universal correspondences:

$$p_1^*(pr_1^* \sigma_F - pr_2^* \sigma_F) = \psi_1^*(p^* \sigma_Z) \text{ in } H^0(P_1, \Omega_{P_1}^2).$$

This immediately implies (47). $\qquad\square$

Corollary 4.9. *The LLSS varieties Z satisfy conjecture 0.4.*

Proof. Indeed, the variety F satisfies conjecture 0.4. This follows either from [9] which provides uniruled divisors and constant cycles Lagrangian surfaces, or explicitly from Proposition 4.4.

Having algebraically coisotropic subvarieties Z_1, Z_2 in F which are of respective codimensions i_1, i_2 and fibered into i_1, resp. i_2-dimensional constant cycles subvarieties of F, their product $Z_1 \times Z_2$ is mapped by ψ onto a codimension $i_1 + i_2$ subvariety of Z, which is fibered into constant cycles subvarieties of dimension $i_1 + i_2$. One just has to check that $Z_1 \times Z_2$ is not contracted by ψ, but because both $\widetilde{F \times F}$ and Z have trivial canonical bundle, the ramification locus of a desingularization $\tilde\psi : \widetilde{F \times F} \to Z$ of ψ is equal to the exceptional divisor of the birational map $\widetilde{F \times F} \to F \times F$. So ψ is of maximal rank where it is defined, and one just has to check that ψ is well defined at the general points of the varieties $Z_1 \times Z_2$ defined above. $\qquad\square$

Remark 4.10. There is a natural uniruled divisor in Z that deserves a special study, namely the branch locus of the desingularization $\tilde\psi : \widetilde{F \times F} \to Z$ of ψ. This branch locus D is a non-empty divisor because Z is simply connected. It is the image of the ramification divisor of $\tilde\psi$ which has to be equal to the exceptional divisor of $\widetilde{F \times F}$ since both $F \times F$ and Z have trivial canonical divisor, and this is why D is uniruled.

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