Irreducible Specht modules for Hecke algebras of type A – revisited

(joint work with Sinéad Lyle)

Set-up

$\mathbb{F}$	field
$\mathfrak{S}_n$	symmetric group on $\{1, \ldots, n\}$
λ	partition of n
	i.e. $\lambda = (\lambda_1, \lambda_2, \ldots)$, $\lambda_i \in \mathbb{Z}_{\geqslant 0}$
	$\lambda_1 \geqslant \lambda_2 \geqslant \ldots,\qquad \lambda_1 + \lambda_2 + \cdots = n.\qquad$ (Write $\lambda \vdash n$.)
S^λ	Specht module for $\mathbb{F}\mathfrak{S}_n$

- $\mathrm{char}(\mathbb{F}) = \infty \quad \rightsquigarrow \quad S^\lambda$ is irreducible; $\qquad \mathrm{Irr}(\mathbb{F}\mathfrak{S}_n) = \{S^\lambda \mid \lambda \vdash n\}$.
- $\mathrm{char}(\mathbb{F}) = p < \infty \quad \rightsquigarrow \quad S^\lambda$ is a p-modular reduction of an irreducible in infinite characteristic, and is not necessarily irreducible.

Main question: For which λ, $\mathbb{F}$ is the Specht module S^λ irreducible?

Fact: Every field is a splitting field for $\mathfrak{S}_n$.

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Main question: For which λ, p is the Specht module S^λ irreducible?

More general set-up

$\mathbb{F}$	field
q	element of $\mathbb{F}^\times$
$\mathcal{H}_n = \mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$	Iwahori–Hecke algebra of $\mathfrak{S}_n$ over $\mathbb{F}$, parameter q
e	minimal such that $1 + q + q^2 + \cdots + q^{e-1} = 0$ in $\mathbb{F}$ (or $e = \infty$)
λ	partition of n
S^λ	Specht module for $\mathcal{H}_n$

- $e = \infty \quad \rightsquigarrow \quad S^\lambda$ is irreducible; $\qquad \operatorname{Irr}(\mathcal{H}_n) = \{S^\lambda \mid \lambda \vdash n\}$.
- $e < \infty \quad \rightsquigarrow \quad S^\lambda$ is not necessarily irreducible.

Main question: For which $\lambda, \mathbb{F}, q$ is the Specht module S^λ irreducible?

In fact, the reducibility of S^λ depends only on λ, $p = \operatorname{char}(\mathbb{F})$ and e.

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Main question: For which λ, p, e is the Specht module S^λ irreducible?

This question is now answered in almost all cases.

Irreducible $\mathcal{H}_n$-modules

Suppose $\lambda \vdash n$.

λ is *e-regular* $\quad\Leftrightarrow\quad \nexists \quad \lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+e-1} > 0.$ (Write $\lambda \vdash_e n$.)

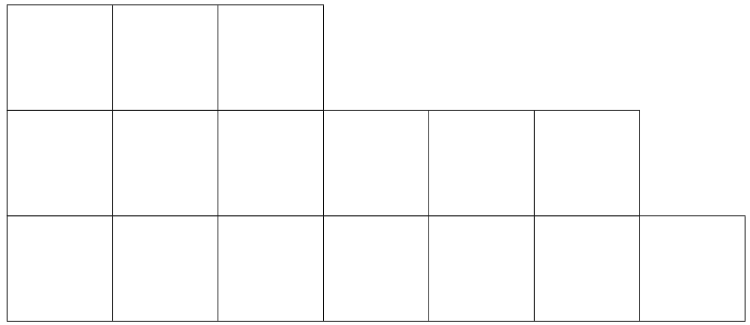
If $\lambda \vdash_e n$, then S^λ has an irreducible cosocle D^λ;

$$\mathrm{Irr}(\mathcal{H}_n) = \{D^\lambda \mid \lambda \vdash_e n\}.$$

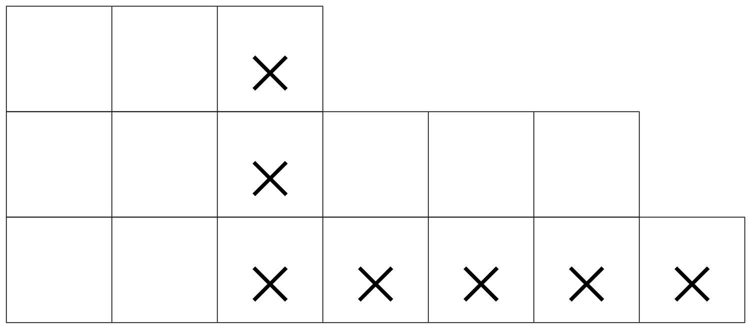
So: if $\lambda \vdash_e n$ and S^λ is irreducible, then we have $S^\lambda = D^\lambda$.

Some combinatorics

Young diagram of a partition: array of boxes in the plane:

$\lambda = (7, 6, 3) \quad \rightsquigarrow \quad [\lambda] =$ 

Hook length of box b: number of boxes directly to the right of or directly above b, including b itself.



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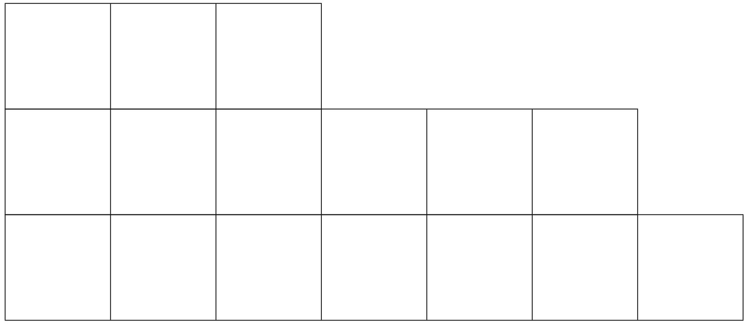
$\lambda = (7,6,3) \quad \rightsquigarrow \quad [\lambda] =$ 

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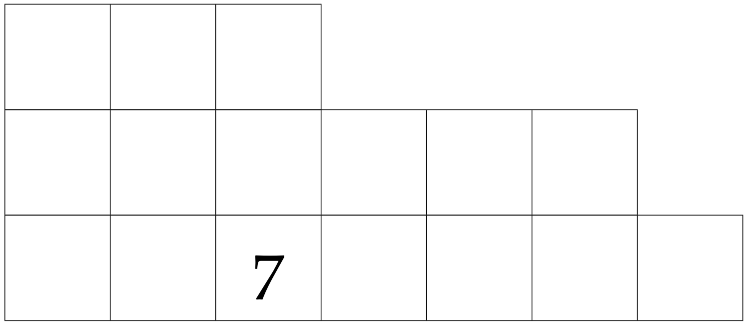

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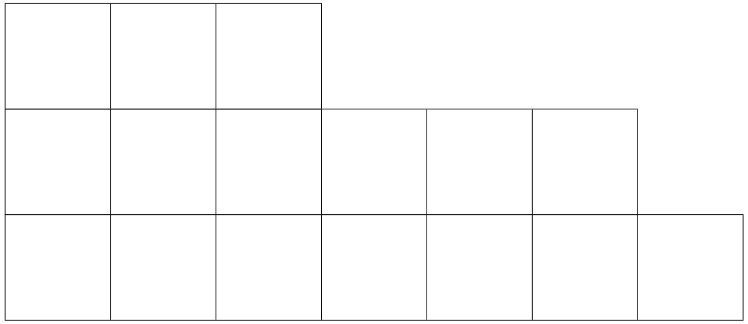

Hook length of box b: number of boxes directly to the right of or directly above b, including b itself.

3	2	1				
7	6	5	3	2	1	
9	8	7	5	4	3	1

(e, p)-power diagram: fill each box with $\nu_{e,p}(h)$, where

h = hook length of the box

$$\nu_{e,p}(h) = \begin{cases} 1 + \nu_p(h/e) & (e \mid h) \\ 0 & (e \nmid h). \end{cases}$$

3	2	1				
7	6	5	3	2	1	
9	8	7	5	4	3	1

$e = 2, p = 3$

0	1	0				
0	2	0	0	1	0	
0	1	0	0	1	0	0

Carter Condition for λ: the entries of the (e,p)-power diagram are constant in each column.

Theorem (James–Mathas)**.** *Suppose $\lambda \vdash_e n$. Then S^λ is irreducible if and only if λ satisfies the Carter Condition.*

Proof. Jantzen–Schaper formula. □

$\lambda = (7,6,3)$, $e = 2$

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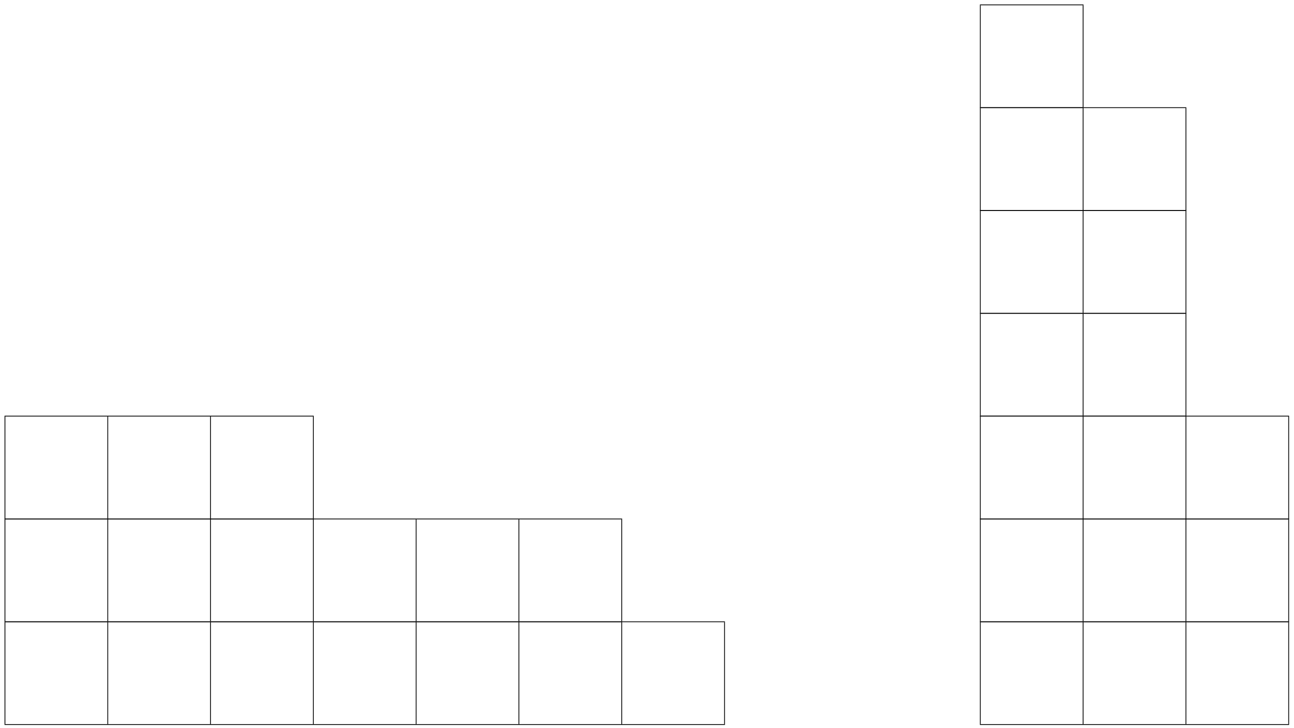
7	6	5	3	2	1	
9	8	7	5	4	3	1

$p > 3$ (including $p = \infty$)

0	1	0				
0	1	0	0	1	0	
0	1	0	0	1	0	0

So the Main Question is answered for e-regular partitions.

Lemma. *Let λ' denote the conjugate (or transpose) partition to λ. Then S^λ is irreducible if and only if $S^{\lambda'}$ is.*



So the Main Question is answered for e-regular partitions.

Lemma. *Let λ' denote the conjugate (or transpose) partition to λ. Then S^{λ} is irreducible if and only if $S^{\lambda'}$ is.*

So the Main Question is answered also for *e-restricted* partitions (i.e. conjugates of e-regular partitions): need entries of the (e, p)-power diagram constant on each *row*.

So consider partitions which are neither e-regular nor e-restricted. At this point, the cases $e = 2$ and $e > 2$ diverge ...

The case $e > 2$

Generalised Carter Condition for λ: for every *non-zero* entry of the (e, p)-power diagram, either all entries in the same row are equal, or all entries in the same column are equal.

Theorem (F, Lyle, James–Lyle–Mathas 2006)**.** *Suppose $e > 2$. Then S^λ is irreducible if and only if λ satisfies GCC.*

$$\lambda = (13, 8, 2^4, 1^5), e = 3, p = 2$$

·												
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1												
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·	·											
1	1											
·	·											
·	·	2	·	·	1	·	·					
·	·	2	·	·	1	·	·	·	·	1	·	·

Ingredients for the proof

Lemma (Brundan–Kleshchev)**.** *Suppose* $\lambda \vdash n$*, and* μ *is a partition obtained by removing all removable boxes of some fixed residue from* $[\lambda]$*. Then*

$$S^\mu \textit{ reducible} \Rightarrow S^\lambda \textit{ reducible}.$$

(*Residue* of box in ith row and jth column: $j - i \pmod{e}$.)

Ingredients for the proof

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$$S^\mu \textit{ reducible} \Rightarrow S^\lambda \textit{ reducible}.$$

Theorem (Carter–Payne(–Lyle))**.** *Suppose* $\lambda \vdash n$ *and* μ *is obtained by replacing some box with a lower box of the same residue. Then*

$$\mathrm{Hom}_{\mathcal{H}_n}(S^\mu, S^\lambda) \neq 0.$$

Theorem (F–Lyle, Lyle–Mathas)**.** *Let* $\bar{\lambda}$ *be obtained by removing the first column from* $[\lambda]$*. If* $\lambda, \mu \vdash n$ *and* $\bar{\lambda}, \bar{\mu} \vdash m$*, then*

$$\dim_{\mathbb{F}} \mathrm{Hom}_{\mathcal{H}_n}(S^\mu, S^\lambda) = \dim_{\mathbb{F}} \mathrm{Hom}_{\mathcal{H}_m}(S^{\bar{\mu}}, S^{\bar{\lambda}}).$$

Theorem (Turner, James–Lyle–Mathas)**.** *[Description of decomposition numbers for* Rouquier blocks.*]*

The case $e = 2$

From now on, assume $e = 2$ (i.e. $q = -1$). The situation here is very different: GCC is neither necessary nor sufficient.

Example. Suppose $p = \infty$, and λ is a *rectangular* partition, i.e. $\lambda = (a^b)$, some a, b. Then S^λ is irreducible.

The symmetric group case is known:

Theorem (James–Mathas)**.** *Suppose* $e = p = 2$*, and* λ *is neither* 2*-regular nor* 2*-restricted. Then* S^λ *is irreducible* $\Leftrightarrow \lambda = (2^2)$.

Proof. Explicit construction of homomorphisms from permutation modules to Specht modules. □

Some computations in the case $p = \infty$

When $p = \infty$, the decomposition numbers for $\mathcal{H}_n$ can be computed via the LLT algorithm. In particular, the reducibility of any Specht module can be checked.

2004: computations by (F–)Mathas . . .

Suppose λ is neither 2-regular nor 2-restricted. Say that λ is an *FM-partition* if:

- $\exists!\ b$ such that $\lambda_b - \lambda_{b+1} \geqslant 2$;
- for any a with $\lambda_a = \lambda_{a+1} > 0$, we have $a \leqslant b - 1 \leqslant \lambda_a$;
- $\lambda_1 > \lambda_2 > \cdots > \lambda_c$, where c is maximal such that $\lambda_{b+c} > 0$;
- all addable boxes of $[\lambda]$, except possibly the highest and lowest, have the same residue;
- if $c > 0$ then all addable boxes of $[\lambda]$ have the same residue.

Conjecture (F(–Mathas)). *Suppose $e = 2$, $p = \infty$, λ is neither 2-regular nor 2-restricted. Then S^λ is irreducible if and only if either λ or λ' is an FM-partition.*

Verified for $n \leqslant 45$.

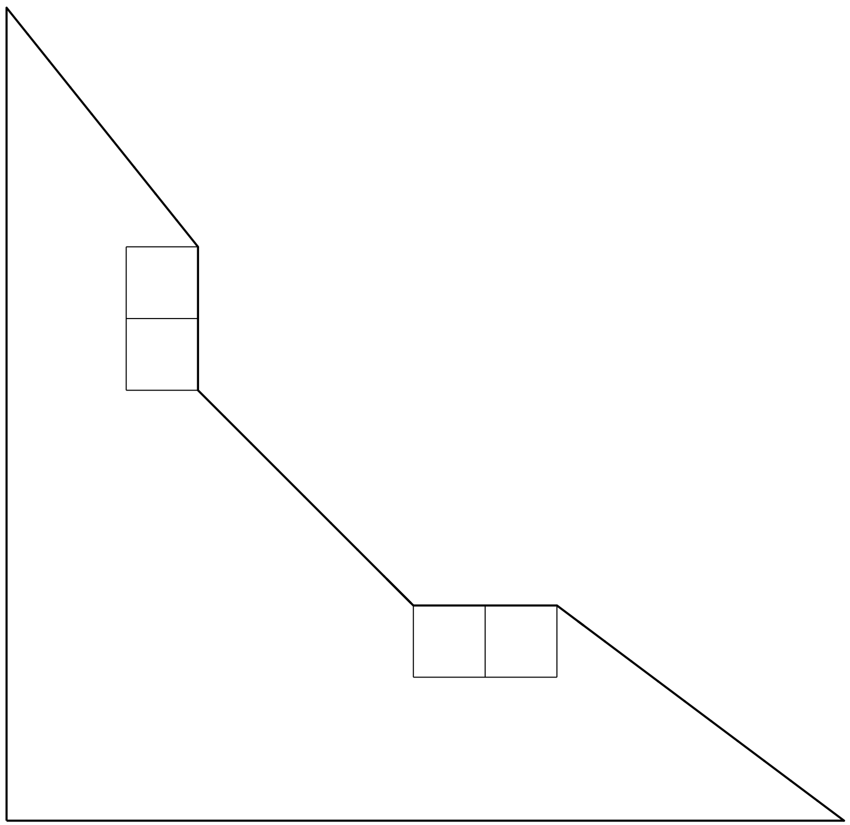
For the prime characteristic case:

Conjecture (F). *Suppose $e = 2$ and $p < \infty$. Then there are only finitely many partitions λ such that λ is neither 2-regular nor 2-restricted and S^λ is irreducible.*

True for $p = 2$ (only one λ), almost proved for $p = 3$ (ten different λs).

Main result

Theorem (F–Lyle). *Suppose $e = 2$, and λ satisfies the following condition: there exist $a < b$ such that $\lambda_a - \lambda_{a+1} \geqslant 2$ and $\lambda_b = \lambda_{b+1} > 0$. Then S^λ is reducible.*



Method of proof

First, assume $p = \infty$: a Specht module in prime characteristic is a modular reduction of a Specht module in infinite characteristic.

Induction on n, using the Brundan–Kleshchev lemma from before. For the difficult cases, two main techniques:

1. Fock space calculations
2. Homomorphisms

Fock space calculations

$\mathcal{U}$	quantum algebra $\mathcal{U}_v(\widehat{\mathfrak{sl}}_2)$ over $\mathbb{Q}(v)$
$\mathcal{F}$	*Fock space*: $\mathcal{U}$-module with $\mathbb{Q}(v)$-basis $\{s(\lambda)\}$ indexed by partitions
M	submodule generated by $s(\varnothing)$

M has a *canonical basis* $\{G(\mu)\}$ indexed by all 2-regular partitions.

Theorem (Ariki)**.** *Write* $G(\mu) = \sum_\lambda d_{\lambda\mu}(v)s(\lambda)$. *Then if* $e = 2$ *and* $p = \infty$,

$$[S^\lambda : D^\mu] = d_{\lambda\mu}(1).$$

Fact: $d_{\lambda\mu}(v)$ is a polynomial with *non-negative integer* coefficients.

So: if S^λ is irreducible, then we must have

$$d_{\lambda\nu}(v) = v^a \qquad \text{(for some particular } \nu\text{)}$$
$$d_{\lambda\mu}(v) = 0 \qquad \text{(for all other } \mu\text{).}$$

M possesses a *bar involution* $m \mapsto \overline{m}$.

Fact: Each $G(\mu)$ is bar-invariant. Moreover, any bar-invariant element of M can be written in the form $\sum_\mu \alpha_\mu(v)G(\mu)$, with $\alpha_\mu(v) \in \mathbb{Q}(v + v^{-1})$.

Corollary. *Suppose $X, Y \in M$ are bar-invariant, and*

$$X = \sum_\lambda a_\lambda(v)s(\lambda), \qquad Y = \sum_\lambda b_\lambda(v)s(\lambda),$$

and that for some particular λ we have $a_\lambda(v) = v^s$, $b_\lambda(v) = v^t$, $s \neq t$. Then (for $e = 2, p = \infty$) S^λ is reducible.

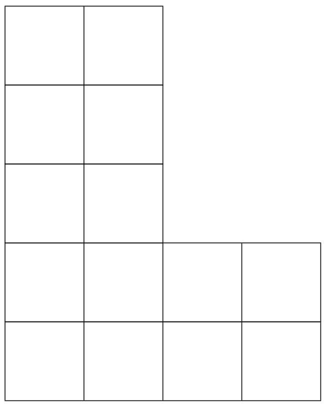
Technique: Construct X and Y, using known $G(\mu)$ and applying Chevalley generators of $\mathcal{U}$.

Homomorphisms

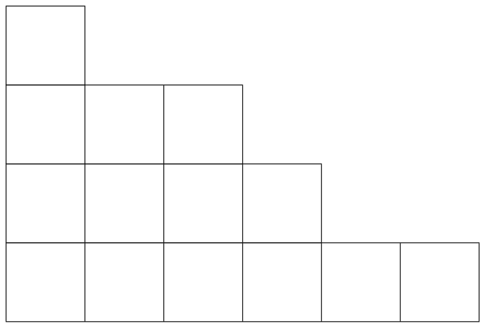
James's regularisation theorem: slide all boxes of $[\lambda]$ south-east as far as possible, to get *regularisation* λ^{reg}. Then $[S^\lambda : D^{\lambda^{\mathrm{reg}}}] = 1$.

So if S^λ is irreducible, then $S^\lambda \cong D^{\lambda^{\mathrm{reg}}}$.

$\lambda = (4^2, 2^3)$



$\lambda^{\mathrm{reg}} = (6, 4, 3, 1)$



Permutation modules

For each λ, have a module M^λ (q-analogue of *permutation module*).

Aim: Construct non-zero homomorphisms $M^\mu \to S^\lambda$.

Why? We have $[M^\mu : D^\nu] = 0$ unless $\nu \trianglerighteq \mu$ (dominance order on partitions).

So if we have $\mathrm{Hom}_{\mathcal{H}_n}(M^\mu, S^\lambda) \neq 0$ and $\lambda^{\mathrm{reg}} \ntrianglerighteq \mu$, then S^λ is reducible.

How do we construct homomorphisms $M^\mu \to S^\lambda$?

- We have $S^\lambda \leqslant M^\lambda$, and we know $\mathrm{Hom}_{\mathcal{H}_n}(M^\mu, M^\lambda)$ explcitly: it has a basis $\{\Theta_T\}$ indexed by *row-standard μ-tableaux of type λ*.
- Suppose $\theta = \sum_T d_T \Theta_T$. Lyle (2006) gives a method to determine whether $\mathrm{Im}(\theta) \leqslant S^\lambda$.

Lemma (F). *Suppose* ξ, ν *are partitions. Put* $l = \text{length}(\nu)$*, and suppose* $\xi_{l-1} \geqslant l$*. Then*

$$\text{Hom}_{\mathcal{H}_n}(M^{\mu}, S^{\lambda}) \neq 0,$$

where

$$\lambda_i = \xi_i + 2\nu_i,$$
$$\mu_i = \xi'_i + 2\nu_i.$$

Proof. We know that $\text{Hom}_{\mathbb{Q}\mathfrak{S}_{|\nu|}}(M^{\nu}, S^{\nu}) \neq 0$; Lyle's method enables us to use a non-zero homomorphism here to construct a non-zero homomorphism $M^{\mu} \to S^{\lambda}$. □

Example: $\lambda = (4^2, 2^2)$

Put $\xi = (2^4)$, $\nu = (1^2)$. Then $\mu = (6^2)$, while $\lambda^{\text{reg}} = (5, 4, 2, 1) \not\trianglerighteq \mu$.

$\mathbb{Q}\mathfrak{S}_2$-homomorphism from M^ν to S^ν given by $\Theta_{T_1} - \Theta_{T_2}$, where

$$T_1 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}, \qquad T_2 = \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}.$$

$\mathcal{H}_{12}$-homomorphism from M^μ to S^λ given by $\Theta_{U_1} - \Theta_{U_2}$, where

$$U_1 = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 3 & 4 \\ \hline\end{array}, \qquad U_2 = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & 1 \\ \hline\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 3 & 4 \\ \hline\end{array}.$$

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References

New

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Old

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