Complex reflection groups and cyclotomic Hecke algebras

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Why should I care?

Complex reflection groups

- have a rich and beautiful theory
- come up in many different contexts

Complex reflection groups

k: a subfield of $\mathbb{C}$
V: a finite dimensional k-vector space

$s \in \mathrm{GL}(V)$ is a *complex reflection* $\Longleftrightarrow$: codim $\ker(s-1) = 1$

i.e., s fixes the hyperplane $H_s := \ker(s-1)$ pointwise.

$W \leq \mathrm{GL}(V)$ is a *complex reflection group* (crg) $\Longleftrightarrow$:
W is finite, generated by reflections.

Examples

- $W \leq \mathrm{GL}_n(\mathbb{Q})$ a Weyl group $\Longrightarrow$ W is a crg.
- $W \leq \mathrm{GL}_n(\mathbb{R})$ a (finite) Coxeter group $\Longrightarrow$ W is a crg.
- $1 \neq \zeta \in k$ with $\zeta^d = 1 \Longrightarrow W = \langle \zeta \rangle \leq k^\times = \mathrm{GL}_1(k)$ is a crg.
- The group

$$W := \left\langle \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}, \frac{\sqrt{-3}}{3} \begin{pmatrix} -\zeta_3 & \zeta_3^2 \\ 2\zeta_3^2 & 1 \end{pmatrix} \right\rangle \leq \mathrm{GL}_2(\mathbb{Q}(\zeta_3)),$$

with $\zeta_3 := \exp(2\pi i/3)$, is a crg of order 72, denoted G_5.
(G_5 has no faithful real reflection representation)

Invariants

$S(V)$: the symmetric algebra of V

So for any basis $\{v_1, \ldots, v_n\}$ of V have $S(V) \cong k[v_1, \ldots, v_n]$.

If $W \leq \mathrm{GL}(V)$ then W acts on $S(V)$. Consider invariants

$$S(V)^W := \{f \in S(V) \mid w.f = f \text{ for all } w \in W\}.$$

Theorem (Shephard–Todd (1954), Chevalley (1955))

Let $W \leq \mathrm{GL}(V)$ be finite. Then the following are equivalent:

(i) *W is generated by reflections*

(ii) *the ring $S(V)^W$ of invariants is a polynomial ring*

Generators of $S(V)^W$

Assume that $S(V)^W$ is a polynomial ring. There exist $n = \dim V$ algebraically independent elements $f_1, \ldots, f_n \in S(V)$ with

$$S(V)^W = k[f_1, \ldots, f_n].$$

The f_i can be chosen to be homogeneous with respect to the natural grading of $S(V)$.

The $(f_i)_i$ are not uniquely determined, but their degrees $d_i = \deg f_i$ are.

These are the *degrees* of the reflection group W.

Clearly $|W| = d_1 \cdots d_n$.

Furthermore, $\sum_{i=1}^n (d_i - 1) = N :=$ number of reflections in W.

Examples

- $W = \mathfrak{S}_n$ in its natural permutation representation on $V = k^n$. Invariants are generated by the elementary symmetric functions

$$f_j := \sum_{i_1 < \ldots < i_j} v_{i_1} \cdots v_{i_j} \qquad (1 \leq j \leq n)$$

  with degrees $1, 2, \ldots, n$, and $d_1 \cdots d_n = n! = |\mathfrak{S}_n|$.
- $W = \langle \zeta \rangle \leq \mathrm{GL}_1(k)$ with $\zeta = \exp(2\pi i/d)$.
  Here $S(V) = k[v]$, $S(V)^W = k[v^d]$
  $\Longrightarrow$ fundamental invariant is v^d, of degree d, and $d = |W|$.
- Recall $G_5 = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}, \frac{\sqrt{-3}}{3} \begin{pmatrix} -\zeta_3 & \zeta_3^2 \\ 2\zeta_3^2 & 1 \end{pmatrix} \right\rangle$, $\zeta_3 = \exp(2\pi i/3)$.
  Here $S(V) = k[v_1, v_2]$, and $S(V)^W = k[f_1, f_2]$ with

$$f_1 := v_1^6 + 20 v_1^3 v_2^3 - 8 v_2^6, \qquad f_2 := 3 v_1^3 v_2^9 + 3 v_1^6 v_2^6 + v_1^9 v_2^3 + v_2^{12},$$

  with degrees $d_1 = 6, d_2 = 12$, $d_1 d_2 = 72 = |W|$.

Parabolic subgroups

$W \leq \mathrm{GL}(V)$ a crg.

For $U \leq V$ a subspace, the fixator

$$C_W(U) := \{w \in W \mid w.v = v \text{ for all } v \in U\}$$

is called a *parabolic subgroup* of W.

Theorem (Steinberg (1964), Lehrer (2004))

Let W be a crg. Every parabolic subgroup of W is generated by the reflections it contains. In particular, it is also a crg.

Examples

For Coxeter groups, these are just the conjugates of the standard parabolic subgroups.
For G_5, there are two non-conjugate parabolic subgroups of order 3.

Regular elements

A vector $v \in V$ is *regular* $:\Longleftrightarrow$ v is not contained in any reflecting hyperplane, i.e., v is not stabilized by any reflection.

An element $w \in W$ is *d-regular* $:\Longleftrightarrow$ w has a regular eigenvector for an eigenvalue ζ which is a primitive dth root of unity.

Denote by $V(w,\zeta)$ the ζ-eigenspace of w in V.

Theorem (Springer (1974))

Let $w \in W$ be d-regular. Then $C_W(w)$ is a crg on $V(w,\zeta)$, with set of degrees

$$\{d_i \mid d \text{ divides } d_i\}.$$

Idea of proof: show that $C_W(w)$ has polynomial invariants on $V(w,\zeta)$.

Examples

- $W = \mathfrak{S}_n$ in its natural permutation representation on $V = k^n$.
  Assume that $d|n$.
  Then the product of n/d disjoint d-cycles is d-regular, with centralizer $C_d \wr \mathfrak{S}_{n/d}$, with degrees

$$\{d_i \mid d \text{ divides } d_i\} = \{d, 2d, \ldots, d \cdot n/d = n\}.$$

- $W = W(F_4)$, a Weyl group. There exist 3-regular elements in W.
  The degrees of $W(F_4)$ are $2, 6, 8, 12$, so the centralizer has degrees $6, 12$: It is the complex reflection group G_5.

So, even if W is a Weyl group, $C_W(w)$ may be a truly complex reflection group.

Eigenspaces

Let $W \leq \mathrm{GL}(V)$ a crg.

Recall: for $\zeta \in k^\times$, $w \in W$,

$$V(w,\zeta) := \{v \in V \mid w.v = \zeta v\}$$

is the eigenspace of W with respect to the eigenvalue ζ.

Have a kind of Sylow theorem for eigenspaces:

Theorem (Springer (1974))

Let W be a crg, ζ a primitive dth root of unity.

(a) $\max_{w\in W} \dim V(w,\zeta) = \#\{i \mid d \text{ divides } d_i\} =: a(d)$.

(b) *For all $w \in W$ there exists $w' \in W$ such that $V(w,\zeta) \subseteq V(w',\zeta)$ and $\dim V(w',\zeta) = a(d)$.*

(c) *The maximal ζ-eigenspaces are conjugate under W.*

Further examples of crg

Examples

- $\mathfrak{S}_n$ acts naturally on $V = k^n = \bigoplus kv_i$.
  Fix $d \geq 2$. In each coordinate have the reflection $v_i \mapsto \zeta_d v_i$.
  Obtain the wreath product $C_d \wr \mathfrak{S}_n$, generated by reflections.
  This is called $G(d, 1, n)$.
  For each divisor e of d, there is a normal reflection subgroup $G(d, e, n)$ of $G(d, 1, n)$ of index e.
- Let $G \leq \mathrm{SL}_2(\mathbb{C})$ finite, $g \in G$. Let ζ be an eigenvalue of g
  $\Longrightarrow \zeta^{-1}g$ is a reflection.
  So, if $G = \langle g_1, \ldots, g_r \rangle$, obtain crg $\langle \zeta_1^{-1}g_1, \ldots, \zeta_r^{-1}g_r \rangle$.
  For example, $G_5 \cong \mathrm{SL}_2(3) \times C_3$.
  (If G is irreducible, then $G/Z(G) \in \{D_n, \mathfrak{A}_4, \mathfrak{S}_4, \mathfrak{A}_5\}$.)
- If $g \in \mathrm{SL}_3(\mathbb{C})$ is an involution, then $-g$ is a reflection.
  ($\mathfrak{A}_5$, $\mathrm{PSL}_2(7)$ and $3.\mathfrak{A}_6$ have faithful 3-dimensional representations and are generated by involutions.)

The classification

Any crg is a direct product of irreducible crg.

Theorem (Shephard–Todd (1954))

Let $W \leq \mathrm{GL}(V)$ be an irreducible crg. Then one of the following holds:

(i) *W is imprimitive and $W = G(de, e, n)$ for some $n, d, e \geq 1$, $de \geq 2$,*

(ii) *$W \cong \mathfrak{S}_n$ ($\cong G(1, 1, n)$), $n \geq 2$, and $\dim V = n - 1$, or*

(iii) *W is one of 34 exceptional groups $G_4, \ldots, G_{37}$, and $\dim V \leq 8$.*

For example, in dimension 2 the dihedral groups lead to $G(de, e, 2)$, while the groups $\mathfrak{A}_4, \mathfrak{S}_4, \mathfrak{A}_5$ lead to 4, 8, resp. 7 exceptional crg.

We have

- $G(1, 1, n) = \mathfrak{S}_n$, $G(2, 1, n) = W(B_n)$, $G(2, 2, n) = W(D_n)$,
- $G(6, 6, 2) = W(G_2)$, $G_{28} = W(F_4)$, $G_{35,36,37} = W(E_{6,7,8})$,
- (Coxeter groups) $G(e, e, 2) = W(I_2(e))$, $G_{23,30} = W(H_{3,4})$.

Consequences of the classification

How many reflections are needed to generate a crg?

Clear: at least $\dim V$ generators are necessary.

Proposition

Let $W \leq \mathrm{GL}(V)$ be an irreducible crg. Then W can be generated by at most $\dim V + 1$ reflections.

A crg is called *well-generated* if $\dim V$ reflections suffice.

In particular any Coxeter group is well-generated.

Steinberg: Any irreducible crg contains a well-generated irreducible crg.

Proposition

Let $W \leq \mathrm{GL}(V)$ be an irreducible crg. Then W has at most three classes of reflecting hyperplanes (two if W is well-generated).

Field of definition

Let $W \leq \mathrm{GL}(V)$ a crg. Over which field(s) can the representations of W be realized?

Let $k_W := \mathbb{Q}(\mathrm{tr}_V(w) \mid w \in W)$, the character field of W on V.

Theorem (Benard (1976), Bessis (1997))

The field k_W is a splitting field for W,
i.e., any (irreducible) representation of W can be realized over k_W.

Examples

- For all Weyl groups W, we have $k_W = \mathbb{Q}$.
- For $W = W(H_4)$, we have $k_W = \mathbb{Q}(\sqrt{5})$.
- For $W = G_5$, we have $k_W = \mathbb{Q}(\sqrt{-3})$.

Field of definition, II

How can we characterize k_W in terms of W?

For well-generated groups, this is possible using only the degrees:

Theorem (M. (1999))

Let W be a well-generated irreducible crg, with degrees $d_1 \leq \ldots \leq d_n$, $\zeta = \exp(2\pi i/d_n)$,

$$G := \text{ setwise stabilizer of } (\zeta^{d_j - 1} \mid 1 \leq j \leq n) \text{ in } \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}).$$

Then $k_W = \mathbb{Q}(\zeta)^G$.

For Weyl groups, the $\zeta^{d_j - 1}$ are the eigenvalues of the Coxeter element

$\Longrightarrow$ k_W is determined by the cofficients of the characteristic polynomial of a Coxeter element.

Automorphisms

Let $\phi \in N_{\mathrm{GL}(V)}(W)$.

Then ϕ stabilizes the set of reflections, of reflecting hyperplanes,...

For some applications (for example in twisted groups of Lie type), replace the crg by the coset $W\phi$.

Similar results as for W hold.
For example, the homogeneous fundamental invariants f_i of W can be chosen to be eigenvectors of ϕ, with eigenvalues ϵ_i, say.

Then the $(d_i, \epsilon_i)_i$ are uniquely determined.

There exist twisted regular elements, which satisfy Springer theory,
a twisted version of the Sylow theorems holds.

Automorphisms, II

The automorphisms of irreducible crg can be classified. In all but *one* case, they come from embeddings into larger reflection groups.

Theorem (M. (2006))

Let $W \leq \mathrm{GL}(V)$ *a crg,* $\phi \in N_{\mathrm{GL}(V)}(W)$ *of finite order,*

$$k_\phi := \mathbb{Q}(\mathrm{tr}_V(w\phi) \mid w \in W).$$

Then every ϕ*-stable irreducible character of* W *has an extension to* $\langle W, \phi \rangle$ *afforded by a representation defined over* k_ϕ*.*

Example

The crg G_5 is normal in G_{14}. This induces non-trivial automorphism of G_5.
(It can also be seen from the graph automorphism of $W(F_4)$.)
Here $k_W = \mathbb{Q}(\zeta_3)$, $k_\phi = \mathbb{Q}(\zeta_3, \sqrt{2})$.

Good presentations

Proposition (Coxeter, ...)

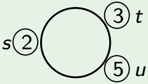
All crg have good, Coxeter-like presentations, where

- *the generators are reflections,*
- *the relations are homogeneous, each involving at most three generators (two if W is well-generated).*

These can be visualized by diagrams.

Examples

For G_{19} :



i.e., $s^2 = 1,\ t^3 = 1,\ u^5 = 1,\ stu = tus = ust$

Good presentations, II

If W is truly complex, then the good presentations satisfy at least one of

- there occur reflections of order > 2, or
- there are homogeneous relations involving > 2 reflections at a time (non-symmetric)

Furthermore, not all parabolic subgroups can be seen from the presentation, in general.

Cyclotomic Hecke algebras

Preliminary definition (as for Iwahori–Hecke algebras):

Let $W \leq \mathrm{GL}(V)$ be a crg, with good presentation

$$W = \langle S \mid R \rangle$$

(where $S \subseteq W$ are reflections and R consists of homogeneous relations).

The *cyclotomic Hecke algebra* $\mathcal{H}(W, \mathbf{u})$ *attached to* W and indeterminates $\mathbf{u} = (u_{s,j} \mid s \in S,\, 1 \leq j \leq o(s))$ is the free associative algebra over $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ on generators $\{\mathbf{s} \mid s \in S\}$ and relations

- $(\mathbf{s} - u_{s,1}) \cdots (\mathbf{s} - u_{s,o(s)}) = 0$ for $s \in S$,
- the homogeneous relations from R.

Problem: W may have several good presentations. Which shall we take?

Example

The 3-dimensional primitive reflection group $G_{24} \cong \mathrm{PSL}_2(7) \times C_2$ can be generated by three reflections of order 2. It has (at least) three good presentations on three reflections:

$$\begin{aligned}
G_{24} =& \langle r, s, t \mid r^2 = s^2 = t^2 = 1, \\
& \quad rsrs = srsr,\ rtr = trt,\ stst = tsts,\ srstrst = rstrstr \rangle, \\
=& \langle r, s, t \mid r^2 = s^2 = t^2 = 1, \\
& \quad rsr = srs,\ rtr = trt,\ stst = tsts,\ tsrtsrtsr = stsrtsrts \rangle, \\
=& \langle r, s, t \mid r^2 = s^2 = t^2 = 1, \\
& \quad rsr = srs,\ rtr = trt,\ stst = tsts,\ strstrstrs = trstrstrst \rangle.
\end{aligned}$$

Are the corresponding cyclotomic Hecke algebras isomorphic?

The braid group

Let $V = \mathbb{C}^n$, $W \leq \mathrm{GL}(V)$ a crg.
To each reflection $s \in W$ is associated its reflecting hyperplane H_s. Let

$$V^{\mathrm{reg}} := V \setminus \bigcup_{s \in W \text{ refl.}} H_s.$$

Theorem of Steinberg:

$$V^{\mathrm{reg}} \longrightarrow V^{\mathrm{reg}}/W$$

is an unramified covering, with Galois group W.

The *braid group* of W is the fundamental group

$$B_W := \pi_1(V^{\mathrm{reg}}/W, x_0).$$

Example

For $W = \mathfrak{S}_n$ in its natural reflection representation, B_W is the Artin braid group on n strings.

The center of B_W

The covering $V^{\text{reg}} \to V^{\text{reg}}/W$ induces an exact sequence

$$1 \longrightarrow P_W := \pi_1(V^{\text{reg}}, x_0) \longrightarrow B_W \longrightarrow W \longrightarrow 1.$$

P_W is the *pure braid group* associated to W.

Let $\boldsymbol{\pi} \in P_W$ be the class of the path

$$[0,1] \longrightarrow V^{\text{reg}}, \quad t \mapsto \exp(2\pi i t)\, x_0,$$

(turning once around each hyperplane).

Theorem (Broué–M.–Rouquier (1998), Bessis (2001,2007))

Let W be irreducible, $W \neq G_{31}$. The center of P_W is generated by $\boldsymbol{\pi}$. Moreover, the exact sequence above restricts to an exact sequence

$$1 \longrightarrow Z(P_W) = \langle \boldsymbol{\pi} \rangle \longrightarrow Z(B_W) \longrightarrow Z(W) \longrightarrow 1.$$

Here $Z(B_W) = \langle \boldsymbol{\beta} \rangle$ with $\boldsymbol{\beta} : t \mapsto \exp(2\pi i t/|Z(W)|)\, x_0$.

Presentations of the braid group

H reflecting hyperplane $\Longrightarrow$ fixator $C_W(H)$ is generated by reflections.
Write $d_H := |C_W(H)|$.
Distinguished reflection: The generator s_H of $C_W(H)$ with non-trivial eigenvalue $\exp(2\pi i/d_H)$.
Braid reflections: Suitable lifts $\mathbf{s}_H \in B_W$ of distinguished $s_H \in W$.

Theorem (Brieskorn, Deligne (1972), Broué–M.–Rouquier (1998), Bessis (2007))

Assume W irreducible. B_W can be generated by at most $\dim V + 1$ *braid reflections, and has a presentation by homogeneous positive* braid relations *in these braid reflections.*
Adding the relations $\mathbf{s}_H^{d_H}$ yields a good presentation of W.

Examples

For G_{24} all three presentations in the previous example come from the braid group.

Springer theory in braid groups

Recall $\boldsymbol{\pi}$, the generator of $Z(P_W)$.

An element $\mathbf{w} \in B_W$ with $\mathbf{w}^d = \boldsymbol{\pi}$ is called a *dth root of $\boldsymbol{\pi}$*.

Recall: d is regular if there exists $w \in W$ with regular ζ_d-eigenvector.

Theorem (Bessis (2007))

Let $W \leq \mathrm{GL}(V)$ be well-generated.

(a) *There exist dth roots of $\boldsymbol{\pi}$ if and only if d is regular.*

(b) *In this case, all dth roots of $\boldsymbol{\pi}$ are conjugate.*

(c) *Let $\mathbf{w} \in B_W$ be a dth root of $\boldsymbol{\pi}$, and w its image in W. Then w is d-regular, and*

$$C_{B_W}(\mathbf{w}) \cong B_{W'}, \qquad \text{where } W' := C_W(w),$$

that is, the centralizer of $\mathbf{w}$ in the braid group is isomorphic to the braid group of the centralizer of w.

Cyclotomic Hecke algebras, II

Let $\mathbf{u} = (u_{s,j} \mid s \in W \text{ dist. reflection}, 1 \leq j \leq o(s))$ be a W-invariant set of indeterminates, $A := \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$.

The (generic) *cyclotomic Hecke algebra attached to W* is the quotient

$$\mathcal{H}(W, \mathbf{u}) = A[B_W] / \left((\mathbf{s} - u_{s,1}) \dots (\mathbf{s} - u_{s,o(s)}) \mid \mathbf{s} \text{ braid-reflection} \right)$$

of the group algebra $A[B_W]$ of the braid group.

This is independent of a choice of presentation!

Examples

- For W a Coxeter group we obtain the usual generic multiparameter Iwahori–Hecke algebra.
- For $W = G_5$,

$$\mathcal{H}(W, \mathbf{u}) = \langle \mathbf{s}, \mathbf{t} \mid \mathbf{stst} = \mathbf{tsts}, \prod_{j=1}^{3} (\mathbf{s} - u_{s,j}) = \prod_{j=1}^{3} (\mathbf{t} - u_{t,j}) = 0 \rangle.$$

Hecke algebras as deformations

From the theorem on presentations of braid groups we get:

Corollary

Under the specialization

$$u_{s,j} \mapsto \exp(2\pi ij/o(s)), \qquad s \in W \text{ dist. refl.}, \ 1 \leq j \leq o(s),$$

$\mathcal{H}(W, \mathbf{u})$ *becomes isomorphic to the group algebra* $\mathbb{C}[W]$ *of* W.

Structure of $\mathcal{H}(W, \mathbf{u})$ (well-known for Coxeter groups (Tits)):

Theorem (Tits, Broué–M. (1993), Ariki–Koike (1993), Ariki (1995))

$\mathcal{H}(W, \mathbf{u})$ *is free as a module over* A *of rank* $|W|$ *(for almost all types).*

How do we find an A-basis of $\mathcal{H}(W, \mathbf{u})$?

Lifting reduced expressions

Choose presentation

$$B_W = \langle \mathbf{S} \mid R \rangle$$

of the braid group, so that

$$W = \langle S \mid R, \text{ order relations} \rangle$$

is a presentation of W, where $S =$ images in W of the $\mathbf{s} \in \mathbf{S}$.
Write T_s for the image of $\mathbf{s}$ in $\mathcal{H}(W, \mathbf{u})$.

For $w \in W$, choose reduced expression

$$w = s_1 \cdots s_r \qquad \text{with } s_i \in S$$

and let

$$\mathbf{w} := \mathbf{s}_1 \cdots \mathbf{s}_r \ \in B_W, \qquad T_\mathbf{w} := T_{s_1} \cdots T_{s_r} \ \in \mathcal{H}(W, \mathbf{u}).$$

Hope: $\{ T_\mathbf{w} \mid w \in W \}$ is an A-basis of $\mathcal{H}(W, \mathbf{u})$.

Bases of $\mathcal{H}(W, \mathbf{u})$

For Coxeter groups $\mathbf{w} \in B_W$ is independent of the choice of reduced expression of $w \in W$, and there is a natural presentation for B_W.

Problem: for crg in general, $\mathbf{w}$ *depends* on the choice of presentation *and* on the choice of reduced expression.

Examples

For $W = G(4,2,2) = \langle s, t, u \mid s^2 = t^2 = u^2 = 1, stu = tus = ust \rangle$, the expressions $sut = uts$ are reduced, but $T_s T_u T_t \neq T_u T_t T_s$.

Proposition (Bremke–M. (1997))

For $W = G(d, 1, n)$, $\{T_{\mathbf{w}} \mid w \in W\}$ is an A-basis of $\mathcal{H}(W, \mathbf{u})$ for any choice of reduced expressions for the $w \in W$.

For exceptional types, still open how to find a nice basis of $\mathcal{H}(W, \mathbf{u})$.

Tits deformation theorem

Recall: have semisimple specialization $\mathbb{C}[W]$ of $\mathcal{H}(W, \mathbf{u})$.

Then Tits' deformation theorem shows:

Corollary

*Assume that $\mathcal{H}(W, \mathbf{u})$ is free over A of rank $|W|$.
Then over a suitable extension field K of Frac(A) we have*

$$\mathcal{H}(W, \mathbf{u}) \otimes_A K \cong K[W].$$

In particular, there is a 1-1 correspondence $\mathrm{Irr}(\mathcal{H}(W, \mathbf{u})) \longleftrightarrow \mathrm{Irr}(W)$.

Splitting fields

Which extension field suffices?

k_W = character field of W. Let $\mu(k_W)$ = group of roots of unity in k_W.

Theorem (M. (1998))

$\mathcal{H}(W, \mathbf{u})$ is split over $K_W := k_W(\mathbf{v})$, where $\mathbf{v} = (v_{s,j})$ with

$$v_{s,j}^{|\mu(k_W)|} = \exp(-2\pi ij/o(s))\, u_{s,j}.$$

Example (Benson–Curtis (1972), Lusztig)

For W a Weyl group, $|\mu(k_W)| = |\mu(\mathbb{Q})| = 2$

$\Longrightarrow$ splitting field for Iwahori–Hecke algebras is obtained by extracting square roots of the indeterminates.

Thus, over K_W, the specialization $v_{s,j} \mapsto 1$ induces a natural bijection

$$\mathrm{Irr}(\mathcal{H}(W, \mathbf{u})) \longrightarrow \mathrm{Irr}(W), \qquad \chi_{\mathbf{v}} \mapsto \chi.$$

Character values

How do we determine a splitting field?
Springer's trick: Find character values on central elements of $\mathcal{H}(W, \mathbf{u})$.

The element $\boldsymbol{\beta} \in B_W$ is central, so acts by a scalar in each irreducible representation $X : \mathcal{H}(W, \mathbf{u}) \longrightarrow \mathrm{GL}_m(K_W)$, with character $\chi_{\mathbf{v}}$.

If $\boldsymbol{\beta} = \mathbf{s}_1 \cdots \mathbf{s}_l$, for braid reflections $\mathbf{s}_i$, then

$$\det X(\boldsymbol{\beta}) = \prod_{i=1}^{l} \det X(\mathbf{s}_i) \qquad \text{is known.}$$

But $\chi_{\mathbf{v}}(\boldsymbol{\beta}) = m \cdot (\det X(\boldsymbol{\beta}))^{1/m}$.

This gives an explicit formula

$$\chi_{\mathbf{v}}(\boldsymbol{\beta}) = \chi(\beta) \cdot (\text{monomial in roots of the } u_{s,j}).$$

Use this to show that certain irrationalities occur.

Automorphisms

Can we lift automorphisms from the reflection group to the Hecke algebra?

Let $\phi \in N_{\mathrm{GL}(V)}(W) \Longrightarrow \phi$ acts on V, on V^{reg}, on V^{reg}/W.

If there is a ϕ-invariant base point x_0
$\Longrightarrow \phi$ also acts on the braid group $B_W = \pi_1(V^{\mathrm{reg}}/W, x_0)$.

$x \in V^{\mathrm{reg}}$ is ϕ-invariant $\Longleftrightarrow x$ is a 1-regular vector for ϕ.

Proposition (M. (2006))

Let $W \leq \mathrm{GL}(V)$ be a crg. In each coset of $W \cdot Z(\mathrm{GL}(V))$ in $N_{\mathrm{GL}(V)}(W)$ there exists a 1-regular element.

Thus, we may lift automorphisms ϕ of W to automorphisms σ_ϕ of the braid group (in general not in a unique way).

Automorphisms, II

In order for the automorphism σ_ϕ to descend to the cyclotomic Hecke algebra, need compatible parameters:

Assume that the parameters $\mathbf{u}$ are also ϕ-invariant
$\Longrightarrow$ the automorphism σ_ϕ of B_W induces an automorphism of $\mathcal{H}(W,\mathbf{u})$, defining an *extended cyclotomic Hecke algebra* $\mathcal{H}(W,\mathbf{u}).\langle\sigma_\phi\rangle$.

Similar statements as before hold for rationality:

Proposition (Digne–Michel (1985), M. (2006))

With the above notation, every σ_ϕ-stable irreducible character of $\mathcal{H}(W,\mathbf{u})$ has an extension to $\mathcal{H}(W,\mathbf{u}).\langle\sigma_\phi\rangle$ realizable over

$$K_\phi := k_\phi((\exp(-2\pi ij/o(s))\, u_{s,j})^{1/|\mu(k_\phi)|} \mid s \in W,\ 1 \leq j \leq o(s)).$$

Symmetrizing forms

We expect cyclotomic Hecke algebras to carry a natural trace form:
There should exist an A-linear form

$$t_{\mathbf{u}} : \mathcal{H}(W, \mathbf{u}) \longrightarrow A$$

with the following properties:

- the bilinear form $\mathcal{H} \times \mathcal{H} \to A$, $(h_1, h_2) \mapsto t_{\mathbf{u}}(h_1 h_2)$, is symmetric and non-degenerate,
- $t_{\mathbf{u}}$ specializes to the canonical trace form on the group algebra of W,
- $t_{\mathbf{u}}(b^{-1})^\vee = \frac{t_{\mathbf{u}}(b\boldsymbol{\pi})}{t_{\mathbf{u}}(\boldsymbol{\pi})}$ for all $b \in B_W$,
- $t_{\mathbf{u}}$ restricted to a parabolic subalgebra has the same properties on that subalgebra.

Rouquier: if it exists, such a $t_{\mathbf{u}}$ is uniquely determined.

Symmetrizing forms, II

For Coxeter groups, such a form can be obtained by setting

$$t_{\mathbf{u}}(T_{\mathbf{w}}) := \begin{cases} 1 & w = 1, \\ 0 & \text{else}, \end{cases}$$

for $w \in W$ (with lifted elements $T_{\mathbf{w}}$ as above).

Problem: for crg, the $T_{\mathbf{w}}$ are not well-defined.

Theorem (Bremke–M. (1997), M.–Mathas (1998))

The algebra $\mathcal{H}(W, \mathbf{u})$ is symmetric over A (for almost all types).

For example for $G(d, 1, n)$, $t_{\mathbf{u}}$ vanishes on $T_{\mathbf{w}}$ for *all* reduced expressions of all $1 \neq w \in W$.

For the proof, take above definition for some basis and check properties.

Schur elements

Let $t_{\mathbf{u}}$ denote the canonical symmetrizing form on $\mathcal{H}(W, \mathbf{u})$.

Write

$$t_{\mathbf{u}} = \sum_{\chi \in \mathrm{Irr}(W)} \frac{1}{S_\chi} \chi_{\mathbf{v}},$$

with *Schur elements* $S_\chi \in K_W$.

Fact: The S_χ are integral over A.

Theorem (Geck–Iancu–M. (2000), M. (1997,2000))

The Schur elements are explicitly known for all types (assuming the existence of the symmetrizing form $t_{\mathbf{u}}$).

For infinite series, determine weights of a Markov trace on $\mathcal{H}(W, \mathbf{u})$.

Constructing representations

For exceptional types, solve linear system of equations

$$\sum_{\chi} \chi_{\mathbf{v}}(T_{\mathbf{w}}) \frac{1}{S_\chi} = t_{\mathbf{u}}(T_{\mathbf{w}}) = \begin{cases} 1 & w = 1, \\ 0 & \text{else}, \end{cases} \qquad (w \in W).$$

How do we know $\chi_{\mathbf{v}}(T_{\mathbf{w}})$ on sufficiently many elements?

Construct representations explicitly.

For small dimensions ($m \leq 6$): take matrices with indeterminate entries, plug into relations, solve non-linear system.

Induction: may assume matrices known for some maximal parabolic subalgebra.

W-graphs

For Coxeter groups W, Lusztig introduced the notion of a *W-graph* for a representation of $\mathcal{H}(W, \mathbf{u})$:
a combinatorial encoding of a representation via a labelled graph, with

- vertices = certain subsets of the set of (standard) generators
- edges = labelled by elements from K_W

Gyoja (1984): W-graphs exist for all representations of Weyl groups.

Suitable generalization makes sense for cyclotomic Hecke algebras as well.

Proposition (M.–Michel (2008))

Models for the irreducible representations of all but five exceptional crg are known.

Use W-graphs, but also Hensel-lifting and Padé-approximation.

Example

For $W = G_5$, with parameters (u, v, w, x, y, z), one Schur element is

$$-\frac{(uy+vx)(vy+ux)(y-z)(uvxy+w^2z^2)(x-z)(v-w)(u-w)}{uvw^4xyz^4}.$$

In fact, the Schur elements always have total degree 0 and are of the form

$$S_\chi = m \cdot \frac{P_1}{P_2},$$

where

- m is an integer in k_W,
- P_1 is a product of cyclotomic polynomials over k_W, evaluated at monomials in the $v_{s,j}$,
- P_2 is a monomial in the $v_{s,j}$.

The spetsial specialization

We are interested in 1-parameter specializations of $\mathcal{H}(W, \mathbf{u})$ through which the specialization to $\mathbb{C}[W]$ factors.

For Iwahori–Hecke algebras, the specialization where

$$(\mathbf{s} - q)(\mathbf{s} + 1) = 0$$

(for all distinguished s) is particularly important.

For cyclotomic Hecke algebras, we may have reflections of order $o(s) > 2$. So consider the *spetsial* specialization $\mathcal{H}(W, q)$ where

$$(\mathbf{s} - q)(\mathbf{s}^{o(s)-1} + \mathbf{s}^{o(s)-2} + \ldots + 1) = 0.$$

By the above, the spetsial algebra $\mathcal{H}(W, q)$ is split semisimple over $k_W(y)$, where $y^{|\mu(k_W)|} = q$.

Families of characters

What about Kazhdan–Lusztig theory for spetsial Hecke algebras: Kazhdan–Lusztig basis, left and 2-sided cells, cell representations?

Kazhdan–Lusztig's combinatorial approach seems not possible.

Theorem (Gyoja (1996), Rouquier (1999))

Let W be a Weyl group. Then two characters of W lie in the same 2-sided cell if and only if they lie in the same block of $\mathcal{H}(W,q) \otimes_A R$, where

$$R := \text{ integral closure of } \mathbb{Z}[q, q^{-1}, (1+q\mathbb{Z}[q])^{-1}] \quad \text{in } k_W(y).$$

Characters inside a fixed 2-sided cell are called a *family* of $\mathrm{Irr}(W)$.

For a crg W, take the above result as *definition* of families in $\mathrm{Irr}(W)$.

Families and Schur elements

How to determine the families?

Recall the form of Schur elements: after the spetsial specialization

$$S_\chi = m_\chi \, y^{a_\chi} \, F_\chi,$$

where $m_\chi \in k_W$ is integral, $a_\chi \in \mathbb{Z}$, $F_\chi \in 1 + yk_W[y]$.

The central element T_π has to act by the same scalar in all irreducible representations of a fixed block.
The explicit knowledge of this scalar gives:

$$2a_\chi + \deg F_\chi$$

is constant on families.

Bad primes

Geck–Rouquier (1997): $\{\chi\}$ is a 1-element family $\iff S_\chi \in R^\times$.

So: $\{\chi\}$ is a 1-element family $\iff m_\chi \in \mathcal{O}_W^\times$ ($\mathcal{O}_W$ ring of integers of k_W).

A prime p is *bad for W* if there exists a Schur element S_χ whose leading coefficient m_χ lies in some prime ideal of $\mathcal{O}_W$ above p.

Only divisors of $|W|$ can be bad.

Examples

- For Weyl groups, these are the usual bad primes.
- For $W = G_5$ have Schur elements

$$2\ q^{-8}(q^4+1)(q^2+q+1)^2, \quad 3\ q^{-1}(q^2+1)^2(q^2+q+1),$$

so $2, 3$ are bad primes. As $|W| = 2^3 3^2$, these are the only ones.

Families, II

Theorem (Broué–Kim(2002), M.–Rouquier(2003), Chlouveraki(2008))

The families of all spetsial cyclotomic Hecke algebras are known.

In fact, Chlouveraki gives an algorithm to determine the families for all 1-parameter specializations of cyclotomic Hecke algebras only using properties of Schur elements.

Example

For $W = G_5$ there are six families, with 1, 2, 2, 3, 5, resp. 8 characters.

Corollary

Both a_χ *and* $\deg F_\chi$ *are individually constant on families.*

Fake degrees

The symmetric algebra $S(V)$, the invariants $S(V)^W$ are naturally graded.

$S(V)_+^W :=$ the invariants of degree at least 1.

$S(V)_W := S(V) \,/\, \big(S(V)_+^W\big)$ the *coinvariant algebra*.

Theorem (Chevalley (1955))

The graded W-module $S(V)_W$ affords the regular representation of W.

The *Poincaré polynomial* of W is the graded dimension

$$\sum_j \dim S(V)_W^j \, q^j = \prod_{j=1}^n \frac{q^{d_j}-1}{q-1}.$$

For $\chi \in \mathrm{Irr}(W)$ the *fake degree* is the graded multiplicity

$$R_\chi := \sum_j \langle \chi, S(V)_W^j \rangle \, q^j.$$

Semipalindromicity

Observation: often, the fake degrees are (semi-)palindromic, that is

$$R_\chi(t) = t^m R_{\bar{\chi}}(t^{-1}) \qquad \text{(some } m \geq 0\text{)}.$$

This is not true, for example, for two characters of $W(E_7)$, and four of $W(E_8)$.

Theorem (M. (1999))

R_χ is semi-palindromic if and only if the character χ_q of $\mathcal{H}(W, q)$ can be realized over $k_W(q)$. More precisely,

$$R_\chi(t) = t^m R_{\delta(\chi)}(t^{-1})$$

for some explicit permutation δ coming from the $\mathrm{Gal}(k_W(y)/k_W(q))$-action on $\mathrm{Irr}(\mathcal{H}(W, q))$.

Rationality of the reflection representation

The spetsial algebra 'knows about' W being well-generated!

For $\chi \in \mathrm{Irr}(W)$ let $D_\chi := S_1/S_\chi$, the *generic degree* of χ.

$\chi \in \mathrm{Irr}(W)$ is *special* if R_χ and D_χ have the same order of zero at $y = 0$.

Proposition (M. (2000))

The following are equivalent:

(i) *W is well-generated.*

(ii) *The reflection character of W is special.*

(iii) *The reflection representation of $\mathcal{H}(W, q)$ can be realized over $k_W(q)$.*

For example, for Coxeter groups the reflection representation of $\mathcal{H}(W, q)$ is always rational.

Finite reductive groups

Let $\mathbf{G}$ be a simple algebraic group defined over $\mathbb{F}_q$ with corresponding Frobenius map $F : \mathbf{G} \to \mathbf{G}$, $G := \mathbf{G}^F$, a finite group of Lie type.
Let W the Weyl group of $\mathbf{G}$.

Lusztig: Ordinary representation theory of G can be described in combinatorial terms only depending on W (actually: on $\mathcal{H}(W, q)$):

- The R_χ are degrees of almost characters
- The $D_\chi = S_1/S_\chi$ are degrees of unipotent characters
- The base change matrix between these two is block-diagonal, where the blocks are just the families in $\mathrm{Irr}(W)$
- This *Fourier matrix* can be obtained from the quantum double of a small finite group C_2^m, $\mathfrak{S}_3$, $\mathfrak{S}_4$, or $\mathfrak{S}_5$.
- This also determines the Frobenius eigenvalues of unipotent characters.

Many of the above notions are available for arbitary crg W!

Spetsial reflection groups

Recall the generic degrees $D_\chi = S_1/S_\chi$, for $\chi \in \mathrm{Irr}(W)$.

Proposition (M. (2000))

Let W be a crg. The following are equivalent:

(i) $S_1 = \prod_i (q^{d_i} - 1)/(q-1)$, *the Poincaré-polynomial of W.*

(ii) $D_\chi \in k_W(q)$ *for all* $\chi \in \mathrm{Irr}(W)$ *(rationality).*

(iii) $D_\chi \in k_W[y]$ *for all* $\chi \in \mathrm{Irr}(W)$ *(integrality).*

(iv) D_χ/R_χ *has no pole at* $y = 0$, *for all* $\chi \in \mathrm{Irr}(W)$.

(v) *For each family* $\mathcal{F} \subset \mathrm{Irr}(W)$, *the* k_W*-subspace of* $k_W(y)$ *spanned by* $\{D_\chi \mid \chi \in \mathcal{F}\}$ *is the same as the one spanned by* $\{R_\chi \mid \chi \in \mathcal{F}\}$.

A crg satisfying the above equivalent conditions is called *spetsial*.

The irreducible spetsial groups are

$\mathfrak{S}_n$, $G(d,1,n)$, $G(e,e,n)$ and

group	4	5	6	7	8	9	10	11	12	13	14	15	16
dim	2	2	2	2	2	2	2	2	2	2	2	2	2
spetsial	$*$		$*$		$*$						$*$		

group	17	18	19	20	21	22	23	24	25	26	27
dim	2	2	2	2	2	2	3	3	3	3	3
spetsial							H_3	$*$	$*$	$*$	$*$

group	28	29	30	31	32	33	34	35	36	37
dim	4	4	4	4	4	5	6	6	7	8
spetsial	F_4	$*$	H_4		$*$	$*$	$*$	E_6	E_7	E_8

All of them are well-generated.

Towards spetses

Spetses is a Greek island in the Aegean sea, lieu of a conference in 1993.

Theorem (M. (1996), Broué–M.–Michel (2009))

Let W be a spetsial crg. Then the fake degrees $\{R_\chi\}$, the generic degrees $\{D_\chi\}$, the families $\mathcal{F}$, can be extended in a well-defined manner to a collection of combinatorial objects:

- *unipotent degrees*
- *Fourier matrices*
- *Frobenius eigenvalues*

satisfying similar properties as the corresponding objects occurring in the representation theory of finite groups of Lie type:

- *there are Harish-Chandra theories*
- ...

All equivalent properties of spetsial crg are required for this to work.

Fusion data

For a family $\mathcal{F} \subset \mathrm{Irr}(W)$, let Ω be the diagonal matrix of Frobenius eigenvalues on $\mathcal{F}$, S the Fourier matrix. Then

- S is symmetric,
- $S^4 = 1, \qquad [S^2, \Omega] = 1, \qquad (\Omega S)^3 = 1.$
  (i.e., S, Ω give an $\mathrm{SL}_2(\mathbb{Z})$-representation),
- Cuntz (2006): for a suitable index i_0, all entries of S in that row are non-zero, and

$$\sum_l \frac{S_{il} S_{jl} \overline{S}_{kl}}{S_{i_0 l}} \in \mathbb{Z} \qquad \text{for all } i, j, k$$

  (Verlinde formula).
  They define structure constants of a $\mathbb{Z}$-based algebra, a generalization of fusion algebras where the structure constants are not necessarily positive.

Unipotent characters for crg G_4 ($\cong \mathrm{SL}_2(3)$)

χ	D_χ	R_χ	Fr_χ	Family
$\phi_{1,0}$	1	1	1	1
$\phi_{2,1}$	$\frac{3-\sqrt{-3}}{6}q\Phi_3'\Phi_4\Phi_6''$	$q\Phi_4$	1	2
$\phi_{2,3}$	$\frac{3+\sqrt{-3}}{6}q\Phi_3''\Phi_4\Phi_6'$	$q^3\Phi_4$	1	2
$Z_3:2$	$\frac{\sqrt{-3}}{3}q\Phi_1\Phi_2\Phi_4$	0	ζ_3^2	2
$\phi_{3,2}$	$q^2\Phi_3\Phi_6$	$q^2\Phi_3\Phi_6$	1	3
$\phi_{1,4}$	$\frac{-\sqrt{-3}}{6}q^4\Phi_3''\Phi_4\Phi_6''$	q^4	1	4
$\phi_{1,8}$	$\frac{\sqrt{-3}}{6}q^4\Phi_3'\Phi_4\Phi_6'$	q^8	1	4
$\phi_{2,5}$	$\frac{1}{2}q^4\Phi_2^2\Phi_6$	$q^5\Phi_4$	1	4
$Z_3:11$	$\frac{\sqrt{-3}}{3}q^4\Phi_1\Phi_2\Phi_4$	0	ζ_3^2	4
G_4	$\frac{1}{2}q^4\Phi_1^2\Phi_3$	0	-1	4

Φ_3', Φ_3'' (resp. Φ_6', Φ_6'') are factors of Φ_3 (resp Φ_6) in $\mathbb{Q}(\zeta_3)$.

The Fourier matrix for G_4

$\mathcal{F}$	1	2	2	2	3	4	4	4	4	4
1	1	.	.	.	.	.	.	.	.	.
2	.	$\frac{3-\sqrt{-3}}{6}$	$\frac{3+\sqrt{-3}}{6}$	$\frac{\sqrt{-3}}{3}$	.	.	.	.	.	.
2	.	$\frac{3+\sqrt{-3}}{6}$	$\frac{3-\sqrt{-3}}{6}$	$-\frac{\sqrt{-3}}{3}$	.	.	.	.	.	.
2	.	$\frac{\sqrt{-3}}{3}$	$-\frac{\sqrt{-3}}{3}$	$\frac{\sqrt{-3}}{3}$	.	.	.	.	.	.
3	.	.	.	.	1	.	.	.	.	.
4	.	.	.	.	.	$-\frac{\sqrt{-3}}{6}$	$\frac{\sqrt{-3}}{6}$	$\frac{1}{2}$	$\frac{\sqrt{-3}}{3}$	$\frac{1}{2}$
4	.	.	.	.	.	$\frac{\sqrt{-3}}{6}$	$-\frac{\sqrt{-3}}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{-3}}{3}$	$\frac{1}{2}$
4	.	.	.	.	.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	.	$-\frac{1}{2}$
4	.	.	.	.	.	$\frac{\sqrt{-3}}{3}$	$-\frac{\sqrt{-3}}{3}$	.	$\frac{\sqrt{-3}}{3}$	.
4	.	.	.	.	.	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	.	$\frac{1}{2}$