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Schanuel's conjecture for non-isoconstant elliptic curves over function fields.

D. Bertrand $^{(*)}$

Abstract: *we discuss functional and number theoretic extensions of Schanuel's conjecture, with special emphasis on the study of elliptic integrals of the third kind.*

Schanuel's conjecture [La] on the layman's exponential function can be viewed as a measure of the defect between an algebraic and a linear dimension. Its functional analogue, be it in Ax's original setting [Ax1] , Coleman's [Co], or Zilber's geometric interpretation [Zi], certainly gives ground to this view-point.

The same remark applies to the elliptic version of the conjecture, and to its functional analogue, as studied by Brownawell and Kubota [BK], and by J. Kirby [K1]. Here, the elliptic curve under consideration is constant. In the same spirit, we discuss in the first section of this note Ax's general theorem [Ax2] on the exponential map on a constant semi-abelian variety G, where transcendence degrees are controlled by the (linear) dimension of a certain "hull". We obtain a similar statement for the universal vectorial extension of G, and refer to the recent work of J. Kirby [K2, K3] for further generalizations of Ax's theorem, involving arbitrary differential fields, multiplicative parametrizations, and uniformity questions.

The naïve number-theoretic analogues of these functional results, however, are clearly false. The first counterexample which comes to mind is provided by periods: Riemann-Legendre relations are quadratic, and cannot be tracked back to hulls of the above type. Furthermore, the theory of mixed motives shows that path integrals have as many reasons to be called periods as closed circuit ones, and we shall show in §2 that they too may obey non-linear constraints. The hopefully correct generalization of Schanuel's conjecture in this context, which is due to André [A2, A3], requires the introduction of a (motivic) Galois group.

In this theory, duality plays a crucial role. Going back to function fields, this makes the hypothesis of constancy of the ambient group sound rather unnatural. For instance, the

$^{(*)}$ Address, AMS Class. and key-words at the end of the paper dual of the one-motive attached to a non constant point on a constant elliptic curve is a non constant semi-abelian surface. And as soon as we allow for such variations, the functional statements cease to hold. Actually, the picture becomes closer to the number theoretic one, at least if we restrict to the "logarithmic" side of the conjecture: transcendence degrees are then controlled by a (linear differential) Galois group, which - not a surprise to model theorists - Manin's kernel theorem can help to compute. This was already noticed in [A1] and [B3] for pencils of abelian varieties (i.e. families of abelian integrals of the second kind), and the third part of the paper extends this approach to pencils of semi-abelian surfaces (elliptic integrals of the 3rd kind).

In fact, the hulls of §1 too can be interpreted as differential Galois groups (now in Kolchin's sense), if we restrict to the "exponential" side of the conjecture. But the specificity of Schanuel's conjecture lies precisely in its blending of exponentials and logarithms, and although we do not investigate this further here, it is likely that a similar blend of Galois groups, possibly with D-structures as in Pillay's theory [Pi], is required. The author can only thank (resp. apologize to) the organisors of the Newton conference for helping him to realize (resp. becoming aware so late of) the relevance of model theory to this circle of problems. He also thanks D. Masser, J. Kirby and Z. Chatzidakis for their comments on an earlier version of the paper.

§1. Constant semi-abelian varieties.

Let (F, ∂) be a differential field of characteristic 0. To give a common framework to the first and third parts of this study, we assume that (F, ∂) is differentially embedded in the field of meromorphic functions over a non empty domain U of the complex plane, and set $\mathcal{O}_F = F \cap \mathcal{O}_U$ (see below for a more algebraic presentation). We further assume that F contains a not necessarily differential subfield K of transcendence degree 1 over $F^\partial = \mathbf{C}$.

Let G be a commutative algebraic group defined over $\mathbf{C}$, and let $exp_G : TG(\mathbf{C}) \to G(\mathbf{C})$ be the exponential map on its Lie algebra, identified with its tangent space TG at the origin. Since exp_G is analytic, it extends to a homomorphism from $TG(\mathcal{O}_F)$ to $G(\mathcal{O}_F)$, whose kernel is easily checked to coincide with that of $\exp_G$. Passing to quotients, we derive an injective homomorphism: $\overline{exp}_G : TG(\mathcal{O}_F)/TG(\mathbf{C}) \to G(\mathcal{O}_F)/G(\mathbf{C})$. Notice that the periods of exp_G are lost in the process.

Suppose now that G is a semi-abelian variety. By rigidity, any algebraic subgroup H of G/F is then defined over $\mathbf{C}$, and $H(F)/H(\mathbf{C})$ embeds into $G(F)/G(\mathbf{C})$. To a given point $y \in G(F)$, we can therefore attach, without specifying fields of definitions, the smallest algebraic subgroup H of G such that $y \in H(F)$ mod $G(\mathbf{C})$. Its connected component though the origin is a semi-abelian subvariety G_y of G, which may be called the *relative hull* of the point y. For instance, the relative hull of a constant point $y \in G(\mathbf{C})$ is trivial; if $G = \mathbf{G}_m^s$ is a torus, a point $y = (y_1, ..., y_s) \in G(F)$ admits G as relative hull iff the classes of $y_1, ..., y_s$ in $F^*/\mathbf{C}^*$ are multiplicatively independent.

The following statement is a direct consequence of Ax's Theorem 3 in [Ax2]. I thank D. Masser for having drawn my attention to this reference. In fact, J. Kirby has recently reproved and extended this theorem in the setting of general differential fields F and general uniformizations. Furthermore, his results involve uniformity statements, a subject we shall not touch upon here. We refer to [K2], [K3] for more comments on these points.

Proposition 1.a ([Ax2], [K3]): *let G be a semi-abelian variety defined over $\mathbf{C}$, let x be a point in $TG(\mathcal{O}_F)$, let $y = exp_G(x)$, and let G_y be the relative hull of y. Then, $tr.deg.(K(x,y)/K) \geq dim(G_y)$.*

Proof of 1.a: if x is constant, the lower bound is trivial; otherwise, we deduce from [Ax2], Theorem 3, that $tr.deg.(\mathbf{C}(x,y)/\mathbf{C}) \geq dim(G_y) + 1$, where 1 stands for the rank of a jacobian matrix. Since $K/\mathbf{C}$ has transcendence degree 1, the claim easily follows.

In [Ax2], Ax assumes that the ambient group G admits no non trivial vectorial subgroup, but his argument readily extends to all G's admitting no non trivial vectorial *quotient*. For later applications, it seems more convenient to state the corresponding result in terms of universal extensions, as follows.

Let $\tilde{G}$ be the universal extension of G. If G is an extension of an abelian variety A by a torus T, this is the pull-back to G of the universal (vectorial) extension $\tilde{A}$ of A, which in turn is an extension of A by the dual $V \simeq \mathbf{G}_a^{dim(A)}$ of $H^1(A, \mathcal{O}_{\mathcal{A}})$, viewed as a vector group [NB : This should not be confused with the prolongation $\tau(A)$ of the standard D-group structure attached to A, which is an extension by TA, here split since A descends to $\mathbf{C}$, cf. [Bu], III, and [Ma]. Recall that $H^1(A, \mathcal{O}_{\mathcal{A}})$ is the tangent space of the dual of A, cf. [Mu], p. 130)]. In particular, the dimension of $\tilde{G}$ is equal to $2dim(A) + dim(T)$. Following a suggestion of Z. Chatzidakis, we may also describe $\tilde{G}$ as the "largest" vectorial extension of G admitting no epimorphism to the additive group $\mathbf{G}_a$. The above Proposition can then be sharpened into

Proposition 1.b: *let G be a semi-abelian variety defined over $\mathbf{C}$, let x be a point in $TG(\mathcal{O}_F)$, let $y = exp_G(x)$, let G_y be the relative hull of y and let $\tilde{G}_y$ be the universal vectorial extension of G_y. Furthermore, let $\tilde{x}$ be a lift of x to $T\tilde{G}(\mathcal{O}_F)$ and let $\tilde{y} = exp_{\tilde{G}}(\tilde{x})$. Then,*

$$tr.deg.(K(\tilde{x}, \tilde{y})/K) \geq dim(\tilde{G}_y).$$

In particular, the equality holds true in either of the following situations:

i) the logarithmic case*, where $\tilde{y}$ is defined over K;*

ii) the exponential case*, where $\tilde{x}$ is defined over K.*

The denomination for these cases come from the classical Schanuel conjecture, where they respectively concern (i) Schneider's problem on the algebraic independence of $\mathbf{Q}$-linearly independent logarithms of algebraic numbers, and (ii) the Lindemann-Weiestrass theorem on the algebraic independence of the exponentials of $\mathbf{Q}$-linearly independent algebraic numbers. That the general inequality implies equalities in these special cases can be seen as follows : if $\tilde{x} \in T\tilde{G}(K)$, then, up to translations by a period and by a K-rational point of TV, it lies in the vector space $T\tilde{G}_y$. Since $exp_{\tilde{G}}$ induces the identity on the vector group $TV \simeq V$, $\tilde{y} = exp_{\tilde{G}}(\tilde{x})$ then differs from an element of $G_y(F)$ by a K-rational point. Its coordinates therefore generate over K a field of transcendence degree at most (and hence equal to) $dim(\tilde{G}_y)$. The argument can be reversed when we start with a point $\tilde{y} \in \tilde{G}(K)$. See Remark 1 below for a more intrinsic reformulation of these equalities.

Proof of 1.b : once again, we may assume that x is not constant, and must then prove that $tr.deg.(\mathbf{C}(\tilde{x}, \tilde{y})/\mathbf{C}) \geq dim(\tilde{G}_y) + 1$. Since $\overline{exp}_{G_y}$ is the restriction of $\overline{exp}_G$ to TG_y, and since two lifts of x to $T\tilde{G}$ differ by an element of $TV \simeq V$, where $exp_{\tilde{G}}$ reduces to the identity, we may also assume that $\tilde{x}$ lies in $T\tilde{G}_y$, and eventually, that $G_y = G$. In this case, any algebraic subgroup $G'/\mathbf{C}$ of $\tilde{G}$ projecting onto G_y coincides with $\tilde{G}$ (in other words, $\tilde{G}$ is an *essential extension* of G): indeed, the quotient $\tilde{G}/G'$ of the universal extension $\tilde{G}$ would otherwise be a non trivial vector group. In particular, $\tilde{G}_y = \tilde{G}$.

Let then $\mathbf{X}$ be the $\mathbf{C}$-algebraic group $T\tilde{G} \times \tilde{G}$, let $\mathbf{A}$ be the analytic subgroup of $\mathbf{X}$ made up by the graph of $exp_{\tilde{G}}$, let $\mathbf{K}$ be the analytic curve defined by the image of $\{\tilde{x}, \tilde{y}\}$, viewed as a map from the complex domain U to $\mathbf{X}(\mathbf{C})$. Up to translation by a constant point, we may assume that $\mathbf{K}$ passes through the origin, and denote by $\mathbf{V}$ its Zariski closure in $\mathbf{X}$ over $\mathbf{C}$, so that $tr.deg.(\mathbf{C}(\tilde{x}, \tilde{y})/\mathbf{C}) = dim\mathbf{V}$. According to [Ax2], Theorem 1, there exists an analytic subgroup $\mathbf{B}$ of $\mathbf{X}$ containing both $\mathbf{A}$ and $\mathbf{V}$ such that $dim\mathbf{B} - dim\mathbf{V} \leq dim\mathbf{A} - dim\mathbf{K}$. We shall prove that $\mathbf{B} = \mathbf{X}$, and consequently, that

$$tr.deg.(\mathbf{C}(\tilde{x}, \tilde{y})/\mathbf{C}) = dim\mathbf{V} \geq dim\mathbf{X} - dim\mathbf{A} + dim\mathbf{K},$$

which is equal to $2dim\tilde{G} - dim\tilde{G} + 1 = dim\tilde{G} + 1 = dim\tilde{G}_y + 1$, as required.

Since $\mathbf{V}$ is a connected algebraic variety passing through the origin, the abstract group it generates in $\mathbf{X}$ is an algebraic subgroup $g(\mathbf{V})$ of $\mathbf{X} = T\tilde{G} \times \tilde{G}$. Since $\mathbf{V}$ contains $\mathbf{K}$, and since $G_y = G$, the image $G' \subset \tilde{G}$ of $g(\mathbf{V})$ under the second projection projects onto G, and therefore coincides with $\tilde{G}$. Let $T' \subset T\tilde{G}$ be the image of $g(\mathbf{V})$ under the first projection.

We can now view $g(\mathbf{V})$ as an algebraic subgroup of $T' \times \tilde{G}$ with surjective images under the two projections. As is well known, any such subgroup of the product $T' \times \tilde{G}$ induces an isomorphism from a quotient of $\tilde{G}$ to a quotient of T'. More precisely, on setting $H = g(\mathbf{V}) \cap (0 \times \tilde{G})$, and $H' = g(\mathbf{V}) \cap (T' \times 0)$, we get an algebraic group isomorphism $\tilde{G}/H \simeq T'/H'$. But if these quotients are not trivial, the second one will admit $\mathbf{G}_a$ among its quotients, and the first one, hence $\tilde{G}$ itself, will share the same property. Again, since $\tilde{G}$ is a universal extension of a semi-abelian variety, this is impossible. Consequently, $\tilde{G}/H = 0$, and $g(\mathbf{V})$, hence $\mathbf{B}$, contains $0 \times \tilde{G}$. Finally, $\mathbf{B}$, which contains $\mathbf{A}$, projects onto $T\tilde{G}$ by the first projection. Hence, $\mathbf{B}$ does coincide with $T\tilde{G} \times \tilde{G} = \mathbf{X}$. (Notice that contrary to Kirby's general setting [K3], the algebraic groups $T\tilde{G}$ and $\tilde{G}$ do not play a symmetric role in this proof; it is likely, however, that Proposition 1.b could be reached by the method of [K3], §5.1.)

Propositions 1.a and b are better expressed in terms of the *logarithmic derivative map* $\partial Log_G : G(F) \to TG(F)$ of the standard D-group structure attached to $G/\mathbf{C}$, cf. [Bu], [Pi], [Ma] - and [BC] for a historical perspective. To make the translation, view y as a section of the constant group scheme $G_U = G \times U$ over U, and the $\mathbf{C}$-vector space $\Omega^1 G$ of invariant differentials on G as a subspace of $H^0(G_U, \underline{\Omega}^1_{G_U})$. Since $(exp_G^*)_0$ is the identity, the requirement $y = exp_G(x)$ becomes: for any $\omega \in \Omega^1 G$, there exists an exact differential $dx_\omega \in d\mathcal{O}_F$ such the differential form $y^*(\omega) - dx_\omega$ on U kills the vector field ∂ :

$$(x, y) \in (TG \times G)(\mathcal{O}_F) \text{ and } y^*(\omega)(\partial) = \partial x_\omega,$$

or more generally, on denoting by ∂Log_G the standard logarithmic derivative on the constant group G : $(x, y) \in (TG \times G)(F)$ and $\partial x = \partial Log_G(y)$. Indeed, the assignment $\omega \mapsto x_\omega$ is a linear form on $\Omega^1 G$ with values in F, defined up to linear forms on $\Omega^1 G$ with values in $F^\partial = \mathbf{C}$, i.e. as an element $x = x(y)$ of $TG(F)/TG(\mathbf{C})$, and the assignment $y \mapsto x(y) : G(F)/G(\mathbf{C}) \to TG(F)/TG(\mathbf{C})$ inverts on its image the map $\overline{exp}_G$ defined above. Keeping in mind that these quotients do not affect fields of definitions over $\mathbf{C}$, and that all these notations should be indexed by ∂, we may then write $x = Log_G(y)$, or more graphically

$$x_\omega(y) = \int^y \omega.$$

These notations remain meaningful for any closed, possibly singular, differential form ω on G, and can be extended to $\tilde{G}$. Proposition 1.b then reads: *let $y \in G(F)$, let $\tilde{y}$ be a lift of y to $\tilde{G}(F)$, and let $\tilde{x} = Log_{\tilde{G}}(\tilde{y})$. Then $tr.deg.(K(\tilde{y}, \tilde{x})/K) \geq dim(\tilde{G}_y)$.*

Remark 1 : we here assume that K is an algebraically closed *differential* subfield of (F, ∂), and consider the two special cases of Proposition 1.b.

i) In the "exponential" one, $\tilde{x}$ is a K-rational point of $T\tilde{G}$, and as explained above, we may assume wlog that it lies in $T\tilde{G}_y$. Set $\tilde{a} = \partial \tilde{x}$. Up to constants, $\tilde{y} = exp_{\tilde{G}}(\tilde{x})$ is then a solution of the differential equation $\partial Log_{\tilde{G}}(\tilde{y}) = \tilde{a}, \quad \tilde{a} \in T\tilde{G}_y(K)$, to which Kolchin's differential Galois theory can be applied: indeed, $\tilde{G}_y$ being here constant, the differential extension $K(\tilde{y})/K$ is a strongly normal one, cf. [Pi], 3.2 and 3.8. In particular, its differential Galois group is an algebraic subgroup of $\tilde{G}_y$. Since its dimension is given by $tr.deg.K(\tilde{y})/K$, the proposition reduces in this case to the relation

$$Aut_\partial(K(\tilde{y})/K) = \tilde{G}_y(\mathbf{C}) \quad .$$

ii) In the "logarithmic" one, $\tilde{y}$ is a K-rational point of G, which may be assumed wlog to lie in $\tilde{G}_y$. Set $\tilde{b} = \partial Log_{\tilde{G}}(\tilde{y})$. Up to constants, $\tilde{x} = Log_{\tilde{G}}(\tilde{y})$ is then a solution of the inhomogeneous linear equation $\partial \tilde{x} = \tilde{b}, \quad \tilde{b} \in T\tilde{G}_y(K)$, to which the standard Picard-Vessiot theory can be applied. In particular, its differential Galois group is a vectorial subgroup of $T\tilde{G}_y$. Since its dimension is given by $tr.deg.K(\tilde{x})/K$, the proposition now reduces to the relation

$$Aut_\partial(K(\tilde{x})/K) = T\tilde{G}_y(\mathbf{C}) \quad .$$

When G is a split product $A \times T$, this can be checked directly, as a slight amendment of the proof of Thm. 3 of [A1] shows(1). For a general study of split products, see [K3].

We now come back to the mixed case, and give a concrete translation of Proposition 1 (see Prop. 5 below for an even more concrete one). Let A be an abelian variety over $\mathbf{C}$, of dimension g, the elements of whose dual $\hat{A} = Pic_0(A) \simeq Ext^1(A, \mathbf{G}_m)$ we identify with the linear equivalence classes of residue divisors of differentials of the third kind on A. Let $\omega_1, ..., \omega_g$ be a basis of Ω^1_A over $\mathbf{C}$, let $\eta_1, ..., \eta_g$ be differential of the second kind on $A/\mathbf{C}$ whose cohomology classes generate a complement of Ω^1_A in $H^{dR}(A/\mathbf{C}) := H^1_{dR}(A/\mathbf{C})$, and let $\xi_{q_1}, ..., \xi_{q_r}$ be differentials of the third kind on $A/\mathbf{C}$, with residue divisors equivalent to $q_1, ..., q_r$ in $\hat{A}(\mathbf{C})$. Denote by G the extension of A by the torus $T = \mathbf{G}^r_m$ parametrized in $Ext(A, T) \simeq \hat{A}^r$ by $q_1, ..., q_r$. Also, consider another torus $T' = \mathbf{G}^{r'}_m$.

Proposition 2: *In the above notations, assume that $q_1, ..., q_r \in \hat{A}(\mathbf{C})$ are linearly independant over $\mathbf{Z}$. Let y be a point of $A(F)$ whose relative hull A_y fills up A, and let $y' = (y'_1, ..., y'_{r'})$ be a point in $T'(F)$, whose relative hull $T'_{y'}$ fills up T'. Then,*

(1) In its appeal to Manin's theorem, the only property requested on the point $y \in G(K)$ is that its class modulo the *constant* sections of (the constant part of) G generate G_y; but this is precisely the definition of our relative hull. See also Footnote 4 below.

$tr.deg._K K(y, y', \int^y \omega_i, \int^y \eta_i, \int^y \xi_{q_j}, \int^{y'_k} \frac{dt}{t}, i = 1, ..., g, j = 1, ...r, k = 1, ..., r') \geq 2g+r+r'$.

Proof: let us first deal with the case $r' = 0$. By Hilbert's Theorem 90 (see [Se]), there exists a $\mathbf{C}$-rational section s of the projection $p : G \to A$ and elements $\Xi_1, ..., \Xi_r$ complementing $\{p^*(\omega_1), ..., p^*(\omega_g)\}$ into a basis of Ω^1_G such that $s^*(\Xi_j) = \xi_{q_j}$ for all j. Then $\mathbf{y} := s(y)$ lies in $G(F)$, projects to $y = p(\mathbf{y})$ in $A(F)$, and satisfies $\int^{\mathbf{y}} \Xi_j = \int^y \xi_{q_j}, \int^{\mathbf{y}} p^*(\omega_i) = \int^y \omega_i$, so that the field of definition over K of $\{\mathbf{y}, \mathbf{x} = Log_G(\mathbf{y})\}$ coincides with $K(y, \int^y \omega_i, \int^y \xi_{q_j})$. Similarly (now by the very definition of the universal extension), the η_i's are pull-backs under a rational section of invariant forms $\tilde{\eta}_i$ on $\tilde{A}$, and the same argument provides a lift $\tilde{\mathbf{y}}$ of $\mathbf{y}$ to $\tilde{G}$ such that $K(\tilde{\mathbf{y}}, Log_{\tilde{G}}\tilde{\mathbf{y}}) = K(y, \int^y \omega_i, \int^y \xi_{q_j}, \int^y \eta_i)$. According to Proposition 1.b, its transcendence degree over K is bounded from below by $dim(\tilde{G_{\mathbf{y}}})$. Now, the semi-abelian subvariety $G_{\mathbf{y}}$ of G projects onto A_y, which fills up A by hypothesis, and is thus an extension of A by a torus $\mathbf{G}^s_m$, parametrized some points $w_1, ..., w_s$ in $\hat{A}(\mathbf{C})$. But (say by [B1], Prop. 1), such a semi-abelian variety can embed in G iff there exists an isogeny $\alpha \in End(A)$ such that $\alpha^*(q_1), ..., \alpha^*(q_r)$ lie in the subgroup of $\hat{A}(\mathbf{C})$ generated by $w_1, ..., w_s$. Since the former are linearly independent over $\mathbf{Z}$, this forces $s = r$, so that the relative hull of $\mathbf{y}$ fills up G, whose universal extension has dimension $2g + r$. (In other words, the hypothesis on the points q_i means that G is an essential extension of A.)

For the general case, we introduce the semi-abelian variety $G \times T'$. A similar argument, combined with the hypothesis on y', shows that the relative hull of the point $(\mathbf{y}, y')$ is $G \times T'$, whence the required lower bound.

The point we made in the introduction about the limits of the functional setting is best illustrated by the following "counterexample" to Proposition 2, with $r = 1$ (and $r' = 0$). We say that an isogeny $f : A \to \hat{A}$ is antisymmetric if its transpose $\hat{f} : A \to \hat{A}$ satisfies $f + \hat{f} = 0$. Instead of the expected lower bound $2g + r + r' = 2g + 1$, we have:

Proposition 3 : *Assume that the abelian variety $A/\mathbf{C}$ admits an antisymmetric isogeny f to $\hat{A}$, and let $y \in A(K)$ be such that $A_y = A$. There exists a differential of the third kind ξ_q on A/K, whose residue divisor lies in the equivalence class of the point $q = f(y) \in \hat{A}(K)$ such that $tr.deg._K K(\int^y \omega_i, \int^y \eta_i, \int^y \xi_q, i = 1, ..., g) = 2g$.*

Proof: in view of Proposition 1.b, the first $2g$ integrals generate over $K = K(y)$ a field of transcendence degree $2g$. As shown by the computational proof given in §2 in the case $g = 1$, the last one can be made to lie in this field (for general g, use the fact that the restriction of the Poincaré bundle to the graph of f is isotrivial).

Remark 2 : in Proposition 3, q is non-torsion, but also *non constant*. In the setting of Proposition 1, this would correspond to a "semi-constant" semi-abelian variety, i.e. a non isoconstant extension G of the constant abelian variety A by the (constant) torus $\mathbf{G}_m$. It would be interesting to construct the corresponding differential equation $\partial Log_{\tilde{G}}\tilde{y} = \partial\tilde{x}$ with the help of a D-group structure on $\tilde{G}$, viewed as an extension, in the category of D-groups, of the standard D-group structure of A by that of $\mathbf{G}_m \times \mathbf{G}_a^g$; see [Pi], [Ma] - and Remark 3.ii below for a slightly different suggestion.

§2. Arithmetic interlude.

We now turn to the number theoretic (i.e. honest) extension of Schanuel's conjecture to the semi-abelian variety $G/\mathbf{C}$. In this case, x lies in $TG(\mathbf{C})$, $y = exp_G(x)$ in $G(\mathbf{C})$, and we want to bound from below the transcendence degree over $\mathbf{Q}$ of the field $k(\tilde{x}, exp_{\tilde{G}}(\tilde{x}))$, where $k = \mathbf{Q}(G)$ denotes the field of definition of G (hence of its universal extension $\tilde{G}$). We shall give a pedestrian approach to the strategy proposed by Y. André in [A3], §23, and take advantage of this walk to write down the *full* period matrix of the "simplest interesting" one-motive. (For a general introduction to one-motives, see [D1].)

The conjecture should cover Schanuel's, and in particular imply the transcendency of π, so that we cannot mod out by the periods of exp_G. Therefore, the lower bound must depend on x, rather than on y[2]. The multiplicative and elliptic cases of the conjecture, as well as Wüstholz's theorem on linear forms in abelian integrals, suggest the introduction of the *Lie hull* of x, denoted by $\mathcal{G}_x$ and defined as smallest algebraic subgroup H of G such that $x \in TH(\mathbf{C})$. Again, there is no need to specify fields of definitions, since all algebraic subgroups of the semi-abelian variety G are defined on a finite extension of k. However, a statement of the type

$$tr.deg.(\mathbf{Q}(G, \tilde{x}, \tilde{y})/\mathbf{Q}) \geq dim(\tilde{\mathcal{G}}_x) \quad (??)$$

is usually false. For instance, let $G = \mathbf{G}_m \times E$, where $E/\overline{\mathbf{Q}}$ is an elliptic curve with complex multiplications, and let ω_1, η_1 be a period and corresponding quasiperiod of E. The point $x = (2\pi i, \omega_1) \in TG(\mathbf{C})$ lifts in $T\tilde{G}$ to a point $\tilde{x}$ of the kernel of $exp_{\tilde{G}}$, which may be represented by the vector $(2\pi i, \eta_1, \omega_1)$. By the CM hypothesis and Legendre relation

(2) An alternative solution consists in replacing the base field $\mathbf{Q}$ by the field of all periods of $\tilde{G}$, as in [B2], Conjecture 2, and [Bn1]. This makes specific cases of the conjecture more difficult to check, but the Lie hull $\mathcal{G}_x$ can then be replaced by the *hull* of y, defined as the connected component G_y of the Zariski closure of $\mathbf{Z}y$ in G. Notice that the inclusion $G_y \subset \mathcal{G}_x$ is often strict. No distinction between the two hulls needed to be made in the relative situation of §1, where we modded out by the (constant) periods of exp_G.

(or Γ-function identities), $\eta_1\omega_1/\pi$ is an algebraic number, so that $\overline{\mathbf{Q}}(\tilde{x},\tilde{y}) = \overline{\mathbf{Q}}(\omega_1,\pi)$ has transcendence degree (at most) 2 . On the other hand, the Lie hull of x is G itself, and its universal vectorial extension $\tilde{\mathcal{G}}_x$ has dimension 3.

Counterexamples not involving vectorial extensions also abound. For instance, consider an abelian 4-fold G of primitive CM type, whose periods satisfy a Shimura relation (cf. [A3], §24.4), and let $x = (\omega_1, ..., \omega_4) \neq 0$ be such a period. Then, $tr.deg.(\mathbf{Q}(x,y)/\mathbf{Q}) \leq 3$, although $\mathcal{G}_x = G$ has dimension 4. But more to the point for our study, we shall now construct a counterexample involving a point y of infinite order on G (and for which $\mathcal{G}_x = G_y = G$).

In the next paragraphs until Conjecture 1, we restrict to the logarithmic case of Schanuel's conjecture, i.e. assume that G *and* $\tilde{y}$ *are defined over a subfield* k *of* $\overline{\mathbf{Q}}$. Let thus E be an elliptic curve defined over the number field k by a Weiestrass equation $Y^2 = 4X^3 - g_2X - g_3$. Let $\wp, \zeta, \sigma$ be the standard Weierstrass functions attached to this model, and let $\omega_1, \omega_2, \eta_1, \eta_2$ be the periods and quasi-periods of $\wp$ and ζ. In particular, exp_E is represented by $(\wp, \wp')$, *quae* functions of the variable z defined by $dz = exp_E^*(dX/Y)$, and $d\zeta = -exp_E^*(XdX/Y)$. We also fix two complex numbers u, v, and assume that their images p, q under exp_E are *non torsion* points of $E(k)$. We do *not* require that p and q be linearly independent over $End(E)$. Denote by G the extension of E by $\mathbf{G}_m$ parametrized by $(-q) - (0)$.

Let us now puncture the curve E at the two points 0 and $-q$, and pinch it at two other k-rational points p_1, p_0 whose difference in the group E is p. The one-motive $M_0 = M(E, -q, p, p_0)$ attached by [De1] to the resulting open singular curve can be described as follows : there is a unique function $f_0 \in k(E)$ with value 1 at p_0 and divisor $(-q + p) - (p) - (-q) + (0)$, and by a well-known description of the set $G(k)$ ([Mu], p. 227), this defines a point y_0 in $G(k)$ lying above p, hence a one-motive $M_0 : \mathbf{Z} \to G : 1 \mapsto y_0$.

The de Rham realization $H^{dR}(M_0/k)$ of M_0 is the k-vector space generated by the differential of a rational function f on E such that $f(p_1)$ differs from $f(p_0)$ (say, by 1), the dfk $\omega = dX/Y$, the cohomology class of the dsk $\eta = XdX/Y$, and the dtk $\xi = \frac{1}{2}\frac{Y - Y(q)}{X - X(q)}\frac{dX}{Y}$. The residue divisor of ξ is $-(0) + (-q)$, and its pullback under exp_E is the logarithmic differential of the function

$$f_v(z) = \frac{\sigma(v+z)}{\sigma(v)\sigma(z)} e^{-\zeta(v)z},$$

whose quasi-periods are given by $e^{\lambda_i(v)}$, with

$$\lambda_i(v) = \eta_i v - \zeta(v)\omega_i, \text{ for } i = 1, 2.$$

The Betti realization of M_0 is the dual of the $\mathbf{Q}$-vector space $H_B(M_0, \mathbf{Q})$ generated by a small loop around the hole $-q$, the two standard loops on the elliptic curve E, and a "loop" from p_0 to p_1 on the pinched curve. Integrating the above differential forms along these loops, we obtain the period matrix of M_0. Not warranting signs, it may be written as

$$\Pi(u, v, \ell_0) := \begin{pmatrix} 2\pi i & \lambda_1(v) & \lambda_2(v) & g(u,v) - \zeta(v)u + \ell_0 \\ 0 & \eta_1 & \eta_2 & \zeta(u) \\ 0 & \omega_1 & \omega_2 & u \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$g(u, v) = \ell n \frac{\sigma(u+v)}{\sigma(u)\sigma(v)}$$

and $e^{\ell_0} = \gamma_0 \in k^*$ can easily be computed in terms p_0, p_1, q, using the triple addition formula for the σ-function ([WW], XX, ex. 20; NB : we modified η in its cohomology class so as to delete an additive k-rational factor from its last period $\int_{p_0}^{p_1} \eta$). It is fun to compute the matrix of cofactors of $\Pi(u, v, \ell_0)$, although the result is not a surprise: dividing by $2\pi i$, we get

$$\Pi'(v, u, \ell_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v & \omega_2 & \omega_1 & 0 \\ \zeta(v) & \eta_2 & \eta_1 & 0 \\ g(v,u) - \zeta(u)v + \ell_0 & \lambda_2(u) & \lambda_1(u) & 2\pi i \end{pmatrix}$$

which after some rearrangement, is the period matrix of the Cartier dual of M_0, given by a point y_0', lying above $-q$, on the extension of $\hat{E}$ by $\mathbf{G}_m$ parametrized by $p \in Pic_0 \hat{E} \simeq E$.

Let now y be an arbitrary point on $G(k)$ projecting onto p, i.e. of the form $y_0\gamma$ for some $\gamma \in \mathbf{G}_m$, let $M(y)$ be the corresponding one-motive, and let $\tilde{y}$ be a lift of y to $\tilde{G}(k)$. A logarithm $\tilde{x}$ of $\tilde{y}$ in $T\tilde{G}(\mathbf{C})$ is given by the last column (without its bottom entry) of the period matrix $\Pi(u, v, \ell)$, where $e^{\ell} = \gamma_0\gamma \in k^*$, and where $\zeta(u)$ should be replaced by $\zeta(u) + \beta$ for some $\beta \in k$ depending on $\tilde{y}$, so that $k(\tilde{x}, \tilde{y})$ coincides with the field

$$k(\tilde{x}) = k(u, \zeta(u), g(u, v) - \zeta(v)u + \ell).$$

Since q has infinite order, the extension G is not isotrivial, and since p too is non torsion, the Lie hull $\mathcal{G}_x$ of the projection x of $\tilde{x}$ to TG fills up G. Therefore, $\tilde{\mathcal{G}}_x$ has dimension 3, and if the statement (??) was correct, the three numbers $u, \zeta(u), g(u, v) - \zeta(v)u + \ell$ would be algebraically independent for any logarithm ℓ of an algebraic number, and elliptic logarithms u, v of non-torsion points on $E(\overline{\mathbf{Q}})$.

We can at last describe our counterexample. Assume that $g_3 = 0$, i.e. that E has complex multiplication by i. (Any CM field would work, modulo a finer choice of the dsk

η.) Then, for any $\alpha \in End(E) = \mathbf{Z}\oplus\mathbf{Z}i$ with norm $N(\alpha) = \alpha\overline{\alpha}$, the functions $\zeta(\alpha z) - \overline{\alpha}\zeta(z)$ and the square of $\sigma(\alpha z)/\sigma(z)^{N(\alpha)}$ lie in the field $k(\wp(z), \wp'(z))$. Suppose now that $v = iu$, the important point being that $\alpha = i$ is totally imaginary. Then, $\zeta(v) = -i\zeta(u)$, and $\sigma(u+v)/\sigma(u)\sigma(v) = -i\sigma((1+i)u)/\sigma(u)^2$ is the square root γ' of an element of k^*, since $N(1+i) = 2$. Choosing $\gamma = (\gamma_0\gamma')^{-1}$ *and* $\ell = -g(u, iu) = -\ell n(\gamma')$, and slightly extending k, we get a point $\tilde{x} \in T\tilde{G}(\mathbf{C})$ with $\tilde{y} = exp_{\tilde{G}}(\tilde{x}) \in \tilde{G}(k)$ and $k(\tilde{x}) = k(u, \zeta(u))$. But this has transcendence degree at most 2 (in fact 2, according to a theorem of Chudnovsky), not 3 !

This example, which translates word for word to the semi-constant situation of Proposition 3, is not mysterious. The one-motive $M = M(y)$ it corresponds to was discovered by Ribet in his study of Galois representations (cf [JR]), and is known to have a degenerate Mumford-Tate group. In general, this group $MT(M)$ is the semi-stabilizer in $GL_{\mathbf{Q}}(H_B(M, \mathbf{Q}))$ of all Hodge cycles occuring in the tensor constructions on $H_B(M, \mathbf{Q})$ and its dual (cf. [De2], p. 43, and [Br]). In the present case (cf. [B4], or more generally [Bn2]), its unipotent radical has

(i) dimension 5 if the points p and q are linearly independent over $End(E)$;

(ii) dimension 3 if there exists $\alpha \in End(E) \otimes \mathbf{Q}$ such that $p = \alpha q$ and $\alpha \neq -\overline{\alpha}$ is not antisymmetric (an automatic condition if E has no CM);

(iii) dimension 3 if $p = \alpha q$ with $\overline{\alpha} = -\alpha$, and y is not a Ribet point;

(iv) dimension 2 in the remaining case.

According to a conjecture of Grothendieck(3), the transcendence degree of the full field of periods of M should be equal to the dimension of $MT(M)$, and an elementary dimension count as in [A3], 23.2.1 (see also the proof of Prop. 1.i above) then implies, *still assuming that k is a number field*:

Proposition 4.a: *recall the notations above, and assume that* $tr.deg.(k(\Pi(u, v, \ell))/k) = dim MT(M)$. *Then, the field* $k(u, \zeta(u), g(u, v) - \zeta(v)u + \ell)$ *has transcendence degree* 3 *in Cases (i,ii, iii). In Case (iv), it coincides with the field* $k(u, \zeta(u), 2c\pi i)$ *for some rational number* c, *and has transcendence degree* 3 *if* $c \neq 0$, *and* 2 *otherwise.*

Proof: let us only treat the last two CM cases, again with $v = iu$. Then, the maximal reductive quotient of $MT(M)$ has dimension 2, while $k(\Pi(u, v, \ell)) = k(2\pi i, \omega_1, u, \zeta(u), \tilde{\ell})$, where

(3) cf. [A3], 23.1.4, 23.3.2. This conjecture actually relates transcendence degrees to motivic Galois groups; in view of [De2] and [Br], Mumford-Tate groups are an acceptable substitute in the case of one-motives.

$\tilde{\ell} := g(u, iu) + \ell$ is a logarithm of an algebraic number $\tilde{\gamma}$. In Case (iii), $dim(MT(M)) = 5$; by the Grothendieck conjecture, we have 5 algebraically independent numbers, any 3 of which must be algebraically independent. In Case (iv), $dim(MT(M)) = 4$, but $\tilde{\gamma}$ is a root of unity and $k(\Pi(u, v, \ell))$ reduces to $k(2\pi i, \omega_1, u, \zeta(u))$. We then have 4 algebraically independent numbers, any 3, or 2, of which must be algebraically independent. Note that in this last case, the transcendence degree of $k(\tilde{x})$ *depends* on the choice of the logarithm x of the point y, although the Lie hull of x always fills up G.

We now drop the assumption that $k \subset \mathbf{C}$ is a number field. The dimension count becomes hopeless, but as suggested in [A3], 23.2.2, a finer approach to the study of any specific period is provided by the $MT(M)$-torsor of all isomorphisms between $H_B(M)^*$ and $H^{dR}(M)$ which, up to homotheties, preserve the cohomology classes of Hodge cycles. The period matrix represents such an isomorphism. For $y \in G(\mathbf{C})$, the choice of a logarithm $x = Log_G(y)$ of y determines a loop γ_x in $H_B(M)$, which projects to a generator of $H_B(M)/H_B(G)$, and which satisfies $g.\gamma_x - \gamma_x \in H_B(G)$ for all $g \in MT(M)$. Define the *Mumford-Tate orbit* MT_x of x as the the Zariski closure in $H_B(G)$ of the orbit of γ_x under this affine action of $MT(M)$.

Conjecture 1 (following André, [A3], 23.4.1): *let G be a semi-abelian variety defined over $\mathbf{C}$, let $\tilde{G}$ be its universal extension, let x be a point in $TG(\mathbf{C})$, let $y = exp_G(x)$, and let MT_x be the Mumford-Tate orbit of x. Let further $\tilde{x}$ be a lift of x to $T\tilde{G}(\mathbf{C})$ and let $\tilde{y} = exp_{\tilde{G}}(\tilde{x})$. Then,*

$$tr.deg.(\mathbf{Q}(G, \tilde{x}, \tilde{y})/\mathbf{Q}) \geq dim(MT_x).$$

A "justification" of the conjecture is given in the proof of Theorem 1 below. Notice that for any semi-abelian variety G, $H^{dR}(G)$ is canonically isomorphic to $\Omega^1_{\tilde{G}}$, so that $dimH_B(G) = dim\tilde{G}$, and that for any $x \in TG(\mathbf{C})$, MT_x is necessarily contained in the Betti homology of the Lie hull $\mathcal{G}_x$ of x. In particular, $dimMT_x \leq dim\tilde{\mathcal{G}}_x$. As shown by the last case of Prop. 4.a), the inequality may be strict. However, if G is isogenous to a *split* product $A \times T$ as in [K2], and if y generates a Zariski dense subgroup of $\mathcal{G}_x$ (i.e. if $G_y = \mathcal{G}_x$), we deduce from [A1], Prop. 1, that the Mumford-Tate orbit MT_x coincides with $H_B(\mathcal{G}_x)$, and Conjecture 1 does imply that $tr.deg._{\mathbf{Q}}\mathbf{Q}(G, \tilde{x}, \tilde{y}) \geq dim(\tilde{\mathcal{G}}_x)$ in this special case.

As a companion to Prop. 4.a, now restricted to Cases (i) and (ii) of its discussion, here is another consequence of Conjecture 1.

Proposition 4.b : *let E be an elliptic curve with complex invariants g_2, g_3, and let u, v, ℓ be complex numbers such that $exp_E(u), exp_E(v)$ are not related by an antisymmetric relation* *over $End(E)$. Assume that Conjecture 1 holds true.Then,*

$$tr.deg.(\mathbf{Q}(g_2, g_3, u, \zeta(u), g(u,v) - \zeta(v)u + \ell, \wp(u), e^{\ell})/\mathbf{Q}) \geq 3;$$

in particular, if g_2, g_3 and $\wp(u)$ are algebraic, the numbers $u, \zeta(u)$ and $\ell n(\sigma(u))$ are algebraically independent; so are the numbers $u, \zeta(u)$ and $\sigma(u)$.

Proof : let γ_x be a loop complementing $H_B(G)$ in $H_B(M)$. In all cases except (iv) (and even in Case (iv), if we avoid a specific line in the choice of γ_x), the orbit of γ_x under the affine action of the unipotent radical of $MT(M)$ already fills up $H_B(G)$, so that the general inequality is clear. The other assertions, which could be checked by dimension count, concern Case (ii), with $v = u$. For the first one, recall that $\sigma(2u)/\sigma(u)^4 = -\wp'(u)$, and choose $-\ell$ as a logarithm of this algebraic number. For the last one, choose $\ell = -g(u,u)$. Notice that in order to reach the values of the σ function, we must here consider a *transcendental* point y on a semiabelian variety G defined over $\overline{\mathbf{Q}}$.

In the same spirit, but back into the functional context of Section 1, here is an application of Prop. 2. We recall that F is a differential field with constant field $\mathbf{C}$, and that K is a subfield of F of transcendence degree 1 over $\mathbf{C}$.

Proposition 5 : *let E be an elliptic curve with complex invariants g_2, g_3 and period lattice Ω, let $v_1, ..., v_n$ be* complex numbers *not lying $\Omega \otimes \mathbf{Q}$, and let $x_1, ..., x_n$ (resp. $x'_1, ..., x'_{r'}$) be elements of F linearly independent over $End(E)$ (resp. $\mathbf{Z}$) modulo $\mathbf{C}$. Then,*

$$tr.deg_K K(x_i, x'_j, \zeta(x_i), \wp(x_i), \sigma(v_i + x_i)/\sigma(x_i)\sigma(v_i), e^{x'_j}; 1 \leq i \leq n, 1 \leq j \leq r') \geq 3n + r'.$$

Proof : for each $i = 1, ..., n$, let G_i be the extension of E by $\mathbf{G}_m$ parametrized by the divisor $(exp_E(-v_i)) - (0)$. Then, $G = G_1 \times ... \times G_n$ is an extension of E^n by $\mathbf{G}_m^r$, with $r = n$, parametrized by $\mathbf{Z}$-linearly independent points $q_1, ..., q_n$ of $(\hat{E})^n(\mathbf{C})$: indeed, their collection can be represented by a diagonal matrix, none of whose diagonal entry is torsion. By hypothesis, the relative hull of the point $y = (exp_E(x_1), ..., exp_E(x_n))$ (resp. $y' = (e^{x'_1}, ..., e^{x'_{r'}})$) fills up E^n (resp. $\mathbf{G}_m^{r'}$). The result follows from Proposition 2, combined with the above computations, on choosing $\ell_i = -g(x_i, v_i)$ for $i = 1, ..., n$.

Remark 3 : i) the "reason" for the validity of Proposition 1 is that the situation it concerns is akin to Case (i) above: in the notations of Prop. 2, the points q_i which parametrize the extension G are constant (since G is constant), while the relative hull G_y of y takes into account only the non-constant "parts" p_j of the points $exp_E(x_j)$. No linear relation over $End(A)$, antisymmetric or not, can then relate the p_j's to the q_i's.

ii) But for the very same reason, we can no longer take $v_i = x_i$ in Prop. 5, and contrary to [BK], the result falls short of the study of the elements $\sigma(x_i)$ themselves. To reach them, "semi-constant" semi-abelian varieties as in Remark 2 seem required. Here, though, is another suggestion: since $\frac{\sigma'}{\sigma} = \zeta$, the couples $(x = Log_E(y), z) \in F \times F^*$ such that $\sigma(x) = z$ are solutions of the system $\frac{\partial z}{z} = ty^*\omega(\partial), \partial t = y^*(\eta)(\partial)$. This may be related to the Manin kernel of the split product of $\tilde{E}$ by $\mathbf{G}_m$, its subgroup $\mathbf{G}_a \times \mathbf{G}_m$ being now endowed with a non standard D-group structure, as in [Pi], end of §2.

§3 . Non-isoconstant semi-abelian surfaces

From now on, $K = \mathbf{C}(S)$ is the field of rational functions on a smooth projective curve S over $\mathbf{C}$, t is a non-constant element of K, and ∂ is the rational vector field d/dt on S. More seriously, we only consider the "logarithmic case" of Schanuel's conjecture, i.e. assume that G and $\tilde{y}$ are defined over K. But we now allow G to be non constant. We start by recalling from [De] and [A1] the general setting of smooth one-motives attached to such datas. This reduces transcendence problems to the computation of an orbit under a Picard-Vessiot group. We then restrict to an elliptic pencil (punctured and pinched as in §2, now along rational sections), describe, in the style of Manin's paper [Mn], the corresponding extensions of its Picard-Fuchs equation, compute their Galois groups with the help of [B5], and apply the result to Schanuel's conjecture.

Let thus $\mathcal{A}$ be an abelian scheme over a non empty Zariski open subset $\mathcal{U}$ of S, let $\mathcal{G}$ be an extension of $\mathcal{A}$ by a constant torus $T_{\mathcal{U}}$ of relative dimension r over $\mathcal{U}$, let $\mathbf{y}$ be a section of $\mathcal{G}$ over $\mathcal{U}$, and let $f : \mathcal{M} \to \mathcal{U}$ be the smooth one-motive over $\mathcal{U}$ attached to the morphism $1 \mapsto \mathbf{y}$ from the constant group scheme $\mathbf{Z}_{\mathcal{U}}$ to $\mathcal{G}$. We denote by $A/K, G/K, y \in G(K), M/K$ the abelian and semi-abelian varieties, point and one-motive over K these datas define at the generic point of S.

The first relative de Rham cohomology sheaf of $\mathcal{M}/\mathcal{U}$ is a locally free $\mathcal{O}_{\mathcal{U}}$-module equipped with a connexion ∇, which, restricted to the generic point and contracted with d/dt, defines a differential operator D on the K-vector space $H^{dR}(M/K)$. The quotient $H^{dR}(G/K)$ of $H^{dR}(M/K)$ by its (trivial rank one) D-submodule $H^{dR}(\mathbf{Z}/K) = (K, \partial)$ is itself an extension of the (trivial) D-module $H^{dR}(T/K) \simeq (K, \partial)^r$ by $H^{dR}(A/K)$.

The first relative Betti homology $R_1 f_* \mathbf{Z} := H_B(\mathcal{M}/\mathcal{U})$ is a constant sheaf over $\mathcal{U}$, whose dual generates over $\mathbf{C}$ the local system of horizontal vectors of ∇. In an analytic neighbourhood U of a point u_0 of $\mathcal{U}$, and relatively to a basis of $H^{dR}(M)$ respecting the above filtrations, its local sections provide a fundamental matrix of solutions for D of the shape

$$\begin{pmatrix} \mathbf{I}_r & \Lambda_1(t) & \Lambda_2(t) & \Gamma(t) \\ 0 & \mathrm{H}_1(t) & \mathrm{H}_2(t) & \mathrm{Z}(t) \\ 0 & \Omega_1(t) & \Omega_2(t) & \mathrm{U}(t) \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

whose entries generate over K a Picard-Vessiot extension $F = F_{u_0}$, for which we set $\mathcal{O}_F = F \cap \mathcal{O}_U$. Its last column (without its bottom entry) represents a logarithm $\tilde{x} = Log_{\tilde{G}}(\tilde{y}) \in T\tilde{G}(\mathcal{O}_F)$ of a K-rational point $\tilde{y}$ lifting y to the universal extension $\tilde{G}$ of G. The field of definition $K(\tilde{x}) = K(\tilde{x}, \tilde{y})$ of $\tilde{x}$ depends only on the image x of $\tilde{x}$ in TG.

Let now $PV(M) = Aut_\partial(F/K)$ be the differential Galois group of the D-module $H^{dR}(M/K)$. For each g in $PV(M)$, $g.\tilde{x} - \tilde{x}$ lies in $H_B(\mathcal{G}/\mathcal{U}) \otimes \mathbf{C}$, and depends only on x. We may therefore define the *Picard-Vessiot orbit* PV_x of x as the Zariski closure of the orbit of $\tilde{x}$ in $H_B(\mathcal{G}/\mathcal{U})$ under this affine action of $PV(M)$.

Theorem 1 : *let G be a semi-abelian variety defined over K, let $\tilde{G}$ be its universal extension, let x be a point in $TG(\mathcal{O}_U)$ such that $y = exp_G(x)$ lies in $G(K)$, and let PV_x be the Picard-Vessiot orbit of x. Let further $\tilde{x}$ be a lift of x to $T\tilde{G}(\mathcal{O}_U)$ such that $\tilde{y} = exp_{\tilde{G}}(\tilde{x})$ lies in $\tilde{G}(K)$. Then,*

$$tr.deg.(K(\tilde{x})/K) = dim(PV_x).$$

Proof : as explained in [Ka], Prop. 2.3.1 and Remark 2.3.3, this is a tautology once one is reminded that a fundamental matrix for D is a generic point of a K-torsor under $PV(M)$. We should point out that by exactly the same argument, the Grothendieck conjecture implies the "logarithmic case" of Conjecture 1, in the form: if G and $\tilde{y}$ are defined over $\overline{\mathbf{Q}}$, then $tr.deg.(\mathbf{Q}(\tilde{x})/\mathbf{Q}) = dim(MT_x)$.

In [A1], Y. André shows that $PV(M)$ is a normal subgroup of the derived group $\mathcal{D}MT(M_{u_0})$ of the Mumford-Tate group of the fiber M_{u_0} of $\mathcal{M}$ above a sufficiently general point $u_0 \in \mathcal{U}$, and gives non-obvious examples where *the inclusion $PV(M) \subset \mathcal{D}MT(M_{u_0})$ is strict.* On the other hand, as soon as $\mathcal{M}$ admits a special fiber M_{u_1} with an abelian Mumford-Tate group, Prop. 2 of [A1] shows that the two groups coincide. Now, at least theoretically, the main theorem of [Bn 2] provides a complete description of $MT(M_{u_0})$. Combined with Theorem 1, this gives a satisfactory answer to the logarithmic case of Schanuel's problem over function fields, under the proviso that $\mathcal{M}$ varies enough in its pencil to ensure both very small and rather large Mumford-Tate groups above various points of the base.

To dispense with this hypothesis, a more direct approach consists in computing the Picard-Vessiot group itself. Manin's kernel theorem provides such a possibility when the abelian variety A is not isoconstant; for abelian integrals of the second kind[(4)], this was already noticed in [A1], Theorem 3 (and less generally in [B3], Thm. 5). We now extend this method to the study of elliptic integrals of the 3rd kind, where in parallel with §2, the description of the D-module $H^{dR}(M/K)$ can be made quite concrete, as follows.

Let E be an elliptic curve defined over the function field $K = \mathbf{C}(S)$ by the Weiestrass equation $Y^2 = 4X^3 - g_2(t)X - g_3(t)$. In the standard non-canonical way (cf. [Mn]), extend the derivation ∂ to the field $K(E)$ and to its space of differentials by setting $\partial x = 0, \partial(f(x,y)dx) = \partial f(x,y)dx$. Let $p(t), q(t)$ be two non torsion points on $E(K)$), possibly linearly dependent over $End(E)$, and consider the differential of the third kind $\xi(t) = \frac{1}{4\pi i}\frac{Y-Y(q)}{X-X(q)}\frac{dX}{Y}$. Since the residues of ξ are the *constant* functions $\pm 1/2\pi i$ of K, the classical formula

$$\forall\ s(t) \in E(K),\ \partial(Res_{s(t)}\xi(t)) = Res_{s(t)}\partial\xi(t)$$

implies that $\partial\xi(t)$ is a differential of the *second* kind on E/K. By Gauss, its cohomology class is killed by a 2nd order fuchsian differential operator L_ξ, and a basis of local solutions of $(L_\xi \circ \partial)y = 0$ in an analytic neighbourhood U of a point u_0 of $\mathcal{U}$ is given by the periods $\lambda_1(t) = \int_{\gamma_1}\xi(t), \lambda_2(t)$ of $\xi(t)$ over loops γ_1, γ_2 of the fiber E_t, and the constant function 1, corresponding to the integral of ξ on a loop around $-q$. Now, the integral $\int_{p_0}^{p_1}\xi(t)$ of ξ between two sections p_0, p_1 differing by p in $E(K)$ is a locally analytic function $\Gamma(t)$, well defined up to the addition of a $\mathbf{Z}$-linear combination of the previous periods, so that $(L_\xi \circ \partial)(\Gamma(t)) := f_{p;p_0}(t)$ is a uniform function on a Zariski open subset of S, with moderate growth at infinity, hence a rational function on S. In brief, $\int_{p_0}^{p_1}\xi(t)$ provides the fourth solution to the 4-th order linear differential operator $(\partial - \frac{\partial f_{p;p_0}}{f_{p;p_0}}) \circ L_\xi \circ \partial \in K[\partial]$. (Manin's paper dealt with the adjoint of the 3rd order operator $L_\xi \circ \partial$.)

From now on, we assume that the j-invariant of E is not constant. Then, L_ξ is equivalent to the standard irreducible Picard-Fuchs equation $L_{E/K}$ attached to the differential dX/Y on E, and the 4-th order operator can be written in the form

$$\mathcal{L}_{p,q;p_0} = \partial_{p,p_0} \circ L_{E/K} \circ \partial_q,$$

where ∂_{p,p_0} and ∂_q are equivalent to ∂. In other words, the section p (resp. q) provides a element N^p (resp. N_q) of the group $Ext(H^{dR}(E/K), \mathbf{1})$ of extensions of the D-module

(4) i.e. when $G = A$. Actually, in this case, even the hypothesis on non-isoconstancy can be dispensed with. See [Ch], bottom of p. 388 and Footnote 1 above - as well as [AV] for an early application to transcendence ! -. However, we do use it, at least formally, in the proof which follows.

$H^{dR}(E/K) \simeq K[\partial]/K[\partial]L_{E/K}$ by the trivial D-module $\mathbf{1} = K[\partial]/K[\partial]\partial$ (resp. of the group $Ext(\mathbf{1}, H^{dR}(E/K))$ and the choice of p_0 then provides a blended extension (in the sense of [B5], Remark 6, and [Ha]) of N^p by N_q. Now comes the main point: since p and q are non-torsion points on the non isoconstant curve E, Manin's theorem (cf. [Mn], [Ch], or [B3], Lemma 8) implies that both extensions N^p and N_q are unsplit; moreover, N_q and the adjoint N_p of N^p are linearly independent over $\mathbf{C}$ in $Ext(\mathbf{1}, H^{dR}(E/K))$ if p and q are linearly independent over $End(E)$. We can then appeal to the purely group theoretic arguments of [B5] to compute the Picard-Vessiot group of $\mathcal{L}_{p,q;p_0}$, as follows.

This PV group is an extension of that of $L_{E/K}$, which is $SL_2(\mathbf{C})$ since $L_{E/K}$ is irreducible and (antisymmetrically) self-adjoint, by its unipotent radical, which, on denoting the solution space of $L_{E/K}$ by $\mathcal{V} \simeq H_B(E_{u_0}) \otimes \mathbf{C}$ and in view of [B5], Thm 3, is isomorphic to

(i) an extension of $\mathcal{V} \times \mathcal{V}$ by $\mathbf{C}$ if p and q are linearly independent over $EndE$;

(ii) the Heisenberg group $\mathcal{H}$ on $\mathcal{V}$ otherwise, i.e. the extension of $\mathcal{V}$ by $\mathbf{C}$ given by the law $(c, v).(c', v') = (c + c' + < v|v' >, v + v')$, where $< | >$ denotes the canonical antisymetric bilinear form induced on $\mathcal{V}$ by the intersection product.

Indeed, the other possibilities mentioned in [B5] (viz. that it becomes abelian, and reduces either to $\mathcal{V} \times \mathbf{C}$ or to $\mathcal{V}$, as in Cases (iii) and (iv) of §2 above) can occur only if the middle operator is symetrically self-adjoint, and the irreducibility of $L_{E/K}$ prevents this. Now, in both Cases (i) and (ii), the Picard-Vessiot orbit of the 4-th solution $\int_{p_0}^{p_1} \xi(t)$ has dimension 3. We therefore deduce from Theorem 1 and the computations of §2 that for any choice $\ell(t), u(t), v(t)$ of analytic functions such that $e^\ell \in K^*$, and $exp_E(u)$, $exp_E(v)$ are non-torsion points on $E(K)$, the functions $u(t), \zeta(u(t)), g(u(t), v(t)) - \zeta(v(t))u(t) + \ell(t)$ are algebraically independent over K; in particular, $u(t), \zeta(u(t))$ and $\ell n\sigma(u(t))$ are algebraically independent. More generally, we obtain the following theorem, which extends Thm 5 of [B3] to the study of $\ell n\sigma$ (but still misses the σ-function itself).

Theorem 2 : *let $g_2(t), g_3(t)$ be algebraic functions such that g_2^3/g_3^2 is not constant, let $\wp_t$ be the Weiestrass function with invariants $g_2(t), g_3(t)$ and period lattice $\Omega(t)$, let $\{u_i; i = 1, ..., n\}$ be holomorphic functions on an open subset of $\mathbf{C}$, linearly independent over $\mathbf{Z}$ modulo $\Omega(t)$, and such that the functions $\wp_t(u_i(t))$ are algebraic. Then, the $3n$ function $u_i(t), \zeta(u_i(t)), \ell n\sigma(u_i(t))$ $(i = 1, ..., n)$ are algebraically independent over $\mathbf{C}(t)$.*

Proof : let E be the corresponding elliptic curve; for $i = 1, ..., n$, set $p_i = q_i = exp_E(u_i)$ and denote by $\mathcal{L}_i$ the differential operator $\mathcal{L}_{p_i,p_i;p_{0,i}}$, for some choice of section $p_{0,i}$. All these are defined over an algebraic extension K of $\mathbf{C}(t)$. The unipotent radical R_u of the differential Galois group over K of the direct sum of the $\mathcal{L}_i$'s naturally embeds in $\mathcal{H}^n$, via the isomorphisms $\phi_i : R_u(PV(\mathcal{L}_i)) \simeq \mathcal{H}$ of Case (ii) above. We claim that the image of R_u coincides with $\mathcal{H}^n$ (which has dimension $2n + n = 3n$). The proposition then easily follows from Thm 1.

For each i, let ψ_i be the composition of ϕ_i with the projection from $\mathcal{H}$ to $\mathcal{V}$. Since the points q_i are linearly independent over $End(E)$, the extensions N_{q_i} defined above are $\mathbf{C}$-linearly independent in $Ext(\mathbf{1}, H^{dR}(E/K))$, and Thm 2 of [B3] (or more generally, Thm 2.2.14 of [Ha]) implies that the image of R_u under $(\psi_1, ..., \psi_n)$ fills up $\mathcal{V}^n$. But since $< | >$ is non degenerate, the derived group of any subgroup of $\mathcal{H}^n$ projecting onto $\mathcal{V}^n$ fills up $\mathbf{C}^n$, so that the only subgroup of $\mathcal{H}^n$ projecting onto $\mathcal{V}^n$ is $\mathcal{H}^n$ itself, and indeed, $R_u = \mathcal{H}^n$.

I shall not attempt here to formulate a Schanuel conjecture for smooth one-motives over an arbitrary base over $\mathbf{C}$, which would extend Proposition 1 and Theorem 1, and parallel Conjecture 1 of §2. The question is of course to find the correct analogue of the Mumford-Tate group. Let me only point out to the probable relevence of the algebraic D-group structure which Pillay's Galois groups [Pi] are endowed with. Already when G is constant, the two sides of Remark 1 show that the expected group should lie in the tangent bundle $T\tilde{G} \times \tilde{G}$ of $\tilde{G}$. In the non constant case, it is not difficult to guess that the prolongation $\tau(\tilde{G})$ of $\tilde{G}$, which in a sense is the natural habitat of Manin kernels (cf. [Ma]), will play a role. The ∂-Hodge structures of [Bu] may also have some bearing on these questions.

We finally mention two further directions of study:

i) the Fourier expansions of the functions $\zeta(z) - \frac{\eta_1}{\omega_1} z$, $\sigma(z) e^{-\frac{\eta_1}{2\omega_1} z^2}$, $\frac{\sigma(z+v)}{\sigma(z)\sigma(v)} e^{-\frac{\eta_1}{\omega_1} vz}$ are building blocks in the theory of q-difference equations. Can q-difference Galois groups and their higher dimensional analogues provide a new insight on Schanuel's conjecture ?

ii) what about characteristic p analogues ? We shall here merely refer to [AMP] for the algebraic independence of the $\mathbf{Z}_p$-powers $f^{\lambda_1}, ..., f^{\lambda_r}$ of a given power series f in $\mathbf{F}_p[[t]]$, and in closer relation to the present study, to [Pa] for a Galois theoretic solution of the logarithmic case of Schanuel's conjecture on powers of the Carlitz module.

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Daniel BERTRAND
Institut de Mathématiques de Jussieu
bertrand@math.jussieu.fr

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