Wadge degrees of $\mathbf{\Delta}_2^0$ omega-powers

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Abstract. We provide, for each natural number n and each class among $D_n(\mathbf{\Sigma}_1^0)$, $\check{D}_n(\mathbf{\Sigma}_1^0)$ and $D_{2n+1}(\mathbf{\Sigma}_1^0) \oplus \check{D}_{2n+1}(\mathbf{\Sigma}_1^0)$, a regular language whose associated omega-power is complete for this class.

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1 Introduction

In the sixties, in order to prove the decidability of the monadic second order theory of one successor over the integers, Büchi studied acceptance of infinite words by finite automata with the now called Büchi acceptance condition, see [Büc62]. Since then, a lot of work has been done on regular ω-languages, accepted by Büchi automata, or by some other variants of automata over infinite words, like Muller or Rabin automata, see [Tho90, Sta97, PP04].

The class of regular ω-languages, those accepted by Büchi or Muller automata, is the ω-Kleene closure of the class of regular finitary languages. Let Σ be a finite alphabet, and L be a finitary language over Σ. The ω-**power** L^∞ of L is the set of infinite words constructible with L by concatenation, i.e., $L^\infty := \{ w_0 w_1 \ldots \in \Sigma^\omega \mid \forall i \in \omega \;\; w_i \in L \}$. The ω-**Kleene closure** of a class $\mathcal{C}$ of languages of finite words over finite alphabets is the class of ω-languages of the form $\bigcup_{1\leq j\leq n} \; K_j \cdot L_j^\infty$, where n is a natural number and the K_j's and the L_j's are in $\mathcal{C}$. We denote here by L^∞ the ω-power of L, as in [Lec05, FL09] and the recent survey paper [FL20], while it is usually denoted by L^ω in theoretical computer science papers, as in [Sta97, Fin01, Fin03, FL07].

Moreover, the operation of taking the ω-power of a finitary language also appears in the characterization of the class of context-free ω-languages as the ω-Kleene closure of the family of context-free finitary languages, see [Sta97].

This shows that the ω-power operation is a fundamental operation over finitary languages in the study of ω-languages, which naturally leads to the question of its complexity. Since the set Σ^ω of infinite words over the finite alphabet Σ can be equipped with the usual Cantor topology, the question of the topological complexity of ω-powers of finitary languages naturally arises, and was asked by Niwinski [Niw90], Simonnet [Sim92], and Staiger [Sta97].

Then the ω-powers have been studied from the perspective of Descriptive Set Theory in several recent papers [Fin01, Fin03, Fin04, Lec05, DF07, FL07, FL09, FL21, FL20].

As noticed by Simonnet in [Sim92], the ω-powers are always analytic sets. It has been proved in [Fin03] that there exists a finitary language L accepted by a one counter automaton such that the ω-power L^∞ is analytic and non Borel. Moreover, we proved in [FL09] that, for every non-null countable ordinal ξ, we can find $\mathbf{\Sigma}^0_\xi$-complete ω-powers, as well as $\mathbf{\Pi}^0_\xi$-complete ω-powers. This shows that, surprisingly, the ω-powers can be very complex. Moreover, some results have been obtained in [FL09] about the Wadge degrees of ω-powers, indicating the location of some ω-powers inside the Wadge hierarchy, which is a great refinement of the Borel hierarchy, see [Wad83].

On the other hand, there were very few results about the Wadge degrees of ω-powers of very low Borel rank. We fill this gap in this paper, studying the ω-powers in the class $\mathbf{\Delta}^0_2$.

Our main result is the following:

Theorem. *Let n be a natural number.*

(a) We can find a regular language $L \subseteq 2^{<\omega}$ such that L^∞ is complete for the class $D_n(\mathbf{\Sigma}^0_1)$, and another one for $\check{D}_n(\mathbf{\Sigma}^0_1)$.

(b) We can find a regular language $L \subseteq 2^{<\omega}$ with the property that L^∞ is complete for the class $D_0(\mathbf{\Sigma}^0_1) \oplus \check{D}_0(\mathbf{\Sigma}^0_1)$, and another one for $D_{2n+1}(\mathbf{\Sigma}^0_1) \oplus \check{D}_{2n+1}(\mathbf{\Sigma}^0_1)$.

Up to our knowledge, the ω-powers we get are the first examples having these topological complexities, even in the non-necessarily regular case (except, for (a) when $n \leq 2$ and $\check{D}_3(\mathbf{\Sigma}^0_1)$, and for (b) when $n=0$, see [Lec05]).

Moreover, our ω-powers are ω-powers of regular finitary languages. The ω-powers of this kind have been studied by Litovsky and Timmerman in [LT87], where they proved that if a regular ω-language $\mathcal{L}$ is an ω-power, then there is a regular finitary language L such that $\mathcal{L}=L^\infty$.

The topological complexity of regular ω-languages is well known. Every regular ω-language is a finite boolean combination of $\mathbf{\Pi}^0_2$ (and hence Borel) sets, and is in particular a $\mathbf{\Delta}^0_3$ set. The trace of the Wadge hierarchy on the ω-regular languages is called the Wagner hierarchy. It has been completely described by Wagner in [Wag79], see also [Sta97, Sel98, Sel08]. Its length is the (countable) ordinal ω^ω.

In particular, a regular ω-language is in the class $\mathbf{\Delta}^0_2$ iff it is in the hierarchy of finite differences of $\mathbf{\Sigma}^0_1$ sets. Our main result implies that, for any Wagner class $\mathbf{\Gamma} \neq \check{\mathbf{\Gamma}}$ with $\mathbf{\Gamma} \subseteq \mathbf{\Delta}^0_2$, we can find $L \subseteq 2^{<\omega}$ regular such that L^∞ is complete for $\mathbf{\Gamma}$. As a consequence, we determine the Wadge-Wagner hierarchy of non self-dual $\mathbf{\Delta}^0_2$ regular ω-powers.

2 Preliminaries

We assume that the reader has a knowledge of basic notions of descriptive set theory, which can be found in [Kec95]. We use in this paper usual notations in this field. In the sequel, $\mathbf{\Gamma}$, $\mathbf{\Lambda}$ will be classes of subsets of Polish spaces. The class of complements of elements of $\mathbf{\Gamma}$ is defined, when X is a Polish space, by $\check{\mathbf{\Gamma}}(X) := \{X \setminus A \mid A \in \mathbf{\Gamma}(X)\}$.

We set $\mathbf{\Sigma}^0_1 := \{O \mid O \text{ is an open subset of a Polish space } X\}$. If $\xi \geq 1$ is a countable ordinal, then $\mathbf{\Pi}^0_\xi := \check{\mathbf{\Sigma}}^0_\xi$ and $\mathbf{\Delta}^0_\xi := \mathbf{\Sigma}^0_\xi \cap \mathbf{\Pi}^0_\xi$. If $\xi \geq 2$ and X is a Polish space, then, inductively,

$$\mathbf{\Sigma}^0_\xi(X) := \{ \bigcup_{n\in\omega} A_n \mid \forall n \in \omega \ \ A_n \in \bigcup_{\eta<\xi} \mathbf{\Pi}^0_\eta(X)\}.$$

The classes $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$ form the **Borel hierarchy**. This hierarchy can be refined, using the classes of differences. If ζ is a countable ordinal and $O := (O_\eta)_{\eta<\zeta}$ is an increasing sequence of subsets of a set X, then $D_\zeta(O) := \{x \in X \mid \exists \eta < \zeta \ \ \text{parity}(\eta) \neq \text{parity}(\zeta) \wedge x \in O_\eta \setminus (\bigcup_{\theta<\eta} O_\theta)\}$. If X is a Polish space, then $D_\zeta(\mathbf{\Gamma})(X) := \{D_\zeta(O) \mid \forall \eta < \zeta \ \ O_\eta \in \mathbf{\Gamma}(X)\}$. The classes $D_\zeta(\mathbf{\Sigma}^0_\xi)$ and $\check{D}_\zeta(\mathbf{\Sigma}^0_\xi)$ form the **Lavrentieff hierarchy**. We set

$$(\mathbf{\Gamma} \oplus \mathbf{\Lambda})(X) := \{(A \cap C) \cup (B \setminus C) \mid C \in \mathbf{\Delta}^0_1(X) \wedge A \in \mathbf{\Gamma}(X) \wedge B \in \mathbf{\Lambda}(X)\}.$$

By [Wag79], the **Wagner hierarchy** of $\mathbf{\Delta}^0_2$ sets is made of the classes $D_n(\mathbf{\Sigma}^0_1)$, $\check{D}_n(\mathbf{\Sigma}^0_1)$, as well as $D_n(\mathbf{\Sigma}^0_1) \oplus \check{D}_n(\mathbf{\Sigma}^0_1)$, where $n \in \omega$, and starts as follows:

$$\begin{array}{ccccc}
D_0(\mathbf{\Sigma}^0_1)=\{\emptyset\} & & D_1(\mathbf{\Sigma}^0_1)=\mathbf{\Sigma}^0_1 & & D_2(\mathbf{\Sigma}^0_1) \ \ \ldots \\
& D_0(\mathbf{\Sigma}^0_1) \oplus \check{D}_0(\mathbf{\Sigma}^0_1) = \mathbf{\Delta}^0_1 & & \mathbf{\Sigma}^0_1 \oplus \mathbf{\Pi}^0_1 & \\
\check{D}_0(\mathbf{\Sigma}^0_1) & & \check{D}_1(\mathbf{\Sigma}^0_1)=\mathbf{\Pi}^0_1 & & \check{D}_2(\mathbf{\Sigma}^0_1) \ \ \ldots
\end{array}$$

We will also consider the class defined, when X is a Polish space, by

$$(\mathbf{\Gamma} \sqcap \mathbf{\Lambda})(X) := \{A \cap B \mid A \in \mathbf{\Gamma}(X) \wedge B \in \mathbf{\Lambda}(X)\}.$$

Recall that $A \in \mathbf{\Gamma}(2^\omega)$ is **$\mathbf{\Gamma}$-complete** if, for any $B \in \mathbf{\Gamma}(2^\omega)$, we can find $f : 2^\omega \to 2^\omega$ continuous such that $B = f^{-1}(A)$. Intuitively, this means that A is part of the most complex sets in $\mathbf{\Gamma}$. The **Wadge class** associated with A is $\{f^{-1}(A) \mid f \text{ continuous}\}$, which by inclusion of classes defines the **Wadge hierarchy** mentioned in the introduction, which refines the Lavrentieff hierarchy. In particular, the Lavrentieff classes have complete sets. By Theorem 22.10 in [Kec95] and its proof, if $\mathbf{\Gamma} \neq \check{\mathbf{\Gamma}}$, then $A \subseteq 2^\omega$ is $\mathbf{\Gamma}$-complete exactly when $A \in \mathbf{\Gamma} \setminus \check{\mathbf{\Gamma}}$.

We now turn to basic notions of automata theory and formal language theory in theoretical computer science, see for instance [Sta97, PP04]. Let Σ be a finite alphabet. Then $\Sigma^{<\omega}$ is the set of finite words over Σ. If $w := a_1 \cdots a_l \in \Sigma^{<\omega}$, then $|w| = l$ is the **length** of w.

Definition 2.1 *A (finite)* **automaton** *is a 5-tuple $\mathcal{A} := (Q, \Sigma, \delta, q_0, F)$, where Q is a finite set of states, Σ is a finite alphabet, $\delta : Q \times \Sigma \to 2^Q$ is a map, $q_0 \in Q$ is the initial state, and $F \subseteq Q$ is the set of final states.*

Let $w := a_1 \cdots a_l \in \Sigma^{<\omega}$ be a finite word over Σ. A **run of** *$\mathcal{A}$ on σ is a sequence $r := (q_i)_{1 \leq i \leq l+1}$ of states with $q_1 = q_0$ and, for each $1 \leq i \leq l$, $q_{i+1} \in \delta(q_i, a_i)$. The* **language accepted** *by $\mathcal{A}$ is $L(\mathcal{A}) := \{w \in \Sigma^{<\omega} \mid \exists r \in Q^{|w|+1} \;\; r$ is a run of $\mathcal{A}$ on w with $q_{l+1} \in F\}$. A language $L \subseteq \Sigma^{<\omega}$ is* **regular** *if $L = L(\mathcal{A})$ for some automaton $\mathcal{A}$.*

Definition 2.2 *A* **Büchi automaton** *is a 5-tuple $\mathcal{A} := (Q, \Sigma, \delta, q_0, F)$, where Q is a finite set of states, Σ is a finite alphabet, $\delta : Q \times \Sigma \to 2^Q$ is a map, $q_0 \in Q$ is the initial state, and $F \subseteq Q$ is the set of final states.*

Let $\sigma := a_1 a_2 \cdots \in \Sigma^\omega$. A **run of** *$\mathcal{A}$ on σ is a sequence $r := (q_i)_{i \geq 1}$ of states with $q_1 = q_0$ and, for each $i \geq 1$, $q_{i+1} \in \delta(q_i, a_i)$. We set $In(r) := \{q \in Q \mid \exists^\infty i \geq 1 \;\; q_i = q\}$. The ω-***language accepted** *by $\mathcal{A}$ is $\mathcal{L}(\mathcal{A}) := \{\sigma \in \Sigma^\omega \mid \exists r \in Q^\omega \;\; r$ is a run of $\mathcal{A}$ on σ with $In(r) \cap F \neq \emptyset\}$. An ω-language $\mathcal{L} \subseteq \Sigma^\omega$ is ω-***regular** *if $\mathcal{L} = \mathcal{L}(\mathcal{A})$ for some Büchi automaton $\mathcal{A}$.*

The usual **concatenation** of two finite words v and w is denoted $v \cdot w$, sometimes just vw. This concatenation is extended to the concatenation of a finite word w and an ω-word σ. The infinite word $w \cdot \sigma$ is then the ω-word such that $(w \cdot \sigma)(k) = w(k)$ if $k \leq |w|$, and $(w \cdot \sigma)(k) = \sigma(k - |w|)$ if $k > |w|$. The concatenation can be extended in an obvious way to infinite sequences of finite words. The concatenation of a set L of finite words with a set $\mathcal{L}$ of infinite words is the set of infinite words $L \cdot \mathcal{L} := \{w \cdot \sigma \mid w \in L \text{ and } \sigma \in \mathcal{L}\}$. The prefix relation is denoted by $\subseteq$: a finite word v is a **prefix** of a finite word w (respectively, an infinite word σ), denoted $v \subseteq w$, if and only if there exists a finite word w' (respectively, an infinite word σ'), such that $w = v \cdot w'$. The ω-**Kleene closure** of the family of regular languages is the class of ω-languages of the form $\bigcup_{1 \leq j \leq n} K_j \cdot L_j^\infty$, for some regular languages K_j and L_j, $1 \leq j \leq n$. As mentioned in the introduction, the class of ω-regular languages is the ω-Kleene closure of the family of regular languages (see [PP04]).

Theorem 2.3 *(Büchi) Let Σ be a finite alphabet, and $\mathcal{L} \subseteq \Sigma^\omega$ be an ω-language. The following are equivalent:*

(a) $\mathcal{L}$ is ω-regular,

(b) we can find $n \in \omega$ and regular languages $K_j,\ L_j \subseteq \Sigma^{<\omega}$, $1 \leq j \leq n$, with the property that $\mathcal{L} = \bigcup_{1 \leq j \leq n}\ K_j \cdot L_j^\infty$.

3 The proof of the main result

We first give an inductive construction of the difference hierarchy.

Lemma 3.1 *Let $\mathbf{\Gamma}$ be a class of subsets of Polish spaces closed under finite intersections and finite unions, and k be a natural number.*

(a) $\check{D}_k(\mathbf{\Gamma}) \sqcap \mathbf{\Gamma} = D_{k+1}(\mathbf{\Gamma})$.

(b) $\check{D}_{2k}(\mathbf{\Gamma}) \sqcap \check{\mathbf{\Gamma}} = \check{D}_{2k+1}(\mathbf{\Gamma})$.

(c) $\check{D}_{2k}(\mathbf{\Gamma}) \sqcap \check{D}_2(\mathbf{\Gamma}) = \check{D}_{2k+2}(\mathbf{\Gamma})$.

(d) $\big(D_{2k+1}(\mathbf{\Gamma}) \oplus \check{D}_{2k+1}(\mathbf{\Gamma})\big) \sqcap \check{D}_2(\mathbf{\Gamma}) = D_{2k+3}(\mathbf{\Gamma}) \oplus \check{D}_{2k+3}(\mathbf{\Gamma})$.

Remark 3.2 The anonymous reviewer of this paper indicated us that "items (a), (b), (c) of Lemma 3.1 were known for quite some time. Namely, they are particular cases of Proposition 8 in Section 5 of [Sel85] about the difference hierarchy over arbitrary bounded distributive lattice. Moreover, the proof of Proposition 8 is essentially the same as even earlier proof for the particular case of difference hierarchy over the c.e. sets in Section 3, Proposition 2 of [Ers68]".

However, in order to keep the paper as self contained as possible for the reader, and because these results are used later in the sequel, we have kept our proofs of items (a), (b), (c), and give them below.

Proof. Fix a Polish space X.

(a) Let $U := (U_\eta)_{\eta<k}$ be an increasing sequence of subsets of X in $\mathbf{\Gamma}$, $A := X \setminus D_k(U)$, and V be a subset of X in $\mathbf{\Gamma}$. We set, for $\eta < k$, $O_\eta := U_\eta \cap V$, and $O_k := V$. Then $(O_\eta)_{\eta<k+1}$ is an increasing sequence of subsets of X in $\mathbf{\Gamma}$, and

$$\begin{aligned} x \in A \cap V &\Leftrightarrow \Big(x \notin \textstyle\bigcup_{\eta<k}\ U_\eta \vee \exists \eta < k\ \ \big(\text{parity}(\eta) = \text{parity}(k) \wedge x \in U_\eta \setminus (\bigcup_{\theta<\eta}\ U_\theta)\big)\Big) \wedge x \in V \\ &\Leftrightarrow x \in D_{k+1}(O), \end{aligned}$$

proving that $A \cap V \in D_{k+1}(\mathbf{\Gamma})$.

Conversely, assume that $D_{k+1}(O) \in D_{k+1}(\mathbf{\Gamma})$. We set $V := O_k$ and, for $\eta < k$, $U_\eta := O_\eta$, which implies that U is an increasing sequence of subsets of X in $\mathbf{\Gamma}$, $D_k(U) \in D_k(\mathbf{\Gamma})$, V is in $\mathbf{\Gamma}$, and $D_{k+1}(O) = \big(X \setminus D_k(U)\big) \cap V$ by the previous computation.

(b) Let $U := (U_\eta)_{\eta<2k}$ be an increasing sequence of subsets of X in $\mathbf{\Gamma}$, $A := X \setminus D_{2k}(U)$, V be a subset of X in $\mathbf{\Gamma}$, and $B := X \setminus V$. We set $O_0 := V$ and, for $\eta < 2k$, $O_{\eta+1} := V \cup U_\eta$, so that $(O_\eta)_{\eta<2k+1}$ is an increasing sequence of subsets of X in $\mathbf{\Gamma}$, and

$$\begin{aligned} x \in A \cap B &\Leftrightarrow \big(x \notin \textstyle\bigcup_{\eta<2k}\ U_\eta \vee \exists j < k\ \ x \in U_{2j} \setminus (\bigcup_{\theta<2j}\ U_\theta)\big) \wedge x \notin V \\ &\Leftrightarrow x \notin D_{2k+1}(O), \end{aligned}$$

proving that $A \cap B \in \check{D}_{2k+1}(\mathbf{\Gamma})$.

Conversely, assume that $D_{2k+1}(O)\in D_{2k+1}(\mathbf{\Gamma})$. We set $B:=X\setminus O_0$ and, for $\eta<2k$, $U_\eta:=O_{\eta+1}$, which implies that U is an increasing sequence of subsets of X in $\mathbf{\Gamma}$, $D_{2k}(U)\in D_{2k}(\mathbf{\Gamma})$, B is in $\check{\mathbf{\Gamma}}$, and $X\setminus D_{2k+1}(O)=\big(X\setminus D_{2k}(U)\big)\cap B$ by the previous computation.

(c) The key fact is that $(C_1\setminus C_0)\cup(D_1\setminus D_0)=(C_1\cup D_1)\setminus(C_0\cap D_0)$ if $C_1\subseteq D_0$ and $D_1\subseteq C_0$. We may assume that $k>0$.

Let $U:=(U_\eta)_{\eta<2k}$ be an increasing sequence of subsets of X in $\mathbf{\Gamma}$, $A:=X\setminus D_{2k}(U)$, $V:=(V_\eta)_{\eta<2}$ be an increasing sequence of subsets of X in $\mathbf{\Gamma}$, and $B:=X\setminus D_2(V)$. We set $O_0:=V_0\cap U_0$, $O_1:=V_1\cap U_1$, $O_{2j}:=(V_0\cap U_{2j})\cup U_{2j-2}\cup(V_1\cap U_{2j-1})$ and

$$O_{2j+1}:=(V_1\cap U_{2j+1})\cup U_{2j-1}$$

if $0<j<k$, $O_{2k}:=V_0\cup U_{2k-2}\cup(V_1\cap U_{2k-1})$ and $O_{2k+1}:=V_1\cup U_{2k-1}$, so that $(O_\eta)_{\eta<2k+2}$ is an increasing sequence of subsets of X in $\mathbf{\Gamma}$, and

$$\begin{aligned}
x\in A\cap B &\Leftrightarrow \big(x\notin U_{2k-1}\vee\exists j<k\ \ x\in U_{2j}\setminus(\textstyle\bigcup_{\theta<2j}\ U_\theta)\big)\wedge x\in\big(V_0\cup(X\setminus V_1)\big)\\
&\Leftrightarrow x\in V_0\setminus U_{2k-1}\vee\exists j<k\ \ x\in(V_0\cap U_{2j})\setminus(\textstyle\bigcup_{\theta<2j}\ U_\theta)\vee x\notin V_1\cup U_{2k-1}\vee\\
&\qquad\qquad\qquad\qquad\qquad\qquad\exists j<k\ \ x\in U_{2j}\setminus(V_1\cup\textstyle\bigcup_{\theta<2j}\ U_\theta)\\
&\Leftrightarrow x\in V_0\cap U_0\vee\exists j\in(0,k)\ x\in\big((V_0\cap U_{2j})\setminus U_{2j-1}\big)\cup\big(U_{2j-2}\setminus(V_1\cup\textstyle\bigcup_{\theta<2j-2}\ U_\theta)\big)\\
&\qquad\qquad\vee x\in\big(V_0\setminus U_{2k-1}\cup U_{2k-2}\setminus(V_1\cup\textstyle\bigcup_{\theta<2k-2}\ U_\theta)\big)\vee x\notin V_1\cup U_{2k-1}\\
&\Leftrightarrow x\in V_0\cap U_0\vee\exists j\in(0,k)\ \ x\in\big((V_0\cap U_{2j})\cup U_{2j-2}\big)\setminus\big((V_1\cap U_{2j-1})\cup\textstyle\bigcup_{\theta<2j-2}\ U_\theta\big)\\
&\qquad\qquad\vee x\in(V_0\cup U_{2k-2})\setminus\big((V_1\cap U_{2k-1})\cup\textstyle\bigcup_{\theta<2k-2}\ U_\theta\big)\vee x\notin V_1\cup U_{2k-1}\\
&\Leftrightarrow x\notin D_{2k+2}(O),
\end{aligned}$$

proving that $A\cap B\in\check{D}_{2k+2}(\mathbf{\Gamma})$.

Conversely, assume that $D_{2k+2}(O)\in D_{2k+2}(\mathbf{\Gamma})$. We set, for $\eta<2k$, $U_\eta:=O_\eta$, which implies that U is an increasing sequence of subsets of X in $\mathbf{\Gamma}$ and $D_{2k}(U)\in D_{2k}(\mathbf{\Gamma})$. We also set, for $\eta<2$, $V_\eta:=O_{2k+\eta}$, which implies that V is an increasing sequence of subsets of X in $\mathbf{\Gamma}$ and $D_2(V)\in D_2(\mathbf{\Gamma})$. Moreover, $X\setminus D_{2k+2}(O)=\big(X\setminus D_{2k}(U)\big)\cap\big(X\setminus D_2(V)\big)$ by the previous computation.

(d) We first check that $(\mathbf{\Gamma}\oplus\check{\mathbf{\Gamma}})\sqcap\check{D}_{2k}(\mathbf{\Gamma})=D_{2k+1}(\mathbf{\Gamma})\oplus\check{D}_{2k+1}(\mathbf{\Gamma})$. Let C be a clopen subset of X, A be a subset of X in $\mathbf{\Gamma}$, B be a subset of X in $\check{\mathbf{\Gamma}}$, and E be a subset of X in $\check{D}_{2k}(\mathbf{\Gamma})$. Then $\big((A\cap C)\cup(B\setminus C)\big)\cap E=\big((E\cap A)\cap C\big)\cup\big((E\cap B)\setminus C\big)$, showing one inclusion by (a) and (b). Conversely, let D be a subset of X in $D_{2k+1}(\mathbf{\Gamma})$, and F be a subset of X in $\check{D}_{2k+1}(\mathbf{\Gamma})$. By (a), we can find a subset E_0 of X in $\check{D}_{2k}(\mathbf{\Gamma})$ and a subset A of X in $\mathbf{\Gamma}$ with $D=E_0\cap A$. By (b), we can find a subset E_1 of X in $\check{D}_{2k}(\mathbf{\Gamma})$ and a subset B of X in $\check{\mathbf{\Gamma}}$ with $F=E_1\cap B$. Note that $(D\cap C)\cup(F\setminus C)=\big((E_0\cap A)\cap C\big)\cup\big((E_1\cap B)\setminus C\big)=\big((A\cap C)\cup(B\setminus C)\big)\cap\big((E_0\cap C)\cup(E_1\setminus C)\big)$, showing the other inclusion. Indeed, $(E_0\cap C)\cup(E_1\setminus C)\in\check{D}_{2k}(\mathbf{\Gamma})$ is true if $k\in 2$, and for $k>2$ by induction, using (c) and this formula again.

From this and (c) we deduce that

$$\begin{aligned}
\big(D_{2k+1}(\mathbf{\Gamma})\oplus\check{D}_{2k+1}(\mathbf{\Gamma})\big)\sqcap\check{D}_2(\mathbf{\Gamma}) &= (\mathbf{\Gamma}\oplus\check{\mathbf{\Gamma}})\sqcap\check{D}_{2k}(\mathbf{\Gamma})\sqcap\check{D}_2(\mathbf{\Gamma})\\
&= (\mathbf{\Gamma}\oplus\check{\mathbf{\Gamma}})\sqcap\check{D}_{2k+2}(\mathbf{\Gamma})=D_{2k+3}(\mathbf{\Gamma})\oplus\check{D}_{2k+3}(\mathbf{\Gamma}),
\end{aligned}$$

finishing the proof. □

Examples. Here are three fundamental examples.

- If $L:=\{w\in 2^{<\omega}\mid 0\subseteq w\vee\exists p\in\omega\ \ 10^p1\subseteq w\}$, then $L^\infty=2^\omega\setminus\{10^\infty\}$ is $D_1(\mathbf{\Sigma}^0_1)$-complete.

- If $L:=\{0\}$, then $L^\infty=\{0^\infty\}$ is $\check{D}_1(\mathbf{\Sigma}^0_1)$-complete.

- If $L:=\{w\in 2^{<\omega}\mid w\subseteq 0^\infty\vee\exists p,q\in\omega\ \ 0^p10^q1\subseteq w\}$, then

$$L^\infty=\{\alpha\in 2^\omega\mid \alpha=0^\infty\vee\exists p\neq q\ \ \alpha(p)=\alpha(q)=1\}$$

is $\check{D}_2(\mathbf{\Sigma}^0_1)$-complete. Indeed, it is enough to see that L^∞ is not $D_2(\mathbf{\Sigma}^0_1)$. In order to see that, we argue by contradiction, which gives O open and C closed with $L^\infty=O\cap C$. Then C must be 2^ω, and thus L^∞ is open. The sequence $(0^n10^\infty)_{n\in\omega}$ gives the desired contradiction.

Notation. We set, for $\alpha\in 2^\omega$ and $\varepsilon\in 2$, $(\alpha)_\varepsilon:=(\alpha(\varepsilon),\alpha(\varepsilon+2),\alpha(\varepsilon+4),\cdots)$. Similarly, if $w\in 2^{<\omega}$ has even length $2l$ and $\varepsilon\in 2$, then we set $(w)_\varepsilon:=\big(w(\varepsilon),w(\varepsilon+2),w(\varepsilon+4),\cdots,w(\varepsilon+2l-2)\big)$. We set, for $L\subseteq 2^{<\omega}$, $L^*:=\{w_1\cdots w_l\mid l\in\omega\wedge\forall i<l\ \ w_{i+1}\in L\}$, and

- $L_0:=\big\{w\in 2^{<\omega}\mid |w|$ is even $\wedge\,(w)_0\in L^*\wedge(0\subseteq(w)_1\vee\exists q\in\omega\ \ 10^q1\subseteq(w)_1)\big\}$,

- $L_1:=\big\{w\in 2^{<\omega}\mid |w|$ is even $\wedge\,(w)_0\in L^*\wedge(w)_1\subseteq 0^\infty\big\}$,

- $L_2:=\big\{w\in 2^{<\omega}\mid |w|$ is even $\wedge\,(w)_0\in L^*\wedge\big((w)_1\subseteq 0^\infty\vee\exists p,q\in\omega\ \ 0^p10^q1\subseteq(w)_1\big)\big\}$.

Lemma 3.3 *Let $L\subseteq 2^{<\omega}$ be a regular language. Then L_0, L_1 and L_2 are also regular.*

Proof. Recall first that the class of regular finitary languages over an alphabet $\Sigma=\{a_1,a_2,\cdots,a_n\}$ is the closure of the class containing the emptyset and the singletons $\{a_i\}$ consisting of a single word of length 1 (we identify here the letter a_i with the word of length 1 containing this single letter), under the operations of union, concatenation, and the star operation $L\mapsto L^*$ over finitary languages. Then the class of finitary regular languages is also closed under intersection (and taking complements). These properties imply that if $L\subseteq 2^{<\omega}$ is a regular language, then L_0, L_1 and L_2 are also regular. $\square$

The next lemma shows that the completeness can be propagated in the difference hierarchy.

Lemma 3.4 *Let k be a natural number.*

(a) If there is $L\subseteq 2^{<\omega}$ such that L^∞ is $\check{D}_k(\mathbf{\Sigma}^0_1)$-complete, then L_0^∞ is $D_{k+1}(\mathbf{\Sigma}^0_1)$-complete.

(b) If there is $L\subseteq 2^{<\omega}$ such that L^∞ is $\check{D}_{2k}(\mathbf{\Sigma}^0_1)$-complete, then L_1^∞ is $\check{D}_{2k+1}(\mathbf{\Sigma}^0_1)$-complete.

(c) If there is $L\subseteq 2^{<\omega}$ such that L^∞ is $\check{D}_{2k}(\mathbf{\Sigma}^0_1)$-complete, then L_2^∞ is $\check{D}_{2k+2}(\mathbf{\Sigma}^0_1)$-complete.

(d) If there is $L\subseteq 2^{<\omega}$ such that L^∞ is complete for the class $D_{2k+1}(\mathbf{\Sigma}^0_1)\oplus\check{D}_{2k+1}(\mathbf{\Sigma}^0_1)$, then L_2^∞ is complete for the class $D_{2k+3}(\mathbf{\Sigma}^0_1)\oplus\check{D}_{2k+3}(\mathbf{\Sigma}^0_1)$.

Proof. (a) Let us check that $L_0^\infty=\{\alpha\in 2^\omega\mid(\alpha)_0\in L^\infty\wedge(\alpha)_1\neq 10^\infty\}$. Assume that $\alpha\in L_0^\infty$, which gives a sequence $(w_i)_{i\in\omega}$ of nonempty words in L_0 with the property that $\alpha=w_0w_1\cdots$. Note that $(\alpha)_\varepsilon=(w_0)_\varepsilon(w_1)_\varepsilon\cdots$ if $\varepsilon\in 2$ since the $|w_i|$'s are even. This implies that $(\alpha)_0\in L^\infty$ and $(\alpha)_1\neq 10^\infty$. Conversely, assume that these two properties hold. If $(\alpha)_1$ has finitely many 1's, then we choose an initial segment of $(\alpha)_0$ in L^* starting a decomposition of $(\alpha)_0$ into words of L of length l large enough to ensure that $(\alpha)_1|l$ contains all the 1's in $(\alpha)_1$, and either $(\alpha)_1(0)=0$, or we can find p,q with $0^p10^q1\subseteq(\alpha)_1|l$.

We set $(w_0)_\varepsilon := (\alpha)_\varepsilon|l$ for each $\varepsilon \in 2$, so that $w_0 \in L_0$. We then consider the rest of this decomposition of $(\alpha)_0$ into words of L, which gives $(w_{i+1})_0$. Setting $(w_{i+1})_1 := 0^{|(w_{i+1})_0|}$, we get $w_{i+1} \in L_0$ and $\alpha = w_0 w_1 \cdots$. If $(\alpha)_1$ has infinitely many 1's, then we construct $w_i \in L_0$, ensuring that $(w_i)_0 \in L^*$ is long enough and $0^p 1 0^q 1 \subseteq (w_i)_1$ for some $p, q \in \omega$. This proves that $\alpha \in L_0^\infty$. By Lemma 3.1.(a), $L_0^\infty \in D_{k+1}(\mathbf{\Sigma}^0_1)$. Assume now that D is a $D_{k+1}(\mathbf{\Sigma}^0_1)$ subset of 2^ω. Lemma 3.1.(a) provides $C \in \check{D}_k(\mathbf{\Sigma}^0_1)$ and $O \in \mathbf{\Sigma}^0_1$ such that $D = C \cap O$. Let $f_0 : 2^\omega \to 2^\omega$ be a continuous map with $C = f_0^{-1}(L^\infty)$, and $f_1 : 2^\omega \to 2^\omega$ be a continuous map with $O = f_1^{-1}(2^\omega \setminus \{10^\infty\})$. Then the map $f : 2^\omega \to 2^\omega$ defined by $\big(f(x)\big)_\varepsilon := f_\varepsilon(x)$ is continuous and satisfies $D = f^{-1}(L_0^\infty)$, showing the completeness of L_0^∞.

(b) Let us check that $L_1^\infty = \{\alpha \in 2^\omega \mid (\alpha)_0 \in L^\infty \wedge (\alpha)_1 = 0^\infty\}$. Assume that $\alpha \in L_1^\infty$, which gives a sequence $(w_i)_{i\in\omega}$ of nonempty words in L_1 with $\alpha = w_0 w_1 \cdots$. Then $(\alpha)_0 \in L^\infty$ and $(\alpha)_1 = 0^\infty$. Conversely, assume that these two properties hold. We set $(w_0)_0 := (\alpha)_0|l_0$, where $l_0 > 0$ and $(w_0)_0 \in L$ starts a decomposition of $(\alpha)_0$ into words of L. We set $(w_0)_1 := 0^{l_0}$, so that $w_0 \in L_1$. We then continue this decomposition of $(\alpha)_0$ into words of L, which gives $(w_{i+1})_0$. We then set $(w_{i+1})_1 := 0^{|(w_{i+1})_0|}$, we get $w_{i+1} \in L_1$ and $\alpha = w_0 w_1 \cdots$, proving that $\alpha \in L_1^\infty$. By Lemma 3.1.(b), $L_1^\infty \in \check{D}_{2k+1}(\mathbf{\Sigma}^0_1)$. Assume now that C is a $\check{D}_{2k+1}(\mathbf{\Sigma}^0_1)$ subset of 2^ω. Lemma 3.1.(b) provides $M \in \check{D}_{2k}(\mathbf{\Sigma}^0_1)$ and $P \in \mathbf{\Pi}^0_1$ such that $C = M \cap P$. Let $f_0 : 2^\omega \to 2^\omega$ be a continuous map with the property that $M = f_0^{-1}(L^\infty)$, and $f_1 : 2^\omega \to 2^\omega$ be a continuous map with $P = f_1^{-1}(\{0^\infty\})$. Then the map $f : 2^\omega \to 2^\omega$ defined by $\big(f(x)\big)_\varepsilon := f_\varepsilon(x)$ is continuous and satisfies $C = f^{-1}(L_1^\infty)$, showing the completeness of L_1^∞.

(c) Let us check that $L_2^\infty = \big\{\alpha \in 2^\omega \mid (\alpha)_0 \in L^\infty \wedge \big((\alpha)_1 = 0^\infty \vee \exists p \neq q \;\; (\alpha)_1(p) = (\alpha)_1(q) = 1\big)\big\}$. Assume that $\alpha \in L_2^\infty$, which gives a sequence $(w_i)_{i\in\omega}$ of nonempty words in L_2 with $\alpha = w_0 w_1 \cdots$. As $(\alpha)_\varepsilon = (w_0)_\varepsilon (w_1)_\varepsilon \cdots$ if $\varepsilon \in 2$, $(\alpha)_0 \in L^\infty$, and $\big((\alpha)_1 = 0^\infty$ or $\exists p \neq q \;\; (\alpha)_1(p) = (\alpha)_1(q) = 1\big)$. Conversely, assume that these two properties hold. If $(\alpha)_1$ has finitely many 1's, then as in (a) we choose l in such a way that $(\alpha)_1|l$ contains all the 1's in $(\alpha)_1$, and either $(w_0)_1 \subseteq 0^\infty$, or $0^p 1 0^q 1$ is a prefix of $(w_0)_1$ for some $p, q \in \omega$. We conclude as in (a). If $(\alpha)_1$ has infinitely many 1's, then we argue as in (a) to see that $\alpha \in L_2^\infty$. By Lemma 3.1.(c), $L_2^\infty \in \check{D}_{2k+2}(\mathbf{\Sigma}^0_1)$. We set

$$O_0 := \{\alpha \in 2^\omega \mid \exists p \neq q \;\; \alpha(p) = \alpha(q) = 1\},$$

and $O_1 := \{\alpha \in 2^\omega \mid \alpha \neq 0^\infty\}$, so that $2^\omega \setminus D_2(O)$ is $\check{D}_2(\mathbf{\Sigma}^0_1)$-complete. Assume now that C is a $\check{D}_{2k+2}(\mathbf{\Sigma}^0_1)$ subset of 2^ω. Lemma 3.1.(c) provides $M \in \check{D}_{2k}(\mathbf{\Sigma}^0_1)$ and $S \in \check{D}_2(\mathbf{\Sigma}^0_1)$ such that $C = M \cap S$. Let $f_0 : 2^\omega \to 2^\omega$ be a continuous map with $M = f_0^{-1}(L^\infty)$, and $f_1 : 2^\omega \to 2^\omega$ be a continuous map with $S = f_1^{-1}\big(2^\omega \setminus D_2(O)\big)$. Then the map $f : 2^\omega \to 2^\omega$ defined by $\big(f(x)\big)_\varepsilon := f_\varepsilon(x)$ is continuous and satisfies $C = f^{-1}(L_2^\infty)$, showing the completeness of L_2^∞.

(d) The proof of (c) shows that

$$L_2^\infty = \big\{\alpha \in 2^\omega \mid (\alpha)_0 \in L^\infty \wedge \big((\alpha)_1 = 0^\infty \vee \exists p \neq q \;\; (\alpha)_1(p) = (\alpha)_1(q) = 1\big)\big\}.$$

By Lemma 3.1.(d), $L_2^\infty \in D_{2k+3}(\mathbf{\Sigma}^0_1) \oplus \check{D}_{2k+3}(\mathbf{\Sigma}^0_1)$. Assume now that C is a subset of 2^ω in $D_{2k+3}(\mathbf{\Sigma}^0_1) \oplus \check{D}_{2k+3}(\mathbf{\Sigma}^0_1)$. Lemma 3.1.(d) provides a subset M of 2^ω in $\big(D_{2k+1}(\mathbf{\Sigma}^0_1) \oplus \check{D}_{2k+1}(\mathbf{\Sigma}^0_1)\big)$, and $S \in \check{D}_2(\mathbf{\Sigma}^0_1)(2^\omega)$ such that $C = M \cap S$. We conclude as in (c). $\square$

Proof of the main result. (a) We argue by induction on n. For the class $D_0(\mathbf{\Sigma}_1^0)$, we can take $L := \emptyset$, so that $L^\infty = \emptyset$. For $\check{D}_0(\mathbf{\Sigma}_1^0)$, we can take $L := 2^{<\omega}$, so that $L^\infty = 2^\omega$. Then, using Lemma 3.3 and inductively, Lemma 3.4.(c) solves our problem for $\check{D}_{2k+2}(\mathbf{\Sigma}_1^0)$. Then Lemma 3.4.(a) solves our problem for $D_{2k+1}(\mathbf{\Sigma}_1^0)$, while Lemma 3.4.(b) solves our problem for $\check{D}_{2k+1}(\mathbf{\Sigma}_1^0)$. Now Lemma 3.4.(a) solves our problem for $D_{2k+2}(\mathbf{\Sigma}_1^0)$.

(b) Note that $D_0(\mathbf{\Sigma}_1^0) \oplus \check{D}_0(\mathbf{\Sigma}_1^0) = \mathbf{\Delta}_1^0$. For $\mathbf{\Delta}_1^0$, we can take $L := \{w \in 2^{<\omega} \mid 0 \subseteq w \vee 1^2 \subseteq w\}$, so that $L^\infty = N_0 \cup N_{1^2}$. Note then that $D_1(\mathbf{\Sigma}_1^0) \oplus \check{D}_1(\mathbf{\Sigma}_1^0) = \mathbf{\Sigma}_1^0 \oplus \mathbf{\Pi}_1^0$. For $\mathbf{\Sigma}_1^0 \oplus \mathbf{\Pi}_1^0$, we can take $L := \{0^2, 0^2 1\} \cup \{w \in 2^{<\omega} \mid \exists p \in \omega \;\; 10^p 1 \subseteq w\}$, so that $L^\infty = \{0^\infty\} \cap (\bigcup_{q\in\omega} N_{0^{2q+2}1}) \cup (N_1 \setminus \{10^\infty\})$. Then, inductively, Lemmas 3.3 and 3.4.(d) solve our problem for $D_{2n+3}(\mathbf{\Sigma}_1^0) \oplus \check{D}_{2n+3}(\mathbf{\Sigma}_1^0)$. $\square$

4 Concluding remarks

We proved that, for any natural number n, we can find a language $L \subseteq 2^{<\omega}$ such that L^∞ is complete for the class $D_n(\mathbf{\Sigma}_1^0)$, and another one for $\check{D}_n(\mathbf{\Sigma}_1^0)$. On the other hand the hierarchy of differences of open sets can be extended to transfinite ranks indexed by countable ordinals, and it is known that the class $\mathbf{\Delta}_2^0$ is actually the union of the classes $D_\xi(\mathbf{\Sigma}_1^0)$, where ξ is a countable ordinal. This naturally leads to the question of the existence of ω-powers located at an infinite level of the hierarchy of differences of open sets.

In the case of $\mathbf{\Delta}_2^0$ regular ω-powers, Wagner's study in [Wag79] shows that they are all located inside the hierarchy of finite differences of open sets. So, in order to determine which Wadge classes of $\mathbf{\Delta}_2^0$ sets $\mathbf{\Gamma}$ have the property that we can find $L \subseteq 2^{<\omega}$ regular such that L^∞ is complete for $\mathbf{\Gamma}$, it remains to solve the question for the classes of the form $D_{2n+2}(\mathbf{\Sigma}_1^0) \oplus \check{D}_{2n+2}(\mathbf{\Sigma}_1^0)$, for $n \in \omega$.

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