Restricted OE from a metric viewpoint

I) ($\exists^+$) sizes for actions by isometries

M: Polish space, d compatible (not nec. complete!)
G: Polish group acting on (M,d) continuously by isometries on the right. $(M,d) \curvearrowleft G$, $\alpha, \beta \in M$, $\alpha \cdot g$

$\rightsquigarrow$ Goal: associate equivalence relations to this act°

eg: - The equivalence relat° $R_{M \curvearrowleft G}$: $(\alpha,\beta) \in R_{M\curvearrowleft G}$ if $\exists g \in G \mid \alpha g = \beta$

- The equivalence relat° "being approximately in the same G-orbit"

$\hookrightarrow$ Define a pseudometric $\tilde{d}$ on M:

$$\tilde{d}(\alpha,\beta) := \inf_{g,h\in G} d(\alpha g, \beta h) = \inf_{g,h\in G} d(\alpha g h^{-1}, \beta)$$
$$= \underline{\inf_{g\in G} d(\alpha g, \beta)}$$

$\rightsquigarrow$ say α, β are (d) approximately in the same G-orbit if $\tilde{d}(\alpha,\beta) = 0$

Note that $\tilde{d}(\alpha,\beta) = 0 \iff \overline{\alpha G} = \overline{\beta \cdot G}$

define $M /\!/ G = \{ \overline{\alpha G} : \alpha \in M \}$

Then $(M/\!/G, \tilde{d})$ is a metric space

Fact: If d is complete, so is $\tilde{d}$

Proof: Take (α_n) $\tilde{d}$-Cauchy
Up to taking a subsequence, $\tilde{d}(\alpha_n, \alpha_{n+1}) < 2^{-n}$
$\rightarrow \forall n$, up to replacing α_{n+1} by some $\alpha_{n+1} g_n$
we actually have $d(\alpha_n, \alpha_{n+1}) < 2^{-n}$
$\rightsquigarrow (\alpha_n)$ is d-Cauchy
$\alpha_n \to \alpha$ $\square$

Rmk: Melleray has shown $\exists (M,d) \curvearrowleft G$ such that $R_{M\curvearrowleft G}$ is universal.

Sizes: $(M,d) \curvearrowleft G$ by isometries continuous
For every $\alpha \in M$, we get a continuous right-inv pseudometric d^0_α on G
given by $d^0_\alpha(g,h) = d(\alpha g, \alpha h)$
$d^0_\alpha(g_1,g_2) = d^0_\alpha(g_1 h, g_2 h)$ $\forall h \in G$

every right pseudo metric δ on G is determined by the associated pseudonorm
$$\rho : G \to \mathbb{R}^+$$
$$g \mapsto \delta(1_G, g)$$
indeed $\delta(g,h) = \delta(1_G, hg^{-1}) = \rho(hg^{-1})$

Def A pseudonorm on a group G is a funct° $\rho : G \to \mathbb{R}^+$
st: (1) $\rho(1_G) = 0$
(2) $\rho(g) = \rho(g^{-1})$ $\forall g \in G$
(3) $\rho(gh) \leq \rho(g) + \rho(h)$ $\forall g,h \in G$

Going back to d^0_α, the pseudonorm is $\rho^0_\alpha(g) = d^0_\alpha(1_G, g)$
$= d(\alpha, \alpha\cdot g)$

We have an equivariance conditi°: • $d^0_{\alpha\cdot g}(h_1, h_2) = d(\alpha\cdot g h_1, \alpha\cdot g h_2)$

$$= d^0_\alpha(g h_1, g h_2)$$

• $\rho^0_{\alpha\cdot g}(h) = d^0_{\alpha\cdot g}(h, 1_G)$

$$= d^0_\alpha(gh, g)$$

$$= d^0_\alpha(ghg^{-1}, 1_G)$$

$$= \rho^0_\alpha(ghg^{-1})$$

$\boxed{\text{Def}}$: A size for $(M,d) \curvearrowleft G$ is a family $(\rho_\alpha)_{\alpha\in M}$ of continuous pseudonorms on G which:

• each ρ_α refine ρ^0_α : $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall g \in G$, if $\rho_\alpha(g) < \delta$ then $\rho^0_\alpha(g) < \varepsilon$.

(equiv: $(G, d_\alpha) \to (G, d^0_\alpha)$ is uniformly continuous) — d_α: right pseudo metric associated to ρ_α

$d_\alpha(g,h) < \delta \Leftrightarrow \rho_\alpha(gh^{-1}) < \delta \ldots$

• Cauchy-equivariance ($\forall \alpha, \beta \in M$): Whenever $(g_i)_{i\in\mathbb{N}}$ is d_α-Cauchy and $\alpha\cdot g_i \longrightarrow \beta$ then $\forall h \in G$, $\rho_\beta(h) = \lim_{i\to\infty} \rho_\alpha(g_i h g_i^{-1})$

(Equivalently : (1) $\alpha \mapsto \rho_\alpha$ is equivariant : $\rho_{\alpha\cdot g}(h) = \rho_\alpha(ghg^{-1})$ ($g_i = g$ constant sequence)

(2) $\forall (g_i)$ d_α-Cauchy & $\alpha\cdot g_i \to \beta$, $\forall h \in G$, $\rho_\beta(h) = \lim_{i\to\infty} \rho_{\alpha\cdot g_i}(h)$)

Def: $\rho = (\rho_\alpha)$ size for $(M,d) \curvearrowleft G$

Say that $\alpha, \beta \in M$ are ρ-approximately in the same G-orbit if $\exists (g_i)$ d_α-Cauchy st $\alpha\cdot g_i \longrightarrow \beta$

We will see this defines an eq rel on M.

Ex: • ρ^0_α is a size : $\|(g_i)$ is d^0_α-Cauchy iff (αg_i) is Cauchy.$\|$

so if $\alpha\cdot g_i \to \beta$ in particular it is d^0_α-Cauchy

$$\rho_{\alpha\cdot g_i}(g) = d(\alpha\cdot g_i g, \alpha g_i) \longrightarrow d(\beta g, \beta) = \rho_\beta(g)$$

by compatibility of d with the topo

Obs: α, β are ρ^0-approx in the same G-orbit iff they are (d)-approx in the same G orbit.

Proof: $\Leftarrow$ if α, β are approx in same G-orbit

$\exists g_i$ st $\alpha\cdot g_i \to \beta$

so (g_i) is d^0_α-Cauchy

$\Rightarrow$ Clear □

• Suppose G admits a complete bi-inv metric δ

call ρ^1 the associated norm

$$\rho^1(ghg^{-1}) = \rho^1(h)$$

Obs: • ρ^1 is a size ($\rho^1_\alpha = \rho^1$)

if (g_i) is d^1_α-Cauchy then $g_i \to g \in G$

in particular if $\beta = \alpha\cdot g$, we have $\alpha\cdot g_i \to \beta$

$\rho^1_\beta(h) = \rho^1_{\alpha\cdot g_i}(h) \ldots$

• α, β are ρ^1-approx in the same G-orbit iff they are in the same G-orbit.

NB: In this setup, note that $G \times M \curvearrowleft G$ with metric

$(h,\alpha)\cdot g = (hg, \alpha\cdot g)$ $\quad d_{G\times M}((g_1,\alpha_1),(g_2,\alpha_2)) = \delta(g_1,g_2) + d(\alpha_1,\alpha_2)$

orbits become closed, $G \times M \hookleftarrow M$

(g, α_1) and (g, α_2) are G-appr in same G-orbit
iff (α_1, α_2) are in the same G-orbit

Q°: Given a size ρ for $(M,d) \curvearrowleft G$, can we find a larger metric space $\tilde{M} \curvearrowleft G$
st being ρ-appr in the same G-orbit "comes from" being appr in the same G-orbit

Def: $\rho = (\rho_\alpha)$ size for $(M,d) \curvearrowleft G$
Say that $\alpha, \beta \in M$ are ρ-approximately in the same G-orbit if $\exists (g_i)$ d_α-Cauchy st $\alpha \cdot g_i \to \beta$

Prop: Let $\hat{G}_\alpha$ denote the pseudo metric complet° of (G, d_α)
Let $g_i \to \hat{g} \in \hat{G}_\alpha$, suppose $\alpha g_i \to \beta$
Then "left multiplicat° by $\hat{g}$ defines an surjective isometry $\hat{G}_\beta \to \hat{G}_\alpha$ which is G-equivt"

Pf: Take $h \in G$, then right multiplicat° by h is a ~~surjective~~ isometry $(G, d_\alpha) \to (G, d_\alpha)$
$\to$ it extends to a surjective isometry $\hat{G}_\alpha \to \hat{G}_\alpha$
$$\hat{g} \longmapsto \hat{g} h$$
we have $d_\alpha(g_i h, g_i h') \to d_\beta(h, h')$

by def, $\lim_{i \to \infty} d_\alpha(g_i h, g_i h') = \hat{d}_\alpha(\hat{g} h, \hat{g} h')$

so $\forall h, h' \in G$ $\boxed{\hat{d}_\alpha(\hat{g} h, \hat{g} h') = d_\beta(h, h')}$

left mult by $\hat{g}$ is an isometry $(G, d_\beta) \to \hat{G}_\alpha$
it extends to the complet° $\hat{G}_\beta \to \hat{G}_\alpha$
+ is G-equivt & since all G orbits are dense, it is surjective □

Denote by $\hat{M}$ the complet° of (M, d). We have $P_\alpha : (\hat{G}_\alpha, \hat{d}_\alpha) \to (\hat{M}, \hat{d})$
$(g \in G \to \alpha \cdot g)$
is uniformly continuous
because d_α refines d°_α

and similarly $P_\beta : \hat{G}_\beta \to \hat{M}$

In the context of above prop, the following commutes:

$$\begin{tikzcd}
\hat{G}_\beta \arrow[rr, "\hat{g}\cdot"] \arrow[dr, "P_\beta"'] & & \hat{G}_\alpha \arrow[dl, "P_\alpha"] \\
& \hat{M} &
\end{tikzcd}$$

($h \in G$ $\quad P_\beta(h) = \beta h$
$P_\alpha(\hat{g} \cdot h) = P_\alpha(\hat{g}) h = \beta h = P_\beta(h)$)

Thm: $\alpha, \beta \in M$ are ρ-approx in the same orbit iff $\exists \pi : \hat{G}_\beta \to \hat{G}_\alpha$ G-equivt
such that $P_\alpha \circ \pi = P_\beta$ $\quad (*)$

Proof: $\Rightarrow$ $g_i \alpha \to \beta$, g_i is d_α-Cauchy $\rightsquigarrow$ $\hat{g} = \lim g_i \in \hat{G}_\alpha$
by what we just did, π = left mult by $\hat{g}$ is as wanted.

$\Leftarrow$ Define $\hat{g} = \pi(1)$. Take $g_i \to \hat{g}$, $g_i \in G$ (in $\hat{G}_\alpha$)
$$\beta = P_\beta(1) \underset{\uparrow \text{by commutativity}}{=} P_\alpha \pi(1) = P_\alpha(\hat{g}) = \lim_i \alpha \cdot g_i$$
□

Cor: Being ρ-approx in the same G-orbit is an equivalence relat°.
Proof: $(*)$ defines an equivalence relat° □

II / Restricted orbit equivalence for pmp act° (of amenable groups)

$\mathcal{R}$: the hyperfinite ergodic pmp equivalence relat° on (X,μ)

Full group of $\mathcal{R}$: $[\mathcal{R}] := \{T \in \mathrm{Aut}(X,\mu) : (x, T(x)) \in \mathcal{R} \ \forall x \in X\}$

endow $[\mathcal{R}]$ with the uniform metric: $d_u(T_1, T_2) = \mu(\{x \in X : T_1(x) \neq T_2(x)\})$

d_u is biinvariant because elts of $[\mathcal{R}]$ preserve μ.

Fact: d_u is complete.

Fix a countable ∞ amenable group Γ.

An arrangement is a group homomorphism $\alpha : \Gamma \to [\mathcal{R}]$
s.t. (1) α is a free act° ($\forall \gamma \in \Gamma \setminus \{1\}$, $\mu(\{x \in X : \alpha(\gamma)x = x\}) = 0$)
(2) $\mathcal{R} = \mathcal{R}_\alpha$ ($:= \{(x, \alpha(\gamma)x) : x \in X, \gamma \in \Gamma\}$)

Denote by $\mathcal{A}$ the set of all arrangements.

then $\mathcal{A} \subseteq [\mathcal{R}]^\Gamma$

Endow $[\mathcal{R}]^\Gamma$ with a metric δ_u given by: fixing an enumeration $\Gamma = \{\gamma_n : n \in \mathbb{N}\}$ once and for all

$$\delta_u((T_\gamma), (U_\gamma)) = \sum \frac{1}{2^n} d_u(T_{\gamma_n}, U_{\gamma_n})$$

then $[\mathcal{R}]^\Gamma \curvearrowleft [\mathcal{R}]$ by conjugacy $(T_\gamma) \cdot T := (T^{-1} T_\gamma T)$

This act° is by isometries, and $\mathcal{A}$ is $[\mathcal{R}]$-invariant.

For restricted OE, $(\mathcal{A}, \delta_u) \curvearrowleft [\mathcal{R}]$ is the $(M,d) \curvearrowleft G$ that we will always consider.

⚠ $\delta_u \restriction \mathcal{A}$ is not complete. In [K-R] an uniformly equiv metric called L^1 metric is defined and claimed to be complete, but it is not! (Lemma 2.1.7)

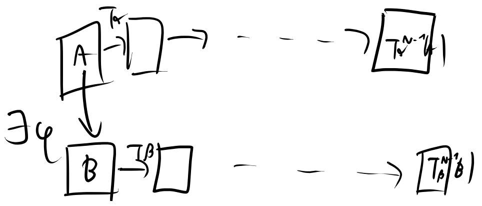
Thm || All arrangements are δ_u-approximately in the same $[\mathcal{R}]$-orbit.
|| Actually all free Γ-act° $\alpha : \Gamma \to [\mathcal{R}]$ are δ_u-approx in the same $[\mathcal{R}]$-orbit.

Proof for $\Gamma = \mathbb{Z}$: Take α, β free $\mathbb{Z}$-act°

Define $T_\alpha := \alpha(1)$
$T_\beta = \beta(1)$

By Rokhlin's lemma, given $\varepsilon > 0$, if we take N st $\frac{1}{N} < \varepsilon$
$\exists A, B \subseteq X$ of equal measure

$\mu(X \setminus (A \sqcup T_\alpha A \cdots \sqcup T_\alpha^{N-1}(A))) < \varepsilon$
$\mu(X \setminus (B \sqcup T_\beta B \cdots \sqcup T_\beta^{N-1}(B))) < \varepsilon$



$$T(x) = \begin{cases} T_\beta^i \varphi T_\alpha^{-i}(x) & \text{if } x \in T_\alpha^i(A), \ 0 \leq i \leq N-1 \\ \psi(x) & \text{where } \psi(X \setminus A \sqcup \cdots) = (X \setminus B \sqcup \cdots) \end{cases}$$

and $\psi \in [[\mathcal{R}]] = \{\varphi$ part dff bij $\forall x \in \mathrm{dom}\,\varphi$ $(x, \varphi(x)) \in \mathcal{R}\}$

$$d_u(T T_\alpha T^{-1}, T_\beta) < 2\varepsilon$$

$\rightsquigarrow \delta_u(\alpha \cdot T^{-1}, \beta)$ can be made as small as we want ... □

Exo: Prove this when Γ is locally finite. (For Γ amenable in general, need the Ornstein-Weiss quasi-tiling theorem)

Next time : Exhaustive OE fits in this framework (Heicklen).

$$\rho_\alpha(T) = \rho^0_\alpha(T) + \sup\{\mu(B) : \forall x, x' \in B \ \forall n \in \mathbb{N} \ (x,x') \in R_\alpha(\Gamma_n) \iff (T(x), T(x')) \in R_\alpha(\Gamma_n)\}$$

$$\Gamma = \bigcup_n \Gamma_n$$