Invent. math. 106, 275–294 (1991)

Inventiones mathematicae

Weights of exponential sums, intersection cohomology, and Newton polyhedra

J. Denef[1] and F. Loeser[2]

[1] University of Leuven, Department of Mathematics, Celestijnenlaan 200B, 3001 Leuven, Belgium
[2] Université Paris 6, and Ecole Polytechnique, Centre de Mathématiques, F-91128 Palaiseau Cedex, France

Oblatum 27-IV-1990

1 Introduction

(1.1) Throughout this paper k always denotes a finite field $\mathbf{F}_q$ with q elements, and ℓ a prime number not dividing q. The algebraic closure of a field K is denoted by $\bar{K}$. Let $\psi: k \to \mathbb{C}^\times$ be a nontrivial additive character, and $\mathfrak{L}_\psi$ the $\bar{\mathbf{Q}}_\ell$-sheaf on $\mathbf{A}_k^1$ associated to ψ and the Artin-Schreier covering $t^q - t = x$. For a morphism $f: X \to \mathbf{A}_k^1$, with X a scheme of finite type over k, one considers the exponential sum $S(f) = \sum_{x \in X(k)} \psi(f(x))$. By Grothendieck's trace formula we have

$$S(f) = \sum_i (-1)^i \operatorname{Tr}(F, H_c^i(X \otimes \bar{k}, f^* \mathfrak{L}_\psi)),$$

where F denotes the (geometric) Frobenius action. In the present paper we will determine the absolute values of the eigenvalues of this Frobenius action when X is a torus and f is nondegenerate with respect to its Newton polyhedron at infinity. Obviously this implies good bounds on the absolute value of $S(f)$.

(1.2) For any commutative ring A, we denote the n-dimensional A-torus $\operatorname{Spec} A[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ by $\mathbf{T}_A^n$. Let $f: \mathbf{T}_A^n \to \mathbf{A}_A^1$ be an A-morphism, thus we can write f as a Laurent polynomial $f = \sum_{i \in \mathbf{Z}^n} c_i x^i$. The *Newton polyhedron* $\Delta_\infty(f)$ *of f at infinity* is the convex hull in $\mathbf{Q}^n$ of $\{i \in \mathbf{Z}^n \mid c_i \neq 0\} \cup \{0\}$. For any face τ of $\Delta_\infty(f)$ we put $f_\tau = \sum_{i \in \tau} c_i x^i$. Call *f nondegenerate with respect to* $\Delta_\infty(f)$ if for every face τ (of any dimension) of $\Delta_\infty(f)$ that does not contain the origin, the subscheme of $\mathbf{T}_A^n$ defined by

$$\frac{\partial f_\tau}{\partial x_1} = \cdots = \frac{\partial f_\tau}{\partial x_n} = 0$$

is empty. When $\Delta_\infty(f)$ has dimension n, we denote its volume by $\operatorname{Vol}(\Delta_\infty(f))$. The first result of the present paper is the following

(1.3) Theorem. *Suppose that* $f\colon \mathbf{T}_k^n \to \mathbf{A}_k^1$ *is nondegenerate with respect to* $\Delta_\infty(f)$ *and that* $\dim \Delta_\infty(f) = n$. *Then*
(a) $H_c^i(\mathbf{T}_{\bar{k}}^n, f^*\mathfrak{L}_\psi) = 0$ *if* $i \neq n$, *and*
(b) $\dim H_c^n(\mathbf{T}_{\bar{k}}^n, f^*\mathfrak{L}_\psi) = n!\,\mathrm{Vol}(\Delta_\infty(f))$.
If in addition the origin is an interior point of $\Delta_\infty(f)$ *then*
(c) $H_c^n(\mathbf{T}_{\bar{k}}^n, f^*\mathfrak{L}_\psi)$ *is pure of weight n (i.e. all eigenvalues of the Frobenius action have absolute value* $q^{n/2}$*).*

This theorem was proved by Adolphson and Sperber [AS1, AS2] for almost all p, and they conjectured it to be true for all p. In contrast with their work, our proof is purely ℓ-adic and based on toroidal compactification.

(1.4) By Deligne's fundamental results [De] each eigenvalue of the Frobenius action on $H_c^n(\mathbf{T}_{\bar{k}}^n, f^*\mathfrak{L}_\psi)$ has absolute value $q^{w/2}$ with $w \in \mathbf{N}$, $w \leqq n$. In the present paper we use intersection cohomology to determine the number e_w of eigenvalues with absolute value $q^{w/2}$ (counting multiplicities). In other words we determine the polynomial

$$E(\mathbf{T}_k^n, f) = \sum_{w=0}^{n} e_w T^w .$$

This problem was posed by Adolphson and Sperber [AS3] who also treated some special cases. To state our formulas we first need some more notation.

(1.5) To any convex polyhedral cone $\sigma \subset \mathbf{Q}^n$ with vertex (at the origin) one associates a polynomial $\alpha(\sigma)$ in one variable T, which only depends on the combinatorial type of σ, and which can be calculated by a recursion formula, see (6.1). If σ is a simplicial cone (i.e. generated by vectors which are linearly independent), then $\alpha(\sigma) = 1$. The polynomials $\alpha(\sigma)$ were first studied by Stanley [S], and are related to intersection cohomology: the coefficient of T^i in $\alpha(\sigma)$ equals the dimension of the cohomology in degree $i - \dim\sigma$ of the stalk at the origin of the intersection complex on the affine toric variety associated to the cone σ, see (6.2). A special role will be played by the value of $\alpha(\sigma)$ at $T = 1$, which will be denoted by $\alpha(\sigma)\,(1)$.

(1.6) For any convex polyhedral cone $\sigma \subset \mathbf{Q}^n$ with vertex we define poly σ as the intersection of σ with a hyperplane in $\mathbf{Q}^n$, not containing the origin, which intersects each one-dimensional face of σ. Note that the convex polytope poly σ is defined up to combinatorial equivalence. Next, let Δ be any convex polyhedron in $\mathbf{Q}^n$ and τ a face of Δ. We denote by $\mathrm{cone}_\Delta \tau$ the convex polyhedral cone in $\mathbf{Q}^n$ generated by $\Delta - \tau = \{x - y \mid x \in \Delta, y \in \tau\}$. Moreover we define $\mathrm{cone}_\Delta^\circ \tau$ as the convex polyhedral cone with vertex obtained by intersecting $\mathrm{cone}_\Delta \tau$ with a plane through the origin which is complementary to the plane generated by $\tau - \tau$. Note that $\mathrm{cone}_\Delta^\circ \tau$ is only defined up to affine equivalence.

(1.7) Let Δ be a convex polytope in $\mathbf{Q}^n$ with integral vertices. For any face τ of Δ we denote by $\mathrm{Vol}(\tau)$ the volume of τ so normalized that a fundamental domain of the lattice $\mathbf{Z}^n \cap$ (affine space of τ) has unit volume. To the polytope Δ we associate a number

(1.7.1)
$$e(\Delta) := (\dim\Delta)!\,\mathrm{Vol}(\Delta) + \sum_{\substack{\tau \text{ face of } \Delta \\ 0 \in \tau \neq \Delta}} (-1)^{\dim\Delta - \dim\tau} (\dim\tau)!\,\mathrm{Vol}(\tau)\,\alpha(\mathrm{cone}_\Delta^\circ \tau)\,(1)\,,$$

and a polynomial $E(\Delta)$ in one variable T which is inductively defined by

$$(1.7.2) \qquad E(\Delta) := e(\Delta)T^{\dim\Delta} - \sum_{\substack{\tau \text{ face of } \Delta \\ 0 \in \tau \neq \Delta}} (-1)^{\dim\Delta - \dim\tau} E(\tau)\alpha(\operatorname{cone}^\circ_\Delta \tau)\,.$$

The polynomial $E(\Delta)$ has degree $\leqq \dim\Delta$ and its coefficient of $T^{\dim\Delta}$ equals $e(\Delta)$.

Now we can state the second result of this paper:

(1.8) Theorem. *Suppose that* $f: \mathbf{T}^n_k \to \mathbf{A}^1_k$ *is nondegenerate with respect to* $\Delta = \Delta_\infty(f)$, *and that* $\dim\Delta = n$. *Then* $E(\mathbf{T}^n_k, f) = E(\Delta)$ *and* $e_n = e(\Delta)$.

Actually Adolphson and Sperber [AS3] conjectured a different formula for the e_w. Our result implies that their conjecture is true for $n \leqq 4$, but false for $w = n = 5$ and a polyhedron $\Delta_\infty(f)$ all whose proper faces are simplices (cf. (8.4)).

Section 2 contains preliminaries on toric schemes over an arbitrary ring. Theorem 1.3 is proved in Sect. 3, assuming some results on tame ramification which are proved in Sect. 4. Section 5 recalls well known facts on intersection cohomology and introduces the formalism of Poincaré polynomials. The intersection cohomology of a toric variety associated to a convex polytope is calculated in Sect. 6. This can be read independently, after 2.1, 2.4 and Sect. 5. Theorem 1.8 is proved in Sect. 7. Section 8 contains explicit formulas to calculate the e_w in special cases, namely when $\Delta_\infty(f)$ is simple at the origin or when $n \leqq 4$. Moreover Theorem 8.5 shows that the smallest d with $w_d \neq 0$ equals the dimension of the smallest face of $\Delta_\infty(f)$ which contains the origin and gives simple formulas for e_d and e_{d+1} (assuming the hypothesis of 1.8). Finally, in Sect. 9 we treat exponential sums on $\mathbf{T}^r_k \times \mathbf{A}^{n-r}_k$.

2 Toric schemes

(2.1) Let A be a commutative ring. To any convex polyhedral cone σ in $\mathbf{Q}^n$ one associates the *affine toric A-scheme* $X_A(\sigma) := \operatorname{Spec} A[\sigma \cap \mathbf{Z}^n]$. Moreover, to any (finite) fan Σ in $\mathbf{Q}^n$ one associates a *toric A-scheme* $X_A(\Sigma)$, obtained by gluing together the schemes $X_A(\check\sigma)$, $\sigma \in \Sigma$, where $\check\sigma$ denotes the dual of σ. This construction is given in [Da, §5] in case A is a field, but it is obvious how to generalize. The scheme $X_A(\Sigma)$ is smooth over A if and only if the fan Σ is regular, it is proper over A if and only if the union of the cones of Σ equals $\mathbf{Q}^n$, cf. [Da, §5]. To any r-dimensional cone $\sigma \in \Sigma$ one associates an $n-r$ dimensional A-torus $X^\sigma_A(\Sigma) := X_A(\operatorname{cospan}\check\sigma) \subset X_A(\check\sigma) \subset X_A(\Sigma)$, where $\operatorname{cospan}\check\sigma$ denotes the largest linear subspace of $\mathbf{Q}^n$ contained in $\check\sigma$ (cf. [Da, 5.7]). These tori form a partition of $X_A(\Sigma)$. Taking $\sigma = \{0\}$, we see that $X_A(\Sigma)$ contains $\mathbf{T}^n_A$ as an open dense subscheme. The closure of $X^\sigma_A(\Sigma)$ in $X_A(\Sigma)$ will be denoted by $\bar X^\sigma_A(\Sigma)$. Note that $\bar X^\sigma_A(\Sigma)$ is a toric A-scheme and equal to the union of all $X^\gamma_A(\Sigma)$ with $\gamma \in \Sigma$, $\sigma \subset \gamma$. Finally for an arbitrary convex polyhedral cone σ and a face τ of σ we put $X^\tau_A(\sigma) := X_A(\tau - \tau) \subset X_A(\operatorname{cone}_\sigma\tau) \subset X_A(\sigma)$. These tori $X^\tau_A(\sigma)$ form a partition of $X_A(\sigma)$ which agrees with the partition introduced above, cf. [Da, 2.7].

(2.2) Definition. Let Y be a scheme over A, and $y \in Y$. We say that Y is *toroidal over A at y* if Y has an etale neighbourhood which is isomorphic over A to an etale neighbourhood of some point in an affine toric A-scheme $X_A(\sigma)$.

The following lemma is well known when A is a field. The general case is proved in a similar way.

(2.3) Lemma. *Let Z be an effective Cartier divisor of $X_A(\Sigma)$ and $z \in Z$. Let σ be the unique cone of Σ with $z \in X_A^\sigma(\Sigma)$. Suppose that the scheme theoretic intersection $Z \cap X_A^\sigma(\Sigma)$ is smooth over A at z, and not equal to $X_A^\sigma(\Sigma)$ at z. Then Z is toroidal over A at z. If in addition Σ is regular, then Z is smooth over A at z.*

Proof. Notice that $r := \dim \sigma < n$, otherwise $X_A^\sigma(\Sigma) = \operatorname{Spec} A$ and the hypothesis about $Z \cap X_A^\sigma(\Sigma)$ would be false. Consider the open subscheme $X_A(\check{\sigma})$ of $X_A(\Sigma)$. We have

$$\check{\sigma} = (\operatorname{cospan} \check{\sigma}) \oplus \omega,\ X_A(\check{\sigma}) = \mathbf{T}_A^{n-r} \times_A X_A(\omega), \text{ and } X_A^\sigma(\Sigma) = \mathbf{T}_A^{n-r} \times \{0\}\ ,$$

for some convex polyhedral cone ω with vertex. Let g be a regular function on an open neighbourhood U of z in $X_A(\check{\sigma})$ which defines Z, and let $y_1, \ldots, y_{n-r}$ be the standard coordinates on $\mathbf{T}_A^{n-r}$. The hypothesis implies that $\dfrac{\partial g}{\partial y_i}(z) \neq 0$ for some i, say $i = 1$. Then the A-morphism

$$\theta : U \to \mathbf{A}_A^1 \times_A \mathbf{T}_A^{n-r-1} \times_A X_A(\omega) : (y_1, \ldots, y_{n-r}, x) \mapsto (g(y, x), y_2, \ldots, y_{n-r}, x)$$

is etale at z. Hence $Z \cap U$ is etale over $\mathbf{T}_A^{n-r-1} \times_A X_A(\omega)$ at z, which proves the first assertion. If Σ is regular, then ω is generated by part of a basis for $\mathbf{Z}^n$ and $X_A(\omega)$ is smooth over A. □

(2.4) Let Δ be a convex polytope in $\mathbf{Q}^n$ of dimension n. The *first meet locus* $F_\Delta(b)$ of a vector $b \in \mathbf{Q}^n$ is the set of all $x \in \Delta$ for which $b \cdot x$ is minimal. It is a face of Δ. For $\sigma \subset \mathbf{Q}^n$ put $F_\Delta(\sigma) = \bigcap_{b \in \sigma} F_\Delta(b)$. Two vectors of $\mathbf{Q}^n$ are called *equivalent* if they have the same first meet locus. The closures of the associated equivalence classes form a fan in $\mathbf{Q}^n$ which is called the *fan associated to* Δ and denoted by $\Sigma(\Delta)$. Note that there is a $1 - 1$ correspondence between the cones $\sigma \in \Sigma(\Delta)$ and the faces τ of Δ, given by $\sigma \mapsto F_\Delta(\sigma)$. Moreover if $\tau = F_\Delta(\sigma)$, then $\check{\sigma} = \operatorname{cone}_\Delta \tau$. The toric A-scheme $X_A(\Sigma(\Delta))$ associated to the fan $\Sigma(\Delta)$ will often be denoted by $X_A(\Delta)$. Note that $X_A(\Delta)$ is proper over A. For $\sigma \in \Sigma(\Delta)$ we will often write $X_A^\sigma(\Delta)$ instead of $X_A^\sigma(\Sigma(\Delta))$. Moreover for a face τ of Δ we will denote by $X_A^\tau(\Delta)$ the unique $X_A^\sigma(\Delta)$ with $F_\Delta(\sigma) = \tau$.

(2.5) Next we discuss a relative version of a well known construction which goes back to Khovanskii [Kh1]. Let $G = \sum_{i \in \mathbf{Z}^n} a_i x^i \in A[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ be a Laurent polynomial over A. The *Newton polyhedron* $\Delta(G)$ of G is the convex hull in $\mathbf{Q}^n$ of $\{i \in \mathbf{Z}^n \mid a_i \neq 0\}$. For each face τ of $\Delta(G)$ we put $G_\tau = \sum_{i \in \tau} a_i x^i$. Assume that $\dim \Delta(G) = n$, and let Σ be any fan which is a subdivision of $\Sigma(\Delta(G))$. We denote by Y the scheme-theoretic closure in $X_A(\Sigma)$ of the locus of $G = 0$ in $\mathbf{T}_A^n$. Note that $X_A(\Sigma)$ and Y are proper over A.

Let σ be an r-dimensional cone of Σ, put $\tau = F_{\Delta(G)}(\sigma)$, and choose any vertex $P \in \tau$. Note that on the open subscheme $X_A(\check{\sigma})$ of $X_A(\Sigma)$, Y is the locus of $x^{-P} G \in A[\check{\sigma} \cap \mathbf{Z}^n]$. (To see this use the argument in the proof of Proposition 3.2 of [Da].) Moreover $Y \cap X_A^\sigma(\Sigma)$ is the locus of $x^{-P} G_\tau \in A[\operatorname{cospan} \check{\sigma} \cap \mathbf{Z}^n]$. Choose a basis $e_1, \ldots, e_{n-r}$ for the lattice $(\operatorname{cospan} \check{\sigma}) \cap \mathbf{Z}^n$, and put $y_i = x^{e_i}$. Then $y_1, \ldots, y_{n-r}$ are coordinates for $X_A^\sigma(\Sigma) \cong \mathbf{T}_A^{n-r}$, and $x^{-P} G_\tau = \tilde{G}_\tau(y_1, \ldots, y_{n-r})$ for a suitable Laurent polynomial $\tilde{G}_\tau$ over A. Thus $Y \cap X_A^\sigma(\Sigma)$ is the locus of $\tilde{G}_\tau(y_1, \ldots, y_{n-r})$ in $\mathbf{T}_A^{n-r}$.

(2.6) Definition. A Laurent polynomial $G(x_1, \ldots, x_n)$ over A is called 0-*non-degenerate* with respect to $\Delta(G)$ if for any face τ of $\Delta(G)$, including $\tau = \Delta(G)$, the subscheme of $\mathbf{T}_A^n$ defined by $G_\tau = 0$ is smooth over A. (cf. [Kh1, §2]).

The following theorem is due to D. Bernstein, A.G. Kushnirenko and A.G. Khovanskii (when $K = \mathbb{C}$).

(2.7) Theorem. *Let $G(x_1, \ldots, x_n)$ be a Laurent polynomial over a field K which is 0-nondegenerate with respect to $\Delta(G)$. Then*

$$\chi(\mathbf{T}_{\bar{K}}^n \cap G^{-1}(0), \mathbf{Q}_\ell) = (-1)^{n-1} n! \operatorname{Vol}(\Delta(G)),$$

where χ denotes the Euler characteristic with respect to ℓ-adic cohomology ($\ell \neq \operatorname{char} K$).

Proof. The theorem is true when K has characteristic zero, by [BKKh, Kh2], and the comparison theorem between singular and etale cohomology. We may suppose that $\dim \Delta(G) = n$, otherwise after a suitable change of coordinates (on the torus) G depends on less than n variables and the Euler characteristic is zero. Let R be a discrete valuation ring with fraction field L of characteristic zero and residue field K. We can lift G to a Laurent polynomial $\hat{G}$ over R which is 0-nondegenerate with respect to $\Delta(\hat{G}) = \Delta(G)$. This is possible because 0-nondegenerateness is a generic condition in characteristic zero, cf. [Kh1, §2]. Choose a *regular* fan Σ which is a subdivision of $\Sigma(\Delta(\hat{G}))$, cf. [Da, §8]. Let V be the subscheme of $\mathbf{T}_R^n$ defined by $\hat{G} = 0$, and let Y be the closure of V in $X_R(\Sigma)$. Note that Y is proper and smooth over R, by (2.3) with $A = R$ and (2.5). Thus $\chi(Y \otimes_R \bar{L}, \mathbf{Q}_\ell) = \chi(Y \otimes_R \bar{K}, \mathbf{Q}_\ell)$. Hence, using the partition of $X_R(\Sigma)$ into tori and induction on n, we obtain $\chi(V \otimes_R \bar{L}, \mathbf{Q}_\ell) = \chi(V \otimes_R \bar{K}, \mathbf{Q}_\ell)$. This yields the theorem since L has characteristic zero. □

3 Proof of Theorem 1.3

(3.1) Proposition. *Let Y be a scheme of pure dimension n over k and $g: Y \to \mathbf{A}_k^1$ a proper k-morphism. Suppose that g is locally acyclic* (*in the sense of* [SGA $4\frac{1}{2}$, p. 54]) *outside a finite number of points, and that $R^i g_* \mathbf{Q}_\ell$ has tame ramification at infinity for each i. Then*

(3.1.1) $$H_c^i(Y \otimes \bar{k}, g^* \mathfrak{L}_\psi) = 0 \quad \textit{for all } i > n, \textit{ and}$$

(3.1.2) *the natural maps $H_c^i(Y \otimes \bar{k}, g^* \mathfrak{L}_\psi) \to H^i(Y \otimes \bar{k}, g^* \mathfrak{L}_\psi)$ are isomorphisms for all i.*

If in addition Y is smooth over k, then

(3.1.3) $$H_c^i(Y \otimes \bar{k}, g^* \mathfrak{L}_\psi) = 0 \quad \textit{for all } i \neq n, \textit{ and}$$

(3.1.4) $$H_c^n(Y \otimes \bar{k}, g^* \mathfrak{L}_\psi) \textit{ is pure of weight } n.$$

Proof. Let $\bar{\eta}$ be a generic geometric point of $\mathbf{A}_k^1$, and $\bar{s}$ an arbitrary geometric point of $\mathbf{A}_k^1$. Since g is proper we have a long exact sequence [SGA 7, XIII. 2.1.8.9]:

$$\cdots \to H^j(Y_{\bar{s}}, \mathbf{Q}_\ell) \xrightarrow{\theta_j} H^j(Y_{\bar{\eta}}, \mathbf{Q}_\ell) \to \mathbf{H}^j(Y_{\bar{s}}, R\Phi(\mathbf{Q}_\ell)) \to \cdots$$

where $R\Phi(\mathbf{Q}_\ell)$ is the complex of vanishing cycles on $Y_{\bar{s}}$ of the constant sheaf $\mathbf{Q}_\ell$. The cohomology sheaves of this complex vanish at each point where g is locally acyclic. Hence

$$\mathbf{H}^j(Y_{\bar{s}}, R\Phi(\mathbf{Q}_\ell)) \cong \bigoplus_{y \in E} (R^j\Phi(\mathbf{Q}_\ell))_y ,$$

where $E \subset Y_{\bar{s}}$ is finite. Since $R^j\Phi(\mathbf{Q}_\ell) = 0$ when $j \geqq n$, we conclude that θ_j is an isomorphism if $j > n$, and surjective if $j = n$ (cf. [SGA 7, I.4.2 and 4.3]). Hence, for $j > n$, $R^j g_* \mathbf{Q}_\ell$ is locally constant, and for $j = n$ it is an extension of a locally constant sheaf by a punctual sheaf. By our hypothesis on tame ramification, these locally constant sheaves are constant on $\mathbf{A}_{\bar{k}}^1$. Thus

(3.1.5) $\qquad H_c^i(\mathbf{A}_{\bar{k}}^1, (R^j g_* \mathbf{Q}_\ell) \otimes \mathfrak{L}_\psi) = 0 \quad$ if $j > n$, $i \geqq 0$, or if $j = n$, $i > 0$.

The tame ramification at infinity also implies (see [Ka, 4.8.2]), that the natural maps

(3.1.6) $\qquad H_c^i(\mathbf{A}_{\bar{k}}^1, (R^j g_* \mathbf{Q}_\ell) \otimes \mathfrak{L}_\psi) \to H^i(\mathbf{A}_{\bar{k}}^1, (R^j g_* \mathbf{Q}_\ell) \otimes \mathfrak{L}_\psi)$

are isomorphisms for all i, j. Hence (3.1.5) also holds for $i \geq 2$ and all j. Assertion (3.1.1) follows now from the Leray spectral sequence

(3.1.7) $\qquad H_c^i(\mathbf{A}_{\bar{k}}^1, (R^j g_* \mathbf{Q}_\ell) \otimes \mathfrak{L}_\psi) \Rightarrow H_c^{i+j}(Y \otimes \bar{k}, g^* \mathfrak{L}_\psi)$.

Assertion (3.1.2) follows directly from the isomorphisms (3.1.6) and the spectral sequences (3.1.7) and

$$H^i(\mathbf{A}_{\bar{k}}^1, (R^j g_* \mathbf{Q}_\ell) \otimes \mathfrak{L}_\psi) \Rightarrow H^{i+j}(Y \otimes \bar{k}, g^* \mathfrak{L}_\psi) .$$

Finally, when Y is smooth, assertions (3.1.3) and (3.1.4) are implied by (3.1.1), (3.1.2) and Poincaré duality. □

The following proposition is well known, see e.g. [Ka, p. 156]:

(3.2) Proposition. *Let Y be a scheme of finite type over k, and $g\colon Y \to \mathbf{A}_k^1$ a k-morphism. Suppose that $R^i g_! \mathbf{Q}_\ell$ has tame ramification at infinity for each i. Then*

$$\chi_c(Y \otimes \bar{k}, g^* \mathfrak{L}_\psi) = \chi_c(Y \otimes \bar{k}, \mathbf{Q}_\ell) - \chi_c(g^{-1}(\bar{\eta}), \mathbf{Q}_\ell) ,$$

where $\bar{\eta}$ is a generic geometric point of $\mathbf{A}_k^1$.

(3.3) Toroidal compactification

Let $f\colon \mathbf{T}_k^n \to \mathbf{A}_k^1$ be nondegenerate with respect to $\Delta = \Delta_\infty(f)$ and assume that $\dim \Delta = n$. For any fan Σ which is a subdivision of $\Sigma(\Delta)$ we will construct a compactification g_Σ of f as follows: Put $A = k[T]$, where T is one variable. Note that $\mathbf{T}_A^n \cong \mathbf{T}_k^n \times \mathbf{A}_k^1$ is an open dense subscheme of $X_A(\Sigma) \cong X_k(\Sigma) \times \mathbf{A}_k^1$. Put

$G = f - T$, and consider it as a Laurent polynomial over A. Notice that $\Delta(G) = \Delta$. We denote by Y_Σ the scheme-theoretic closure in $X_A(\Sigma)$ of the locus of $G = 0$ in $\mathbf{T}_A^n$, and by g_Σ the proper morphism $g_\Sigma\colon Y_\Sigma \to \mathbf{A}_k^1$ induced by the projection $X_A(\Sigma) \to \operatorname{Spec} A = \mathbf{A}_k^1$. Because the graph of f is the locus of $G = 0$ in $\mathbf{T}_A^n$, we see that $\mathbf{T}_k^n$ is an open subscheme of Y_Σ and that the restriction of g_Σ to $\mathbf{T}_k^n$ is f. Thus g_Σ is a compactification of f. For $\sigma \in \Sigma$ we put $Y_\Sigma^\sigma := Y_\Sigma \cap X_A^\sigma(\Sigma)$ and $\bar{Y}_\Sigma^\sigma := Y_\Sigma \cap \bar{X}_A^\sigma(\Sigma)$. Notice that $\bar{Y}_\Sigma^\sigma$ has pure dimension n-$\dim\sigma$, if it is nonempty.

(3.4) Key Lemma. *Assume the notation of* (3.3). *Suppose for each $\sigma \in \Sigma$ with $0 \in F_\Delta(\sigma)$ that $\sigma \in \Sigma(\Delta)$. Then Y_Σ is toroidal over $k[T]$ at all but a finite number of points. In particular g_Σ is locally acyclic outside a finite number of points.*

Proof. The assertion about the local acyclicity follows directly from the first assertion and [SGA 4 1/2, Th. finitude 2.16, p. 243], because the first assertion implies that, at all but a finite number of points, g_Σ is locally a projection $X \times \mathbf{A}_k^1 \to \mathbf{A}_k^1$, with X a toric k-scheme.

Let $\sigma \in \Sigma$ be arbitrary, and put $r = \dim\sigma$, $\tau = F_\Delta(\sigma)$. We will use notation from (3.3) and (2.5). In view of Lemma 2.3 and the material in (2.5) we have to show that the locus of $\tilde{G}_\tau = 0$ in $\mathbf{T}_{k[T]}^{n-r}$ is smooth over $k[T]$ outside a finite number of points. If $0 \notin \tau$ then $\tilde{G}_\tau = \tilde{f}_\tau$ and the above mentioned locus is smooth over $k[T]$ because f is nondegenerate. Thus suppose that $0 \in \tau$. Then we can take $P = 0$ in (2.5). Hence $\tilde{G}_\tau = \tilde{f}_\tau - T$ and $\tilde{f}_\tau(y) = f_\tau(x)$. Clearly $\tilde{f}_\tau(y_1, \ldots, y_{n-r})$ is nondegenerate with respect to $\Delta_\infty(\tilde{f}_\tau)$. Moreover $\dim \Delta_\infty(\tilde{f}_\tau) = \dim\tau = n - r$ because $\sigma \in \Sigma(\Delta)$. The assertion follows now from Lemma 3.5 below.

(3.5) Lemma. *Let $f\colon \mathbf{T}_K^n \to \mathbf{A}_K^1$ be a K-morphism, where K is any field. Suppose that f is nondegenerate with respect to $\Delta_\infty(f)$ and that* $\dim \Delta_\infty(f) = n$. *Then f is smooth outside a finite number of points.*

Proof. Put $\Delta = \Delta_\infty(f)$ and $\Sigma = \Sigma(\Delta)$. Let τ_0 be the smallest face of Δ which contains the origin, and let σ_0 be the unique cone of Σ with $F_\Delta(\sigma_0) = \tau_0$. We have $\mathbf{T}_K^n \subset X_K(\check{\sigma}_0) \subset X_K(\Sigma)$. Let V be the locus in $\mathbf{T}_K^n$ of $x_i \dfrac{\partial f}{\partial x_i} = 0$, $i = 1, \ldots, n$. We denote by $\bar{V}$ the closure of V in $X_K(\Sigma)$. Since $\bar{V}$ is proper and $X_K(\check{\sigma}_0)$ is affine, it suffices to prove that $\bar{V} \subset X_K(\check{\sigma}_0)$. Hence it suffices to show that $\bar{V} \cap X_K^\sigma(\Sigma) = \emptyset$ for any $\sigma \in \Sigma$ with $0 \notin F_\Delta(\sigma)$. But this follows from the nondegenerateness of f, because an argument similar to (2.5) shows that $\bar{V} \cap X_K^\sigma(\Sigma) = \emptyset$ whenever the locus in $\mathbf{T}_K^n$ of $x_1 \dfrac{\partial f_\tau}{\partial x_1} = \cdots = x_n \dfrac{\partial f_\tau}{\partial x_n} = 0$, with $\tau = F_\Delta(\sigma)$, is empty. □

(3.6) Remark. Assume the hypothesis of the Key Lemma 3.4, and let $\sigma \in \Sigma$. Then the restriction of g_Σ to $\bar{Y}_\Sigma^\sigma$ is locally acyclic outside a finite set E of points. Moreover if $0 \notin F_\Delta(\sigma)$ then we can take E empty. These assertions are proved in the same way as (3.4).

(3.7) Remark. The scheme Y_Σ is always toroidal over k at all points. Moreover if Σ is regular then Y_Σ is smooth over k. This follows from the argument in (3.4) and a straightforward variant of (2.3).

(3.8) Let Y be a scheme of finite type over k, and $\mathfrak{F}$ a constructible $\bar{\mathbf{Q}}_\ell$-sheaf on Y. Let J be a finite index set, and Y_j, $j \in J$, closed subschemes of Y. We use the

following notation, for $I \subset J$:

$$Y_I = \bigcap_{j \in I} Y_j, \text{ with the convention that } Y_\varnothing = Y\,,$$

$$\mathring{Y} = Y \backslash \bigcup_{j \in J} Y_j, \text{ and } \mathring{Y}_I = Y_I \backslash \bigcup_{j \notin I} Y_j.$$

With this notation we have:

(3.8.1) Lemma. *Suppose for all* $I \subset I' \subset J$ *with* $\varnothing \neq Y_{I'} \subsetneqq Y_I$, *that* $\dim Y_{I'} < \dim Y_I$. *If* $H^i_c(Y_I \otimes \bar{k}, \mathfrak{F}) = 0$ *for all* $I \subset J$, $i > \dim Y_I$, *then* $H^i_c(\mathring{Y} \otimes \bar{k}, \mathfrak{F}) = 0$ *for all* $i > \dim Y$.

Proof. Let $1 \in J$ and put $Y' = \bigcup_{j \neq 1} Y_j$. By induction on $\# J$, the lemma is true for $\mathring{Y}$ replaced by $Y \backslash Y'$ and for Y, $\mathring{Y}$ replaced by Y_1, $Y_1 \backslash (Y_1 \cap Y')$. If $Y_1 = Y$ then there is nothing to prove. If $Y_1 \subsetneqq Y$ then $\dim Y_1 < \dim Y$ and the lemma follows from the exact sequence

$$\cdots \to H^{i-1}_c((Y_1 \backslash (Y_1 \cap Y')) \otimes \bar{k}, \mathfrak{F}) \to H^i_c(\mathring{Y} \otimes \bar{k}, \mathfrak{F}) \to H^i_c((Y \backslash Y') \otimes \bar{k}, \mathfrak{F}) \to \cdots.$$

□

Now we can prove Theorem 1.3, using the results on tame ramification from Sect. 4 below.

(3.9) Proof of Theorem 1.3

(a) Put $\Sigma = \Sigma(\Delta_\infty(f))$. Apply Lemma 3.8.1 to $Y = Y_\Sigma$, $\mathfrak{F} = g^*_\Sigma \mathfrak{L}_\psi$, and the family of closed subschemes $\bar{Y}^\sigma_\Sigma$ with $\dim \sigma = 1$. We recall from (3.3) that $\bar{Y}^\sigma_\Sigma$ has pure dimension n-$\dim \sigma$ whenever it is nonempty. This implies that the first hypothesis in Lemma 3.8.1 is indeed satisfied. The second hypothesis also holds because of Proposition 3.1.1, the Key Lemma 3.4, (3.6), and Corollary 4.4 below. Indeed the assumption in the Key Lemma holds by our special choice for Σ. Thus Lemma 3.8.1 yields assertion (a) of Theorem 1.3 when $i > n$. The case $i < n$ is trivial because $\mathbf{T}^n_k$ is affine and smooth.

(b) follows directly from (a) using Proposition 3.2, Theorem 2.7 and Theorem 4.2 below.

(c) Now we assume that the origin is an interior point of $\Delta = \Delta_\infty(f)$. This implies that any subdivision Σ of $\Sigma(\Delta)$ satisfies the assumption of the Key Lemma 3.4. Thus we can take for Σ a regular subdivision (cf. [Da, §8]), so that Y_Σ is smooth over k by (3.7). Hence by (3.1.4) for $Y = Y_\Sigma$ and (4.4), it suffices to show that $H^i_c(D \otimes \bar{k}, g^*_\Sigma \mathfrak{L}_\psi) = 0$ for all i, where $D = Y_\Sigma \backslash \mathbf{T}^n_k$. But this follows from the Mayer-Vietoris sequence because $D = \bigcup_{0 \notin F_\Delta(\sigma)} \bar{Y}^\sigma_\Sigma$, the origin being an interior point of Δ. Indeed $H^i_c(\bar{Y}^\sigma_\Sigma \otimes \bar{k}, g^*_\Sigma \mathfrak{L}_\psi) = 0$ if $0 \notin F_\Delta(\sigma)$, because then $R^i(g_\Sigma|_{\bar{Y}^\sigma_\Sigma})_* \mathbf{Q}_\ell$ is locally constant for all i by the second assertion in (3.6) and hence constant on $\mathbf{A}^1_{\bar{k}}$ by (4.4). □

4 Tame ramification

(4.1) Proposition. *Let R be a discrete valuation ring of characteristic zero with residue field k. Let $\mathfrak{F}$ be a $\mathbf{Q}_\ell$-sheaf on $\mathbf{A}^1_R$ which is locally constant outside a closed* *subscheme $S \subset \mathbf{A}^1_R$ which is proper over R. Then $\mathfrak{F}|_{\mathbf{A}^1_k}$ has tame ramification at infinity.*

Proof. This is a direct consequence of [SGA 1, XIII.2.3.a] with $X = \mathbf{P}^1_R \backslash S$, $U = \mathbf{A}^1_R \backslash S$. Indeed $\mathbf{P}^1_R \backslash S$ is open in $\mathbf{P}^1_R$, because S is proper over R. □

(4.2) Theorem. *Suppose that $f: \mathbf{T}^n_k \to \mathbf{A}^1_k$ is nondegenerate with respect to $\Delta_\infty(f)$. Then $R^i f_! \mathbf{Q}_\ell$ has tame ramification at infinity, for all i.*

Proof. We may assume that $\dim \Delta_\infty(f) = n$, otherwise after a suitable change of coordinates on the torus, f depends only on $r = \dim \Delta_\infty(f)$ variables and we can apply the Künneth formula.

Let R be a discrete valuation ring of characteristic zero with residue field k. We can lift f to a Laurent polynomial $\hat{f}$ over R which is nondegenerate with respect to $\Delta_\infty(f) = \Delta_\infty(\hat{f})$. This is possible because nondegenerateness with respect to $\Delta_\infty(\hat{f})$ is a generic condition in characteristic zero, cf. [Ko, 6.1]. Choose a *regular* fan Σ which is a subdivision of $\Sigma(\Delta_\infty(f))$, cf. [Da, §8]. Put $\hat{A} = R[T]$, where T is one variable. Let $\hat{Y}$ be the scheme-theoretic closure in $X_{\hat{A}}(\Sigma)$ of the locus of $\hat{f} - T = 0$ in $\mathbf{T}^n_{\hat{A}}$, and $\hat{g}$: $\hat{Y} \to \mathbf{A}^1_R$ be induced by the projection $X_{\hat{A}}(\Sigma) \to \operatorname{Spec} \hat{A}$. Clearly $\mathbf{T}^n_R$ is an open subscheme of $\hat{Y}$, and $\hat{g}$ is a compactification of $\hat{f}$: $\mathbf{T}^n_R \to \mathbf{A}^1_R$ (compare with (3.3)). The map $\hat{g}$ is smooth outside $\hat{g}^{-1}(S)$, for some $S \subset \mathbf{A}^1_R$ which is proper over R. Indeed this can be verified by an argument which is entirely similar to the proof of the Key Lemma 3.4, but now using the second assertion of Lemma 2.3 and Lemma 4.3 below instead of (3.5). This argument also shows that we can choose S such that the restriction of $\hat{g}$ to $\hat{Y} \cap \bar{X}^\sigma_{\hat{A}}(\Sigma)$ is smooth outside $\hat{g}^{-1}(S)$, for all $\sigma \in \Sigma$. Hence, outside $\hat{g}^{-1}(S)$, the divisor $\hat{Y} \backslash \mathbf{T}^n_R$ of $\hat{Y}$ has normal crossings relative to $\mathbf{A}^1_R$. Thus by [SGA7, XIII.2.1.11 and 2.1.8.9] we conclude that $R^i \hat{f}_! \mathbf{Q}_\ell$ is locally constant outside S. The theorem follows now from Proposition 4.1. □

(4.3) Lemma. *Let R be any commutative ring, and $f: \mathbf{T}^n_R \to \mathbf{A}^1_R$ an R-morphism which is nondegenerate with respect to $\Delta_\infty(f)$. Then there exists a closed subscheme $S \subset \mathbf{A}^1_R$ which is proper over R, such that f is smooth outside $f^{-1}(S)$.*

Proof. We may assume that $\dim \Delta_\infty(f) = n$, because otherwise after a suitable change of coordinates f depends on less than n variables. Put $\Delta = \Delta_\infty(f)$ and $\Sigma = \Sigma(\Delta)$. Let τ_0 be the smallest face of Δ which contains the origin, and let σ_0 be the unique cone of Σ with $F_\Delta(\sigma_0) = \tau_0$. We have $\mathbf{T}^n_R \subset X_R(\check{\sigma}_0) \subset X_R(\Sigma)$. Notice that $\Delta \subset \check{\sigma}_0$, hence $f \in R[\check{\sigma}_0 \cap \mathbf{Z}^n]$ and f extends to a morphism f': $X_R(\check{\sigma}_0) \to \mathbf{A}^1_R$. Let V be the locus in $\mathbf{T}^n_R$ of $x_i \dfrac{\partial f}{\partial x_i} = 0$, $i = 1, \ldots, n$. We denote by $\bar{V}$ the closure of V in $X_R(\Sigma)$. Since $\bar{V}$ is proper over R, it suffices to prove that $\bar{V} \subset X_R(\check{\sigma}_0)$, because then we can take $S = f'(\bar{V})$. Hence it suffices to show that $\bar{V} \cap X^\sigma_R(\Sigma) = \varnothing$ for any $\sigma \in \Sigma$ with $0 \notin F_\Delta(\sigma)$. But this follows from the nondegenerateness of f, as in the proof of Lemma 3.5. □

(4.4) Corollary. *Suppose that $f: \mathbf{T}^n_k \to \mathbf{A}^1_k$ is nondegenerate with respect to $\Delta_\infty(f)$ and that $\dim \Delta_\infty(f) = n$. Let Σ be any subdivision of $\Sigma(\Delta_\infty(f))$, $\sigma \in \Sigma$, and g_Σ: $Y_\Sigma \to \mathbf{A}^1_k$ the compactification of f associated to Σ as in* (3.3). *Then $R^i(g_\Sigma)_* \mathbf{Q}_\ell$ and $R^i(g_\Sigma|_{\bar{Y}^\sigma_\Sigma})_* \mathbf{Q}_\ell$ have tame ramification at infinity for all i.*

Proof. This is an easy consequence of Theorem 4.2, of some of the material in the proof of (3.4), and of Lemma 4.5 below, with Y_I of the form $\bar{Y}^\sigma_\Sigma$ and $\mathring{Y}_I$ of the form Y^σ_Σ. □

(4.5) Lemma. *Assume the notation of* (3.8), *and let* $g\colon Y \to \mathbf{A}_k^1$ *be a k-morphism. If* $R^i(g|_{\mathring{Y}_I})_! \mathfrak{F}$ *has tame ramification at infinity for all* $I \subset J$ *and all* i, *then* $R^i g_! \mathfrak{F}$ *has tame ramification at infinity for all* i.

Proof. Use induction on $\# J$ and the long exact sequence associated to the cohomology of a closed subscheme (cf. the proof of (3.8.1)). □

5 Preliminaries on intersection cohomology and weights

All schemes considered are assumed to be separated.

(5.1) The intersection complex

Let X be a scheme of finite type over $k = \mathbf{F}_q$ of pure dimension n. Let $\mathfrak{F}$ be a constructible locally constant $\bar{\mathbf{Q}}_\ell$-sheaf on some dense open smooth subscheme U of X. Then there exists a unique object $K^\bullet \in D_c^b(X, \bar{\mathbf{Q}}_\ell)$ (in the derived category, see [De, 1.1.3]) with $K^\bullet|_U = \mathfrak{F}[n]$ such that *both* $K^\bullet$ and its Verdier dual $\mathfrak{D}K^\bullet$ satisfy

(5.1.1) $$H^i K^\bullet = 0 \quad \text{for all } i < -n, \text{ and}$$

(5.1.2) $$\dim \operatorname{Supp} H^i K^\bullet < -i \quad \text{for all } i > -n\,,$$

see [BBD, Proposition 1.4.14]. This unique object is called the *intersection complex on X associated to* $\mathfrak{F}$ and is denoted by $I_X^\bullet(\mathfrak{F})$. When $\mathfrak{F} = \bar{\mathbf{Q}}_\ell$ we will write $I_X^\bullet$ instead of $I_X^\bullet(\bar{\mathbf{Q}}_\ell)$. The hypercohomology $\mathbf{H}^i(X \otimes \bar{k}, I_X^\bullet[-n])$ is called the intersection cohomology of X.

(5.2) Poincaré duality

The unique characterization (5.1) of $I_X^\bullet(\mathfrak{F})$ directly implies that

(5.2.1) $$\mathfrak{D}(I_X^\bullet(\mathfrak{F})) = I_X^\bullet(\check{\mathfrak{F}})(n)\,,$$

where $\check{\mathfrak{F}}$ denotes $\underline{\mathrm{Hom}}(\mathfrak{F}, \bar{\mathbf{Q}}_\ell)$. Hence Verdier duality yields the Poincaré duality

(5.2.2) $$\mathbf{H}^i(X \otimes \bar{k}, I_X^\bullet(\mathfrak{F})[-n]) \cong \mathrm{Hom}(\mathbf{H}_c^{2n-i}(X \otimes \bar{k}, I_X^\bullet(\check{\mathfrak{F}})[-n](n)), \bar{\mathbf{Q}}_\ell)\,.$$

(5.3) Let X, Y, Z be pure-dimensional schemes of finite type over k. We will frequently use the following well known facts which follow directly from the unique characterization 5.1 (see [GM]):

(5.3.1) Let $\pi\colon X \to Y$ *be a locally trivial fibration with respect to the etale topology. Suppose that the fibers of f are smooth of pure dimension d. Then* $I_X^\bullet = \pi^* I_Y^\bullet[d]$.

(5.3.2) Let $Z \subset X$ *be a closed subscheme of* X. *Assume that locally for the etale topology at each closed point of* Z, *the pair* (Z, X) *is isomorphic to the pair* $(Z, Z \times Y)$, *with* Y *smooth of pure dimension* d. *Then* $I_Z^\bullet = I_X^\bullet[-d]|_Z$.

We will use the following fundamental theorem of Gabber (see [BBD, Corollaire 5.3.2]):

(5.4) Purity Theorem. *Assume the notation of* (5.1) *and the terminology of* [De, 6.2]. *If* $\mathfrak{F}$ *is pure of weight* w, *then* $I_X^\bullet(\mathfrak{F})$ *is pure of weight* $w + n$.

(5.5) Poincaré polynomials

Let $C^\bullet \in D_c^b(\operatorname{Spec} k, \bar{\mathbf{Q}}_\ell)$, thus we can consider $C^\bullet$ as a complex of $\bar{\mathbf{Q}}_\ell$ vector spaces with $\operatorname{Gal}(\bar{k}, k)$ action. Suppose that $C^\bullet$ is mixed in the sense of [De, 6.2], then we associate to $C^\bullet$ the Laurent polynomial in one variable T.

$$P(C^\bullet) = \sum_{w \in \mathbf{Z}} \left(\sum_{i \in \mathbf{Z}} (-1)^i c_{w,i} \right) T^w ,$$

where $c_{w,i}$ is the number of eigenvalues with absolute value $q^{w/2}$ of the (geometric) Frobenius action on $H^i(C^\bullet)$ (counting multiplicities). We call $P(C^\bullet)$ the Poincaré polynomial of $C^\bullet$. It is easy to see that P is additive on triangles and multiplicative on tensor products, i.e. $P(C_1^\bullet \otimes C_2^\bullet) = P(C_1^\bullet)P(C_2^\bullet)$. Moreover the Poincaré duality 5.2.2 implies

$$P(R\Gamma(X, I_X^\bullet)) = T^{2\dim X} P(R\Gamma_c(X, I_X^\bullet))(T^{-1}) , \tag{5.5.1}$$

for any pure-dimensional scheme X of finite type over k.

(5.6) If F is a Laurent polynomial in one variable T and $m \in \mathbf{Z}$, then $\operatorname{trunc}_{\leqq m}(F)$ denotes the Laurent polynomial obtained from F by omitting all monomials of degree $> m$.

6 Intersection cohomology of toric varieties

(6.1) The polynomials α *and* β

For any convex polyhedral cone σ in $\mathbf{Q}^n$ with vertex, and for any convex polytope Δ in $\mathbf{Q}^n$ we define in an inductive way polynomials $\alpha(\sigma)$ and $\beta(\Delta)$ in one variable T by

$$\alpha(\sigma) = \operatorname{trunc}_{\leqq \dim\sigma - 1}((1 - T^2)\beta(\operatorname{poly}\sigma)), \quad \text{if } \dim\sigma > 0 , \tag{6.1.1}$$

$$\beta(\Delta) = (T^2 - 1)^{\dim\Delta} + \sum_{\substack{\tau \text{ face of } \Delta \\ \tau \neq \Delta}} (T^2 - 1)^{\dim\tau}\alpha(\operatorname{cone}_\Delta^\circ \tau) , \tag{6.1.2}$$

$$\alpha(\{0\}) = 1 . \tag{6.1.3}$$

Here $\operatorname{poly}\sigma$ and $\operatorname{cone}_\Delta^\circ \tau$ are as defined in (1.6), and trunc as in (5.6). Clearly $\alpha(\sigma)$, $\beta(\Delta)$ only depend on the combinatorial type of σ, Δ. Moreover they only contain even powers of T. These polynomials were first studied by Stanley [S]. One verifies by induction that $\alpha(\sigma) = 1$ when σ is a simplicial cone.

(6.2) Theorem. *Let* σ *be a convex polyhedral cone in* $\mathbf{Q}^n$ *with vertex, and* Δ *a convex polytope in* $\mathbf{Q}^n$, *both of dimension* n. *As always let* $k = \mathbf{F}_q$ *and assume the notation of*

(2.1), (2.4) *and* (5.1). *Then for all i we have*

(6.2.1) $$\dim H^i((I^\bullet_{X_k(\sigma)}[-n])_0) = \textit{coefficient of } T^i \textit{ in } \alpha(\sigma), \textit{ and}$$

(6.2.2) $$\dim \mathbf{H}^i(X_{\bar{k}}(\Delta), I^\bullet_{X_k(\Delta)}[-n]) = \textit{coefficient of } T^i \textit{ in } \beta(\Delta)\,,$$

where the subscript 0 *denotes the stalk at the origin. Moreover*

(6.2.3) $$(I^\bullet_{X_k(\sigma)})_0 \textit{ is pure of weight } n\,.$$

Assertion (6.2.2) for $k = \mathbb{C}$ was stated without proof by Stanley [S] and attributed to J.N. Bernstein, A.G. Khovanskii, and R.D. MacPherson. In this section we give a complete proof of Theorem 6.2. After a preliminary version of our paper was written, we received a preprint of Fieseler [F] proving (6.2.1) and (6.2.2) for $k = \mathbb{C}$. His proof is based on equivariant Morse theory and the decomposition theorem. Note that (6.2) for finite k formally implies the case $k = \mathbb{C}$.

(6.3) Assuming (6.2.3) and using (5.4) and [De, 6.2.6], we see that the assertions (6.2.1) and (6.2.2) are equivalent with respectively

(6.3.1) $$P((I^\bullet_{X_k(\sigma)})_0) = (-1)^n\alpha(\sigma), \text{ and}$$

(6.3.2) $$P(R\Gamma(X_{\bar{k}}(\Delta), I^\bullet_{X_k(\Delta)})) = (-1)^n\beta(\Delta)\,.$$

(6.4) Lemma. *Let $\Delta \subset \mathbf{Q}^n$ be a convex polyhedral cone or an n-dimensional convex polytope, and τ a face of Δ. Then $I^\bullet_{X_k(\Delta)}$ is constant on the torus $X^\tau_{\bar{k}}(\Delta)$ (notation from* (2.1) *and* (2.4)) *with fiber*

(6.4.1) $$(I^\bullet_{X_k(\omega_\tau)})_0\,[\dim\tau]\,,$$

where ω_τ is a cone which is affinely equivalent with $\mathrm{cone}^\circ_\Delta\tau$.

Proof. Put $\gamma = \mathrm{cone}_\Delta\tau$. Note that $X^\tau_k(\Delta) = X_k(\mathrm{cospan}\,\gamma)$ is a closed subscheme of $X_k(\gamma)$ which is open in $X_k(\Delta)$. Thus we have to study the restriction of $I^\bullet_{X_k(\gamma)}$ to $X^\tau_k(\Delta)$. Clearly $\gamma = \mathrm{cone}_\Delta\tau = \mathbf{Q}^{\dim\tau}\oplus\omega_\tau$ and $X_k(\gamma)\cong\mathbf{T}^{\dim\tau}_k\times X_k(\omega_\tau)$, for some convex polyhedral cone ω_τ which is affinely equivalent with $\mathrm{cone}^\circ_\Delta\tau$. Consider now the following diagram

$$\begin{array}{ccccc} X^\tau_k(\Delta) & \hookrightarrow & X_k(\gamma) & & \\ \| \wr & & \| \wr & & \\ \mathbf{T}^{\dim\tau}_k\times\{0\} & \xhookrightarrow{i} & \mathbf{T}^{\dim\tau}_k\times X_k(\omega_\tau) & \xrightarrow{\pi} & X_k(\omega_\tau)\,. \end{array}$$

By (5.3.1) the intersection complex on $\mathbf{T}^{\dim\tau}_k\times X_k(\omega_\tau)$ equals $\pi^* I^\bullet_{X_k(\omega_\tau)}[\dim\tau]$. Thus the restriction of this complex to $\mathbf{T}^{\dim\tau}_{\bar{k}}\times\{0\}$ is constant with fiber (6.4.1) because $\pi\circ i$ is the constant map onto the origin. □

(6.5) Lemma. *Let X be a pure dimensional scheme of finite type over k. Let $b\in X(k)$, $X_0 = X\setminus\{b\}$ and $h: X\times\mathbf{A}^1_k\to X$ a k-morphism. Suppose that*

(i) $h^{-1}(b) = X\times\{0\}\cup\{b\}\times\mathbf{A}^1_k$, *and*

(ii) $\tilde{h}_0: X_0\times\mathbf{T}^1_k\to X_0\times\mathbf{T}^1_k: (x,t)\mapsto(h(x,t),t)$ *is an isomorphism of k-schemes.*

Then $\mathbf{H}^i(X\otimes\bar{k}, I^\bullet_X)\cong H^i((I^\bullet_X)_b)$, for all i.

Proof. It suffices to show for all i that $\mathbf{H}^i(X\otimes\bar{k}, j_! I^\bullet_{X_0}) = 0$, where j is the open immersion $j: X_0 \hookrightarrow X$. Let θ be the morphism $\theta: X\hookrightarrow X\times\mathbf{A}^1_k: x\mapsto(x,1)$.

From (ii) it follows that $h \circ \theta: X \to X$ is an isomorphism. Hence it suffices to prove that $\mathbf{H}^i(X \times \mathbf{A}^1_{\bar{k}}, h^* j_! I^\bullet_{X_0}) = 0$ for all i, because $h \circ \theta$ induces an isomorphism $\mathbf{H}^i(X \otimes \bar{k}, j_! I^\bullet_{X_0}) \to \mathbf{H}^i(X \otimes \bar{k}, (h \circ \theta)^* j_! I^\bullet_{X_0})$ which factors through $\mathbf{H}^i(X \times \mathbf{A}^1_{\bar{k}}, h^* j_! I^\bullet_{X_0})$. Let $\pi_0: X_0 \times \mathbf{T}^1_k \to X_0$ be the projection. We have

$$(h^* j_! I^\bullet_{X_0})|_{X_0 \times \mathbf{T}^1_k} = (j \pi_0 \tilde{h}_0)^* j_! I^\bullet_{X_0} = \tilde{h}_0^* \pi_0^* I^\bullet_{X_0} = \tilde{h}_0^* I^\bullet_{X_0 \times \mathbf{T}^1_k}[-1]$$
$$= I^\bullet_{X_0 \times \mathbf{T}^1_k}[-1] = I^\bullet_{X_0} \boxtimes \bar{\mathbf{Q}}_\ell ,$$

because of (5.3.1) and (ii). Moreover (i) implies that $h^* j_! I^\bullet_{X_0}$ is zero outside $X_0 \times \mathbf{T}^1_k$. Hence $h^* j_! I^\bullet_{X_0} = j_! I^\bullet_{X_0} \boxtimes u_! \bar{\mathbf{Q}}_\ell$, where u is the open immersion $\mathbf{T}^1_k \subsetneq \mathbf{A}^1_k$. Since $H^i(\mathbf{A}^1_{\bar{k}}, u_! \bar{\mathbf{Q}}_\ell) = 0$ for all i, the lemma follows now from the Künneth formula. □

Actually we will only need the following corollary, which appears without proof in [KL, 4.5.a].

(6.6) Corollary. *Let X be a pure dimensional closed subscheme of $\mathbf{A}^m_k$ containing the origin. Suppose that X is invariant under the action of $\mathbf{T}^1_k$ on $\mathbf{A}^m_k$ defined by $\lambda(y_1, \ldots, y_m) = (\lambda^{a_1} y_1, \ldots, \lambda^{a_m} y_m)$, where $a_1, \ldots, a_m \in \mathbb{N} \setminus \{0\}$. Then $\mathbf{H}^i(X \otimes \bar{k}, I^\bullet_X) \cong H^i((I^\bullet_X)_0)$, for all i.*

Proof. Apply Lemma 6.5 with $b = 0$ and $h((y_1, \ldots, y_m), \lambda) = (\lambda^{a_1} y_1, \ldots, \lambda^{a_m} y_m)$.

(6.7) Lemma. *Let $X = X_k(\sigma)$, where σ is an n-dimensional convex polyhedral cone in $\mathbf{Q}^n$ with vertex. Then we have:*

(a) *$H^i((I^\bullet_X)_0) \cong \mathbf{H}^i(X \otimes \bar{k}, I^\bullet_X)$, for all i,*

(b) *$H^i((I^\bullet_X)_0) \cong \mathbf{H}^i(X \otimes \bar{k} \setminus \{0\}, I^\bullet_X)$, for all $i < 0$, and*

(c) *$P((I^\bullet_X)_0) = \mathrm{trunc}_{\leqq n-1} P(R\Gamma(X \otimes \bar{k} \setminus \{0\}, I^\bullet_X))$, when $n \geqq 1$.*

Proof. (a) Choose generators $b_1, \ldots, b_m \neq 0$ for the semi-group $\sigma \cap \mathbf{Z}^n$, and a vector $\gamma \in \mathbf{Z}^n$ in the interior of $\check{\sigma}$. Consider $X_k(\sigma)$ as a closed subscheme of $\mathbf{A}^m_k$ by setting $y_1 = x^{b_1}, \ldots, y_m = x^{b_m}$. Apply now Corollary 6.6 with $a_i = b_i \cdot \gamma > 0$, $i = 1, \ldots, m$.

(b) By (a) and Poincaré duality 5.2.2, assertion (b) is equivalent with

$$\mathbf{H}^i_c(X \otimes \bar{k}, I^\bullet_X) \cong \mathbf{H}^i_c(X \otimes \bar{k} \setminus \{0\}, I^\bullet_X), \text{ for all } i \geqq 1 .$$

But this last assertion follows directly from the long exact sequence associated to the pair $(X \setminus \{0\}, X)$. Indeed $H^i((I^\bullet_X)_0) = 0$ for all $i \geqq 0$, when $n \geqq 1$, by (5.1.2).

(c) By the Purity Theorem 5.4, $H^i((I^\bullet_X)_0)$ is mixed of weight $\leqq n + i$. Thus because of assertion (b) and the last sentence in its proof, it suffices to prove that $\mathbf{H}^i(X \otimes \bar{k} \setminus \{0\}, I^\bullet_X)$ is mixed of weight $\geqq n + i$, for all i. But this follows directly from (5.4), [De], and Poincaré duality. Note that this argument also proves that $(I^\bullet_{X_k(\sigma)})_0$ is pure of weight n. □

Remark. Combining (6.4) with Lemma 4.5(b) of [KL] and using induction, one gets a direct simple proof of the Purity Theorem 5.4 for toric varieties.

(6.8) Proof of Theorem 6.2

Assertion (6.2.3) follows from the proof of Lemma 6.7(c). Hence by (6.3) it suffices to prove (6.3.1) and (6.3.2). For this, we use induction on n. Using the decomposition of $X_k(\sigma)$ into tori, the additivity and multiplicativity of P, Lemma 6.4, and the induction hypothesis we obtain

$$\begin{aligned} P(R\Gamma_c(X_{\bar{k}}(\sigma)\setminus\{0\}, I^\bullet_{X_k(\sigma)})) &= \sum_{\substack{\tau \text{ face of } \sigma \\ \tau \neq \{0\}}} P(R\Gamma_c(X^\tau_{\bar{k}}(\sigma), I^\bullet_{X_k(\sigma)})) \\ &= \sum_{\tau \neq \{0\}} (-1)^{\dim\tau} P(R\Gamma_c(\mathbf{T}^{\dim\tau}_{\bar{k}}, \bar{\mathbf{Q}}_\ell)) P((I^\bullet_{X_k(\omega_\tau)})_0) \\ &= (-1)^n \sum_{\tau \neq \{0\}} (T^2-1)^{\dim\tau}\alpha(\mathrm{cone}^\circ_\sigma(\tau)) . \end{aligned}$$

Indeed α only depends on the combinatorial type, $\dim\omega_\tau = n - \dim\tau$, and $P(R\Gamma_c(\mathbf{T}^{\dim\tau}_{\bar{k}}, \bar{\mathbf{Q}}_\ell)) = (T^2-1)^{\dim\tau}$. Hence, by the definition (6.1.2) of β we get

(6.8.1) $$P(R\Gamma_c(X_{\bar{k}}(\sigma)\setminus\{0\}, I^\bullet_{X_k(\sigma)})) = (-1)^n(T^2-1)\beta(\mathrm{poly}\,\sigma) .$$

The induction hypothesis implies

$$P(R\Gamma(X_{\bar{k}}(\mathrm{poly}\,\sigma), I^\bullet_{X_k(\mathrm{poly}\,\sigma)})) = (-1)^{n-1}\beta(\mathrm{poly}\,\sigma) .$$

Hence (5.5.1) yields $T^{2(n-1)}\beta(\mathrm{poly}\,\sigma)(T^{-1}) = \beta(\mathrm{poly}\,\sigma)$, since $X_k(\mathrm{poly}\,\sigma)$ is proper. Using (5.5.1) again and (6.8.1) we obtain

$$\begin{aligned} P(R\Gamma(X_{\bar{k}}(\sigma)\setminus\{0\}, I^\bullet_{X_k(\sigma)})) &= (-1)^n(1-T^2)T^{2(n-1)}\beta(\mathrm{poly}\,\sigma)(T^{-1}) \\ &= (-1)^n(1-T^2)\beta(\mathrm{poly}\,\sigma) . \end{aligned}$$

Assertion (6.3.1) follows now directly from Lemma 6.7(c) and the definition (6.1.1). To prove (6.3.2) we use the partition of $X_k(\Delta)$ into tori and Lemma 6.4, yielding

$$\begin{aligned} P(R\Gamma(X_{\bar{k}}(\Delta), I^\bullet_{X_k(\Delta)}) &= \sum_{\tau \text{ face of } \Delta} P(R\Gamma_c(X^\tau_{\bar{k}}(\Delta), I^\bullet_{X_k(\Delta)})) \\ &= \sum_\tau (-1)^{\dim\tau} P(R\Gamma_c(\mathbf{T}^{\dim\tau}_{\bar{k}}, \bar{\mathbf{Q}}_\ell)) P((I^\bullet_{X_k(\omega_\tau)})_0) \\ &= (-1)^n \sum_\tau (T^2-1)^{\dim\tau}\alpha(\mathrm{cone}^\circ_\Delta(\tau)) = (-1)^n\beta(\Delta) . \qquad \square \end{aligned}$$

7 Proof of Theorem 1.8

(7.1) Proposition. *Let Y be a scheme of pure dimension n over k and $g\colon Y \to \mathbf{A}^1_k$ a proper k-morphism. Suppose that, outside a finite number of points, g is locally acyclic relative to $I^\bullet_Y$ (in the sense of* [SGA $4\frac{1}{2}$, p. 242]), *and that $R^i g_* I^\bullet_Y$ has tame ramification at infinity for each i. Then*

(a) $\mathbf{H}^i_c(Y \otimes \bar{k}, I^\bullet_Y[-n] \otimes g^*\mathfrak{L}_\psi) = 0$, *for all $i \neq n$, and*
(b) $\mathbf{H}^n_c(Y \otimes \bar{k}, I^\bullet_Y[-n] \otimes g^*\mathfrak{L}_\psi)$ *is pure of weight n.*

Proof. Straightforward adaptation of the proof of Proposition 3.1 replacing everywhere the constant sheaf $\mathbf{Q}_\ell$ by the complex $I^\bullet_Y[-n]$. Indeed now we have the Poincaré duality 5.2.2 with $\mathfrak{F} = g^*\mathfrak{L}_\psi$. (Note that $I^\bullet_Y(g^*\mathfrak{L}_\psi) = I^\bullet_Y \otimes g^*\mathfrak{L}_\psi$.) For (b) we also need the Purity Theorem 5.4 and Deligne's result [De] to insure that the weights are $\leqq n$. Nevertheless we still have to show that $R^j\Phi(I^\bullet_Y[-n]) = 0$ when

$j \geqq n$. This follows directly from the fact [B, 2.3.9] that $R\Phi(F^\bullet[-1])$ is perverse whenever $F^\bullet$ is perverse. Since we did not find a complete proof of this fact in the literature we give an alternative argument: Applying Proposition 4.4.2 of [BBD] to $I_Y^\bullet[-1]|_{Y_{\bar{\eta}}}$ yields $R^j\Psi(I_Y^\bullet) = 0$ for all $j \geqq 0$. But $R^j\Phi(I_Y^\bullet) = R^j\Psi(I_Y^\bullet)$ when $j \geqq 0$, because of (5.1.2) and the triangle

$$(I_Y^\bullet)|_{Y_{\bar{s}}} \to R\Psi(I_Y^\bullet) \to R\Phi(I_Y^\bullet)\,. \qquad \square$$

(7.2) Let $f\colon \mathbf{T}_k^n \to \mathbf{A}_k^1$ be nondegenerate with respect to $\Delta = \Delta_\infty(f)$ and assume that $\dim \Delta = n$. For any subdivision Σ of $\Sigma(\Delta)$ we constructed in (3.3) a compactification $g_\Sigma\colon Y_\Sigma \to \mathbf{A}_k^1$ of f. From now on we take $\Sigma = \Sigma(\Delta)$ and put $Y = Y_{\Sigma(\Delta)}$, and $g = g_{\Sigma(\Delta)}$. Thus $Y \subset X_A(\Delta)$, where $A = k[T]$. For a face τ of Δ we put $Y^\tau = Y \cap X_A^\tau(\Delta)$. Note that the Y^τ form a partition of Y.

(7.3) Lemma. *Let Y and g be as in* (7.2). *Then, outside a finite number of points, g is locally acyclic relative to $I_Y^\bullet$.*

Proof. From the Key Lemma 3.4 it follows that, at all but a finite number of points, g is locally a projection $X \times \mathbf{A}_k^1 \to \mathbf{A}_k^1$, with X a toric k-scheme. We have $I_{X\times\mathbf{A}_k^1}^\bullet = I_X^\bullet[1] \boxtimes \bar{\mathbf{Q}}_\ell$, by (5.3.1). The lemma follows now from the fact [SGA $4\frac{1}{2}$, Th. finitude 2.16 p. 243] that any scheme X over k is universally locally acyclic relative to any complex in $D_c^b(X, \bar{\mathbf{Q}}_\ell)$. $\square$

(7.4) Lemma. *Let $Y \subset X_A(\Delta)$ be as in* (7.2). *Then locally for the etale topology at each point of Y, the pair $(Y, X_A(\Delta))$ is k-isomorphic with the pair $(Y, Y \times \mathbf{A}_k^1)$.*

Proof. The Cartier divisor Y intersects each stratum $X_A^\sigma(\Delta)$ transversally (over k). Apply now the same ideas as in the proof of Lemma 2.3.

(7.5) Lemma. *Assume the notation of* (7.2). *Let τ be a face of Δ which does not contain the origin. Then $\mathbf{R}^i(g|_{Y^\tau})_! I_Y^\bullet$ is constant on $\mathbf{A}_{\bar{k}}^1$, and $\mathbf{H}_c^i(Y^\tau \otimes \bar{k}, I_Y^\bullet \otimes g^*\mathfrak{L}_\psi) = 0$, for all i.*

Proof. From (2.5) it follows that Y^τ is the locus in $X_A^\tau(\Delta)$ of $\tilde{G}_\tau = 0$, where $G = f - T$. Because $0 \notin \tau$, we have $\tilde{G}_\tau = \tilde{f}_\tau$. Since $\tilde{f}_\tau$ does not contain the variable T, we see that $Y^\tau = Z \times \mathbf{A}_k^1$, for some closed subscheme Z of $X_k^\tau(\Delta)$. We consider the following diagram of natural maps, where $h = g|_{Y^\tau}$:

$$\begin{array}{ccccc}
X_A(\Delta) = X_k(\Delta) \times \mathbf{A}_k^1 & \hookleftarrow & Y^\tau = Z \times \mathbf{A}_k^1 & \xrightarrow{h} & \mathbf{A}_k^1 \\
\downarrow \pi & & \downarrow \pi_0 & & \downarrow \rho \\
X_k(\Delta) & \hookleftarrow & Z & \xrightarrow{\lambda} & \operatorname{Spec} k
\end{array}$$

From Lemma 7.4 and (5.3.2) it follows that $I_Y^\bullet = I_{X_A(\Delta)}^\bullet[-1]|_Y$. Hence by (5.3.1)

$$I_Y^\bullet|_{Y^\tau} = I_{X_A(\Delta)}^\bullet[-1]|_{Z\times\mathbf{A}_k^1} = (\pi^* I_{X_k(\Delta)}^\bullet)|_{Z\times\mathbf{A}_k^1} = \pi_0^*(I_{X_k(\Delta)}^\bullet|_Z)\,.$$

Thus base change yields

$$\mathbf{R}^i h_! I_Y^\bullet = \mathbf{R}^i h_! \pi_0^*(I_{X_k(\Delta)}^\bullet|_Z) = \rho^* \mathbf{R}^i \lambda_!(I_{X_k(\Delta)}^\bullet|_Z)\,,$$

which proves the first assertion of the lemma. The second assertion follows now directly from the Leray spectral sequence for h. $\square$

(7.6) Let f and Δ be as in (7.2), and τ a face of Δ of dimension d with $0 \in \tau$. Then we can write $f_\tau = \tilde{f}_\tau(x^{e_1}, \ldots, x^{e_d})$, where $\tilde{f}_\tau$ is a Laurent polynomial in d variables and

$e_1, \ldots, e_d$ form a basis for the lattice $\mathbf{Z}^n \cap$ (affine space of τ). Clearly $\tilde{f}_\tau$ is non-degenerate with respect to $\Delta_\infty(\tilde{f}_\tau)$. Moreover there is a volume preserving bijection between the faces of $\Delta_\infty(\tilde{f}_\tau)$ and τ.

(7.7) Lemma. *Assume the notation of* (7.2) *and* (7.6). *Let τ be a face of Δ which contains the origin. Then $Y^\tau \cong \mathbf{T}_k^{\dim \tau}$ with $g|_{Y^\tau}$ corresponding to $\tilde{f}_\tau$. Moreover $I_Y^\bullet$ is constant on $Y^\tau \otimes \bar{k}$ with fiber* (6.4.1).

Proof. Let $\sigma \in \Sigma(\Delta)$ be such that $F_\Delta(\sigma) = \tau$. Notice that $Y^\tau = Y \cap X_\Delta^\sigma(\Delta)$ is a closed subscheme of $Y \cap X_\Delta(\check{\sigma})$ which is open in Y. From (2.5) (taking the origin for the vertex $P \in \tau$) it follows that $Y \cap X_\Delta(\check{\sigma})$ is the locus of $f(x) - T = 0$ in $X_\Delta(\check{\sigma})$. Hence $Y \cap X_\Delta(\check{\sigma}) \cong X_k(\check{\sigma})$. Under this isomorphism Y^τ corresponds to $X_k^\tau(\Delta) \cong \mathbf{T}_k^{\dim \tau}$, and $I_Y^\bullet|_{Y^\tau}$ to $I_{X_k(\Delta)}^\bullet|_{X_k^\tau(\Delta)}$. Apply now Lemma 6.4. □

(7.8) Lemma. *Let Y and g be as in* (7.2). *Then $\mathbf{R}^i g_* I_Y^\bullet$ has tame ramification at infinity for all i.*

Proof. By Lemma 4.5 (for complexes) it suffices to show that $\mathbf{R}^i(g|_{Y^\tau})_! I_Y^\bullet$ has tame ramification at infinity, for each face τ of Δ. When $0 \notin \tau$, this is clear by Lemma 7.5. When $0 \in \tau$, this follows from Lemma 7.7, Theorem 4.1, and the spectral sequence

$$R^i(g|_{Y^\tau})_! H^j(I_Y^\bullet|_{Y^\tau}) \Rightarrow \mathbf{R}^{i+j}(g|_{Y^\tau})_! I_Y^\bullet . \qquad \square$$

(7.9) Proof of Theorem 1.8

We use the notation of (7.2), in particular $g\colon Y \to \mathbf{A}_k^1$ is the toroidal compactification of f associated to $\Sigma(\Delta)$. Proposition 7.1 applies to this situation, because of Lemmas 7.3 and 7.8. Hence using the formalism of Poincaré polynomials 5.5 we have $P(R\Gamma_c(Y \otimes \bar{k}, I_Y^\bullet[-n] \otimes g^*\mathfrak{L}_\psi)) = bT^n$ for some $b \in \mathbf{Z}$. From the additivity of P we get

$$(7.9.1) \qquad P(R\Gamma_c(\mathbf{T}_{\bar{k}}^n, f^*\mathfrak{L}_\psi)) = bT^n - \sum_{\substack{\tau \text{ face of } \Delta \\ \tau \neq \Delta}} P(R\Gamma_c(Y^\tau \otimes \bar{k}, I_Y^\bullet[-n] \otimes g^*\mathfrak{L}_\psi)) .$$

By Lemma 7.7, (6.3.1), and the multiplicativity of P we have for any face τ of Δ with $0 \in \tau$ that

$$(7.9.2) \quad P(R\Gamma_c(Y^\tau \otimes \bar{k}, I_Y^\bullet[-n] \otimes g^*\mathfrak{L}_\psi)) = P(R\Gamma_c(\mathbf{T}_{\bar{k}}^{\dim \tau}, \tilde{f}_\tau^*\mathfrak{L}_\psi))\alpha(\operatorname{cone}_\Delta^\circ \tau) .$$

Indeed $\dim \omega_\tau = n - \dim \tau$ and α only depends on the combinatorial type. Combining (7.9.1) and (7.9.2) and Lemma 7.5, we obtain

$$P(R\Gamma_c(\mathbf{T}_{\bar{k}}^n, f^*\mathfrak{L}_\psi)) = bT^n - \sum_{\substack{\tau \text{ face of } \Delta \\ 0 \in \tau \neq \Delta}} P(R\Gamma_c(\mathbf{T}_{\bar{k}}^{\dim \tau}, \tilde{f}_\tau^*\mathfrak{L}_\psi))\alpha(\operatorname{cone}_\Delta^\circ \tau) .$$

Together with Theorem 1.3 this yields

$$(7.9.3) \qquad E(\mathbf{T}_k^n, f) = (-1)^n bT^n - \sum_{0 \in \tau \neq \Delta} (-1)^{n - \dim \tau} E(\mathbf{T}_k^{\dim \tau}, \tilde{f}_\tau)\alpha(\operatorname{cone}_\Delta^\circ \tau) .$$

By Theorem 1.3, the value of $E(\mathbf{T}_k^{\dim \tau}, \tilde{f}_\tau)$ at $T = 1$ equals $(\dim \tau)! \operatorname{Vol}(\tau)$. Evaluating (7.9.3) at $T = 1$ we get $(-1)^n b = e(\Delta)$. Hence $E(\mathbf{T}_k^n, f) = E(\Delta)$ by (7.9.3) and induction on n. From (6.1.1) it follows that $\deg E(\tau)\alpha(\operatorname{cone}^\circ_\Delta \tau) < n$ when $\tau \neq \Delta$. Thus $e_n = e(\Delta)$. □

8 Explicit formulas

(8.1) Throughout this section let $f\colon \mathbf{T}^n_k \to \mathbf{A}^1_k$ be nondegenerate with respect to $\Delta = \Delta_\infty(f)$ and assume that $\dim \Delta = n$. We will give explicit formulas for the e_w (defined in 1.4) in some special cases. Put

$$V_i = \sum_{\substack{\tau \text{ face of } \Delta \\ 0 \in \tau,\, \dim \tau = i}} \operatorname{Vol}(\tau)\ .$$

For any face τ of Δ we denote by $F_\tau(i)$ the number of i-dimension faces of Δ containing τ. Let τ_0 be the smallest face of Δ containing the origin. We say that Δ is *simple at the origin* if $F_{\tau_0}(n-1) = n - \dim \tau_0$. We will write $F_0(i)$ instead of $F_{\tau_0}(i)$.

(8.2) Theorem. *Assume the notation of* (8.1) *and suppose that Δ is simple at the origin. Then for $w = 0, 1, \ldots, n$ we have*

$$e_w = \sum_{i=0}^{w} (-1)^{w-i} i! \binom{n-i}{n-w} V_i\ .$$

This theorem was proved by Adolphson and Sperber [AS 3] for almost all p, by using p-adic methods. Note that Δ is simple at the origin if $n \leqq 2$.

Proof. The hypothesis implies that $\alpha(\operatorname{cone}^\circ_\Delta \tau) = 1$ for any face τ of Δ with $0 \in \tau$. The case $w = n$ follows directly from Theorem 1.8. When $w < n$ induction on n reduces us to prove that for any face σ of Δ, with $0 \in \sigma$ and $\ell := \dim \sigma \leqq w$, we have

$$(-1)^{n-1} \sum_{k=w}^{n-1} (-1)^k F_\sigma(k) \binom{k-\ell}{k-w} = \binom{n-\ell}{n-w}\ .$$

But this follows easily from

$$F_\sigma(k)\binom{k-\ell}{k-w} = \binom{n-\ell}{n-k}\binom{k-\ell}{k-w} = \binom{n-\ell}{n-w}\binom{n-w}{k-w}\ . \qquad \square$$

(8.3) For any convex polyhedral cone σ of dimension $n \leqq 4$ with vertex we have

$$\alpha(\sigma) = 1 + (\lambda - n)T^2\ ,$$

where λ denotes the number of faces of σ with dimension $n-1$. A rather long calculation using Theorem 1.8 yields the following formulas for $n = 3, 4$.

Case $n = 3$

$$e_3 = 6V_3 - 2V_2 + V_1 - (F_0(1) - 2)V_0\ ,$$

$$e_2 = 2V_2 - 2V_1 + (2F_0(1) - 3)V_0\ ,$$

$$e_1 = V_1 - F_0(1)V_0\ ,$$

$$e_0 = V_0\ .$$

Case $n = 4$

$$e_4 = 24V_4 - 6V_3 + 2V_2 - (W_1 - 2V_1) + (F_0(3) - 3)V_0 ,$$
$$e_3 = 6V_3 - 4V_2 + (2W_1 - 3V_1) - (F_0(3) + F_0(2) - 6)V_0 ,$$
$$e_2 = 2V_2 - W_1 + (F_0(2) + F_0(1) - 4)V_0 ,$$
$$e_1 = V_1 - F_0(1)V_0 ,$$
$$e_0 = V_0 ,$$

where

$$W_1 := \sum_{\substack{\tau \text{ face of } \Delta \\ 0 \in \tau, \dim \tau = 1}} F_\tau(3)\mathrm{Vol}(\tau) .$$

(8.4) Adolphson and Sperber [AS3, Conjecture 1.13] conjectured a formula for the e_w. The above formulas (8.3) show that their conjecture is true for $n \leqq 4$. Actually they proved this for $n \leqq 3$ for almost all p (unpublished). We will now show that their conjecture is false for some 5-dimensional simplicial Newton polyhedra (i.e. all whose proper faces are simplices), and $w = 5$.

Assume the origin is a vertex of Δ and suppose their conjecture is true for $w = n$ and all Newton polyhedra which are combinatorially equivalent with Δ. Applying this to $m\Delta$, $m \in \mathbb{N} \setminus \{0\}$, we get a polynomial in m with constant term $(-1)^n(F_0(n-1) - n + 1)$. Comparing with Theorem 1.8 we obtain

$$\alpha(\mathrm{cone}_\Delta 0)(1) = F_0(n-1) - n + 1 . \tag{8.4.1}$$

But this is not true in general. Indeed for $n = 5$ and Δ simplicial one has

$$\alpha(\mathrm{cone}_\Delta 0)(1) = 6 + 5F_0(4) - 3F_0(2) .$$

Hence (8.4.1) implies $3f_1 - 4f_3 - 10 = 0$, where f_i is the number of i-dimensional faces of the 4-dimensional polytope poly $\mathrm{cone}_\Delta 0$. But this last equation does not hold for all 4-dimensional simplicial polytopes, since it is independent of the Dehn-Sommerville relations (cf. [MS, p. 103]).

(8.5) Theorem. *Assume the notation of* (8.1), *in particular τ_0 is the smallest face of Δ containing the origin. Put $d = \dim \tau_0$. Then*

(a) $e_w = 0$ *if* $w < d$,

(b) $e_d = d!\mathrm{Vol}(\tau_0)$, *and*

(c) $e_{d+1} = (d+1)!\,V_{d+1} - F_0(d+1)d!\mathrm{Vol}(\tau_0)$.

Proof. Using Euler's theorem on polytopes and induction one verifies that the constant term of $\alpha(\sigma)$ and $\beta(\Delta)$ is 1. Assertion (a) follows from Theorem 1.8 and induction. Then (b) is obtained by induction and Euler's theorem for poly $\mathrm{cone}^\circ_\Delta \tau_0$. To prove (c), use again induction, Euler's theorem and the fact that $\alpha(\sigma)$ only contains even powers of T. We leave the details to the reader. □

Remark. Assume the notation of Theorem 8.5. An analysis of the proof of Theorems 8.5, 1.8 and 6.2 shows that the eigenvalues of weight d of $H^n_c(\mathbf{T}^n_{\bar{k}}, f^*\mathfrak{L}_\psi)$ are precisely the eigenvalues of $H^d_c(\mathbf{T}^d_{\bar{k}}, \tilde{f}^*_{\tau_0}\mathfrak{L}_\psi)$, where $\tilde{f}_{\tau_0}$ is as in (7.6).

9 Exponential sums on $\mathbf{T}_k^r \times \mathbf{A}_k^{n-r}$

(9.1) Throughout this section let $V = \mathbf{T}_k^r \times \mathbf{A}_k^{n-r}$, with $0 \leqq r \leqq n$, and let $f\colon V \to \mathbf{A}_k^1$ be a k-morphism. The Newton polyhedron $\Delta_\infty(f)$ of f at infinity is defined as before. Call f nondegenerate with respect to $\Delta_\infty(f)$ if its restriction to $\mathbf{T}_k^n$ is such. For any $B \subset \{1, 2, \ldots, n\}$ we put $\mathbf{Q}_B^n = \{(t_1, \ldots, t_n) \in \mathbf{Q}^n | t_i = 0 \text{ for each } i \in B\}$. We call f *commode* if $\dim(\Delta_\infty(f) \cap \mathbf{Q}_B^n) = n - \#B$ for all $B \subset \{r+1, \ldots, n\}$. When $r = 0$ this means that f contains for each $i = 1, \ldots, n$ a monomial $a_i x_i^{m_i}$ with $m_i \in \mathbb{N} \setminus \{0\}$, $a_i \in k \setminus \{0\}$. Finally we define $E(V,f)$ as in (1.4) with $\mathbf{T}_k^n$ replaced by V.

(9.2) Theorem. *Suppose that $f\colon V \to \mathbf{A}_k^1$ is commode and nondegenerate with respect to $\Delta = \Delta_\infty(f)$. Then*

(a) $H_c^i(V \otimes \bar{k}, f^*\mathfrak{L}_\psi) = 0$ *if* $i \neq n$, *and*

(b) $\dim H_c^n(V \otimes \bar{k}, f^*\mathfrak{L}_\psi) = \sum_{B \subset \{r+1, \ldots, n\}} (-1)^{\#B}(n - \#B)!\operatorname{Vol}(\Delta \cap \mathbf{Q}_B^n)$.

If $r = 0$ or if the origin is an interior point of $\Delta \cap \mathbf{Q}_{\{r+1, \ldots, n\}}^n$, then

(c) $H_c^n(V \otimes \bar{k}, f^*\mathfrak{L}_\psi)$ *is pure of weight n.*

A somewhat weaker version of this theorem was proved by Adolphson and Sperber [AS1, AS2] for almost all p, and they conjectured the theorem for all p when $r = 0$. The condition that f is commode cannot be removed. Indeed a straightforward calculation with Gaussian sums shows that $|\sum_{x,y,z \in k} \psi(z^2 + xz + yz)| = q^2$, which would contradict (9.2.a). We will prove the theorem in (9.4) by partitioning V into tori.

(9.3) Lemma. *Assume the notation of* (3.8) *and let $n \in \mathbf{N}$. If $H_c^i(\mathring{Y}_I \otimes \bar{k}, \mathfrak{F}) = 0$ for all $I \subset J$, $i > n - \#I$, then $H_c^i(Y \otimes \bar{k}, \mathfrak{F}) = 0$ for all $i > n$.*

Proof. Let $1 \in J$. By induction on $\#J$ we have $H_c^i((Y \setminus Y_1) \otimes \bar{k}, \mathfrak{F}) = 0$ when $i > n$ and $H_c^i(Y_1 \otimes \bar{k}, \mathfrak{F}) = 0$ when $i > n - 1$. The lemma follows now from the long exact sequence associated to the pair (Y_1, Y). □

(9.4) Proof of Theorem 9.2

(a) Because of Theorem 1.3(a) and the hypothesis that f is commode we can apply Lemma 9.3 with $J = \{r+1, \ldots, n\}$ and Y_I the locus of $(x_i = 0)_{i \in I}$ in V, so that the $\mathring{Y}_I$ are tori. This yields assertion (a) if $i > n$. When $i < n$ the assertion is clear since V is smooth and affine.

(b) follows directly from (a), Theorem 1.3(b), and the additivity of the Euler characteristic.

(c) Clearly $E(V,f) \equiv E(\mathbf{T}_k^n, f) \bmod T^n$. Hence by Theorem 1.8 it suffices to prove that $\dim H_c^n(V \otimes \bar{k}, f^*\mathfrak{L}_\psi)$ equals $e(\Delta)$. But this follows from (b) and (1.7.1). Indeed the hypothesis implies that all faces τ of Δ which contain the origin are of the form $\Delta \cap \mathbf{Q}_B^n$ with $B \subset \{r+1, \ldots, n\}$, and $\operatorname{cone}_\Delta^\circ \tau$ is simplicial for such τ.

(9.5) Theorem. *Suppose that $f\colon V \to \mathbf{A}_k^1$ is commode and nondegenerate with respect to $\Delta = \Delta_\infty(f)$. Then*

$$E(V,f) = e(\Delta)T^n - \sum (-1)^{n - \dim\tau} E(\tau)\alpha(\operatorname{cone}_\Delta^\circ \tau)\,,$$

where the summation runs over all proper faces τ of Δ which contain the origin and are not of the form $\Delta \cap \mathbf{Q}_B^n$ with $B \subset \{r+1, \ldots, n\}$.

Proof. Using the decomposition of V into tori and the additivity of Poincaré polynomials we get

$$E(V,f) = E(\mathbf{T}_k^n,f) + \sum_{\tau = \Delta \cap \mathbf{Q}_B^n} (-1)^{n-\dim\tau} E(\mathbf{T}_k^{\dim\tau}, f_\tau)\,,$$

where B runs over all nonempty subsets of $\{r+1,\ldots,n\}$. The corollary follows now from (1.7.2) and Theorem 1.8, since $\operatorname{cone}_\Delta^\circ \tau$ is simplicial whenever $\tau = \Delta \cap \mathbf{Q}_B^n$, $B \subset \{r+1,\ldots,n\}$. □

(9.6) Remark. Assertion (c) of Theorem 9.2 can be proved without using Theorem 1.8, by adapting the argument in (3.9c). In the present case $\Sigma(\Delta)$ contains the positive octant of $\mathbf{Q}^{n-r}$. Choose a regular subdivision Σ of $\Sigma(\Delta)$ which still contains this octant. Then Σ satisfies the assumption of the Key Lemma 3.4 and V is an open subscheme of Y_Σ. To prove the assertion, we proceed now as in (3.9c) replacing $\mathbf{T}_k^n$ by V.

References

[AS1] Adolphson, A., Sperber, S.: Exponential sums and Newton polyhedra. Bull. Am. Math. Soc. **16**, 282–286 (1987)

[AS2] Adolphson, A., Sperber, S.: Exponential sums and Newton polyhedra, cohomology and estimates. Ann. Math. **130**, 367–406 (1989)

[AS3] Adolphson, A., Sperber, S.: Exponential sums on $\mathbf{G}_m^n$. Invent. Math. **101**, 63–79 (1990)

[B] Brylinski, J.-L.: (Co-)Homologie d'Intersection et Faisceaux Pervers. Séminaire Bourbaki, n° 585 (1981/82); Astérisque **92–93** (1982)

[BBD] Beilinson, A., Bernstein, J., Deligne, P.: Faisceaux pervers. Astérisque **100** (1983)

[BKKh] Bernstein, D., Kushnirenko, A.G., Khovanskii, A.G.: Newton polyhedra. Usp. Mat. Nauk **31**, 201–202 (1976)

[Da] Danilov, V.I.: The geometry of toric varieties. Russ. Math. Surv. **33**, 97–154 (1978)

[De] Deligne, P.: La conjecture de Weil II. Publ. Math. Inst. Hautes Étud. Sci. **52**, 137–252 (1980)

[F] Fieseler, K.-H.: Rational intersection cohomology of projective toric varieties. (Preprint)

[GM] Goresky, M., MacPherson, R.: Intersection homology II. Invent. Math. **72**, 77–130 (1983)

[Ka] Katz, N.: Sommes exponentielles. Astérisque **79** (1980)

[Kh1] Khovanskii, A.G.: Newton polyhedra and toroidal varieties. Funct. Anal. Appl. **11**, 289–296 (1977)

[Kh2] Khovanskii, A.G.: Newton polyhedra and the genus of complete intersections. Funct. Anal. Appl. **12**, 38–46 (1978)

[KL] Kazhdan, D., Lusztig, G.: Schubert varieties and Poincaré duality. (Proc. Symp. Pure Math., vol. 36, pp. 185–203) Providence, RI: Am. Math. Soc. 1980

[Ko] Kouchnirenko, A.G.: Polyèdres de Newton et nombres de Milnor. Invent. Math. **32**, 1–31 (1976)

[MS] McMullen, P., Shephard, G.C.: Convex polytopes and the upper bound conjecture. (Lond. Math. Soc. Lect. Note Ser. vol. 3) Cambridge: Cambridge Univeristy Press 1971

[S] Stanley, R.: Generalized H-vectors, intersection cohomology of toric varieties, and related results. In: Nagata, N., Matsumura, H. (eds.) Commutative Algebra and Combinatorics (Adv. Stud. Pure Math., vol. 11, pp. 187–213) Amsterdam New York: North-Holland 1987

[SGA1] Grothendieck, A.: Revêtements étales et groupe fondamental. (Lect. Notes Math., vol. 224) Berlin Heidelberg New York: Springer 1971

[SGA4½] Deligne, P.: Cohomologie étale. (Lect. Notes Math., vol. 569) Berlin Heidelberg New York: Springer 1977

[SGA7] Grothendieck, A., Deligne, P., Katz, N.: Groupes de monodromie en géometrie algébrique. (Lect. Notes Math., vols. 288, 340) Berlin Heidelberg New York: Springer 1972, 1973