On extensions of Presburger arithmetic,

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Abstract :

Semenov proved the decidability of $Th < \mathbb{N}, +, f >$, where f is effectively compatible with addition. He showed that this theory admits quantifier elimination in a language containing the Presburger predicates and a logarithmic function. In particular, $Th < \mathbb{N}, +, 2^x >$ is decidable. We give a detailed proof of this result.
We examine the relationships between $Th < \mathbb{N}, +, P_2 >$, $Th < \mathbb{N}, +, V_2)$ and $Th < \mathbb{N}, +, 2^x >$, where $P_2(x)$ iff x is a power of 2 and $V_2(x) = y$ iff y is the largest power of 2 dividing x.

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§ 1. Introduction

We are going to present extensions of Presburger's result of the decidability of the theory of $\mathbb{Z}$-groups. Let us recall his result to fix notations. He proved that the theory of totally ordered abelian groups with a least strictly positive element admits quantifier elimination (q.e.) in $\{+,-,\leq,0,1,D_n; n \in \omega\}$, the predicate D_n is defined by $D_n(x)$ iff $\exists y\ n.y = x$.
As a corollary, we have that $Th\langle \mathbb{N},+\rangle$ is decidable.

The main result we want to present is the decidability of $\langle \mathbb{N},+,2^x\rangle$. We give an axiomatization T for $Th\langle \mathbb{N},+,2^x\rangle$ and we prove that T admits q.e. in an extended language containing the congruence predicates and a logarithmic function. Those results are for the most part due to Semenov.
He introduced the concept of a function f on $\mathbb{N}$ effectively compatible with addition and he proved the decidability of $\langle \mathbb{N},+,f\rangle$, using a quantifier elimination result. (See theorem 2, p. 617 $[S]_2$).

Definitions : (See $[S]_2$, p. 616).
(1) An f-sum is a sum of the form $\sum_i a_i\ f(x + b_i)$, with $a_i, b_i \in \mathbb{Z}$.
(2) f is effectively compatible with addition if for every m the values of f are periodic modulo m and if every f-sum A is either bounded or there exists Δ s.t. $A(x+\Delta) > f(x)$ or $\exists \Delta(-A(x+\Delta) > f(x))$.
If f is effectively compatible with addition, then either $f(x)-x$ is bounded or $\forall c \exists \Delta \forall x\ \ f(x+\Delta) > c\ f(x) + cx$.

A first result which extends Preburger's result is the result of Büchi that $Th_{\omega} < \mathbb{N}, S >$, the weak monadic second-order theory of $\mathbb{N}$ with the successor function S is decidable. (See [B]).

Büchi proved his result by showing that

(i) the 2-recognizable sets are definable in $Th_{\omega} < \mathbb{N}, S >$,

(ii) the theories $Th_{\omega} < \mathbb{N}, S >$ and $Th < \mathbb{N}, +, V_2 >$ are bi-interpretable, where $V_2(x)$ is the greatest power of 2 dividing x.

(iii) the definable subsets of $< \mathbb{N}, +, V_2 >$ are 2-recognizable.

Then he concluded by using Kleene's result that the empty problem for finite automata is decidable.

Comments

A subset of $\mathbb{N}$ is 2-recognizable iff it is accepted by a finite automaton with alphabet {0,1}.

To bi-interpret $Th_{\omega} < \mathbb{N}, S >$ and $Th < \mathbb{N}, +, V_2 >$, one interprets the finite set n by the number $[n] = \sum_{i \in n} \chi(i) 2^i$, where χ is the characteristic function of n, the addition by the relation $A(n_1, n_2, n_3)$ iff $[n_1] + [n_2] = [n_3]$ (see [R], p. 617 3.4) and $V_2(n)$ by the least power of 2 belonging to n.

A power of 2 is interpreted by a singleton.

Conversely, we can define the relation $\in$ as follows : R(x,y) iff x is a power of 2 and x belongs to the binary expansion of $y \equiv (V_2(x) = x \ \& \ \exists z \ \exists t ((y = z + x + t) \ \& \ (z < x) \ \& \ (t = 0 \lor V_2(t) \geq x)))$.] .

From the fact that Th $<\mathbb{N},+,V_2>$ is decidable follows that Th $<\mathbb{N},+,P_2>$, where $P_2(n)$ iff n is a power of 2, and Th $<\mathbb{N},+,\lambda_2>$, where $\lambda_2(n)$ is the greatest power of 2 less than or equal to n, are decidable.

Van den Dries gave a model theoretic proof of this latter result. More precisely, he gave a universal axiomatization T for Th $<\mathbb{N},+,P_2>$ in the language $\{+,\dot{-},\leqslant,0,1,\lambda_2,P_2,./n,n\in\omega\}$ where $./n(x) = y$ iff $\bigvee_{0\leqslant k<n} x = ny + k$. He proved that T is model-complete with prime model $<\mathbb{N},+,P_2>$. T can be axiomatized as follows :

(1) Axioms for $\{\mathbb{N},+,0,1,\leqslant,\dot{-}\}$, together with : for each natural number $n > 0$, $\forall x \forall y \ (x/n = y \leftrightarrow \bigvee_{0\leqslant k<n} ny + k = x)$.

(2) $P(1)$ & $\forall x \ (P_2(x) \rightarrow x > 0)$

(3) $\forall x \ (P_2(x) \leftrightarrow P_2(2x))$

(4) $\forall x \forall y \ (P_2(x)$ & $x < y < 2x \rightarrow \neg P_2(y))$

(5) $\forall x \ (x > 0 \rightarrow (\lambda x \leqslant x < 2\lambda x$ & $P_2(\lambda x)))$ & $\lambda(0) = 0$

(6) For each natural number n and n-tuple of odd prime powers $q_1,\ldots,q_n$:

$$\forall y \ (P_2(y) \rightarrow \bigvee_{r=1}^{k} \bigwedge_{i=1}^{n} q_i(y - (k_{i,r} - q_i)/q_i) = y - (k_{i,r} - q_i)$$

where $(k_{i,r})_{1\leqslant r\leqslant n}$ is the n-tuple of natural numbers between 0 and q_i to which is congruent a power of 2, and and k is the ppcm of the orders of the subgroups of powers of 2 in $(\mathbb{Z}/q_i\ \mathbb{Z} - \{0\},.)$

The proof that Th $<\mathbb{N},+,V_2>$ is decidable is not model-theoretic. Can we give a "reasonable" language in which this theory admits q.e. ?

The theory of $<\mathbb{N},+,V_2>$ has more expressive power than Th $<\mathbb{N},+,\lambda_2>$. Semenov exhibited a family of 2-recognizable subsets of $\mathbb{N}$ which are not definable in $<\mathbb{N},+,P_2>$ (see $[S]_1$ Corollary 4, p. 418).

On the other hand, Cherlin noted that the theory of $<\mathbb{N},+,V_2,2^x>$ is undecidable. So the graph of 2^x is not definable in $<\mathbb{N},+,V_2>$. Conversely $V_2(x)$ is not definable in $<\mathbb{N},+,2^x>$.

To prove the undecidability of Th $\langle \mathbb{N},+,V_2,2^x\rangle$, one interprets either the relation $\in$ on $\mathbb{N}$ as follows : let x,y belong to $\mathbb{N}$, then $x \in y$ iff $R(2^x,y)$ (with R defined as above), or the theory of binary relations on finite subsets of $\mathbb{N}$.
The binary relation A on a subset of n elements is coded by

$$\sum_{\substack{(x,y)\text{ s.t. }A(x,y)\\ x \leqslant x,y < 2^n}} 2^{x+2^y}.$$

From the undecidability of Th$\langle \mathbb{N},+,V_2,2^x\rangle$, one can deduce that Th$\langle \mathbb{N},+,v_2\rangle$, where v_2 is the 2-adic valuation, is undecidable. One defines $2^x = y$ by ($v_2(y) = x$ and y is the least z such that $v_2(z) = x$) and one defines $V_2(x) = y$ by $y = 2^{v_2(x)}$.
A last remark about the complexity of those theories is that the theories Th$\langle \mathbb{N},+,V_2\rangle$ and Th $\langle \mathbb{N},+,2^x\rangle$ are non elementary recursive. This is due respectively to Myers and Semenov (There will be a survey article on these questions by Compton and Henson).

§ 2. Decidability of Th$\langle \mathbb{N},+,2^x\rangle$

1. First we present an axiom system T for the theory of $\langle \mathbb{N},+,2^x\rangle$ in the language $L = \{+,\dot{-},0,1, ,2^x,D_n ; n \in \mathbb{N}, \ell(x),\lambda(x)\}$.
Let T be the following set of axioms :

(1) $\forall x \ \forall y \ \forall z \quad ((x+y)+z = x+(y+z))$

$\forall x \quad (x + 0 = 0 + x = x)$

$\forall x \ \forall y \ \forall z \quad (x + z = y + z \to x = y)$

$\forall x \ \forall y \quad (x + y = y + x)$

$\forall x \ \forall y \quad (x \leqslant y \leftrightarrow \exists u \ \ x + u = y)$

$\forall x \ \forall y \quad (x \leqslant y \vee y \leqslant x)$

$\forall x \quad (x \geqslant 0 \ \& \ x \neq 0 \to x \geqslant 1) \ \& \ 0 \neq 1$

$\forall x \ \exists y \quad (\bigvee_{o \leqslant k < n} x = ny + k)$

$\forall x \quad (\exists y \ \ ny = x \leftrightarrow D_n(x))$

$\forall x \ \forall y \ \forall u \quad (x \dot{-} y = u \leftrightarrow ((x \geqslant y \ \& \ x = y+u) \vee (x \leqslant y \ \& \ u = 0))$

(2) $\forall x \ (\lambda x \leqslant x < 2\lambda x)$

(3) $\forall x \forall y \ (x \leqslant y \rightarrow \ell x \leqslant \ell y)$

(4) $\ell(1) = 0$

(5) $\forall x \ (x \geqslant 1 \rightarrow \ell(2x) = \ell(x) + 1)$

(6) $\forall x \ (x \geqslant 1 \rightarrow 2^{\ell x} = \lambda x)$

(7) $\forall x \ (\ell(2^x) = x)$

(8) $\forall x \ (2^{x+1} = 2^x + 2^x)$

(9) $\forall x \ (x \geqslant 1 \rightarrow 2^{x-1} \geqslant x)$

(10) Let m be an odd natural number and φ the Euler function,

$$\forall x \ (D_{\varphi(m)}(x) \rightarrow D_m(2^x + (m-1)))$$

Remark :

We may replace axiom (8) by $\forall x \ (\lambda(2x) = 2\lambda x)$.

Properties of models of T

We are going to list a series of properties we can deduce from the axioms.

1. $\lambda(0) = 0$ since $\lambda x \leqslant x$ [axiom 2]
2. $\lambda(1) = 1$ since $\lambda(1) \leqslant 1$ and $1 \lneqq 2\lambda(1)$ [axiom 2]
3. $2^0 = 1$ since $2^{\ell(1)} = \lambda(1) = 1$ [axioms 6, 4 and property 2]
4. $\lambda(2^x) = 2^x$ if $x \geqslant 1$ and $\lambda(2^0) = \lambda(1) = 1 = 2^0$
5. $\lambda(2x) = 2\lambda(x)$ [$\lambda(2x) = 2^{\ell(2x)} = 2^{\ell x+1} = 2^{\ell x}.2 = 2\lambda x$ if $x \geqslant 1$ and $\lambda(0) = 2\lambda(0) = 0$].
6. $(2^{\ell x} < y < 2^{\ell x+1}) \rightarrow y \neq 2^{\ell y}$ [$\ell x \leqslant \ell y \leqslant \ell x+1$. So either $\ell x = \ell y$, so $2^{\ell y} = 2^{\ell x}$ and $2^{\ell y} \neq y$, or $\ell x + 1 = \ell y$, so $2^{\ell y} = 2^{\ell x+1}$ which implies that $2^{\ell y} > y$ ✳]
7. $\lambda(\lambda x) = \lambda x$ [$\lambda x = 2^{\ell x}$ and $\lambda(\lambda x) = \lambda(2^{\ell x}) = 2^{\ell x}$ (property 4))
8. $\ell(x) = \ell(\lambda x)$ [$\lambda x = \lambda(\lambda x)$ i.e. $2^{\ell x} = 2^{\ell(\lambda x)}$. So $\ell x = \ell(\lambda x)$]
9. Let n,m belong to $\mathbb{N}$ and $N \geqslant \ell n - \ell m + 1$, then $x \geqslant 2N$ implies that $nx \leqslant m\, 2^x$. [$2^N.\lambda m \geqslant 2.\lambda n$, and $nx \leqslant 2\lambda n\, x$ so $nx \leqslant 2\lambda n\, x \leqslant 2^N \lambda m\, x$

$$\leqslant \lambda m\, 2^x \leqslant m\, 2^x \text{ (see } (\star)).$$

($\star$) By axiom 9, $x \geqslant 1 \rightarrow 2^{x-1} \geqslant x$. So $x \geqslant N+1$ implies that $2^{(x-N)-1} \geqslant x-N$. So if $2(x-N) \geqslant x$ i.e. $x \geqslant 2N$, then $2^x \geqslant 2^N.x$]

Let us prove now the remark. It suffices to show that axiom 8 follows from the other axioms and property 5.
We will use property 4 which has been proved without using axiom 8.
First we prove that $\ell(2^{x+1}) = \ell(2.2^x)$
$[\ell(2^{x+1}) = x + 1$ and $\ell(2.2^x) = \ell(2^x) + 1 = x + 1\ ]$
Then, $2^{\ell(2^{x+1})} = 2^{\ell(2.2^x)}$ and

$$\begin{aligned} 2^{\ell(2^{x+1})} &= \lambda(2^{x+1}) = 2^{x+1} \\ 2^{\ell(2.2)^x} &= \lambda(2.2^x) \\ &= 2\lambda(2^x) = 2.2^x. \end{aligned}$$

In the following we are going to prove that Th(N) admits quantifier elimination in L, where $N = \langle \mathbb{N}, +, \dot{-}, 0, 1, \leqslant, 2^x, D_n ; n \in \mathbb{N}, \ell(x), \lambda(x) \rangle$ and N satisfies T.
By inspection of the proof, we see that we only used the axioms of T.
So we get the following results.

<u>Proposition 1</u> : Th(N) admits q.e..
<u>Corollary 1</u> : Th($\langle \mathbb{N}, +, 2^x \rangle$) is decidable.
<u>Proposition</u> 2 : T admits q.e.
<u>Corollary 2</u> : T is complete and decidable.
<u>Proof</u> : N is the prime model of T. □

<u>Proof of Proposition 1</u>

We show that any 1-existential L-formula $\exists x\ \theta(x,\overline{y})$ is equivalent to an open formula.

<u>1^{er} step</u> :
By adding possibly more (quantified) variables, we transform $\exists x\ \theta(x,\overline{y})$ into an existential formula $\exists x \exists \overline{x}\ \theta_o(x,\overline{x},\overline{y})$, where θ_o is a conjunct of congruence conditions on the L-terms in the $\overline{y}$-variables and inequations between terms which are of the following forms :

(i) $\sum_i a_i 2^{cx_i} + \sum_j b_j x_j + d$, with $a_i, b_j, d \in \mathbb{Z}$ and $c \in \mathbb{N}$.
(we will call such terms S-terms).

(ii) $t(\overline{y})$, where $t(.)$ is an L-term and $\overline{y}$ are the non quantified variables.

We replace in θ - any term of the form $2^{t(x,\overline{y})}$, where $t(x,\overline{y})$ is not the variable x by 2^{x_j} with x_j a new (quantifier) variable and by the atomic formula $x_j = t(x,\overline{y})$.

- any term of the form $\ell(t)$ by a new (quantified) variable x_i and by the atomic formula $2^{x_i} \leqslant t < 2^{x_i+1}$.

- any atomic formula of the form $D_n(t \dot{-} s)$ by $(t \leqslant s) \vee (t > s \;\&\; \exists z \;\; t = s + nz)$ where t or s are L-terms where x appears.

- any inequation of the form $s_1(x,\overline{x})+t_1(\overline{y}) \leqslant s_2(x,\overline{x})+t_2(\overline{y})$ by $(s_1 \geqslant s_2 \;\&\; t_2 \geqslant t_1 \;\&\; s_1 \dot{-} s_2 \leqslant t_2 \dot{-} t_1) \vee (s_1 \leqslant s_2 \;\&\; t_1 \leqslant t_2) \vee (s_1 \leqslant s_2 \;\&\; t_1 \geqslant t_2 \;\&\; t_1 \dot{-} t_2 \leqslant s_2 \dot{-} s_1)$.

2^e step :

We will eliminate the largest (for the order) quantified variable.

Let $\chi_i(x,\overline{x})$ be $x_{i(0)} \leqslant \dots \leqslant x_{i(n)}$ where $x_o = x$, $\overline{x} = (x_1,\dots,x_n)$ where i is a permutation of $\{0,1,\dots,n\}$. Let S_{n+1} be the set of permutations on $\{0,1,\dots,n\}$.

We have $\theta_o(x,\overline{x},\overline{y}) \leftrightarrow \bigvee_{i\in S_{n+1}} (\chi_i(x,\overline{x}) \wedge \theta_o(x,\overline{x},\overline{y})$

and $\exists x \exists \overline{x}\; \theta_o(x,\overline{x},\overline{y}) \leftrightarrow \bigvee_{i\in S_{n+1}} \exists x_{i(o)} \dots \exists x_{i(n)} (\chi_i(x,\overline{x}) \wedge \theta_o(x,\overline{x},\overline{y}))$.

From now on we will deal with the 1-existential formula $\exists x_{i(n)}\; \chi_i(x,\overline{x}) \wedge \theta_o(x,\overline{x},\overline{y})$ and we show how to eliminate this quantifier.

3^e step :

We denote $x_{i(n)}$ by x_o. We distinguish between two different ways x_o can occur in the system.

I) x_o occurs linearly in every inequation of the conjunct in θ_o. (In the process we will possibly replace x_i, $i \geqslant 1$ by dx_i but we can choose the same d for every x_i in each inequation).

We may assume that the system of inequations looks like :

$$\bigwedge_{i,j,k} (f_j(\overline{x}) + g_j(\overline{y}) \leq d_k\, x_o \leq f_i(\overline{x}) + g_i(\overline{y})),$$ where $f_j(\overline{x})$, $f_i(\overline{x})$ are S-terms and $g_j(\overline{y})$ and $g_i(\overline{y})$ are $\mathcal{L}$-terms, $d_k \in \mathbb{Z}$. (⋆)

(Indeed $f_j(\overline{x}) + d_k x_o \leq g_j(\overline{y}) \leftrightarrow [(d_k x_o \leq g_j(\overline{y}) \dot{-} f_j(\overline{x})$ & $f_j(\overline{x}) \leq g_j(\overline{y}))$ v $(f_j(\overline{x}) \geq g_j(\overline{y})$ & $f_j(\overline{x}) \dot{-} g_j(\overline{y}) \leq (-d_k).x_o])$

Let d be the least common positive multiple of the d_k.

Let d_o be an odd natural number s.t. $\exists n \in \mathbb{N}\ d = d_o . 2^n$

For each x_i, $1 \leq i \leq n$, the following disjunct holds :

$$\bigvee_{0 \leq k_i < d\varphi(d_o)} (x_i \dot{-} n = k_i + d\varphi(d_o)x_i' \ \&\ x_i \geq n) \ \mathrm{v} \bigvee_{0 \leq \ell < n} x_i = \ell.$$

We replace each inequation by a disjunct of inequations which are obtained by either replacing x_i by $k_i + n + \varphi(d_o).dx_i'$ or by ℓ with $0 \leq \ell < n$.

We again obtain S-terms since the coefficient of the x_i' is $d.\varphi(d_o)$. So we obtain a disjunct of systems each looking as (⋆) with $\overline{x}$ replaced by $\overline{x}'$. Then we multiply out each inequation of (⋆) in order to have d as a coefficient for x_o.

Before pursuing, we consider a special case. Namely we eliminate x_o in one inequation, say $f_j(\overline{x}') + g_j(\overline{y}) \leq dx_o \leq f_i(\overline{x}') + g_i(\overline{y})$. (1)

(1) is equivalent to :

$$((f_i+g_i)-(f_j+g_j) \geq d) \mathrm{v} (\bigvee_{0 \leq c < d} [(f_i+g_i)-(f_j+g_j) = c \ \&$$

$$\bigvee_{0 \leq c_j < d} f_j + g_j \underset{d}{\equiv} c_j \ \& \bigvee_{0 \leq c_k < c} c_j + c_k \underset{d}{\equiv} 0])$$ and the

congruence condition $f_j + g_j \underset{d}{\equiv} c_j$ is equivalent to a congruence condition on g_j of the form $g_j \underset{d}{\equiv} z_j$, where z_j is determined by the following :

Either x_i is a constant $\ell < n$, so $2^{x_i} = 2^\ell$, or x_i is equal to $k_i + n + d\varphi(d_o)x_i'$, so $2^{x_i} = 2^{k_i} 2^n (2^{\varphi(d_o) dx_i'} =$

$$= 2^{k_i} 2^n (1 + d_o z) \text{ for some } z$$
$$= 2^{k_i+n} + 2^{k_i}.d.z$$
$$\underset{d}{\equiv} 2^{k_i+n}$$

So in both cases, $2^{x_i} \equiv_d 2^{u_i}$ with $u_i = k_i + n$ if $x_i \geq n$ and $u_i = \ell$ if $x_i < n$.

So $f_j(\overline{x}') = \sum_\ell a_\ell 2^{e' x_\ell} + \sum_t b_t x_t + e$

$$\equiv_d \sum_\ell a_\ell 2^{e' u_\ell} + \sum_t b_t u_t + e$$

So let $z_j \equiv_d c_j - (\sum_\ell a_\ell 2^{e' u_\ell} + \sum_t b_t u_t + e)$.

Let $i \in \{1,\ldots,p\}$ and $j \in \{1,\ldots,q\}$ index $f_i(\overline{x}')$ and $f_j(\overline{x}')$. Let S_p (S_q) be the set of permutations on p (respectively q) elements.

The system (★) of inequations is equivalent to :

$$\bigvee_{\sigma \in S_p,\ \tau \in S_q} f_{\tau(1)}(\overline{x}') + g_{\tau(1)}(\overline{y}) \geq \ldots \geq f_{\tau(q)}(\overline{x}') + g_{\tau(q)}(\overline{y}) \ \& \ f_{\tau(1)} + g_{\tau(1)} \leq f_{\sigma(1)} + g_{\sigma(1)} \ \&$$

$$f_{\sigma(1)}(\overline{x}') + g_{\sigma(1)}(y) \leq \ldots \leq f_{\sigma(p)}(\overline{x}') + g_{\sigma(p)}(\overline{y}) \ \&$$

$$[(f_{\sigma(1)} + g_{\sigma(1)}) \dot{-} (f_{\tau(1)} + g_{\tau(1)}) \geq d$$

$$\vee (\bigvee_{0 \leq c < d} [(f_{\sigma(1)} + g_{\sigma(1)}) - (f_{\tau(1)} + g_{\tau(1)}) = c \ \&$$

$$\bigvee_{z_\tau \in C_\tau} g_{\tau(1)}(\overline{y}) \equiv_d z_\tau]),$$

where C_τ is determined as follows :

let $f_{\tau(1)}(\overline{x}) = \sum_\ell a \ 2^{e' x_\ell} + \sum_t b_t x_t + e$

let C'_τ be the set of natural numbers such that if $f_{\tau(1)} + g_{\tau(1)} \equiv_d c'_\tau$ then there exist c', $0 \leq c' \leq c$ s.t. $c'_\tau + c' \equiv_d 0$.

Now let $u_\ell = n + k_\ell$ with $0 \leq k_\ell < d.\varphi(d_o)$ if $x_\ell \geq n$, and $u_\ell = s$, $0 \leq s < n$, if $x_\ell < n$.

So $z_\tau \equiv_d c'_\tau - \sum_\ell a_\ell 2^{e' u_\ell} + \sum_t b_t u_t + e$ and C_τ is the set of those z_τ depending on C'_τ.

II) x_o occurs in an inequation in an exponential term.
Let $a_o 2^{dx_o} + \sum_{i=1}^{n} 2^{dx_i} a_i + \sum_{j=o}^{n} b_j x_j + c \leq t(\bar{y})$ be such an inequation, where $t(\bar{y})$ is an L-term, $d \in \mathbb{N}$, $a_i, b_j, c \in \mathbb{Z}$.
Denote such an inequation by $\tau(x_o, \bar{x}, \bar{y})$.
We are going to replace such a formula by a boolean combination of inequations between S-terms in $x_o, x_1, \ldots, x_n$ where x_o occurs linearly and L-terms in $\bar{y}$. We will assume that $d = 1$.

Case $a_o > 0$

In this case, we will distinguish four cases :

(1) $\lambda a_o \cdot 2^{x_o} \leq \frac{1}{2}.\lambda(t(\bar{y}))$

(2) $\lambda a_o \cdot 2^{x_o} = \lambda(t(\bar{y})) \leftrightarrow x_o = \ell(t(\bar{y})) - \ell(a_o)$

(3) $\lambda a_o \cdot 2^{x_o} = 2.\lambda(t(\bar{y})) \leftrightarrow x_o = \ell(t(\bar{y})) + 1 - \ell(a_o)$

(4) $\lambda a_o \cdot 2^{x_o} > 2.\lambda(t(\bar{y}))$

In subcases (2) and (3), we substitute x_o by an L-term in $\bar{y}$. In subcases (1) and (4) we estimate $a_o 2^{x_o} + \sum_{i=1}^{n} 2^{x_i} a_i + \sum_{j=o}^{n} b_j x_j + c$.

Let $J_1 = \{j \in \{0,\ldots,n\} / b_j \geq 0\}$. If $J_1 \neq \emptyset$, let $b_+ = 2(\ell(\sum_{J_1} b_j) + 3)$ and otherwise let $b_+ = 0$.
If $J - J_1 \neq \emptyset$, let $b_- = 2(\ell(-\sum_{J-J_1} b_j) + 4)$, otherwise let $b_- = 0$.

Let $\Delta = \ell(\sum_i |a_i|) + 3$. Let $c_+ = \ell(c) + 3$ if $c > 0$ and $c_+ = 0$ otherwise.
Let $c_- = \ell(-c) + 4$ if $c < 0$ and $c_- = 0$, otherwise.

(1) If $x_o \geq \max\{b_+, c_+\}$ and if $x_i \leq x_o - \Delta$ for all $0 \leq i \leq n$, then $\tau(x_o, \bar{x}, \bar{y})$ holds.
We can express that as follows :

$$(x_o \leq \ell(t(\bar{y})) - \ell(a_o) - 1) \;\&\; ((x_o \leq b_+) \vee (x_o \leq c_+) \vee [(x_o \geq b_+ \;\&\; x_o \geq c_+) \;\&\; (\bigwedge_{i=1}^{n} x_i + \Delta \leq x_o \vee (\bigvee_{i=1}^{n} \bigvee_{0 \leq k < \Delta} (x_i + k = x_o \;\&\; \tau(x_i + k, \bar{x}, \bar{y}))))]).$$

We obtain this result by the following estimations :

(a) $\sum_J b_j x_j \geq 0$

$\sum_J b_j x_j \leq \sum_{J_1} b_j x_j \leq (\sum_{J_1} b_j).x_o \leq 2^{x_o - 2}$ if

$x_o \geq 2(\ell(\sum_{J_1} b_j) + 3)$ (see Property 9).

Now (1) $\sum_{i=1}^{n} 2^{x_i} a_i + a_o 2^{x_o} + c + \sum_J b_j x_j \leq \sum 2^{x_i} |a_i| +$

$a_o 2^{x_o} + c + \sum_{J_1} b_j x_j$.

Suppose $x_o \geq b_+$ and $x_i \leq x_o - \Delta$, $1 \leq i \leq n$

$$(1) \leq (\sum_i |a_i| \; 2^{-\Delta} + a_o) 2^{x_o} + c + 2^{x_o} . 2^{-2}$$

$$\leq \left(\frac{\Sigma |a_i|}{2\lambda(\Sigma |a_i|)} . 2^{-2} + a_o\right) 2^{x_o} + c + 2^{x_o} . 2^{-2}$$

$$\leq 2^{x_o}(a_o + \tfrac{1}{4} + \tfrac{1}{4}) + c$$

(a.1) $\sum_J b_j x_j \geq 0$ and $c \geq 0$.

So $2^{x_o} \geq 2\lambda c . 2^2 \geq 4.c$

So if $x_o \geq \max \{b_+, c_+\}$ and $x_i \leq x_o - \Delta$, $1 \leq i \leq n$, then

$$\sum_{i=1}^{n} 2^{x_i} a_i + a_o 2^{x_o} + c + \sum_J b_j x_j \leq 2^{x_o}(a_o + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4})$$
$$\neq$$
$$\leq 2^{x_o}(a_o + 1)$$
$$\neq$$
$$\leq 2^{x_o}.2 \lambda a_o$$
$$\neq$$
$$< \lambda t(\bar{y}) \leq t(\bar{y})$$

(a.2) $\sum_J b_j x_j \geq 0$ and $c \leq 0$.

Suppose $x_o \geq b_+$ and $x_i - \Delta \leq x_o$, $\forall i \quad 0 \leq i \leq n$.

$$\text{So } \sum_{i=1}^{n} 2^{x_i} a_i + a_o 2^{x_o} + c + \sum_J b_j x_j \leqslant 2^{x_o}(a_o + \frac{1}{4} + \frac{1}{4})$$

$$\leqslant 2^{x_o}(a_o + \frac{1}{2})$$

$$\lneqq t(\overline{y})$$

(b) $\sum_J b_j x_j \leqslant 0.$

Suppose that $x_o \geqslant c_+$ and $x_i - \Delta \leqslant x_o$ $\forall i \quad 0 \leqslant i \leqslant n$

Then,

$$\sum_{i=1}^{n} 2^{x_i} a_i + a_o 2^{x_o} + c + \sum_J b_j x_j \leqslant \sum_{i=1}^{n} 2^{x_i} a_i + a_o 2^{x_o} + c$$

$$\leqslant (\sum_i |a_i| \, 2^{-\Delta} + a_o) 2^{x_o} + c$$

$$\leqslant (\frac{\Sigma |a_i|}{2\lambda \Sigma |a_i|} \, 2^{-2} + a_o) 2^{x_o} + c$$

$$\leqslant 2^{x_o}(a_o + \frac{1}{4}) + 2^{x_o} . \frac{1}{4}$$

$$\leqslant 2^{x_o}(a_o + \frac{1}{2})$$

$$< t(\overline{y})$$

(4) If $x_o \geqslant \max\{b_-, c_-\}$ and if $x_i \leqslant x_o - \Delta$, $1 \leqslant i \leqslant n$, then $\tau(x_o, \overline{x}, \overline{y})$ doesn't hold.

We can express that as follows :

$$(x_o \geqslant \ell(t(\overline{y})) - \ell(a_o) + 2) \;\&\; ((x \leqslant b_-) \vee (x_o \leqslant c_-) \vee$$
$$(x_o \geqslant b_- \;\&\; x_o \geqslant c_- \;\&\; (\bigvee_{0 \leqslant k < \Delta} (x_i + k = x_o) \;\&\; \tau(x_i + k, \overline{x}, \overline{y}))) .$$

We obtain this result by the following estimations.

(a) $\sum_J b_j x_j \leqslant 0$

$$\sum_J b_j x_j \geqslant \sum_{J-J_1} b_j x_j \geqslant -2^{x_o - 3} \quad , \text{ if } x_o \geqslant 2(\ell(-\sum_{J-J_1} b_j) + 4).$$

(See Property 9).

Suppose $x_i + \Delta \leqslant x_o, \forall i \quad 1 \leqslant i \leqslant n$ and $x_o \geqslant b_-$, then

$$\sum_{i=1}^{n} a_i 2^{x_i} + a_o 2^{x_o} + c + \sum_J b_j x_j \geqslant \sum_{i=1}^{n} 2^{x_i} a_i + a_o 2^{x_o} + c + \sum_{J-J_1} b_j x_j$$

$$\geqslant -\sum_{i=1}^{n} 2^{x_o - \Delta} |a_i| + a_o 2^{x_o} + c + \sum_{J-J_1} b_j x_j$$

$$\geqslant 2^{x_o}\left(a_o - \frac{\Sigma\, |a_i|}{2\,(\Sigma |a_i|)} . 2^{-2}\right) + c - 2^{x_o - 3}$$

If $c \geqslant 0$

$$\geqslant 2^{x_o}(a_o - \tfrac{1}{4}) - 2^{x_o - 3} \underset{\neq}{\geqslant} 2^{x_o}(a_o - \tfrac{1}{2})$$

$$\underset{\neq}{\geqslant} 2^{x_o} \frac{\lambda a_o}{2}$$

$$\underset{\neq}{\geqslant} 2\, \lambda(t(\overline{y}))$$

$$\underset{\neq}{\geqslant} t(\overline{y})$$

If $c \leqslant 0$, then suppose $x_o \geqslant c_-$. So $2^{x_o} \geqslant \lambda(-c).2^4$ i.e.

$$-2^{x_o} \underset{\neq}{\leqslant} c\,.\,2^3$$

$$\sum_{i=1}^{n} a_i\, 2^{x_i} + a_o\, 2^{x_o} + c + \sum_J b_j x_j \geqslant 2^{x_o}(a_o - \tfrac{1}{4}) - \tfrac{1}{8} . 2^{x_o} - \tfrac{1}{8} 2^{x_o}$$

$$\geqslant 2^{x_o}(a_o - \tfrac{1}{2})$$

$$\geqslant 2^{x_o} \frac{\lambda a_o}{2} \geqslant 2\lambda(t(\overline{y}))$$

$$> t(\overline{y})$$

(b) $\sum_J b_j x_j \geqslant 0$.

Suppose $x_i + \Delta \leqslant x_o \quad \forall i \quad 1 \leqslant i \leqslant n$ and $x_o \geqslant c_-$.
Then,

$$\sum_{i=1}^{n} a_i 2^{x_i} + a_o 2^{x_o} + c + \sum_{J} b_j x_j \geqslant \sum_{i=1}^{n} a_i 2^{x_i} + a_o 2^{x_o} + c$$

$$\geqslant 2^{x_o}(a_o - \tfrac{1}{4}) - \tfrac{1}{8}\, 2^{x_o}$$

$$\geqslant 2^{x_o}(a_o - \tfrac{1}{4} - \tfrac{1}{8})$$

$$\underset{\neq}{\geqslant} 2^{x_o} \frac{\lambda a_o}{2} \geqslant 2\lambda(t(\overline{y}))$$

$$> t(\overline{y})$$

Let $N = \max\{b_+, c_+, b_-, c_-\}$.

If $a_o > 0$,

$$\begin{aligned}
\tau(x_o,\overline{x},\overline{y}) \leftrightarrow\ & (x_o = \ell(t(\overline{y})) - \ell(a_o)\ \&\ \tau(\ell(t(\overline{y})) - \ell(a_o), \overline{x}, \overline{y}))\ v \\
& (x_o = \ell(t(\overline{y})) - \ell(a_o) + 1\ \&\ \tau(\ell(t(\overline{y})) - \ell(a_o) + 1, \overline{x}, \overline{y})\ v \\
& \bigvee_{0 \leqslant k < N} (x_o = k\ \&\ \tau(k, \overline{x}, \overline{y}))\ v \\
& (x_o \geqslant N\ \&\ (x_o \leqslant \ell(t(\overline{y})) - \ell(a_o) - 1\ \&\ [\ (\bigwedge_{i=1}^{n} x_i + \Delta \leqslant x_o)\ v \\
& \bigvee_{i=1}^{n} \bigvee_{0 \leqslant k < \Delta} (x_i + k = x_o\ \&\ \tau(x_i + k, \overline{x}, \overline{y})\ ]\ )\ v \\
& (x_o \geqslant \ell(t(\overline{y})) + 2 - \ell(a_o)\ \& \\
& \bigvee_{i=1}^{n} \bigvee_{0 \leqslant k < \Delta} x_i + k = x_o\ \&\ \tau(x_i + k, \overline{x}, \overline{y}))).
\end{aligned}$$

<u>Case $a_o \leq 0$.</u>

As in case $a_o > 0$, let $b_+ = 2(\ell(\sum_{J_1} b_j) + 3)$ if $J_1 \neq \emptyset$ and $b_+ = 0$, otherwise.

Let $c_+ = \ell(c) + 3$, if $c > 0$ and $c_+ = 0$ if $c \leqslant 0$.

Let $N' = \max\{b_+, c_+\}$. Let $\Delta' = \ell(\Sigma|a_i|) + 2 - \ell(-a_o)$.

Then if $x_o \geqslant N'$ and if $x_i \leqslant x_o - \Delta'$, $1 \leqslant i \leqslant n$, then $\tau(x_o, \overline{x}, \overline{y})$ holds.

We obtain this result making the following estimations :

(a.1) $\sum_J b_j x_j \geq 0$ and $c \geq 0$. Suppose moreover that $x \geq N'$ and $x_i \leq x_o - \Delta'$, $1 \leq i \leq n$.

So $\sum_J b_j x_j \leq 2^{x_o - 2}$ and $c < 2^{x_o-2}$.

Therefore, $a_o 2^{x_o} + \sum_i a_i 2^{x_i} + \sum_J b_j x_j + c$

$$\leq a_o 2^{x_o} + \sum_i |a_i| 2^{x_o - \Delta'} + 2^{x_o - 2} + c$$

$$< 2^{x_o}(a_o + \sum_i |a_i|\ 2^{-\Delta'} + \frac{1}{4} + \frac{1}{4})$$

$$< 2^{x_o}(a_o + \frac{\lambda(-a_o)}{2} \cdot \frac{\sum_i |a_i|}{2\lambda(\Sigma|a_i|)} + \frac{1}{2})$$

$$< 2^{x_o}(a_o + \frac{1}{2} + \frac{\lambda(-a_o)}{2} \leq 0$$

$$< t(\bar{y})$$

(a.2) $\sum_J b_j x_j \geq 0$ & $c \leq 0$. Suppose $x_o \geq b_+$ and $x_i + \Delta' \leq x_o$ $\forall i$ $1 \leq i \leq n$.

So $\sum_J b_j x_j \leq \sum_{J_1} b_j x_j \leq 2^{x_o - 2}$

Thus $a_o 2^{x_o} + \sum_i a_i 2^{x_i} + \sum_J b_j x_j + c \leq a_o 2^{x_o} + \sum_i |a_i| 2^{x_i} + 2^{x_o - 2}$

$$\leq 2^{x_o}(a_o + \frac{\lambda(-a_o)}{2} + \frac{1}{4}) \leq 0$$

$$\leq t(\bar{y})$$

(a.3) $\sum_J b_j x_j \leq 0$. Suppose $x_o \geq c_+$ and $x_i + \Delta' \leq x_o$ for all i $1 \leq i \leq n$.

Therefore,

$a_o 2^{x_o} + \sum_i a_i 2^{x_i} + \sum_J b_j x_j + c \leq a_o 2^{x_o} + \frac{\lambda(-a_o)}{2} 2^{x_o} + \frac{1}{4} 2^{x_o}$

$$\leq (a_o + \frac{\lambda(-a_o)}{2} + \frac{1}{4}) 2^{x_o}$$

$$< 0 \leq t(\bar{y})$$

So, if $a_o < 0$,

$$\tau(x_o,\bar{x},\bar{y}) \leftrightarrow [\bigvee_{0\leqslant k<N'} (x_o = k \;\&\; \tau(k,\bar{x},\bar{y}))\ v\ (x_o \geqslant N' \;\&\; \bigwedge_{i=1}^{n} x_i + \Delta \leqslant x_o)\ v\ (x_o \geqslant N' \;\&\; (\bigvee_{i=1}^{n} \bigvee_{0\leqslant k<\Delta'} x_i+k = x_o$$

$$\text{et } \tau(x_i+k,\bar{x},\bar{y})))].$$

□

§ 3. Decidability of Th $<\mathbb{N},+,f>$, where f is effectively compatible with addition.

Let T_f be the following theory :

(1) axioms for $<\mathbb{N},+,\dot{-},\leqslant,0,1,D_n;n \in \omega>$ (see axioms (1) for T).

(2) $\forall x\ \forall y\ (x < y \to f(x) < f(y))$

(3) $\forall x\ \forall y\ (x \geqslant \Delta \to (f^{-1}(x) = y \leftrightarrow f(y) \leqslant x < f(y+1)),$ where $\Delta \in \mathbb{N}$

(4) $\forall x\ (x \geqslant 1 \to f(x) \geqslant c.f(x-1) \geqslant x),\ c > 1,\ c \in \mathbb{R}.$

or

(4)' $\forall c\ \exists \Delta\ \forall x\ (f(x+\Delta) > cf(x) + cx)$

(5) the values of f are periodic modulo m, for every m.

Let us make two remarks :

1. Axiom (4) implies that $f(x) \geqslant c^n f(x-n)$ and that $(x \geqslant \frac{c^n \cdot n}{c^n - m} \to f(x) \geqslant mx)$, where n is sufficiently large in order that $c^n - m > 0$.

2. If f is effectively compatible with addition and if $f(x) - x$ is unbounded, then f satisfies axiom(4)'.

Along the lines of Proposition 2, one can prove :

Proposition 3 : T_f admits q.e. in $\{+,\dot{-},0,1,\leqslant,f,f^{-1},D_n;n \in \omega\}$. □

An example of such function f is $f : n \mapsto n.2^n$.

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