5. A <u>prederivator</u> is just a strict 2-functor $\mathbb{D}$ from the 2-category Cat of small categories or a convenient full sub-2-category Dia of Cat to the category of categories, contravariant on functors and natural transformations.

$$I \longmapsto \mathbb{D}(I)$$

$$I \xrightarrow{u} J \quad \longmapsto \quad \mathbb{D}(I) \xleftarrow{u^* := u^*_{\mathbb{D}} := \mathbb{D}(u)} \mathbb{D}(J)$$

$$\begin{tikzcd} I \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & J \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} \mathbb{D}(I) & & \mathbb{D}(J) \arrow[ll, bend right, "u^*_{\mathbb{D}}"'] \arrow[ll, bend left, "v^*_{\mathbb{D}}"] \arrow[ll, phantom, "\Uparrow \alpha^* := \alpha^*_{\mathbb{D}} := \mathbb{D}(\alpha)"] \end{tikzcd}$$

The most important example for us is the following. Let (C, W) be a localizer and define $\mathbb{D} := \mathbb{D}_{(C,W)}$ as follows

$\mathbb{D}(I) := W^{-1}_{I^\circ} C^{I^\circ}$ where $I^\circ := I^{op}$ denotes the category opposite to I

$u^*_{\mathbb{D}} := \overline{(u^\circ)^*}$ $\qquad u : I \longrightarrow J, \quad u^\circ : I^\circ \longrightarrow J^\circ$

$$(u^\circ)^* : C^{J^\circ} \longrightarrow C^{I^\circ}$$

$$\overline{(u^\circ)^*} : \underbrace{W^{-1}_{J^\circ} C^{J^\circ}}_{\substack{\shortparallel \\ \mathbb{D}(J)}} \longrightarrow \underbrace{W^{-1}_{I^\circ} C^{I^\circ}}_{\substack{\shortparallel \\ \mathbb{D}(I)}}$$

$$\alpha^*_{\mathbb{D}} := \overline{(\alpha^\circ)^*} \qquad \begin{tikzcd} I \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & J \end{tikzcd}, \quad \begin{tikzcd} I^\circ \arrow[r, bend left, "u^\circ"] \arrow[r, bend right, "v^\circ"'] \arrow[r, phantom, "\Uparrow\alpha^\circ"] & J^\circ \end{tikzcd}$$

$$\begin{tikzcd} C^{I^\circ} & C^{J^\circ} \arrow[l, bend right, "(u^\circ)^*"'] \arrow[l, bend left, "(v^\circ)^*"] \arrow[l, phantom, "\Uparrow(\alpha^\circ)^*"] \end{tikzcd}$$

and by the 2-universal property of localization

$$\begin{tikzcd} W_{I^\circ}^{-1} C^{I^\circ} & W_{J^\circ}^{-1} C^{J^\circ} \arrow[l, bend right, "u^*_{\mathbb{D}} = \overline{(u^\circ)^*}"'] \arrow[l, bend left, "v^*_{\mathbb{D}} = \overline{(v^\circ)^*}"] \arrow[l, phantom, "\Uparrow\overline{(\alpha^\circ)^*}"] \end{tikzcd}$$

A <u>derivator</u> is a prederivator satisfying a list of axioms Der 1 to Der 5 that will be explained later. A <u>Grothendieck localizer</u> is a localizer (C, W) such that the prederivator $\mathbb{D}_{(C,W)}$ is a derivator. A <u>Quillen localizer</u> is a localizer (C, W) such that there exists a closed model structure on C with W as the class of weak equivalences.

~~The~~ Cisinski's theorem that I will explain in the next lecture can be roughly stated as "every Quillen localizer* is a Grothendieck localizer"

Fix a prederivator $\mathbb{D} : \mathrm{Dia}^{o} \to \mathrm{CAT}$, where Dia is a full sub-2-category of Cat containing finite partially ordered sets and satisfying all stability conditions needed. Typically Dia = Cat or the full subcategory of finite categories, or finite direct categories, or finite posets.

We will try to explain the axioms of the derivators. The most important are Der 2, Der 3, Der 4.

We will say that the prederivator $\mathbb{D}$ <u>admits left Kan extensions</u> if for every $u : I \to J$ in Dia, the functor $u^* : \mathbb{D}(J) \to \mathbb{D}(I)$ has a left adjoint that will then be denoted by $u_! : \mathbb{D}(I) \to \mathbb{D}(J)$. This is what is known as Der 3d, the "right" part of the axiom Der 3 (because of right exactness properties of $u_!$)

<u>Example</u> If $(\mathcal{C}, W)$ is a localizer admitting homotopy left Kan extensions, then the prederivator $\mathbb{D}_{(\mathcal{C},W)} : \mathrm{Cat}^{o} \to \mathrm{CAT}$ admits left Kan extensions

Dually we will say that the prederivator $\mathbb{D}$ <u>admits right Kan extensions</u> if for every $u : I \to J$ in Dia, the functor $u^* : \mathbb{D}(J) \to \mathbb{D}(I)$ has a right adjoint $u_* : \mathbb{D}(I) \to \mathbb{D}(J)$

So the axiom Der 3 says that the prederivator $\mathbb{D}$ admits left and right Kan extensions

<u>Example</u> Let K be a commutative ring, $C(K)$ the category of complexes of K-modules, and Qis_K the class of quasi-isomorphisms of $C(K)$. Consider the prederivator (which in fact is a derivator), $\mathbb{D}er_K := \mathbb{D}_{(C(K), Qis_K)}$ associated to the localizer $(C(K), Qis_K)$. Observe that $\mathbb{D}er_K(e)$, where e is the point category, is simply the derived category of K-modules, and more generally, for every small category I, $\mathbb{D}er_K(I)$ is canonically isomorphic to the derived category of presheaves on I with value K-modules.

Let G be a finite group and $\Lambda = K[G]$ the ring of G. If we consider G as a category with one object, the category of presheaves on G with value K-modules is exactly the category of K-modules endowed with a right action of G, or equivalently the category of right Λ-modules.

Recall that for any K-module M endowed with a right action of G, the homology groups $H_i(G, M)$ (resp. $H^i(G,M)$ cohomology groups) are defined by

$$H_i(G,M) = \mathrm{Tor}_i^{\Lambda}(M,K) = H_i(M \overset{L}{\otimes} K) \quad , i \geq 0$$

$$(\text{resp. } H^i(G,M) = \mathrm{Ext}^i_{\Lambda}(K, M) = H^i \, R\mathrm{Hom}(K,M) \quad , i \geq 0 \;)$$

where K is considered as a $K[G]$-bimodule by the K-algebra morphism $K[G] \to K$ defined by $g \mapsto 1$, g in G.

As it is easily verified

$$M \otimes K = \begin{matrix}\text{coinvariants of}\\ M \text{ under the}\\ \text{action of } G\end{matrix} = \varinjlim_{G^\circ} M$$

$$\left(\text{resp. } \operatorname{Hom}(K, M) = \begin{matrix}\text{invariants of}\\ M \text{ under the}\\ \text{action of } G\end{matrix} = \varprojlim_{G^\circ} M \right)$$

and so

$$H_i(G, M) = H_i\left(\underrightarrow{\text{holim}}_{G^\circ} M\right) = H_i\left((p_G)_!(M)\right)$$

$$\left(\text{resp. } H^i(G, M) = H^i\left(\underleftarrow{\text{holim}}_{G^\circ} M\right) = H^i\left((p_G)_*(M)\right)\right)$$

and $(p_G)_!(M)$ (resp. $(p_G)_*(M)$) can be considered as the "total" homology (resp. cohomology) of the group G with coefficient the right G-module M (recall that $p_G : G \to e$ denotes the functor of G to the point category)

Grothendieck's philosophy is that for a general derivator $\mathbb{D}$, for a category I in the domain of definition $\mathcal{D}ia$ of $\mathbb{D}$ and an object X of $\mathbb{D}(I)$, $(p_I)_!(X)$ (resp. $(p_I)_*(X)$) can be considered as the homology (resp. cohomology) object of I with coefficient X, and often uses the notation

$$H_*(I, X) := (p_I)_!(X) \quad \left(\text{resp. } H^*(I, X) := (p_I)_*(X)\right)$$

(For a general derivator $\mathbb{D}$ the "$*$" sign in H_* or H^* has no particular meaning and the symbols H_*, H^* must be taken as a whole)

In order to explain axiom Der 4 consider first a "2-square" in Dia

$$\mathcal{D} = \begin{tikzcd} A' \arrow[r, "v"] \arrow[d, "u'"'] & A \arrow[d, "u"] \\ B' \arrow[r, "w"'] \arrow[ur, phantom, "\Nearrow \alpha"] & B \end{tikzcd} \qquad wu' \xrightarrow{\alpha} uv$$

By contravariant 2-functoriality we deduce a 2-square in CAT

$$\begin{tikzcd} \mathbb{D}(A') & \mathbb{D}(A) \arrow[l, "v^*"'] \\ \mathbb{D}(B') \arrow[u, "u'^*"] & \mathbb{D}(B) \arrow[l, "w^*"] \arrow[u, "u^*"'] \arrow[ul, phantom, "\Swarrow \alpha^*"] \end{tikzcd} \qquad v^*u^* \xrightarrow{\alpha^*} u'^* w^*$$

and if the prederivator $\mathbb{D}$ admits left Kan extensions, using the unit $1_{\mathbb{D}(A)} \xrightarrow{\eta} u^*u_!$ and the counit $u'_! u'^* \xrightarrow{\varepsilon'} 1_{\mathbb{D}(B')}$ of the adjunctions $(u_!, u^*)$ and $(u'_!, u'^*)$ we get an arrow

$$\begin{tikzcd} u'_! v^* \arrow[r, "c_{\mathcal{D}}"] \arrow[d, "u'_! v^* * \eta"'] & w^* u_! \\ u'_! v^* u^* u_! \arrow[r, "u'_! * \alpha^* * u_!"'] & u'_! u'^* w^* u_! \arrow[u, "\varepsilon' * w^* u_!"'] \end{tikzcd}$$

$$c_{\mathcal{D}} = (\varepsilon' * w^* u_!)(u'_! * \alpha^* * u_!)(u'_! v^* * \eta)$$

known as the <u>homological</u> (or <u>left</u>) change base morphism

$$\begin{tikzcd} \mathbb{D}(A') \arrow[d, "u'_!"'] & \mathbb{D}(A) \arrow[l, "v^*"'] \arrow[d, "u_!"] \arrow[dl, phantom, "\Searrow c_{\mathcal{D}}"] \\ \mathbb{D}(B') & \mathbb{D}(B) \arrow[l, "w^*"] \end{tikzcd}$$

A particular case: let $I \xrightarrow{u} J$ an arrow in Dia and j an object of J. There is a 2-square

$$\begin{tikzcd} j\backslash I \arrow[r, "k"] \arrow[d, "p"'] \arrow[dr, phantom, "\nearrow\alpha"] & I \arrow[d, "u"] \\ e \arrow[r, "j"'] & J \end{tikzcd} \qquad \begin{tikzcd} (i, j \xrightarrow{\xi} u(i)) \arrow[r, mapsto] \arrow[d, mapsto] & i \arrow[d, mapsto] \\ * \arrow[r, mapsto] & j \arrow[r, "\xi"] & u(i) \end{tikzcd}$$

defined by

$$\alpha_{(i,\xi)} = \xi$$

The axiom Der 4d says that the corresponding base change morphism is an isomorphism

$$p_! k^* \xrightarrow{\sim} j^* u_!$$

We will say that the prederivator $\mathbb{D}$ is <u>cocomplete</u>, if it admits left Kan extensions <u>and</u> satisfies this ~~left~~ right Der 4 axiom. Let us explain a little more this last axiom.

If A is a small category, let denote by $p_A : A \to e$ the unique functor from A to the point category e. Define in the framework of derivators the homotopy colimit functor

$$\underrightarrow{\mathrm{holim}}_A : \mathbb{D}(A^\circ) \to \mathbb{D}(e)$$

by

$$\underrightarrow{\mathrm{holim}}_A = (p_{A^\circ})_! \,.$$

(The reason of the choice of the opposite category A° is for compatibility reasons with the homotopy colimits in the framework of localizers $(\mathcal{C}, W)$ where homotopy colimits are defined for covariant functors whereas the categories $\mathbb{D}_{(\mathcal{C},W)}(I)$ are localizations of presheaves)

If $I \xrightarrow{u} J$ is an arrow in $\mathcal{D}ia$ and X an object of $\mathbb{D}(J)$, when there is no ambiguity about the functor u, the object $u^*(X)$ of $\mathbb{D}(I)$ is sometimes denoted by $X|I$

For I in $\mathcal{D}ia$, X in $\mathbb{D}(I)$ and i an object of I, define $X(i) := i^*(X)$ where $i : e \to I$ denotes also the functor defined by the object i.

With these notations the axiom Der 4 says that for every arrow $I \xrightarrow{u} J$ in $\mathcal{D}ia$, every object X of $\mathbb{D}(I)$ and every object j of J the canonical morphisme

$$\underrightarrow{\mathrm{holim}}_{(j\backslash I)^{\circ}} X|(j\backslash I) \xrightarrow{\sim} u_!(X)(j)$$

is an isomorphisme of $\mathbb{D}(e)$

<u>Example</u> If $(\mathcal{C},W)$ is a cocomplete localizer, then the prederivator $\mathbb{D}_{(\mathcal{C},W)}$ is cocomplete

This is now tautological if we observe that for any functor $u : I \to J$ and any object j of J, the category $(j\backslash I)^{\circ}$ is canonicaly isomorphic to the category I°/j (in category theorist's notations $(j \downarrow u)^{\circ} \simeq (u^{\circ} \downarrow j)$)

The axiom Der 4d is the dual axiom. Consider a 2-square in Dia

$$\mathcal{D} = \begin{tikzcd} A' \arrow[r, "v"] \arrow[d, "u'"'] & A \arrow[d, "u"] \arrow[dl, phantom, "\swArrow\alpha"] \\ B' \arrow[r, "w"'] & B \end{tikzcd} = \begin{tikzcd} A' \arrow[r, "u'"] \arrow[d, "v"'] & B' \arrow[d, "w"] \arrow[dl, phantom, "\swArrow\alpha"] \\ A \arrow[r, "u"'] & B \end{tikzcd} \qquad uv \xrightarrow{\alpha} wu'$$

By 2-functoriality we deduce a 2-square in CAT

$$\begin{tikzcd} \mathbb{D}(A') & \mathbb{D}(A) \arrow[l, "v^*"'] \\ \mathbb{D}(B') \arrow[u, "u'^*"] & \mathbb{D}(B) \arrow[l, "w^*"] \arrow[u, "u^*"'] \arrow[ul, phantom, "\neArrow\alpha^*"] \end{tikzcd} \qquad u'^* w^* \xrightarrow{\alpha^*} v^* u^*$$

and if the prederivator $\mathbb{D}$ admits right Kan extensions, using the unit $1_{\mathbb{D}(B')} \xrightarrow{\eta'} u'_* u'^*$ and the counit $u^* u_* \xrightarrow{\varepsilon} 1_{\mathbb{D}(A)}$ of the adjunctions (u'^*, u'_*) and (u^*, u_*) we get an arrow

$$\begin{tikzcd} w^* u_* \arrow[r, "c'_{\mathcal{D}}"] \arrow[d, "\eta' * w^* u_*"'] & u'_* v^* \\ u'_* u'^* w^* u_* \arrow[r, "u'_* * \alpha^* * u_*"'] & u'_* v^* u^* u_* \arrow[u, "u'_* v^* * \varepsilon"'] \end{tikzcd}$$

$$c'_{\mathcal{D}} = (u'_* v^* * \varepsilon)(u'_* * \alpha^* * u_*)(\eta' * w^* u_*)$$

Known as the cohomological (or right) change base ~~functor~~ morphism. If $\mathbb{D}$ admits both left and right Kan extensions, then $c'_{\mathcal{D}}$ is simply the "transpose" morphism of the left base change morphism $c_{\mathcal{D}} : v_! u'^* \to u^* w_!$

Let $I \xrightarrow{u} J$ an arrow in Dia and j an object of J. The 2-square

$$\begin{tikzcd} I/j \arrow[r, "\ell"] \arrow[d, "q"'] & I \arrow[d, "u"] \arrow[dl, phantom, "\swarrow \beta"] \\ e \arrow[r, "j"'] & J \end{tikzcd}
\qquad
\begin{tikzcd} (i, u(i) \xrightarrow{s} j) \arrow[r, mapsto] \arrow[dd, mapsto] & i \arrow[d, mapsto] \\ & u(i) \arrow[d, "s"] \\ * \arrow[r, mapsto] & j \end{tikzcd}$$

defined by

$$\beta_{(i,s)} = s$$

induces a base change morphism

$$j^* u_* \longrightarrow q_* \ell^*$$

The axiom Der 4d says that this morphism is an iso.

If we introduce a notation for the homotopy limit functor

$$\underleftarrow{holim}_A := (p_{A^\circ})_* : \mathbb{D}(A) \longrightarrow \mathbb{D}(e)$$

the axiom Der 4d says that for every $I \xrightarrow{u} J$ in Dia, every object X of $\mathbb{D}(I)$ and every object j of J, the canonical morphisme

$$u_*(X)(j) \longrightarrow \underleftarrow{holim}_{(I/j)^\circ} X \big| (I/j)$$

is an isomorphism

Let us now explain the Der 2 axiom

Recall that for I in Dia, X an object of $\mathbb{D}(I)$ and i an object of I we defined

$$X(i) := i^*(X) \in \mathrm{Ob}\,\mathbb{D}(e)$$

where $i : e \to I$ denotes the functor from the point category e to I defined by the object i. Observe that a morphism $\alpha : i \to i'$ in I defines a natural transformation denoted by the same symbol

$$\begin{tikzcd} e \arrow[r, bend left=50, "i"{name=U}] \arrow[r, bend right=50, "i'"'{name=D}] & I \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$$

and by contravariant 2-functoriality $\alpha^* : i'^* \to i^*$ and

$$\alpha^*_X : X(i') = i'^*(X) \to i^*(X) = X(i) .$$

It is easy to verify that this defines a functor

$$\mathrm{dia}(X) : I^\circ \longrightarrow \mathbb{D}(e)$$

called the <u>underlying functor</u> or <u>underlying diagram</u> of X, functorially in X, so that we get a functor

$$\mathrm{dia}_I : \mathbb{D}(I) \longrightarrow \underline{\mathrm{Hom}}(I^\circ, \mathbb{D}(e)) = \mathbb{D}(e)^{I^\circ}$$

We will say that the prederivator $\mathbb{D}$ is <u>conservative</u> if for every I in Dia the functor dia_I is conservative, and this is the Der 2 axiom. This means exactly that if I is in Dia and $f : X \to X'$ an arrow of $\mathbb{D}(I)$ such that for every object i of I, $i^*(f) : X(i) \to X'(i)$ is an isomorphism, then f is an isomorphism.

An example of a consequence of this axiom:

<u>Proposition</u> Let $\mathbb{D}$ be a conservative and cocomplete prederivator of domain Dia. If $u : I \to J$ is a fully faithful functor in Dia, then $u_!$ is fully faithful

<u>Proof</u> It's enough to prove that the counit $1_{\mathbb{D}(I)} \to u^* u_!$ of the adjunction $(u_! ; u^*)$ is an isomorphism and by conservativity that for any object i de I, $i^* \to i^* u^* u_! = u(i)^* u_!$ is an isomorphism (where $i : e \to I$ and $u(i) : e \to J$ denotes also the functors defined by the objects i and $u(i)$)

By axiom Der 4 d, there is a canonical isomorphism

$$p_! k^* \xrightarrow{\sim} u(i)^* u_! \qquad \begin{array}{ccc} i\backslash I \simeq u(i)\backslash I & \xrightarrow{k} & I \\ p \big\downarrow & & \big\downarrow u \\ e & \xrightarrow[u(i)]{} & J \end{array}$$

and as u is fully faithful $u(i)\backslash I \simeq i\backslash I$, which has an initial object $(i, 1_i)$. The pair $((i,1_i), p)$ (where $(i,1_i) : e \to i\backslash I$ denote also the functor defined by the object $(i,1_i)$) is an adjoint pair, and by 2-fonctoriality $((i,1_i)^*, p^*)$ is an adjoint pair and $p_! \simeq (i,1_i)^*$. Therefore

$$p_! k^* \simeq (i,1_i)^* k^* = (k(i,1_i))^* = i^*$$

which proves the proposition modulo the isomorphism compatibility "easy" verifications.

In what follows, we will fix a prederivator $\mathbb{D}$ of domain $\mathcal{D}ia$ and we will suppose that $\mathbb{D}$ is <u>complete</u>, <u>cocomplete</u> and <u>conservative</u>, i.e. that it satisfies axioms Der 2, Der 3, Der 4

An arrow $I \xrightarrow{u} J$ in $\mathcal{D}ia$ is called a $\mathbb{D}$-equivalence if the canonical arrow induced by u

$$\begin{tikzcd} (p_I)_!\, p_I^* \arrow[r, "\sim"] \arrow[d, phantom, "\wr"] & (p_J)_!\, p_J^* \\ (p_J)_!\, u_!\, u^* (p_J)_! \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} I \arrow[rr, "u"] \arrow[dr, "p_I"'] & & J \arrow[dl, "p_J"] \\ & e & \end{tikzcd}$$

is an isomorphism. This by "transposition" is equivalent to ask that the arrow

$$\begin{tikzcd} (p_J)_*\, p_J^* \arrow[r, "\sim"] \arrow[dr] & (p_I)_*\, p_I^* \arrow[d, phantom, "\wr"] \\ & (p_J)_*\, u_*\, u^* (p_J)^* \end{tikzcd}$$

is an isomorphism. In Grothendieck's homological (resp. cohomological) notation this means for every X in $\mathbb{D}(e)$ that the morphism

$$H_*(I, p_I^* X) \longrightarrow H_*(J, p_J^* X)$$

(resp. $H^*(J, p_J^* X) \longrightarrow H^*(I, p_I^* X)$)

induced by u is an isomorphism, i.e that u induces isomorphism in homology (resp cohomology) with constant coefficients

<u>Examples</u> 1) If $\mathbb{D} = \mathrm{HOT}$ is the prederivator (which is a derivator) associated to the localizer (Top, W) of topological spaces and weak equivalences of topological spaces then $\mathbb{D}$-equivalences are exactly usual weak equivalences in $\mathcal{C}at$, i.e. functors between small categories such that the topological realization of the nerve is a weak equivalence (and even a homotopy equivalence) of topological spaces.

2) If C is a complete and cocomplete category which is <u>not</u> a preordered set and if $\mathbb{D}$ is the prederivator (which is a derivator) defined by the localizer $(C, \mathrm{iso}(C))$, i.e. $\mathbb{D}(I) = C^{I^\circ}$ = presheaves on I with values in C, then $\mathbb{D}$-equivalences are functors $u : I \to J$ between small categories inducing bijection of connected components.

3) If $\mathbb{D} = \mathbb{D}\mathrm{er}_K$, K commutative ring, then $I \xrightarrow{u} J$ is a $\mathbb{D}$-equivalence if and only if u induces an isomorphism of homology (or cohomology) groups with coefficients constant K-modules

The $\mathbb{D}$-equivalences "satisfy the theorem A of Quillen":

Let

$$\begin{tikzcd} I \arrow[rr, "u"] \arrow[dr, "v"'] & & J \arrow[dl, "w"] \\ & K & \end{tikzcd}$$

be a commutative triangle in Dia and consider the commutative diagram

$$\begin{tikzcd} I \arrow[rr, "u"] \arrow[dr, "v"] \arrow[ddr, bend right, "p_I"'] & & J \arrow[dl, "w"'] \arrow[ddl, bend left, "p_J"] \\ & K \arrow[d, "p_K"] & \\ & e & \end{tikzcd}$$

We will say that u is a $\mathbb{D}$-equivalence colocally (resp. locally) on K if the canonical map

$$\begin{tikzcd} v_! p_I^* \arrow[r, "\sim"] \arrow[d, "\wr"] & w_! p_J^* \\ w_! u_! u^* p_J^* \arrow[ur] & \end{tikzcd} \qquad \left(\text{resp. } \begin{tikzcd} w_* p_J^* \arrow[r, "\sim"] \arrow[dr] & v_* p_I^* \arrow[d, "\wr"] \\ & w_* u_* u^* p_J^* \end{tikzcd}\right)$$

is an isomorphism. If u is a $\mathbb{D}$-equivalence colocally (resp. locally) on K, then by applying $(p_K)_!$ (resp. $(p_K)_*$)

$$\begin{tikzcd} (p_K)_! v_! p_I^* \arrow[r] \arrow[d, "\wr"] & (p_K)_! w_! p_J^* \arrow[d, "\wr"] \\ (p_I)_! p_I^* & (p_J)_! p_J^* \end{tikzcd} \qquad \left(\text{resp. } \begin{tikzcd} (p_K)_* w_* p_J^* \arrow[r] \arrow[d, "\wr"] & (p_K)_* v_* p_I^* \arrow[d, "\wr"] \\ (p_J)_* p_J^* & (p_I)_* p_I^* \end{tikzcd}\right)$$

is an isomorphism, so u is a (global) $\mathbb{D}$-equivalence.

Let us characterize arrows in Dia that are, for example, $\mathbb{D}$-equivalences locally on K.

Observe that as $\mathbb{D}$ is a conservative prederivator u is a $\mathbb{D}$-equivalence locally on K if and only if for every object k of K the arrow

$$k^* w_* p_J^* \longrightarrow k^* v_* p_I^* \qquad (*)$$

is a isomorphism. On the other hand, by Der 4g we have canonical isomorphisms

$$k^* w_* \simeq (p_{J/k})_* b^* \quad \text{and} \quad k^* v_* \simeq (p_{I/k})_* a^*$$

$$\begin{tikzcd} J/k \arrow[r, "b"] \arrow[d, "p_{J/k}"'] \arrow[dr, phantom, "\swarrow"] & J \arrow[d, "w"] \\ e \arrow[r, "k"'] & K \end{tikzcd} \qquad \begin{tikzcd} I/k \arrow[r, "a"] \arrow[d, "p_{I/k}"'] \arrow[dr, phantom, "\swarrow"] & I \arrow[d, "v"] \\ e \arrow[r, "k"'] & K \end{tikzcd}$$

and so

$$k^* w_* p_J^* \simeq (p_{J/k})_* b^* p_J^* = (p_{J/k})_* p_{J/k}^*$$

and

$$k^* v_* p_I^* \simeq (p_{I/k})_* a^* p_I^* = (p_{I/k})_* p_{I/k}^*$$

and the map $(*)$ is identified to the canonical map

$$(p_{J/k})_* p_{J/k}^* \longrightarrow (p_{I/k})_* p_{I/k}^* .$$

In conclusion u is a $\mathbb{D}$-equivalence locally on K if and only if for every object k of K the functor $I/k \to J/k$ induced by u is a $\mathbb{D}$-equivalence. Dually, u is a $\mathbb{D}$-equivalence colocally on K if and only if for every object k of K the functor $k\backslash I \to k\backslash J$ induced by u is a $\mathbb{D}$-equivalence

So the fact that a colocal (resp. local) $\mathbb{D}$-equivalence is a $\mathbb{D}$-equivalence is a relative version of Quillen's A theorem, the precise Quillen's situation corresponding to the case when $K = J$ and $w = 1_J$, in which case we will say that u is $\mathbb{D}$-coaspheric (resp. $\mathbb{D}$-aspheric)

So $u : I \longrightarrow J$ in Dia is called $\mathbb{D}$-<u>coaspheric</u> (resp. $\mathbb{D}$-<u>aspheric</u>) if the following equivalent conditions are satisfied:

a) $u_! \, p_I^* \longrightarrow p_J^*$ (resp $p_J^* \longrightarrow u_* \, p_I^*$) is an isomorphism;

b) $(p_J)_* \longrightarrow (p_I)_* u^*$ (resp. $(p_I)_! \, u^* \longrightarrow (p_J)_!$) is a isomorphism;

c) for every object j of J, the functor $j\backslash I \longrightarrow j\backslash J$ (resp. $I/j \longrightarrow J/j$) induced by u is a $\mathbb{D}$-equivalence

The arrows in (b) are obtained by "transposition" of the arrows in (a). The conditions in (b) are "cofinality conditions":

$$\underleftarrow{\mathrm{holim}}_{J^\circ} \longrightarrow \underleftarrow{\mathrm{holim}}_{I^\circ} u^* \quad (\text{resp. } \underrightarrow{\mathrm{holim}}_{I^\circ} u^* \longrightarrow \underrightarrow{\mathrm{holim}}_{J^\circ}).$$

If the $\mathbb{D}$-equivalences are the arrows in Dia inducing bijection of connected components (as in the case of the derivator $I \mapsto C^{I^\circ}$) then these conditions are the classical cofinality conditions. So the notions of asphericity and coasphericity are the homotopy analogues (in the framework of derivators) of the notions of cofinality.

<u>Exercise</u> Prove that if (u, v) is an adjoint pair of arrows in $\mathcal{D}ia$, then u is $\mathbb{D}$-aspherical and v is $\mathbb{D}$-coaspherical. Deduce that if I is in $\mathcal{D}ia$, and if i is an initial (resp. a final) object of I then the map $e \longrightarrow I$ defined by the object i is $\mathbb{D}$-aspherical (resp. $\mathbb{D}$-coaspherical)

An important consequence of the fact that $\mathbb{D}$-equivalences satisfy the theorem A of Quillen is the following theorem

<u>Theorem</u> (Grothendieck) An arrow $u : I \to J$ in $\mathcal{D}ia$ is a $\mathbb{D}$-equivalence if and only if $u^{o} : I^{o} \to J^{o}$ is a $\mathbb{D}$-equivalence

As a consequence we have:

$$u \ \mathbb{D}\text{-aspherical} \iff u^{o} : \mathbb{D}\text{-coaspherical}$$

and for a commutative triangle in $\mathcal{D}ia$

$$\begin{tikzcd} I \arrow[rr, "u"] \arrow[dr] & & J \arrow[dl] \\ & K & \end{tikzcd}$$

u $\mathbb{D}$-equivalence locally on K $\iff$

$\iff$ u^{o} $\mathbb{D}$-equivalence colocally on K

A very important fact is the following.

<u>Theorem</u> (Cisinski) If $\mathcal{D}ia = \mathcal{C}at$ then every usual weak equivalence (map in $\mathcal{C}at$ whose topological realization of the nerve is an homotopy equivalence) is a $\mathbb{D}$-equivalence

A category I in $\mathbb{D}$ia is called $\mathbb{D}$<u>aspheric</u> if the map $p_I : I \longrightarrow e$ to the point category is a $\mathbb{D}$-equivalence.

<u>Proposition</u> Let I in $\mathcal{D}$ia. The following conditions are equivalent:

a) I is $\mathbb{D}$-aspheric;

b) p_I is $\mathbb{D}$-aspheric;

c) p_I is $\mathbb{D}$-coaspheric;

d) $(p_I)_! \, p_I^* \longrightarrow 1_{\mathbb{D}(e)}$ is an isomorphism;

e) $1_{\mathbb{D}(e)} \longrightarrow (p_I)_* \, p_I^*$ is an isomorphism;

f) p_I^* is fully faithful

The exercise p. 32 implies that if I has an initial of a final object then I is $\mathbb{D}$-aspheric

More generally, Cisinski's theorem (p. 32) implies that if the classifying space of I is a contractible space then I is $\mathbb{D}$-aspheric.

$\mathbb{D}$-exact squares, $\mathbb{D}$-proper, $\mathbb{D}$-smooth functors.

A 2-square in $\mathcal{D}ia$

$$\mathcal{D} = \begin{tikzcd} I' \arrow[r, "v"] \arrow[d, "u'"'] \arrow[dr, phantom, "\swArrow \alpha"] & I \arrow[d, "u"] \\ J' \arrow[r, "w"'] & J \end{tikzcd}$$

is called $\mathbb{D}$-exact, or is said to satisfy the base change property for $\mathbb{D}$,

if the homological base change map

$$v_! u'^* \xrightarrow{c_{\mathcal{D}}} u^* w_!$$

or equivalently the cohomological base change map

$$w^* u_* \xrightarrow{c'_{\mathcal{D}}} u'_* v^*$$

is an isomorphism.

Proposition The following conditions are equivalent:

a) $\mathcal{D}$ is $\mathbb{D}$-exact;

b) for every object i of I the map $i\backslash I' \longrightarrow u(i)\backslash J'$ induced by u' is a $\mathbb{D}$-equivalence locally on J';

c) for every object j' of J' the map $I'/j' \longrightarrow I/w(j')$ induced by v is a $\mathbb{D}$-equivalence colocally on I.

Proof As $\mathbb{D}$ is a conservative prederivator, the map $c_{\mathcal{D}}$ is an isomorphism if and only if for every object i of I

$$i^* v_! u'^* \longrightarrow i^* u^* w_! = (u(i))^* w_! \qquad (1)$$

is an isomorphism. Using Der 3g we have canonical isomorphisms

$$i^* v_! \simeq (p_{i\backslash I'})_!\, k^* \,, \qquad (u(i))^* w_! \simeq (p_{u(i)\backslash J'})_!\, l^*$$

$$\begin{tikzcd} i\backslash I' \arrow[r, "k"] \arrow[d, "p_{i\backslash I'}"'] \arrow[dr, phantom, "\Nearrow"] & I' \arrow[d, "v"] \\ e \arrow[r, "i"'] & I \end{tikzcd} \qquad \begin{tikzcd} u(i)\backslash J' \arrow[r, "l"] \arrow[d, "p_{u(i)\backslash J'}"'] \arrow[dr, phantom, "\Nearrow"] & J' \arrow[d, "w"] \\ e \arrow[r, "u(i)"'] & J \end{tikzcd}$$

so the arrow (1) is identified to

$$(p_{i\backslash I'})_!\,(u'k)^* = (p_{i\backslash I'})_!\, k^* u'^* \longrightarrow (p_{u(i)\backslash J'})_!\, l^*$$

obtained by transposition of

$$l_*\,(p_{u(i)\backslash J'})^* \longrightarrow (u'k)_*\,(p_{i\backslash I'})^* \qquad (2)$$

As by definition this map is an isomorphism if and only if $i\backslash u'$ is a $\mathbb{D}$-equivalence

$$\begin{tikzcd} i\backslash I' \arrow[r, "i\backslash u'"] \arrow[d, "k"'] \arrow[dr, "u'k"] & u(i)\backslash J' \arrow[d, "l"] \\ I' \arrow[r, "u'"'] & J' \end{tikzcd} \qquad \begin{tikzcd} (i', i \xrightarrow{s} v(i')) \arrow[r, mapsto] \arrow[d] & (u'(i'), u(i) \xrightarrow{\alpha_{i'}\, u(s)} w(u'(i'))) \arrow[d] \\ i' \arrow[r, mapsto] & u'(i') \end{tikzcd}$$

locally on J', this proves the equivalence of (a) and (b).

The equivalence of (a) and (c) is proved dually by using the map $c'_{\mathbb{D}}$ obtained by transposition of $c_{\mathbb{D}}$.

If $\mathcal{C} = $
$$\begin{tikzcd} I' \arrow[r, "v"] \arrow[d, "u'"'] & I \arrow[d, "u"] \\ J' \arrow[r, "w"'] & J \end{tikzcd}$$
is a commutative square, it can be considered as a 2-square in two different ways

$$\mathcal{D} = \begin{tikzcd} I' \arrow[r, "v"] \arrow[d, "u'"'] & I \arrow[d, "u"] \arrow[dl, phantom, "\swarrow 1_{uv}"] \\ J' \arrow[r, "w"'] & J \end{tikzcd} \quad \text{and} \quad \mathcal{D}' = \begin{tikzcd} I' \arrow[r, "v"] \arrow[d, "u'"'] & I \arrow[d, "u"] \arrow[dl, phantom, "\nearrow 1_{uv}"] \\ J' \arrow[r, "w"'] & J \end{tikzcd}$$

and the corresponding base change maps in the two situations have nothing to do each other.

$$v_! u'^* \xrightarrow{c_{\mathcal{D}}} u^* w_! \qquad u'_! v^* \xrightarrow{c_{\mathcal{D}'}} w^* u_!$$

$$w^* u_* \xrightarrow{c'_{\mathcal{D}}} u'_* v^* \qquad u^* w_* \xrightarrow{c'_{\mathcal{D}'}} v_* u'^*$$

By convention, we will always consider the commutative square $\mathcal{C}$ as the 2-square $\mathcal{D}$ and <u>never</u> as $\mathcal{D}'$. So the commutative square $\mathcal{C}$ is $\mathbb{D}$-exact if

$$v_! u'^* \longrightarrow u^* w_! \quad \text{or equivalently} \quad w^* u_* \longrightarrow u'_* v^*$$

is an isomorphism.

Grothendieck, by analogy to the cohomological base change properties of proper and smooth maps in algebraic geometry, gave the following definitions

A map $I \xrightarrow{u} J$ (resp. $J' \xrightarrow{w} J$) in Dia is called $\mathbb{D}$-<u>proper</u> (resp. $\mathbb{D}$-<u>smooth</u>) if every cartesian square of the form

$$\begin{tikzcd} I' \arrow[r, "v"] \arrow[d, "u'"'] & I \arrow[d, "u"] \\ J' \arrow[r, "w"'] & J \end{tikzcd}$$

is $\mathbb{D}$-exact, and this property remains true after any base change.

Theorem (Grothendieck) Let $u : I \to J$ a map in $\mathcal{D}ia$. The following conditions are equivalent:

a) u is $\mathbb{D}$-proper;

b) for every diagram of cartesian squares of the form:

$$\begin{tikzcd} I'' \arrow[r] \arrow[d] & I' \arrow[r] \arrow[d] & I \arrow[d, "u"] \\ J'' \arrow[r] & J' \arrow[r] & J \end{tikzcd}$$

the left square is $\mathbb{D}$-exact;

c) for every diagram of cartesian squares of the form

$$\begin{tikzcd} I'' \arrow[r, "v"] \arrow[d] & I' \arrow[r] \arrow[d, "u'"] & I \arrow[d, "u"] \\ J'' \arrow[r, "w"'] & J' \arrow[r] & J \end{tikzcd}$$

if w is $\mathbb{D}$-coaspheric then v is $\mathbb{D}$-coaspheric

d) for every object j of J, the canonical map

$$I_j \longrightarrow I/j \ , \quad i \longmapsto (i, u(i) \xrightarrow{1_j} j)$$

where I_j is the fiber of u over j, is $\mathbb{D}$-coaspheric.

Proof (a) $\Leftrightarrow$ (b) is a translation of the definition

(b) $\Rightarrow$ (c). It's enough to prove that if a commutative square

$$\begin{tikzcd} I' \arrow[r, "v"] \arrow[d, "u'"'] & I \arrow[d, "u"] \\ J' \arrow[r, "w"'] & J \end{tikzcd}$$

is $\mathbb{D}$-exact and if w is $\mathbb{D}$-coaspheric then v is $\mathbb{D}$-coaspheric.

As the square is $\mathbb{D}$-exact the base change map $v_! u'^* \xrightarrow{c} u^* w_!$ is an isomorphism and as w is $\mathbb{D}$-coaspheric the canonical map $w_! p_{J'}^* \xrightarrow{a} p_J^*$ is an isomorphism. Observe that the canonical map $v_! p_{I'}^* \to p_I^*$ is equal to $(u^* * a)(c * p_{J'}^*)$

$$v_! \, p_{I'}^{*} = v_! \, u'^{*} p_{J'}^{*} \xrightarrow[\sim]{c * p_{J'}^{*}} u^{*} w_! \, p_{J'}^{*} \xrightarrow[\sim]{u^{*} * a} u^{*} p_J^{*} = p_I^{*}$$

Therefore the map v is $\mathbb{D}$-coaspherical

(c) $\Rightarrow$ (d). Consider the diagram of cartesian squares (for j an object of J)

$$\begin{tikzcd} I_j \arrow[r] \arrow[d] & I/j \arrow[r] \arrow[d] & I \arrow[d] \\ e \arrow[r, "(j,1_j)"'] & J/j \arrow[r] & J \end{tikzcd}$$

As $(j, 1_j)$ is a final object of J/j, the functor $(j,1_j) : e \to J/j$ has a left adjoint and is therefore $\mathbb{D}$-coaspherical. The condition (c) implies then that $I_j \longrightarrow I/j$ is $\mathbb{D}$-coaspherical

(d) $\Rightarrow$ (a). It's not hard to prove, and is left as an exercise that the condition (d) is equivalent to the condition that for every cartesian square of the form

$$\begin{tikzcd} I' \arrow[r] \arrow[d] & I \arrow[d, "u"] \\ J' = \{0 \to 1\} \arrow[r] & J \end{tikzcd}$$

the inclusion $I'_0 \hookrightarrow I$ of the fiber of u' over 0 is a $\mathbb{D}$-coaspherical map.

which implies that the condition (d) is stable under base change.

So it is enough to prove that the condition (d) implies that every cartesian square of the form

$$(*) \quad \begin{tikzcd} I' \arrow[r, "v"] \arrow[d, "u'"'] & I \arrow[d, "u"] \\ J' \arrow[r, "w"'] & J \end{tikzcd}$$

is $\mathbb{D}$-exact

Using the proposition characterizing the $\mathbb{D}$-exact squares, it's enough to prove that for every object j' of J', the map

$$I'/j' \longrightarrow I/w(j')$$

induced by v is a $\mathbb{D}$-equivalence colocally over I, or better that this map is $\mathbb{D}$-coaspheric. Observe that in the commutative square

$$\begin{tikzcd}
I'_{j'} \arrow[r] \arrow[d] & I_{w(j')} \arrow[d] \\
I'/j' \arrow[r] & I/w(j')
\end{tikzcd}$$

the upper horizontal arrow is an isomorphism (as the square $*$ is cartesian) and that the vertical arrows are $\mathbb{D}$-coaspheric (as u and u' satisfy the condition (d): u by hypothesis and u' by stability of this condition by base change). This implies that the lower horizontal arrow is $\mathbb{D}$-coaspheric by an easy lemma (exercise):

Let $A \xrightarrow{f} B \xrightarrow{g} C$ in Dia. If f and gf are $\mathbb{D}$-coaspheric then g is $\mathbb{D}$-coaspheric.

Hint

$$\begin{tikzcd}
(gf)_! \, p_A^* \arrow[rr] \arrow[d, phantom, "\wr"] & & p_C^* \\
g_! f_! \, p_A^* \arrow[dr] & & \\
& g_! \, p_B^* \arrow[uur] &
\end{tikzcd}$$

$\square$

There is a dual characterization of $\mathbb{D}$-smooth functor and as a consequence $\boxed{u \ \mathbb{D}\text{-proper} \iff u^{\circ} \ \mathbb{D}\text{-smooth}}$!

The other axioms

Der 1 i) The category $\mathbb{D}(\emptyset)$ is equivalent to the point category.

ii) Let I, J in Dia and

$$\begin{tikzcd} I \arrow[dr, "i"] & \\ & I \amalg J \\ J \arrow[ur, "j"'] & \end{tikzcd}$$

the canonical maps. Then

$$\mathbb{D}(I \amalg J) \xrightarrow{(i^*, j^*)} \mathbb{D}(I) \times \mathbb{D}(J)$$

is an equivalence of categories

Exercise i) Prove that Der 1 (i) joined with Der 3g (resp. Der 3d) implies that for every I in Dia, $\mathbb{D}(I)$ has a final (resp. initial) object

ii) Prove that Der 1 (ii) joined with Der 3g (resp. Der 3d) implies that for every I in Dia, $\mathbb{D}(I)$ is stable under binary products (resp. coproducts)

In order to state Der 5, some preliminaries are needed. Let I and J in Dia. An object j in J defines a map

$$j_I : I \longrightarrow I \times J, \quad i \longmapsto (i, j),$$

hence a functor

$$j_I^* : \mathbb{D}(I \times J) \longrightarrow \mathbb{D}(I).$$

Observe that an arrow $\alpha : j \longrightarrow j'$ of J defines a natural transformation

$$\begin{tikzcd} I \arrow[r, bend left, "j_I"] \arrow[r, bend right, "j'_I"'] \arrow[r, phantom, "\Downarrow \alpha_I"] & I \times J \end{tikzcd} \qquad \alpha_{I,i} := (1_i, \alpha), \quad i \in \operatorname{Ob} I$$

and by 2-functoriality, a natural transformation

$$\begin{tikzcd} \mathbb{D}(I \times J) \arrow[r, bend left=50, "j_I^*"] \arrow[r, bend right=50, "j_I'^*"'] \arrow[r, phantom, "\Uparrow \alpha_I^*"] & \mathbb{D}(I) \end{tikzcd}$$

defining a functor

$$\mathbb{D}(I \times J) \times \mathbb{J}^{\circ} \longrightarrow \mathbb{D}(I)$$

or equivalently a functor

$$\mathbb{D}(I \times J) \xrightarrow{dia_{I,J}} \underline{\mathrm{Hom}}(\mathbb{J}^{\circ}, \mathbb{D}(I)) = \mathbb{D}(I)^{J^{\circ}}$$

generalizing the functor

$$dia_J : \mathbb{D}(J) \simeq \mathbb{D}(e \times J) \longrightarrow \mathbb{D}(e)^{J^{\circ}}$$

We can now state

<u>Der 5</u>) For every I in Dia <u>if J is the category $\{0 \to 1\}$</u>, then the functor

$$\mathbb{D}(I \times J) \xrightarrow{dia_{I,J}} \mathbb{D}(I)^{J^{\circ}}$$

is full and essentially surjective

<u>Remark</u> The functor $dia_{I,J}$ (or dia_I) is <u>never</u> (outside very trivial situations) faithful. Sometimes a stronger form of axiom Der 5 is needed, asking that $dia_{I,J}$ is full and essentially surjective for every <u>free and finite</u> category J. This stronger form is verified if $\mathbb{D} = \mathbb{D}_{(C,W)}$, with (C,W) a Quillen localizer

Pointed Derivators

Recall that if J is a category, a sieve of J is a full subcategory $J' \hookrightarrow J$ such that if $j \to j'$ is an arrow of J with j' in J', then j is also in J'. A functor $u : I \to J$ is called an open immersion if it induces an isomorphism of I onto a sieve of J. The notions of cosieve and closed immersion are defined dually.

A pointed derivator is a derivator satisfying the following axiom

Der 6 If $u : I \to J$ is an open (resp. a closed) immersion, in Dia, then the functor $u_!$ (resp u_*) has a left (resp. right) adjoint $u^?$ (resp. $u^!$).

Exercise i) (Easy) Prove that if $\mathbb{D}$ is a pointed derivator, then for every I in Dia the category $\mathbb{D}(I)$ has a null object (an object that is both initial and final).

ii) (Super-difficult). Conversely prove that if for every I in Dia the category $\mathbb{D}(I)$ has a null object and if $\mathbb{D}$ is a derivator the $\mathbb{D}$ satisfies the axiom Der 6 (some strong stability conditions on Dia are needed)

Triangulated derivators

Denote $\square$ the category

$$\square = \begin{tikzcd} (0,0) & (0,1) \arrow[l] \\ (1,0) \arrow[u] & (1,1) \arrow[l] \arrow[u] \end{tikzcd}$$

and $\ulcorner$, $\lrcorner$ the subcategories

$$\ulcorner = \begin{tikzcd} (0,0) & (0,1) \arrow[l] \\ (1,0) \arrow[u] & \end{tikzcd} \qquad \lrcorner = \begin{tikzcd} & (0,1) \\ (1,0) & (1,1) \arrow[l] \arrow[u] \end{tikzcd}$$

of $\square$. Let $i_\ulcorner : \ulcorner \longrightarrow \square$, $i_\lrcorner : \lrcorner \longrightarrow \square$ the inclusion functors

Let $\mathbb{D}$ be any conservative cocomplete (resp. complete) prederivator (for example a derivator). An object X of $\mathbb{D}(\square)$ is called (homotopically) cocartesian (resp. cartesian) if the adjunction morphism

$$i_{\ulcorner !}\, i_\ulcorner^* X \longrightarrow X \qquad (\text{resp. } X \longrightarrow i_{\lrcorner *}\, i_\lrcorner^* X)$$

is an isomorphism.

Exercise Let $\mathbb{D}$ be a conservative cocomplete (resp complete) prederivator, denote by $k : e \to \square$ the functor defined by the object $(1,1)$ (resp. $(0,0)$) of $\square$ and p the projection $p = p_\ulcorner : \ulcorner \to e$ (resp. $p = p_\lrcorner : \lrcorner \to e$). For every object X of $\mathbb{D}(\square)$

define a canonical morphism, functorial in X,

$$p_!(X|\ulcorner) = p_! i_\ulcorner^* X \longrightarrow k^* X \quad (\text{resp. } k^* X \longrightarrow p_* i_\lrcorner^* X = p_*(X|\lrcorner))$$

and prove that X is cocartesian (resp. cartesian) if and only if this map is an isomorphism. (This means that an object X of $\mathbb{D}(\square)$ is cocartesian (resp. cartesian) if and only if its "lower right vertex" (resp. "upper left vertex") is the "homotopy pushout" (resp. "homotopy pullback") of its "upper left corner" (resp. "lower right corner").)

A <u>triangulated derivator</u> is a pointed derivator $\mathbb{D}$ satisfying the following axiom:

<u>Der 7</u>) An object of $\mathbb{D}(\square)$ is cocartesian if and only if it is cartesian

So a triangulated derivator is a prederivator satisfying all the axioms Der 1 to Der 7.

<u>Examples</u> 1) If $(\mathcal{C}, W)$ is a <u>stable</u> <u>Quillen</u> <u>localizer</u>, i.e. if there exists a stable Quillen model category structure on $\mathcal{C}$ with W as the class of weak equivalences, then the prederivator $\mathbb{D}_{(\mathcal{C},W)}$ is a triangulated derivator. This is a tautological consequence of the non-triangulated case (if we admit the difficult part of exercise p. 42) as a stable model category is by definition a model category with

a null object, such that a commutative square is homotopy cocartesian if and only if it is homotopy cartesian. Particular examples of stable model categories are the model category of spectra, giving rise to the triangulated derivator of spectra $\mathbb{S}p$, and the model category of unbounded complexes of a Grothendieck category $\mathcal{A}$, giving rise to the unbounded derived triangulated derivator $\mathbb{D}er_{\mathcal{A}}$ of $\mathcal{A}$.

2) In the last lecture of this course, Bernhard Keller will prove that if $\mathcal{E}$ is an exact category, the localizer of bounded complexes of $\mathcal{E}$ and quasi-isomorphisms defines a triangulated derivator with domain finite direct categories, the bounded derived triangulated derivator $\mathbb{D}er^{b}_{\mathcal{E}}$ of $\mathcal{E}$.

Theorem Let $\mathbb{D}$ be a triangulated derivator of domain $\mathcal{D}ia$. For every I in $\mathcal{D}ia$ the category $\mathbb{D}(I)$ has a canonical structure of a triangulated category such that for every arrow $u: I \longrightarrow J$ in $\mathcal{D}ia$, u^* is an exact functor, in the triangulated sens (is a triangulated functor)

We will not prove this theorem; we will simply describe the triangulated structure on $\mathbb{D}(e)$, e point category (which is not a real restriction as it is not hard to prove that if $\mathbb{D}$ is a triangulated derivator then for any I in Dia, $J \mapsto \mathbb{D}(I \times J)$ defines equally well a triangulated derivator) without proving that $\mathbb{D}(e)$ is an additive category (which is the most difficult part) nor proving inversibility of the suspension functor and axioms TR I to TR IV.

Definition of the suspension functor

As for a triangulated derivator an object of $\mathbb{D}(\square)$ is cocartesian if and only if it is cartesian, such an object will be simply called bicartesian.

Denote

$$i_0 : e \longrightarrow \ulcorner \;, \quad i_1 : e \longrightarrow \square$$

the functors defined respectively by the objects $(0,0)$ and $(1,1)$ of $\ulcorner$ and $\square$.

The suspension functor

$$S : \mathbb{D}(e) \longrightarrow \mathbb{D}(e)$$

is defined by

$$S := i_1^* \, i_{\ulcorner !} \, i_{0*}$$

In order to understand this definition, observe that for every object X of $\mathbb{D}(e)$ the object $i_{\ulcorner !} i_{0*} X$ of $\mathbb{D}(\square)$ is bicartesian (every object of the form $i_{\ulcorner !} Y$ is bicartesian; as $i_\ulcorner$ is fully faithful, $Y \to i_\ulcorner^* i_{\ulcorner !} Y$ is an isomorphism (proposition on page 26'); therefore $i_{\ulcorner !} Y \to i_{\ulcorner !} i_\ulcorner^* i_{\ulcorner !} Y$ and $i_{\ulcorner !} i_\ulcorner^* i_{\ulcorner !} Y \to i_{\ulcorner !} Y$ are isomorphisms) and that there are canonical isomorphisms

$$\operatorname{dia}(i_{0*}X) \simeq \begin{tikzcd} X \arrow[r] \arrow[d] & 0 \\ 0 & \end{tikzcd}$$

and

$$\operatorname{dia}(i_{\ulcorner !}\, i_{0*}X) \simeq \begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & S(X) \end{tikzcd}$$

where 0 is some fixed null object of $\mathbb{D}(e)$.

Let now T a bicartesian object of $\mathbb{D}(\square)$ such that $(0,1)^* T$ and $(1,0)^* T$ are null objects of $\mathbb{D}(e)$:

$$\operatorname{dia}(T) = \begin{tikzcd} T_{00} \arrow[r] \arrow[d] & T_{01} \simeq 0 \arrow[d] \\ 0 \simeq T_{10} \arrow[r] & T_{11} \end{tikzcd}$$

By the conservativity axiom the adjunction morphism

$$i_{\ulcorner}^* T \longrightarrow i_{0*} i_0^* i_{\ulcorner}^* T = i_{0*}(T_{00})$$

is an isomorphism. As T is in particular a cocartesian object of $\mathbb{D}(\square)$ the adjunction morphism

$$i_{\ulcorner !}\, i_{\ulcorner}^* T \longrightarrow T$$

is also an isomorphism. We deduce isomorphisms

$$T_{11} = i_1^* T \longleftarrow i_1^* i_{\ulcorner !}\, i_{\ulcorner}^* T \longrightarrow i_1^* i_{\ulcorner !}\, i_{0*}(T_{00}) = S(T_{00})$$

hence an isomorphism

$$\Theta_T : (1,1)^* T = T_{11} \longrightarrow S(T_{00}) = S((0,0)^* T)$$

<u>Definition of distinguished triangles</u>

Denote by $\square\square$ the category

$$\square\square := \begin{array}{ccccc} (0,0) & \longleftarrow & (0,1) & \longleftarrow & (0,2) \\ \uparrow & & \uparrow & & \uparrow \\ (1,0) & \longleftarrow & (1,1) & \longleftarrow & (1,2) \end{array}$$

and

$$j_i : \square \longrightarrow \square\square \ , \qquad i = 0, 1, 2$$

the functor defined by

$$j_i(k,\ell) = (k, \delta_i(\ell))$$

where δ_i is the injection $\{0,1\} \to \{0,1,2\}$ that avoids i. An object T of $\mathbb{D}(\square\square)$ is called bicartesian if $j_i^* T$ is bicartesian for $i = 0,1,2$.

A <u>triangle</u> in $\mathbb{D}(e)$ is just a diagram in $\mathbb{D}(e)$ of the form

$$X \longrightarrow Y \longrightarrow Z \longrightarrow S(X).$$

The notions of morphism and isomorphism of triangles are defined in the evident way.

Let T a bicartesian object of $\mathbb{D}(\square\square)$ such that $(1,0)^* T$ and $(0,2)^* T$ are null objects of $\mathbb{D}(e)$:

$$\mathrm{dia}(T) = \begin{array}{ccccc} T_{00} & \xrightarrow{f} & T_{01} & \longrightarrow & T_{02} \simeq 0 \\ \downarrow & & \downarrow g & & \downarrow \\ 0 \simeq T_{10} & \longrightarrow & T_{11} & \xrightarrow[h]{} & T_{12} \end{array}$$

By definition of bicartesian objects of $\mathbb{D}(\square\square)$,

$j_1^*(T)$ is a bicartesian object of $\mathbb{D}(\square)$. Observe that

$$\operatorname{dia} j_1^* T = \begin{tikzcd} T_{00} \arrow[r] \arrow[d] & T_{02} \simeq 0 \arrow[d] \\ 0 \simeq T_{10} \arrow[r] & T_{12} \end{tikzcd}.$$

So there is a canonical isomorphism

$$\Theta := \Theta_{j_1^* T} : T_{12} \xrightarrow{\sim} S(T_{00})$$

and we get a triangle

$$T_{00} \xrightarrow{f} T_{01} \xrightarrow{g} T_{11} \xrightarrow{\Theta h} S(T_{00}).$$

Such a triangle is calle a <u>standard distinguished triangle</u>. A <u>distinguished triangle</u> is a triangle isomorphic to a standard distinguished triangle