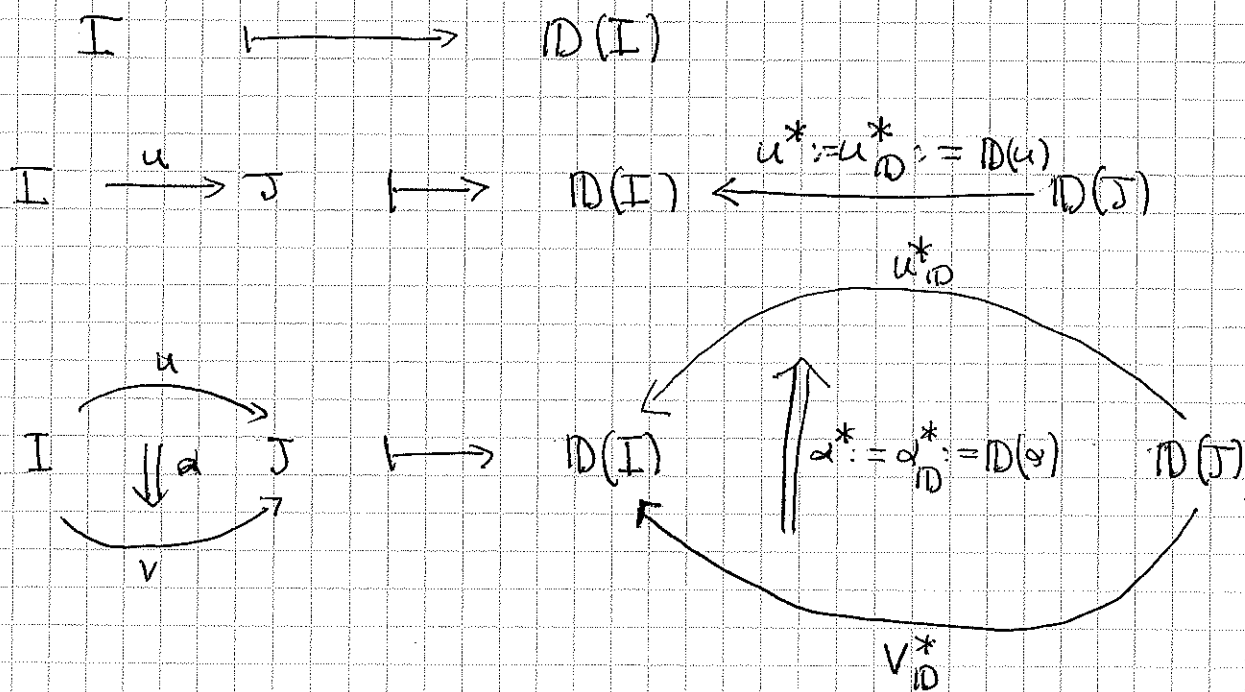


5. A prederivator is just a strict 2-functor  $\mathbb{D}$  from the 2-category  $\text{Cat}$  of small categories or a convenient full sub-2-category  $\text{Dia}$  of  $\text{Cat}$  to the category of categories, contravariant on functors and natural transformations.



The most important example for us is the following. Let  $(\mathcal{C}, W)$  be a localizer and define  $\mathbb{D} := \mathbb{D}_{(\mathcal{C}, W)}$  as follows

$$\mathbb{D}(I) := W_{I^{\circ}}^{-1} \mathcal{C}^{I^{\circ}}$$

where  $I^{\circ} := I^{op}$  denotes the category opposite to  $I$

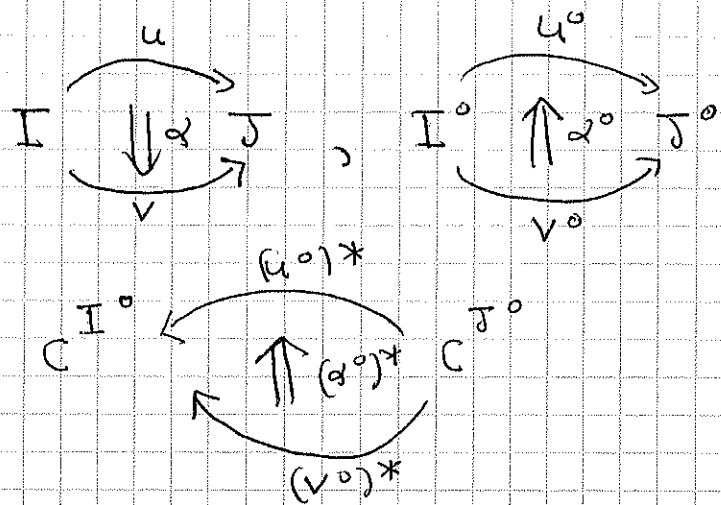
$$u_{\mathbb{D}}^* := \overline{(u^{\circ})^*}$$

$$u: I \rightarrow J, \quad u^{\circ}: I^{\circ} \rightarrow J^{\circ}$$

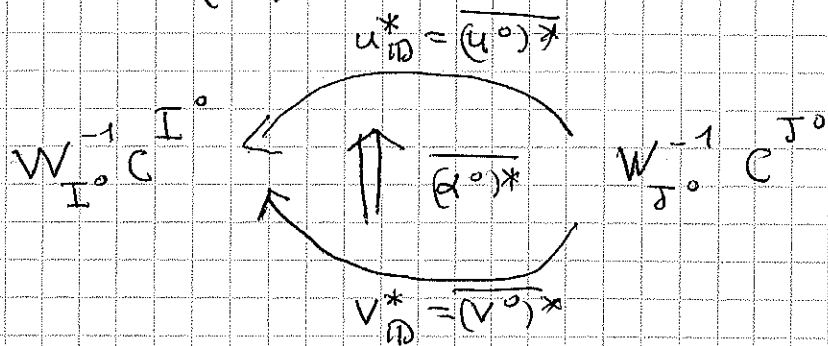
$$(u^{\circ})^*: \mathcal{C}^{J^{\circ}} \rightarrow \mathcal{C}^{I^{\circ}}$$

$$\overline{(u^{\circ})^*}: \underbrace{W_{J^{\circ}}^{-1} \mathcal{C}^{J^{\circ}}}_{\mathbb{D}(J)} \rightarrow \underbrace{W_{I^{\circ}}^{-1} \mathcal{C}^{I^{\circ}}}_{\mathbb{D}(I)}$$

$$\alpha_{\mathbb{D}}^* := \overline{(\alpha^0)^*}$$



and by the 2-universal property of localization



A derivator is a prederivator satisfying a list of axioms Der 1 to Der 5 that will be explained later. A Grothendieck localizer is a localizer  $(C, W)$  such that the prederivator  $\mathbb{D}_{(C, W)}$  is a derivator. A Quillen localizer is a localizer  $(C, W)$  such that there exists a closed model structure on  $C$  with  $W$  as the class of weak equivalences.

~~The~~ Cisinski's theorem that I will explain roughly in the next lecture can be stated as "every Quillen localizer\* is a Grothendieck localizer"

Fix a prederivator  $\mathbb{D}: \text{Dia}^{\circ} \rightarrow \text{CAT}$ , where  $\text{Dia}$  is a full sub-2-category of  $\text{Cat}$  containing finite partially ordered sets and satisfying all stability conditions needed. Typically  $\text{Dia} = \text{Cat}$  or the full subcategory of finite categories, or finite direct-categories, or finite posets.

We will try to explain the axioms of the derivators. The most important are Der 2, Der 3, Der 4.

We will say that the prederivator  $\mathbb{D}$  admits left Kan extensions if for every  $u: I \rightarrow J$  in  $\text{Dia}$ , the functor  $u^*: \mathbb{D}(J) \rightarrow \mathbb{D}(I)$  has a left adjoint that will then be denoted by  $u_! : \mathbb{D}(I) \rightarrow \mathbb{D}(J)$ . This is what is known as Der 3, the "~~left~~<sup>right</sup>" part of the axiom Der 3 (because of right exactness properties of  $u_!$ ).

Example If  $(\mathcal{C}, \mathcal{W})$  is a localizer admitting homotopy left Kan extensions, then the prederivator  $\mathbb{D}_{(\mathcal{C}, \mathcal{W})}: \text{Cat}^{\circ} \rightarrow \text{CAT}$  admits left Kan extensions.

Dually we will say that the prederivator  $\mathbb{D}$  admits right Kan extensions if for every  $u: I \rightarrow J$  in  $\text{Dia}$ , the functor  $u^*: \mathbb{D}(J) \rightarrow \mathbb{D}(I)$  has a right adjoint  $u_*: \mathbb{D}(I) \rightarrow \mathbb{D}(J)$ .

So the axiom Der 3 says that the prederivator  $\mathbb{D}$  admits left and right Kan extensions.

Example Let  $K$  be a commutative ring,  $\mathcal{C}(K)$  the category of complexes of  $K$ -modules, and  $\mathcal{Q}is_K$  the class of quasi-isomorphisms of  $\mathcal{C}(K)$ . Consider the prederivator (which in fact is a derivator),  $\mathbb{D}er_K := \mathbb{D}_{(\mathcal{C}(K), \mathcal{Q}is_K)}$  associated to the localizer  $(\mathcal{C}(K), \mathcal{Q}is_K)$ . Observe that  $\mathbb{D}er_K(\mathbb{e})$ , where  $\mathbb{e}$  is the point category, is simply the derived category of  $K$ -modules, and more generally, for every small category  $I$ ,  $\mathbb{D}er_K(I)$  is canonically isomorphic to the derived category of presheaves on  $I$  with value  $K$ -modules.

Let  $G$  be a finite group and  $\Lambda = K[G]$  the ring of  $G$ . If we consider  $G$  as a category with one object, the category of presheaves on  $G$  with value  $K$ -modules is exactly the category of  $K$ -modules endowed with a right action of  $G$ , or equivalently the category of right  $\Lambda$ -modules.

Recall that for any  $K$ -module  $M$  endowed with a right action of  $G$ , the homology groups  $H_i(G, M)$  (resp.  $H^i(G, M)$  cohomology groups) are defined by

$$H_i(G, M) = \text{Tor}_i^\Lambda(M, K) = H_i(M \otimes_\Lambda K), \quad i \geq 0$$

$$\text{(resp. } H^i(G, M) = \text{Ext}_\Lambda^i(K, M) = H^i R\text{Hom}(K, M), \quad i \geq 0 \text{)}$$

where  $K$  is considered as a  $K[G]$ -bimodule by the  $K$ -algebra morphism  $K[G] \rightarrow K$  defined by  $g \mapsto 1$ ,  $g$  in  $G$ .

As it is easily verified

$$M \otimes K = \begin{matrix} \text{coinvariants of} \\ M \text{ under the} \\ \text{action of } G \end{matrix} = \varinjlim_{G^o} M$$

$$\text{(resp. } \text{Hom}(K, M) = \begin{matrix} \text{invariants of} \\ M \text{ under the} \\ \text{action of } G \end{matrix} = \varprojlim_{G^o} M)$$

and so

$$H_i(G, M) = H_i(\varinjlim_{G^o} M) = H_i((P_G)_!(M))$$

$$\text{(resp. } H^i(G, M) = H^i(\varprojlim_{G^o} M) = H^i((P_G)_*(M))$$

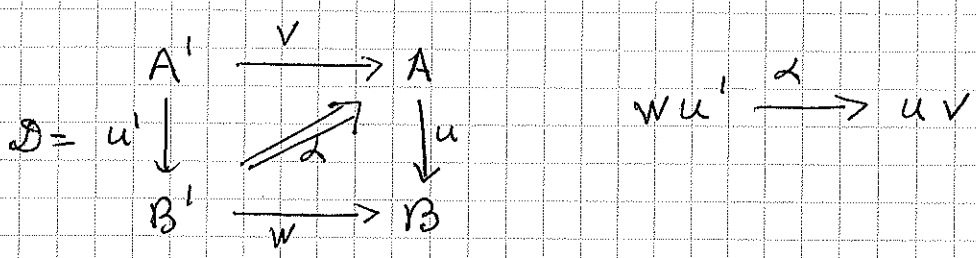
and  $(P_G)_!(M)$  (resp.  $(P_G)_*(M)$ ) can be considered as the "total" homology (resp. cohomology) of the group  $G$  with coefficient the  $\mathbb{Z}$ -right  $G$ -module  $M$  (recall that  $P_G: G \rightarrow e$  denotes the functor of  $G$ -co. the point category)

Grothendieck's philosophy is that for a general derivator  $\mathbb{D}$ , for a category  $I$  in the domain of definition  $\text{Dia}$  of  $\mathbb{D}$  and an object  $X$  of  $\mathbb{D}(I)$ ,  $(P_I)_!(X)$  (resp.  $(P_I)_*(X)$ ) can be considered as the homology (resp. cohomology) object of  $I$  with coefficient  $X$ , and often uses the notation

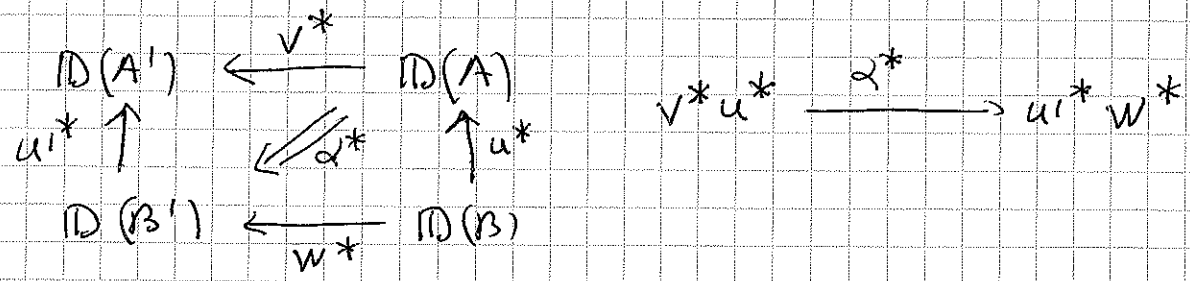
$$H_*(I, X) := (P_I)_!(X) \quad \text{(resp. } H^*(I, X) := (P_I)_*(X))$$

(For a general derivator  $\mathbb{D}$  the "\*" sign in  $H_*$  or  $H^*$  has no particular meaning and the symbols  $H_*$ ,  $H^*$  must be taken as a whole)

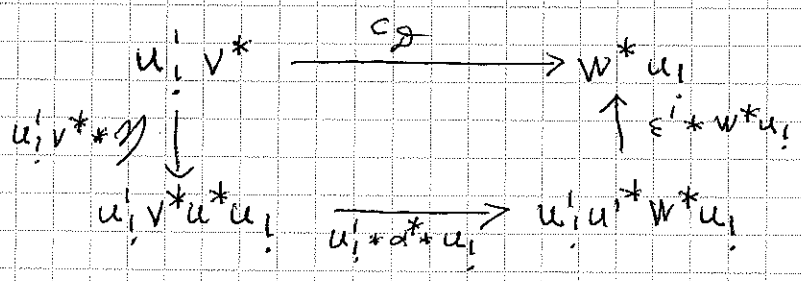
In order to explain axiom Der 4 consider first a "2-square" in  $\mathcal{D}oa$



By contravariant 2-functoriality we deduce a 2-square in  $CAT$

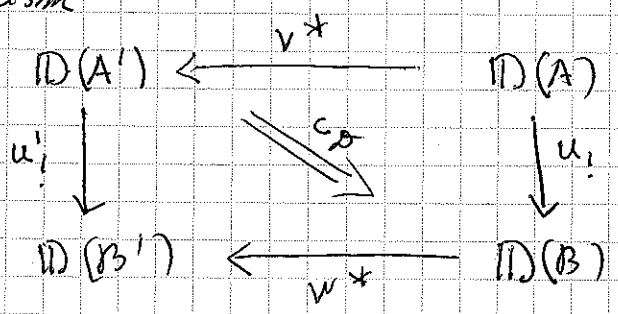


and if the prederivator  $\mathbb{D}$  admits left Kan extensions, using the unit  $1 \xrightarrow{\eta} u^* u_1$  in  $\mathbb{D}(A)$  and the counit  $u_1 u_1^* \xrightarrow{\epsilon^1} 1$  in  $\mathbb{D}(B')$  of the adjunctions  $(u_1, u^*)$  and  $(u_1^*, u^*)$  we get an arrow



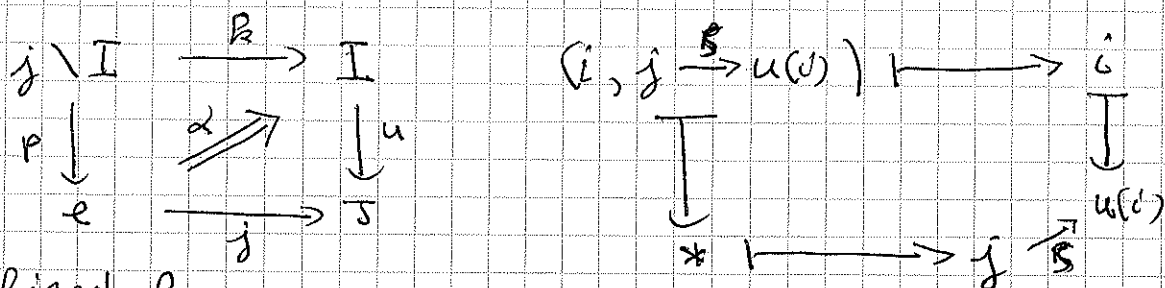
$$c_D = (\epsilon^1 * w^* u_1) (u_1^* \alpha^* u_1) (u_1^* v^* \eta)$$

known as the homological (or left) change of base morphism





A particular case: let  $I \xrightarrow{u} J$  an arrow in  $\mathcal{D}$  and  $j$  an object of  $\mathcal{J}$ . There is a 2-square



defined by

$$\alpha_{(i, \mathcal{S})} = \mathcal{S}$$

The axiom Der 4 says that the corresponding base change morphism is an isomorphism

$$p_! R^* \xrightarrow{\sim} j^* u_!$$

We will say that the prederivator  $\mathbb{D}$  is cocomplete, if it admits left Kan extensions and satisfies this ~~left~~ <sup>right</sup> Der 4 axiom. Let us explain a little more this last axiom.

If  $A$  is a small category, let denote by  $p_A: A \rightarrow e$  the unique functor from  $A$  to the point category  $e$ . Define in the framework of derivators the homotopy colimit functor

$$\text{holim}_{\rightarrow A} : \mathbb{D}(A^{\circ}) \rightarrow \mathbb{D}(e)$$

by

$$\text{holim}_{\rightarrow A} = (p_{A^{\circ}})_!$$

(The reason of the choice of the opposite category  $A^{\circ}$  is for compatibility reasons with the homotopy colimits in the framework of localizers  $(\mathcal{C}, \mathcal{W})$  where <sup>homotopy</sup> colimits are defined for covariant

functions whereas the categories  $D_{(C,W)}(I)$  are localizations of presheaves)

If  $I \xrightarrow{u} J$  is an arrow in  $Dna$  and  $X$  an object of  $D(J)$ , when there is no ambiguity about the functor  $u$ , the object  $u^*(X)$  of  $D(I)$  is sometimes denoted by  $X|I$

For  $I$  in  $Dna$ ,  $X$  in  $D(I)$  and  $i$  an object of  $I$ , define  $X(i) := i^*(X)$  where  $i: e \rightarrow I$  denotes also the functor defined by the object  $i$ .

With these notations the axiom Der 4~~5~~ says that for every arrow  $I \xrightarrow{u} J$  in  $Dna$ , every object  $X$  of  $D(I)$  and every object  $j$  of  $J$  the canonical morphism

$$\text{holim}_{(J \setminus I)^\circ} X|_{(J \setminus I)} \xrightarrow{\sim} u_!(X)(j)$$

is an isomorphism of  $D(e)$

Example If  $(C,W)$  is a cocomplete localizer, then the prederivator  $D_{(C,W)}$  is cocomplete

This is now tautological if we observe that for any functor  $u: I \rightarrow J$  and any object  $j$  of  $J$ , the category  $(J \setminus I)^\circ$  is canonically isomorphic to the category  $I^\circ/j$  (in category theory's notations  $(J \downarrow u)^\circ \simeq (u^\circ \downarrow j)$ )



The axiom Der 4 of is the dual axiom. Consider a 2-square in  $\mathcal{D}$  via

$$\mathcal{D} = \begin{array}{ccc} A' & \xrightarrow{v} & A \\ u' \downarrow & \swarrow \alpha & \downarrow u \\ B' & \xrightarrow{w} & B \end{array} = \begin{array}{ccc} A' & \xrightarrow{u'} & B' \\ v \downarrow & \swarrow \alpha & \downarrow w \\ A & \xrightarrow{u} & B \end{array} \quad uv \xrightarrow{\alpha} wu'$$

By 2-functoriality we deduce a 2-square in  $\text{CAT}$

$$\begin{array}{ccc} \mathbb{D}(A') & \xleftarrow{v^*} & \mathbb{D}(A) \\ u'^* \uparrow & \swarrow \alpha^* & \uparrow u^* \\ \mathbb{D}(B') & \xleftarrow{w^*} & \mathbb{D}(B) \end{array} \quad u'^* w^* \xrightarrow{\alpha^*} v^* u^*$$

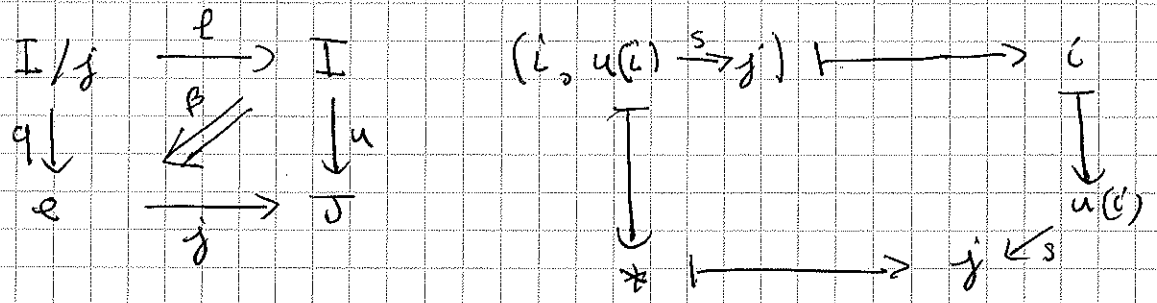
and if the prederivator  $\mathbb{D}$  admits right Kan extensions, using the unit  $1 \xrightarrow{\eta'} u'_* u'^*$  and the counit  $u^* u_* \xrightarrow{\epsilon} 1_{\mathbb{D}(A)}$  of the adjunctions  $(u'^*, u'_*)$  and  $(u^*, u_*)$  we get an arrow

$$\begin{array}{ccc} w^* u_* & \xrightarrow{c'_D} & u'_* v^* \\ \eta'^* w^* u_* \downarrow & & \uparrow u'_* v^* \epsilon \\ u'_* u'^* w^* u_* & \xrightarrow{u'_* \alpha^* u_*} & u'_* v^* u^* u_* \end{array}$$

$$c'_D = (u'_* v^* * \epsilon) (u'_* * \alpha^* + u_*) (\eta'^* w^* u_*)$$

Known as the cohomological (or right) change  $(\leftarrow)$  base morphism. If  $\mathbb{D}$  admits both left and right Kan extensions, then  $c'_D$  is simply the "transpose" morphism of the left base change morphism  $c_D : v_! u'^* \rightarrow u^* w_!$

Let  $I \xrightarrow{u} J$  an arrow in  $\mathcal{D}ia$  and  $j$  an object of  $\overline{J}$ . The 2-square



defined by

$$\beta_{(e,s)} = s$$

induces a base change morphism

$$j^* u_* \longrightarrow q_* \ell^*$$

The axiom Der 4g says that this morphism is an iso.

If we introduce a notation for the homotopy limit functor

$$\text{holim}_A := (p_{A^0})_* : \mathbb{D}(A) \longrightarrow \mathbb{D}(e)$$

The axiom Der 4g says that for every  $I \xrightarrow{u} J$  in  $\mathcal{D}ia$ , every object  $X$  of  $\mathbb{D}(I)$  and every object  $j$  of  $\overline{J}$ , the canonical morphism

$$u_*(X)(j) \longrightarrow \text{holim}_{(I/j)^0} X / (I/j)$$

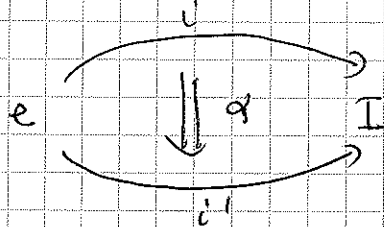
is an isomorphism.

Let us now explain the Der 2 axiom

Recall that for  $I$  in  $\text{Dia}$ ,  $X$  an object of  $\text{ID}(I)$  and  $i$  an object of  $I$  we defined

$$X(i) := i^*(X) \in \text{Ob ID}(e)$$

where  $i: e \rightarrow I$  denotes the functor from the point category  $e$  to  $I$  defined by the object  $i$ . Observe that a morphism  $\alpha: i \rightarrow i'$  in  $I$  defines a natural transformation denoted by the same symbol



and by contravariant 2-functoriality  $\alpha^*: i'^* \rightarrow i^*$  and

$$\alpha_X^*: X(i') = i'^*(X) \rightarrow i^*(X) = X(i).$$

It is easy to verify that this defines a functor

$$\text{dia}(X): I^\circ \longrightarrow \text{ID}(e)$$

called the underlying functor or underlying diagram of  $X$ , functorially in  $X$ , so that we get a functor

$$\text{dia}_I: \text{ID}(I) \longrightarrow \underline{\text{Hom}}(I^\circ, \text{ID}(e)) = \text{ID}(e)^{I^\circ}$$

We will say that the prederivator  $\text{ID}$  is conservative if for every  $I$  in  $\text{Dia}$  the functor  $\text{dia}_I$  is conservative, and this is the Der 2 axiom. This means exactly

that if  $I$  is in  $\text{Dia}$  and  $f: X \rightarrow X'$  an arrow of  $\text{ID}(I)$  such that for every object  $i$  of  $I$ ,  $i^*(f): X(i) \rightarrow X'(i)$  is an isomorphism, then  $f$  is an isomorphism.

An example of a consequence of this axiom:

Proposition Let  $\mathbb{D}$  be a conservative and cocomplete prederivator of domain  $\mathcal{D} \text{ via}$ . If  $u: \mathbb{I} \rightarrow \mathbb{J}$  is a fully faithful functor in  $\mathcal{D} \text{ via}$ , then  $u_!$  is fully faithful

Proof It's enough to prove that the counit  $1_{\mathbb{I} \times \mathbb{J}} \rightarrow u^* u_!$  of the adjunction  $(u_!, u^*)$  is an isomorphism and by conservativity that for any object  $i$  of  $\mathbb{I}$ ,  $i^* \rightarrow i^* u^* u_! = u(i)^* u_!$  is an isomorphism (where  $i: e \rightarrow \mathbb{I}$  and  $u(i): e \rightarrow \mathbb{J}$  denotes also the functors defined by the objects  $i$  and  $u(i)$ )

By axiom Der 4d, there is a canonical isomorphism

$$\begin{array}{ccc}
 p_! R^* \xrightarrow{\sim} u(i)^* u_! & & i \setminus \mathbb{I} \simeq u(i) \setminus \mathbb{I} \xrightarrow{p_!} \mathbb{I} \\
 & & \downarrow p \quad \downarrow u \\
 & & e \xrightarrow{u(i)} \mathbb{J}
 \end{array}$$

and as  $u$  is fully faithful  $u(i) \setminus \mathbb{I} \simeq i \setminus \mathbb{I}$ , which has an initial object  $(i, 1_i)$ . The pair  $((i, 1_i), p)$  (where  $(i, 1_i): e \rightarrow i \setminus \mathbb{I}$  denote also the functor defined by the object  $(i, 1_i)$ ) is an adjoint pair, and by 2-functoriality  $((i, 1_i)^*, p^*)$  is an adjoint pair and  $p_! \simeq (i, 1_i)^*$ . Therefore

$$p_! R^* \simeq (i, 1_i)^* R^* = (R(i, 1_i))^* = i^*$$

which proves the proposition modulo the isomorphism compatibility <sup>heavy</sup> verifications.

In what follows, we will fix a prederivator  $\mathbb{D}$  of domain  $\mathbb{D}ia$  and we will suppose that  $\mathbb{D}$  is complete, cocomplete and conservative, i.e. that it satisfies axioms Der 2, Der 3, Der 4

An arrow  $I \xrightarrow{u} J$  <sup>in  $\mathbb{D}ia$</sup>  is called a  $\mathbb{D}$ -equivalence if the canonical arrow induced by  $u$

$$\begin{array}{ccc} (P_I)_! P_I^* & \xrightarrow{\sim} & (P_J)_! P_J^* \\ \downarrow \scriptstyle{!} & \nearrow & \downarrow \scriptstyle{!} \\ (P_J)_! u_! u^* (P_J)_! & & \end{array} \quad \begin{array}{ccc} I & \xrightarrow{u} & J \\ \downarrow \scriptstyle{!} & & \downarrow \scriptstyle{!} \\ P_I & \xrightarrow{u} & P_J \end{array}$$

is an isomorphism. This by "transposition" is equivalent to ask that the arrow

$$\begin{array}{ccc} (P_J)_* P_J^* & \xrightarrow{\sim} & (P_I)_* P_I^* \\ \downarrow \scriptstyle{*} & \nearrow & \downarrow \scriptstyle{*} \\ (P_J)_* u_* u^* (P_J)_* & & \end{array}$$

is an isomorphism. In Grothendieck's homological (resp. cohomological) notation for every  $X$  in  $\mathbb{D}(a)$  this means that the morphism

$$H_* (I, P_I^* X) \longrightarrow H_* (J, P_J^* X)$$

$$\text{(resp. } H^* (J, P_J^* X) \longrightarrow H^* (I, P_I^* X) \text{)}$$

induced by  $u$  is an isomorphism, i.e. that  $u$  induces isomorphism in homology (resp. cohomology) with constant coefficients

Examples 1) If  $\mathbb{D} = \text{HOT}$  is the prederivator (which is a derivator) associated to the localizer  $(\text{Top}, W)$  of topological spaces and weak equivalences of topological spaces then  $\mathbb{D}$ -equivalences are exactly usual weak equivalences in  $\text{Cat}$ , i.e. functors between small categories such that the topological realization of the nerve is a weak equivalence (and even a homotopy equivalence) of topological spaces.

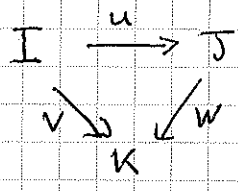
2) If  $\mathcal{C}$  is a complete and cocomplete category which is not a preordered set and if  $\mathbb{D}$  is the prederivator (which is a derivator) defined by the localizer  $(\mathcal{C}, \text{Iso}(\mathcal{C}))$ , i.e.  $\mathbb{D}(\mathbb{I}) = \mathcal{C}^{\mathbb{I}^{\circ}}$  = presheaves on  $\mathbb{I}$  with values in  $\mathcal{C}$ , then  $\mathbb{D}$ -equivalences are functors  $u: \mathbb{I} \rightarrow \mathbb{J}$  between small categories inducing bijection of connected components.

3) If  $\mathbb{D} = \text{Der}_K$ ,  $K$  commutative ring, then  $\mathbb{I} \xrightarrow{u} \mathbb{J}$  is a  $\mathbb{D}$ -equivalence if and only if  $u$  induces an isomorphism of homology (or cohomology) groups with coefficients constant  $K$ -modules

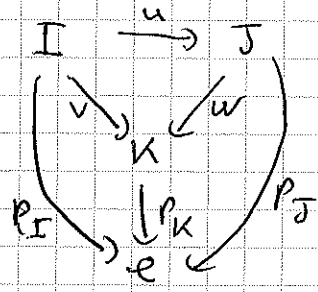


The  $\mathbb{D}$ -equivalences "satisfy the theorem A of Quillen":

Let



be a commutative triangle in  $\mathbb{D}ia$  and consider the commutative diagram



We will say that  $u$  is a  $\mathbb{D}$ -equivalence colocally (resp. locally) on  $K$  if the canonical map

$$\begin{array}{ccc}
 v_! p_I^* & \xrightarrow{\sim} & w_! p_J^* & \text{(resp. } w_* p_J^* \xrightarrow{\sim} v_* p_I^* \text{)} \\
 \downarrow \wr & & \downarrow \wr & \\
 w_! u_! u^* p_J^* & & & w_* u_* u^* p_J^*
 \end{array}$$

is an isomorphism. If  $u$  is a  $\mathbb{D}$ -equivalence colocally (resp. locally) on  $K$ , then by applying  $(p_K)_!$  (resp.  $(p_K)_*$ )

$$\begin{array}{ccc}
 (p_K)_! v_! p_I^* & \xrightarrow{\sim} & (p_K)_! w_! p_J^* & \text{(resp. } (p_K)_* w_* p_J^* \xrightarrow{\sim} (p_K)_* v_* p_I^* \text{)} \\
 \downarrow \wr & & \downarrow \wr & \\
 (p_I)_! p_I^* & & (p_J)_! p_J^* & (p_I)_* p_I^* \quad (p_J)_* p_J^* \quad (p_I)_* p_I^*
 \end{array}$$

is an isomorphism, so  $u$  is a (global)  $\mathbb{D}$ -equivalence

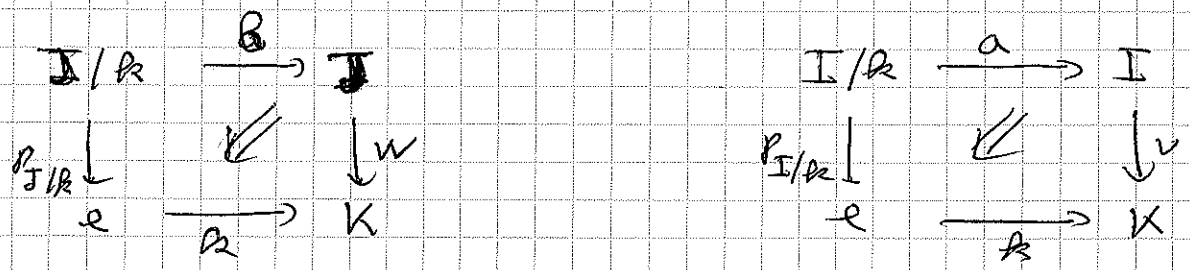
Let us characterize arrows in  $\mathbb{D}ia$  that are, for example,  $\mathbb{D}$ -equivalences locally on  $K$

Observe that as  $\mathbb{D}$  is a conservative prederivation  $u$  is a  $\mathbb{D}$ -equivalence locally on  $K$  if and only if for every object  $k$  of  $K$  the arrow

$$R^* W_* P_J^* \longrightarrow R^* V_* P_I^* \quad (*)$$

is a isomorphism. On the other hand, by Der 4g we have canonical isomorphisms

$$R^* W_* \simeq (P_{J/R})_* b^* \quad \text{and} \quad R^* V_* \simeq (P_{I/R})_* a^*$$



and so

$$R^* W_* P_J^* \simeq (P_{J/R})_* b^* P_J^* = (P_{J/R})_* P_{J/R}^*$$

and

$$R^* V_* P_I^* \simeq (P_{I/R})_* a^* P_I^* = (P_{I/R})_* P_{I/R}^*$$

and the map (\*) is identified to the canonical map

$$(P_{J/R})_* P_{J/R}^* \longrightarrow (P_{I/R})_* P_{I/R}^*$$

In conclusion  $u$  is a  $\mathbb{D}$ -equivalence locally on  $K$  if and only if for every object  $k$  of  $K$  the functor  $I/R \rightarrow J/R$  induced by  $u$  is a  $\mathbb{D}$ -equivalence. Dually,  $u$  is a  $\mathbb{D}$ -equivalence colocally on  $K$  if and only if for every object  $k$  of  $K$  the functor  $k \setminus I \rightarrow k \setminus J$  induced by  $u$  is a  $\mathbb{D}$ -equivalence

So the fact that a colocal (resp. local)  $\mathbb{D}$ -equivalence is a  $\mathbb{D}$ -equivalence is a relative version of Quillen's A theorem, the precise Quillen's

situation corresponding to the case when  $K=J$  and  $w = 1_J$ , in which case we will say that  $u$  is  $\mathbb{D}$ -coaspheric (resp.  $\mathbb{D}$ -aspheric)

So  $u: I \rightarrow J$  in  $\mathbb{D}a$  is called  $\mathbb{D}$ -coaspheric (resp.  $\mathbb{D}$ -aspheric) if the following equivalent conditions are satisfied:

- $u_* p_I^* \rightarrow p_J^*$  (resp.  $p_J^* \rightarrow u_* p_I^*$ ) is an isomorphism;
- $(p_J)_* \rightarrow (p_I)_* u^*$  (resp.  $(p_I)_! u^* \rightarrow (p_J)_!$ ) is a isomorphism;
- for every object  $j$  of  $J$ , the functor  $j \backslash I \rightarrow j \backslash J$  (resp.  $I / j \rightarrow J / j$ ) induced by  $u$  is a  $\mathbb{D}$ -equivalence

The arrows in (b) are obtained by "transposition" of the arrows in (a). The conditions in (b) are "cofinality conditions":

$$\begin{array}{ccc} \xleftarrow{\text{holim}}_{I^0} & \xrightarrow{\text{holim}}_{I^0} u^* & \text{(resp. } \xrightarrow{\text{holim}}_{I^0} u^* \rightarrow \xrightarrow{\text{holim}}_{J^0} \text{)} \end{array}$$

If the  $\mathbb{D}$ -equivalences are the arrows in  $\mathbb{D}a$  inducing projection of connected components (as in the case of the derivator  $I \mapsto C^{I^0}$ ) then these conditions are the classical cofinality conditions. So the notions of asphericity and coasphericity are the homotopy analogues (in the framework of derivators) of the notions of cofinality.

Exercise Prove that if  $(u, v)$  is an adjoint pair of arrows in  $\mathcal{D}ia$ , then  $u$  is  $\mathbb{N}$ -aspheric and  $v$  is  $\mathbb{N}$ -coaspheric. Deduce that if  $I$  is in  $\mathcal{D}ia$ , and if  $i$  is an initial (resp. a final) object of  $I$  then the map  $e \rightarrow I$  defined by the object  $i$  is  $\mathbb{N}$ -aspheric (resp.  $\mathbb{N}$ -coaspheric).

An important consequence of the fact that  $\mathbb{N}$ -equivalences satisfy the theorem A of Quillen is the following theorem

Theorem (Criswell) An arrow  $u: I \rightarrow J$  in  $\mathcal{D}ia$  is a  $\mathbb{N}$ -equivalence if and only if  $u^\circ: I^\circ \rightarrow J^\circ$  is a  $\mathbb{N}$ -equivalence

As a consequence we have:

$$u \text{ } \mathbb{N}\text{-aspheric} \iff u^\circ: \mathbb{N}\text{-coaspheric}$$

and for a commutative square in  $\mathcal{D}ia$

$$\begin{array}{ccc} I & \xrightarrow{u} & J \\ & \searrow & \swarrow \\ & K & \end{array}$$

$$u \text{ } \mathbb{N}\text{-equivalence locally on } K \iff$$

$$\iff u^\circ \text{ } \mathbb{N}\text{-equivalence colocally on } K$$

A very important fact is the following.

Theorem (Cosansini) If  $\mathcal{D}ia = \mathcal{C}at$  then every usual weak equivalence (map in  $\mathcal{C}at$  whose topological realization of the nerve is an homotopy equivalence) is a  $\mathbb{N}$ -equivalence

A category  $I$  in  $Dva$  is called  $\mathbb{D}$ -aspheric if the map  $p_I: I \rightarrow e$  to the point category is a  $\mathbb{D}$ -equivalence.

Proposition Let  $I$  in  $Dva$ . The following conditions are equivalent:

- a)  $I$  is  $\mathbb{D}$ -aspheric;
- b)  $p_I$  is  $\mathbb{D}$ -aspheric;
- c)  $p_I$  is  $\mathbb{D}$ -coaspheric;
- d)  $(p_I)_! p_I^* \rightarrow 1_{D(e)}$  is an isomorphism;
- e)  $1_{D(e)} \rightarrow (p_I)_* p_I^*$  is an isomorphism;
- f)  $p_I^*$  is fully faithful

The exercise p. 32 implies that if  $I$  has an initial or a final object then  $I$  is  $\mathbb{D}$ -aspheric

More generally, Cisinski's theorem (p. 32) implies that if the classifying space of  $I$  is a contractible space then  $I$  is  $\mathbb{D}$ -aspheric.

D-exact squares, D-proper, D-smooth functors.

A 2-square in  $D\text{Cat}$

$$D = \begin{array}{ccc} I' & \xrightarrow{v} & I \\ w' \downarrow & \swarrow \alpha & \downarrow u \\ J' & \xrightarrow{w} & J \end{array}$$

is called D-exact, or is said to satisfy the base change property for D,

if the homological base change map

$$V_! u'^* \xrightarrow{c_D} u^* W_!$$

or equivalently the cohomological base change map

$$W^* u_* \xrightarrow{c'_D} u'_* V^*$$

is an isomorphism.

Proposition The following conditions are equivalent:

- a)  $D$  is D-exact;
- b) for every object  $i$  of  $I$  the map  $i \setminus I' \rightarrow u(i) \setminus J'$  induced by  $u'$  is a D-equivalence locally on  $J'$ ;
- c) for every object  $j'$  of  $J'$  the map  $I'/j' \rightarrow I/W(j')$  induced by  $v$  is a D-equivalence colocally on  $I$ .

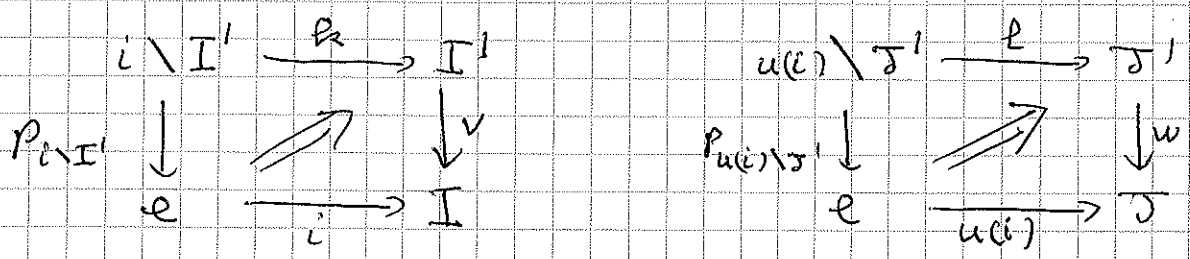
Proof As  $D$  is a conservative prederivator, the map  $c_D$  is an isomorphism if and only if for every object  $i$  of  $I$

$$i^* V_! u'^* \longrightarrow i^* u^* W_! = (u(i))^* W_! \tag{1}$$



is an isomorphism. Using Der 3g we have canonical isomorphisms

$$i^* v_! \cong (p_{i \setminus I'})_! \mathbb{R}^* , \quad (u(i))^* w_! \cong (p_{u(i) \setminus J'})_! \mathbb{L}^*$$



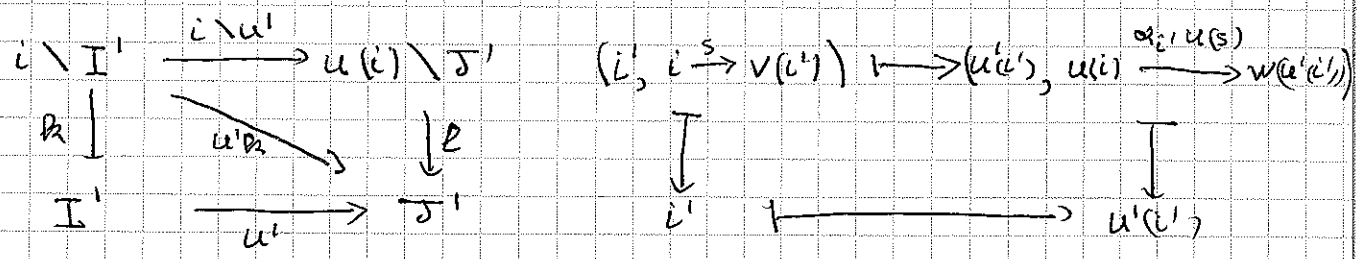
so the arrow (1) is identified to

$$(p_{i \setminus I'})_! (u^! \mathbb{R})^* = (p_{i \setminus I'})_! \mathbb{R}^* u^! \longrightarrow (p_{u(i) \setminus J'})_! \mathbb{L}^*$$

obtained by transposition of

$$\mathbb{L}^* (p_{u(i) \setminus J'})^* \longrightarrow (u^! \mathbb{R})_* (p_{i \setminus I'})^* \quad (2)$$

As by definition this map is an isomorphism if and only if  $i \setminus u^!$  is a D-equivalence



locally on  $J'$ , this proves the equivalence of (a) and (b).

The equivalence of (a) and (c) is proved dually by using the map  $c'_D$  obtained by transposition of  $c_D$ .

If  $\begin{array}{ccc} I' & \xrightarrow{v} & I \\ u' \downarrow & & \downarrow u \\ J' & \xrightarrow{w} & J \end{array}$  is a commutative square, it can be considered as a 2-square in two different ways

$$\mathcal{D} = \begin{array}{ccc} I' & \xrightarrow{v} & I \\ u' \downarrow & \swarrow \tau_{uv} & \downarrow u \\ J' & \xrightarrow{w} & J \end{array} \quad \text{and} \quad \mathcal{D}' = \begin{array}{ccc} I' & \xrightarrow{v} & I \\ u' \downarrow & \searrow \tau_{uv} & \downarrow u \\ J' & \xrightarrow{w} & J \end{array}$$

and the corresponding base change maps in the two situations have nothing to do each other.

$$\begin{array}{ccc} v_! u'^* & \xrightarrow{c_{\mathcal{D}}} & u^* w_! \\ w^* u_* & \xrightarrow{c'_{\mathcal{D}}} & u'_* v^* \end{array} \quad \begin{array}{ccc} u'_! v^* & \xrightarrow{c_{\mathcal{D}'}} & w^* u_! \\ u^* w_* & \xrightarrow{c'_{\mathcal{D}'}} & v_* u'^* \end{array}$$

By convention, we will always consider the commutative square  $\mathcal{C}$  as the 2-square  $\mathcal{D}$  and never as  $\mathcal{D}'$ . So the commutative square  $\mathcal{C}$  is  $\mathbb{D}$ -exact if

$$v_! u'^* \rightarrow u^* w_! \quad \text{or equivalently} \quad w^* u_* \rightarrow u'_* v^*$$

is an isomorphism.

Grothendieck, by analogy to the cohomological base change properties of proper and smooth maps in algebraic geometry, gave the following definitions

A map  $I \xrightarrow{u} J$  (resp.  $J' \xrightarrow{w} J$ ) in  $\mathbb{D}\text{Set}$  is called  $\mathbb{D}$ -proper (resp.  $\mathbb{D}$ -smooth) if every cartesian square of the form

$$\begin{array}{ccc} I' & \xrightarrow{v} & I \\ u' \downarrow & & \downarrow u \\ J' & \xrightarrow{w} & J \end{array}$$

is  $\mathbb{D}$ -exact, and this property remains true after any base change.

Theorem (Grothendieck) Let  $u: I \rightarrow J$  a map in  $\mathcal{D}(\mathcal{A})$ . The following conditions are equivalent:

- a)  $u$  is  $\mathbb{D}$ -proper;
- b) for every diagram of cartesian squares of the form

$$\begin{array}{ccccc} I'' & \longrightarrow & I' & \longrightarrow & I \\ \downarrow & & \downarrow & & \downarrow u \\ J'' & \longrightarrow & J' & \longrightarrow & J \end{array}$$

the left square is  $\mathbb{D}$ -exact;

- c) for every diagram of cartesian squares of the form

$$\begin{array}{ccccc} I'' & \xrightarrow{v} & I' & \longrightarrow & I \\ \downarrow & & \downarrow u' & & \downarrow u \\ J'' & \xrightarrow{w} & J' & \longrightarrow & J \end{array}$$

if  $w$  is  $\mathbb{D}$ -coaspheric then  $v$  is  $\mathbb{D}$ -coaspheric

- d) for every object  $j$  of  $J$ , the canonical map

$$I_j \longrightarrow I/j, \quad i \longmapsto (i, u(i) \xrightarrow{1_j} j)$$

where  $I_j$  is the fiber of  $u$  over  $j$ , is  $\mathbb{D}$ -coaspheric.

Proof (a)  $\Leftrightarrow$  (b) is a translation of the definition (b)  $\Rightarrow$  (c). It's enough to prove that if

a commutative square

$$\begin{array}{ccc} I' & \xrightarrow{v} & I \\ u' \downarrow & & \downarrow u \\ J' & \xrightarrow{w} & J \end{array}$$

is  $\mathbb{D}$ -exact and if  $w$  is  $\mathbb{D}$ -coaspheric then  $v$  is  $\mathbb{D}$ -coaspheric.

As the square is  $\mathbb{D}$ -exact the base change map

$$v_! u'^* \xrightarrow{c} u^* w_!$$

is an isomorphism and as  $w$  is

$\mathbb{D}$ -coaspheric the canonical map  $w_! p_{J'}^* \xrightarrow{a} p_J^*$  is

an isomorphism. Observe that the canonical map

$$v_! p_{I'}^* \longrightarrow p_I^* \quad \text{is equal to} \quad (u^* a)(c^* p_{J'}^*)$$

$$V_I P_I^* = V_I u_I^* P_J^* \xrightarrow{c^* P_J^*} u^* W_I P_J^* \xrightarrow{u^* * a} u^* P_J^* = P_I^*$$

Therefore the map  $v$  is  $\mathbb{D}$ -coaryheric

(c)  $\Rightarrow$  (d). Consider the diagram of cartesian squares (for  $j$  an object of  $\mathcal{J}$ )

$$\begin{array}{ccccc} I_j & \longrightarrow & I/j & \longrightarrow & I \\ \downarrow & & \downarrow & & \downarrow \\ e & \xrightarrow{(\beta, \gamma)} & \mathcal{J}/j & \longrightarrow & \mathcal{J} \end{array}$$

As  $(\beta, \gamma)$  is a final object of  $\mathcal{J}/j$ , the functor  $(\beta, \gamma): e \rightarrow \mathcal{J}/j$  has a left adjoint and is therefore  $\mathbb{D}$ -coaryheric. The condition (c) implies then that  $I_j \rightarrow I/j$  is  $\mathbb{D}$ -coaryheric

(d)  $\Rightarrow$  (a). It's not hard to prove, and is left as an exercise that the condition (d) is equivalent to the condition that for every cartesian square of the form

$$\begin{array}{ccc} I' & \longrightarrow & I \\ \downarrow & & \downarrow u \\ \mathcal{J}' = \{0 \rightarrow 1\} & \longrightarrow & \mathcal{J} \end{array} \quad \begin{array}{l} \text{the inclusion } I'_0 \hookrightarrow I \\ \text{of the fiber of } u' \text{ over } 0 \\ \text{is a } \mathbb{D}\text{-coaryheric map} \end{array}$$

which implies that the condition (d) is stable under base change.

So it is enough to prove that the condition (d) implies that every cartesian square of the form

$$\begin{array}{ccc} I' & \xrightarrow{v} & I \\ u' \downarrow & & \downarrow u \\ \mathcal{J}' & \xrightarrow{w} & \mathcal{J} \end{array} \quad \text{is } \mathbb{D}\text{-exact}$$

Using the proposition characterizing the  $\mathbb{D}$ -exact squares, it's enough to prove that for every object  $y'$  of  $\mathcal{Y}'$ , the map

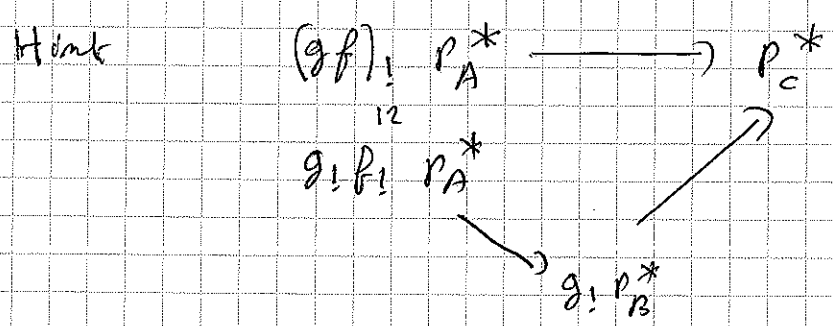
$$I'/y' \longrightarrow I/w(y')$$

induced by  $v$  is a  $\mathbb{D}$ -equivalence locally over  $I$ , or better that this map is  $\mathbb{D}$ -coaspheric. Observe that in the commutative square

$$\begin{array}{ccc} I/y' & \longrightarrow & I/w(y') \\ \downarrow & & \downarrow \\ I'/y' & \longrightarrow & I/w(y') \end{array}$$

the upper horizontal arrow is an isomorphism (as the square  $*$  is cartesian) and that the vertical arrows are  $\mathbb{D}$ -coaspheric (as  $u$  and  $u'$  satisfy the condition (d) :  $u$  by hypothesis and  $u'$  by stability of this condition by base change). This implies that the lower horizontal arrow is  $\mathbb{D}$ -coaspheric by an easy lemma (exercise) :

Let  $A \xrightarrow{f} B \xrightarrow{g} C$  in  $\mathcal{D}ia$ . If  $f$  and  $gf$  are  $\mathbb{D}$ -coaspheric then  $g$  is  $\mathbb{D}$ -coaspheric.



There is a dual characterization of  $\mathbb{D}$ -smooth functors and as a consequence  $u$   $\mathbb{D}$ -proper  $\Leftrightarrow u^o$   $\mathbb{D}$ -smooth ! □

## The other axioms

Der 1 i) The category  $\mathbb{D}(\emptyset)$  is equivalent to the point category.

ii) Let  $I, J$  in  $\mathcal{D}(\alpha)$  and

$I \xrightarrow{i} I \amalg J$       be canonical maps. Then

$$J \xrightarrow{j} I \amalg J \quad \mathbb{D}(I \amalg J) \xrightarrow{(i^*, j^*)} \mathbb{D}(I) \times \mathbb{D}(J)$$

is an equivalence of categories

Exercise i) Prove that Der 1 (i) joined with Der 3g (resp. Der 3d) implies that for every  $I$  in  $\mathcal{D}(\alpha)$ ,  $\mathbb{D}(I)$  has a final (resp. initial) object

ii) Prove that Der 1 (ii) joined with Der 3g (resp. Der 3d) implies that for every  $I$  in  $\mathcal{D}(\alpha)$ ,  $\mathbb{D}(I)$  is stable under binary products (resp. co-products)

In order to state Der 5, some preliminaries are needed. Let  $I$  and  $J$  in  $\mathcal{D}(\alpha)$ . An object  $j$  in  $J$  defines a map

$$j_I: I \longrightarrow I \times J, \quad i \longmapsto (i, j),$$

hence a functor

$$j_I^*: \mathbb{D}(I \times J) \longrightarrow \mathbb{D}(I).$$

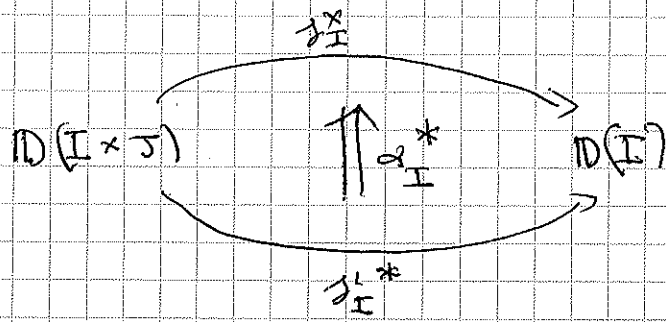
Observe that an arrow  $\alpha: j \rightarrow j'$  of  $J$  defines a natural transformation

$$\begin{array}{ccc} & \xrightarrow{j_I} & \\ I & \downarrow \alpha_I & I \times J \\ & \xrightarrow{j'_I} & \end{array}$$

$$\alpha_{I, i} = \Pi_{i, \alpha}, \quad \forall \text{obj } I$$



and by 2-functoriality, a natural transformation



defining a functor

$$D(I \times J) \times \mathbb{I}^0 \longrightarrow D(I)$$

or equivalently a functor

$$D(I \times J) \xrightarrow{\text{dia}_{I,J}} \text{Hom}(\mathbb{I}^0, D(I)) = D(I)^{\mathbb{I}^0}$$

generalizing the functor

$$\text{dia}_J : D(J) = D(e \times J) \longrightarrow D(e)^{\mathbb{I}^0}$$

We can now state

Der 5 For every  $I$  in  $D$  if  $J$  is the category  $\{0 \rightarrow 1\}$ , then the functor

$$D(I \times J) \xrightarrow{\text{dia}_{I,J}} D(I)^{\mathbb{I}^0}$$

is full and essentially surjective

Remark The functor  $\text{dia}_{I,J}$  (or  $\text{dia}_I$ ) is never (outside very trivial situations) faithful. Sometimes

a stronger form of axiom Der 5 is needed, asking that  $\text{dia}_{I,J}$  is full and essentially surjective

for every free and finite category  $J$ . This stronger form is verified if  $D = D_{(C,W)}$ , with  $(C,W)$  a Quillen localizer

## Pointed Derivators

(42)

Recall that if  $\mathcal{J}$  is a category, a sieve of  $\mathcal{J}$  is a full subcategory  $\mathcal{J}' \hookrightarrow \mathcal{J}$  such that if  $f \rightarrow g'$  is an arrow of  $\mathcal{J}$  with  $g'$  in  $\mathcal{J}'$ , then  $f$  is also in  $\mathcal{J}'$ . A functor  $u: \mathcal{I} \rightarrow \mathcal{J}$  is called an open immersion if it induces an isomorphism of  $\mathcal{I}$  onto a sieve of  $\mathcal{J}$ . The notions of cosieve and closed immersion are defined dually.

A pointed derivator is a derivator satisfying the following axiom

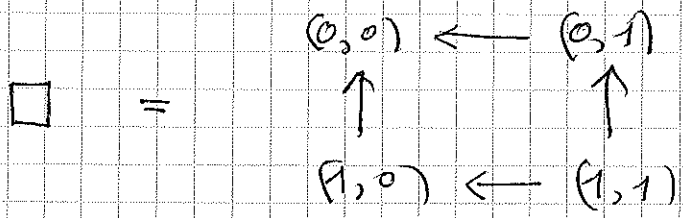
Der 6 If  $u: \mathcal{I} \rightarrow \mathcal{J}$  is an open (resp. a closed) immersion in  $\mathcal{D}ia$ , then the functor  $u_!$  (resp.  $u_*$ ) has a left (resp. right) adjoint  $u^?$  (resp.  $u^!$ ).

Exercise 1) (Easy) Prove that if  $\mathbb{D}$  is a pointed derivator, then for every  $\mathcal{I}$  in  $\mathcal{D}ia$  the category  $\mathbb{D}(\mathcal{I})$  has a null object (an object that is both initial and final).

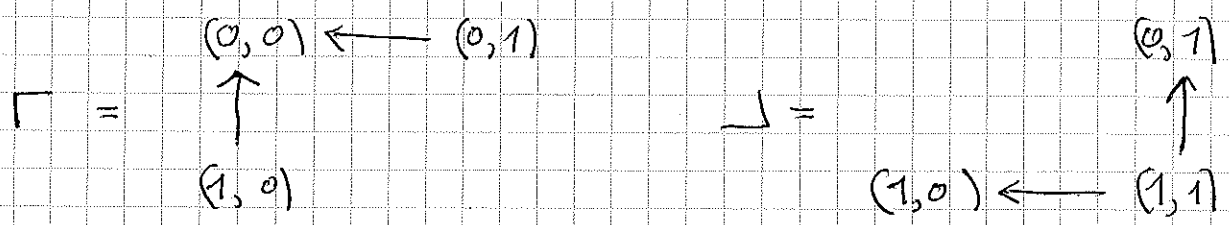
ii) (Super-difficult). Conversely prove that if for every  $\mathcal{I}$  in  $\mathcal{D}ia$  the category  $\mathbb{D}(\mathcal{I})$  has a null object and if  $\mathbb{D}$  is a derivator the  $\mathbb{D}$  satisfies the axiom Der 6 (some strong stability conditions on  $\mathcal{D}ia$  are needed)

Triangulated derivators

Denote  $\square$  the category



and  $\Gamma, \downarrow$  the subcategories



of  $\square$ . Let  $i_\Gamma: \Gamma \rightarrow \square, i_\downarrow: \downarrow \rightarrow \square$  the inclusion functors

Let  $\mathbb{D}$  be any conservative cocomplete (resp. complete) prederivator (for example a derivator).

An object  $X$  of  $\mathbb{D}(\square)$  is called (homotopically) co cartesian (resp. cartesian) if the adjunction morphism

$$i_\Gamma! i_\Gamma^* X \rightarrow X \quad (\text{resp. } X \rightarrow i_\downarrow * i_\downarrow^* X)$$

is an isomorphism.

Exercise Let  $\mathbb{D}$  be a conservative cocomplete (resp. complete) prederivator, denote by  $R: e \rightarrow \square$  the functor defined by the object  $(1,1)$  (resp.  $(0,0)$ ) of  $\square$  and  $p$  the projection  $p = p_\Gamma: \Gamma \rightarrow e$  (resp.  $p = p_\downarrow: \downarrow \rightarrow e$ ). For every object  $X$  of  $\mathbb{D}(\square)$

define a canonical morphism, functorial in  $X$ ,

$$p_!(X|\Gamma) = p_! L_{\Gamma}^* X \rightarrow R^* X \quad (\text{resp. } R^* X \rightarrow p_* L_{\Gamma}^* X = p_*(X|\Gamma))$$

and prove that  $X$  is cocartesian (resp. cartesian)

if and only if this map is an isomorphism.

(This means that an object  $X$  of  $\mathcal{D}(\square)$  is cocartesian (resp. cartesian) if and only if its "lower right vertex" (resp. "upper left vertex") is the "homotopy pushout" (resp. "homotopy pullback") of its "upper left corner" (resp. "lower right corner").)

A triangulated derivator is a pointed derivator  $\mathcal{D}$  satisfying the following axiom:

Der 7] An object of  $\mathcal{D}(\square)$  is cocartesian if and only if it is cartesian.

So a triangulated derivator is a prederivator satisfying all the axioms Der 1 to Der 7.

Examples 1) If  $(C, W)$  is a stable Quillen localizer, i.e. if there exists a stable Quillen model category structure on  $C$  with  $W$  as the class of weak equivalences, then the prederivator  $\mathcal{D}_{(C, W)}$  is a triangulated derivator. This is a tautological consequence of the non-triangulated case (if we admit the difficult part of exercise p. 42) as a stable model category is by definition a model category with

a null object, such that a commutative square is homotopy cocartesian if and only if it is homotopy cartesian. Particular examples of stable model categories are the model category of spectra, giving rise to the triangulated derivator of spectra  $\mathbb{D}_{\text{Sp}}$ , and the model category of unbounded complexes of a Grothendieck category  $\mathcal{A}$ , giving rise to the unbounded derived triangulated derivator  $\mathbb{D}\text{er}_{\mathcal{A}}$  of  $\mathcal{A}$ .

2) In the last lecture of this course, Bernhard Keller will prove that if  $\mathcal{E}$  is an exact category, the localizer of bounded complexes of  $\mathcal{E}$  and quasi-isomorphisms defines a triangulated derivator with domain for the direct categories, the bounded derived triangulated derivator  $\mathbb{D}\text{er}_{\mathcal{E}}^{\text{b}}$  of  $\mathcal{E}$ .

Theorem Let  $\mathbb{D}$  be a triangulated derivator of domain  $\text{Dia}$ . For every  $I$  in  $\text{Dia}$  the category  $\mathbb{D}(I)$  has a canonical structure of a triangulated category such that for every arrow  $u: I \rightarrow J$  in  $\text{Dia}$ ,  $u^*$  is an exact functor, in the triangulated sense ( $u^*$  is a triangulated functor).

We will not prove this theorem; we will simply describe the triangulated structure on  $\mathbb{D}(\mathcal{E})$ ,  $\mathcal{E}$  point category (which is not a real restriction as it is not hard to prove that if  $\mathbb{D}$  is a triangulated

derivation then for any  $I$  in  $\mathcal{D}(\mathcal{A})$ ,  $J \mapsto \mathbb{D}(I \otimes J)$  defines equally well a triangulated derivation) without proving that  $\mathbb{D}(\mathcal{R})$  is an additive category (which is the most difficult part) nor proving universality of the suspension functor and axioms TR I to TR IV.

Definition of the suspension functor

As for a triangulated derivation an object of  $\mathbb{D}(\square)$  is bicartesian if and only if it is cartesian, such an object will be simply called bicartesian.

Denote

$$i_0: \mathcal{R} \rightarrow \Gamma, \quad i_1: \mathcal{R} \rightarrow \square$$

the functors defined respectively by the objects  $(0,0)$  and  $(1,1)$  of  $\Gamma$  and  $\square$ .

The suspension functor

$$S: \mathbb{D}(\mathcal{R}) \rightarrow \mathbb{D}(\mathcal{R})$$

is defined by

$$S := i_1^* i_{\Gamma!} i_0^*$$

In order to understand this definition, observe that for every object  $X$  of  $\mathbb{D}(\mathcal{R})$  the object  $i_{\Gamma!} i_0^* X$  of  $\mathbb{D}(\square)$  is bicartesian (every object of the form  $i_{\Gamma!} Y$  is bicartesian, as  $i_{\Gamma}$  is fully faithful,  $Y \rightarrow i_{\Gamma}^* i_{\Gamma!} Y$  is an isomorphism (proposition on page 26)); therefore  $i_{\Gamma!} Y \rightarrow i_{\Gamma!} i_{\Gamma}^* i_{\Gamma!} Y$  and  $i_{\Gamma!} i_{\Gamma}^* i_{\Gamma!} Y \rightarrow i_{\Gamma!} Y$  are isomorphisms) and that there are canonical isomorphisms



$$\text{dia}(i_0^* X) \cong \begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \\ 0 & & \end{array}$$

and

$$\text{dia}(i_1^* i_0^* X) \cong \begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & S(X) \end{array}$$

where  $0$  is some fixed null object of  $\mathcal{D}(\mathcal{C})$ .

Let now  $T$  a bicartesian object of  $\mathcal{D}(\square)$  such that  $(0,1)^* T$  and  $(1,0)^* T$  are null objects of  $\mathcal{D}(\mathcal{C})$ :

$$\text{dia}(T) = \begin{array}{ccc} T_{00} & \longrightarrow & T_{01} \cong 0 \\ \downarrow & & \downarrow \\ 0 \cong T_{10} & \longrightarrow & T_{11} \end{array}$$

By the conservativity axiom the adjunction morphism

$$i_1^* T \longrightarrow i_0^* i_1^* i_1^* T = i_0^*(T_{00})$$

is an isomorphism. As  $T$  is in particular a cocartesian object of  $\mathcal{D}(\square)$  the adjunction morphism

$$i_1^* i_1^* T \longrightarrow T$$

is also an isomorphism. We deduce isomorphisms

$$T_{11} = i_1^* T \longleftarrow i_1^* i_1^* i_1^* T \longrightarrow i_1^* i_1^* i_0^*(T_{00}) = S(T_{00})$$

hence an isomorphism

$$\theta_T : (1,1)^* T = T_{11} \longrightarrow S(T_{00}) = S((0,0)^* T)$$

## Definition of distinguished triangles

Denote by  $\square$  the category

$$\square := \begin{array}{ccccc} (0,0) & \leftarrow & (0,1) & \leftarrow & (0,2) \\ & \uparrow & \uparrow & & \uparrow \\ (1,0) & \leftarrow & (1,1) & \leftarrow & (1,2) \end{array}$$

and

$$j_i : \square \rightarrow \square, \quad i = 0, 1, 2$$

the functor defined by

$$j_i(R, \ell) = (R, \mathcal{S}_i(\ell))$$

where  $\mathcal{S}_i$  is the injection  $\{0, 1\} \rightarrow \{0, 1, 2\}$  that avoids  $i$ . An object  $T$  of  $\mathbb{D}(\square)$  is called bicartesian if  $j_i^* T$  is bicartesian for  $i=0, 1, 2$ .

A triangle in  $\mathbb{D}(\mathcal{A})$  is just a diagram in  $\mathbb{D}(\mathcal{A})$  of the form  $X \rightarrow Y \rightarrow Z \rightarrow S(X)$ .

The notions of morphism and isomorphism of triangles are defined in the evident way.

Let  $T$  a bicartesian object of  $\mathbb{D}(\square)$  such that  $(1,0)^* T$  and  $(0,2)^* T$  are null objects of  $\mathbb{D}(\mathcal{A})$ :

$$\text{dia}(T) = \begin{array}{ccccc} T_{00} & \xrightarrow{f} & T_{01} & \longrightarrow & T_{02} \cong 0 \\ & & \downarrow g & & \downarrow \\ 0 \cong T_{10} & \longrightarrow & T_{11} & \xrightarrow{h} & T_{12} \end{array}$$

By definition of bicartesian objects of  $\mathbb{D}(\square)$ ,

$j_1^*(T)$  is a bicartesian object of  $\mathcal{D}(\square)$ . Observe that

$$\text{dia } j_1^* T = \begin{array}{ccc} T_{00} & \longrightarrow & T_{02} \simeq 0 \\ \downarrow & & \downarrow \\ 0 \simeq T_{10} & \longrightarrow & T_{12} \end{array}$$

So there is a canonical isomorphism

$$\Theta = \Theta_{j_1^* T} : T_{12} \xrightarrow{\sim} S(T_{00})$$

and we get a triangle

$$T_{00} \xrightarrow{f} T_{01} \xrightarrow{g} T_{11} \xrightarrow{\Theta R} S(T_{00})$$

Such a triangle is called a standard distinguished triangle. A distinguished triangle is a triangle isomorphic to a standard distinguished triangle.