Stability conditions on Dia

D 0 Every finite partially ordered set is in Dia.

D 1 Dia is stable under finite sums and fiber products.

D 2d If I is in Dia and i an object of I, then $i\backslash I$ is in Dia.

D 2g If I is in Dia and i an object of I, then I/i is in Dia.

D 3 If I is in Dia, then I° is in Dia.

D 4d If $u : I \to J$ is a cofibered category on J (in Grothendieck's sens) and if J and the fibers of u are in Dia, then I is in Dia

D 4g If $u : I \to J$ is a fibered category on J (in Grothendieck's sens) and if J and the fibers of u are in Dia, then I is in Dia.

(D3) $\Longrightarrow$ ((D 2d) $\Longleftrightarrow$ (D 2g) and (D 4d) $\Longleftrightarrow$ (D 4g))

(D 4g) $\Longrightarrow$ (D 2g) and (D 4d) $\Rightarrow$ D 2d if discrete categories (small) are in Dia.

(D 1) and (D 2d) $\Longrightarrow$ If $I \to J$ is in Dia and j an object of J, then $j\backslash I$ is in Dia

(D 1) and (D 2g) $\Longrightarrow$ If $I \to J$ is in Dia and j an object of J, then I/j is in Dia

Axioms of Derivators

Let Dia a full 2-subcategory of Cat satisfying all stability conditions needed and

$$\mathbb{D} : \mathrm{Dia}^{o} \longrightarrow \mathrm{CAT}$$

a 2-functor (a prederivator). The prederivator $\mathbb{D}$ is called a derivator if the following five axioms are satisfied

Der 1 i) The category $\mathbb{D}(\emptyset)$ is equivalent to the point category e.

ii) Let I, J in Dia and

$$\begin{tikzcd} I \arrow[dr, "i"] & \\ & I \amalg J \\ J \arrow[ur, "j"'] & \end{tikzcd}$$

the canonical maps. Then

$$\mathbb{D}(I \amalg J) \xrightarrow{(i^*, j^*)} \mathbb{D}(I) \times \mathbb{D}(J)$$

is an equivalence of categories

Der 2 For every I in Dia the family of functors

$$i^* : \mathbb{D}(I) \longrightarrow \mathbb{D}(e) \quad , \quad i \in \mathrm{Ob}\, I ,$$

where $i : e \longrightarrow I$ denotes also the functor defined by the object i, is conservative

Der 3d For every $u : I \longrightarrow J$ in Dia the functor $u^* : \mathbb{D}(J) \to \mathbb{D}(I)$ has a left adjoint $u_! : \mathbb{D}(I) \to \mathbb{D}(J)$

Der 3g For every $u : I \longrightarrow J$ in Dia the functor $u^* : \mathbb{D}(J) \longrightarrow \mathbb{D}(I)$ has a right adjoint $u_* : \mathbb{D}(I) \longrightarrow \mathbb{D}(J)$,

Der 4d For every $u : I \longrightarrow J$ in $\mathcal{D}ia$ and every object j of J the canonical map

$$j^* u_! \longleftarrow p_! k^* \quad \text{induced by the "2-square"} \quad \begin{tikzcd} j\backslash I \arrow[r, "k"] \arrow[d, "p"'] \arrow[dr, phantom, "\Nearrow"] & I \arrow[d, "u"] \\ e \arrow[r, "j"'] & J \end{tikzcd}$$

is an isomorphism

Der 4g For every $u : I \longrightarrow J$ in $\mathcal{D}ia$ and every object j of J the canonical map

$$j^* u_* \longrightarrow q_* l^* \quad \text{induced by the "2-square"} \quad \begin{tikzcd} I/j \arrow[r, "l"] \arrow[d, "q"'] \arrow[dr, phantom, "\Swarrow"] & I \arrow[d, "u"] \\ e \arrow[r, "j"'] & J \end{tikzcd}$$

is an isomorphism

Der 5 For every I in $\mathcal{D}ia$ the canonical functor

$$\mathbb{D}(I \times \{0 \to 1\}) \longrightarrow \underline{\mathrm{Hom}}(\{0 \to 1\}^{\circ}, \mathbb{D}(I))$$

is full and essentially surjective

Additional axioms for triangulated derivators

Der 6 If $u : I \longrightarrow J$ is an open (resp. a closed) immersion in $\mathcal{D}ia$, then the functor $u_!$ (resp. u_*) has a left (resp. right) adjoint $u^?$ (resp. $u^!$)

Der 7 An object of $\mathbb{D}(\square) = \mathbb{D}\begin{pmatrix} (0,0) \leftarrow (0,1) \\ \uparrow \qquad\quad \uparrow \\ (1,0) \leftarrow (1,1) \end{pmatrix}$ is cocartesian if and only if it is cartesian.

<u>Results on $\mathbb{D}$-equivalences, $\mathbb{D}$-aspheric maps, etc.</u>

A) $u : I \to J$ in Dia. Equivalent conditions

a) u $\mathbb{D}$-equivalence;

b) $p_{I!}\, p_I^* \longrightarrow p_{J!}\, p_J^*$ isomorphism;

c) $p_{J*}\, p_J^* \longrightarrow p_{I*}\, p_I^*$ isomorphism;

B) $u : I \to J$ in Dia. Equivalent conditions

a) u $\mathbb{D}$-aspheric

b) $p_J^* \longrightarrow u_*\, p_I^*$ isomorphism;

c) $(p_I)_!\, u^* \longrightarrow (p_J)_!$ isomorphism;

d) $\forall j \in \mathrm{Ob}\, J$, $I/j \to J/j$ is a $\mathbb{D}$-equivalence.

C) $u : I \to J$ in Dia. Equivalent conditions

a) u $\mathbb{D}$-coaspheric

b) $u_!\, p_I^* \longrightarrow p_J^*$ isomorphism;

c) $(p_J)_* \longrightarrow (p_I)_*\, u^*$ isomorphism;

d) $\forall j \in \mathrm{Ob}\, J$, $j\backslash I \longrightarrow j\backslash J$ is a $\mathbb{D}$-equivalence

D) $\begin{tikzcd} I \arrow[r, "u"] \arrow[dr, "v"'] & J \arrow[d, "w"] \\ & K \end{tikzcd}$ commutative triangle in Dia

Equivalent conditions:

a) u is a $\mathbb{D}$-equivalence locally on K;

b) $w_*\, p_J^* \longrightarrow v_*\, p_I^*$ isomorphism;

c) $(p_I)_!\, v^* \longrightarrow (p_J)_!\, w^*$ isomorphism;

d) $\forall k \in \mathrm{Ob}\, K$, $I/k \to J/k$ is a $\mathbb{D}$-equivalence.

E) $\begin{tikzcd} I \arrow[r, "u"] \arrow[dr, "v"'] & J \arrow[d, "w"] \\ & K \end{tikzcd}$ commutative triangle in Dia

Equivalent conditions:

a) u is a $\mathbb{D}$-equivalence colocally on K;

b) $v_! \, p_I^* \longrightarrow w_! \, p_J^*$ isomorphism;

c) $(p_J)_* \, w^* \longrightarrow (p_I)_* \, v^*$ isomorphism;

d) $\forall \, k \in \operatorname{Ob} K$, $k\backslash I \longrightarrow k\backslash J$ is a $\mathbb{D}$-equivalence.

F) I in Dia. Equivalent conditions

a) I $\mathbb{D}$-aspheric;

b) p_I $\mathbb{D}$-equivalence;

c) p_I $\mathbb{D}$-aspheric;

d) p_I $\mathbb{D}$-coaspheric;

e) $(p_I)_! \, p_I^* \longrightarrow 1_{\mathbb{D}(e)}$ isomorphism;

f) $1_{\mathbb{D}(e)} \longrightarrow (p_I)_* \, p_I^*$ isomorphism;

g) p_I^* fully faithfull.

G) Duality:

a) $u: I \to J$ $\mathbb{D}$-equivalence $\Leftrightarrow$ $u^\circ: I^\circ \to J^\circ$ $\mathbb{D}$-equivalence;

b) $u: I \to J$ $\mathbb{D}$-aspheric $\Leftrightarrow$ $u^\circ: I^\circ \to J^\circ$ $\mathbb{D}$-coaspheric;

c) $\begin{tikzcd} I \arrow[r, "u"] \arrow[dr] & J \arrow[d] \\ & K \end{tikzcd}$ commutative triangle

u $\mathbb{D}$-equivalence locally on K $\Leftrightarrow$

$\Leftrightarrow$ u° $\mathbb{D}$-equivalence colocally on K

d) I $\mathbb{D}$-aspheric $\Leftrightarrow$ I° $\mathbb{D}$-aspheric

H) $\mathcal{D} = $
$$\begin{tikzcd} I' \arrow[r, "v"] \arrow[d, "u'"'] \arrow[dr, phantom, "\swarrow\alpha"] & I \arrow[d, "u"] \\ J' \arrow[r, "w"'] & J \end{tikzcd}$$
2-square in Dia

Equivalent conditions:

a) $\mathcal{D}$ is $\mathbb{D}$-exact;

b) $c_{\mathcal{D}} : v_! u'^* \to u^* w_!$ isomorphism;

c) $c'_{\mathcal{D}} : w^* u_* \to u'_* v^*$ isomorphism;

d) $\forall i \in \mathrm{Ob}\, I$, $i\backslash I' \to u(i)\backslash J'$ is a $\mathbb{D}$-equivalence locally on J';

e) $\forall j' \in \mathrm{Ob}\, J'$, $I'/j' \to I/w(j')$ is a $\mathbb{D}$-equivalence colocally on I.

I) $u : I \to J$ in Dia. Equivalent conditions:

a) u is $\mathbb{D}$-proper;

b) for every diagram of cartesian squares of the form
$$\begin{tikzcd} I'' \arrow[r] \arrow[d] & I' \arrow[r] \arrow[d] & I \arrow[d, "u"] \\ J'' \arrow[r] & J' \arrow[r] & J \end{tikzcd}$$
the left square is $\mathbb{D}$-exact;

c) for every diagram of cartesian squares of the form
$$\begin{tikzcd} I'' \arrow[r, "v"] \arrow[d] & I' \arrow[r] \arrow[d] & I \arrow[d, "u"] \\ J'' \arrow[r, "w"'] & J' \arrow[r] & J \end{tikzcd}$$
if w is $\mathbb{D}$-coaspheric then v is $\mathbb{D}$-coaspheric.

d) $\forall j \in \mathrm{Ob}\, J$, $I_j \to I/j$ is $\mathbb{D}$-coaspheric

e) $\forall i \in \mathrm{Ob}\, I$, $i\backslash I \to u(i)\backslash J$ has $\mathbb{D}$-aspheric fibers

f) $\forall i \in \mathrm{Ob}\, I$, $i\backslash I \to u(i)\backslash J$ is a $\mathbb{D}$-equivalence and remains a $\mathbb{D}$-equivalence after any base change

J) $u : I \to J$ in $\mathcal{D}ia$. Equivalent conditions:

a) u is $\mathbb{D}$-smooth

b) for every diagram of cartesian squares of the form

$$\begin{tikzcd} I'' \arrow[r] \arrow[d] & J'' \arrow[d] \\ I' \arrow[r] \arrow[d] & J' \arrow[d] \\ I \arrow[r, "u"] & J \end{tikzcd}$$

the upper square is $\mathbb{D}$-exact;

c) for every diagram of cartesian squares of the form

$$\begin{tikzcd} I'' \arrow[r, "v"] \arrow[d] & I' \arrow[r] \arrow[d] & I \arrow[d, "u"] \\ J'' \arrow[r, "w"] & J' \arrow[r] & J \end{tikzcd}$$

is w is $\mathbb{D}$-aspheric then v is $\mathbb{D}$-aspheric;

d) $\forall j \in \mathrm{Ob}\, J$, $I_j \to j\backslash I$ is $\mathbb{D}$-aspheric;

e) $\forall i \in \mathrm{Ob}\, I$ $I/i \to J/u(i)$ has $\mathbb{D}$-aspheric fibers;

f) $\forall i \in \mathrm{Ob}\, I$ $I/i \to J/u(i)$ is a $\mathbb{D}$-equivalence and remains a $\mathbb{D}$-equivalence after any base change.

K) Duality: $u : I \to J$ in $\mathcal{D}ia$

$$u : I \to J \ \mathbb{D}\text{proper} \iff u^\circ : I^\circ \to J^\circ \ \mathbb{D}\text{-smooth}.$$