# Higher algebra and topological quantum field theory

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Notes for a talk at the workshop
on Lurie's classification of topological quantum field theories [Lur09a].

**Abstract**

We present Baez and Dolan's paper [BD95]. We present some results informally and refer to Lurie [Lur09a] for their precise statements.

# Contents

# 1 Homotopy

## 1.1 Simplicial sets

Let $\Delta$ be the category whose objects are finite ordered sets $[n] = [0, \ldots, n-1]$ and whose morphisms are nondecreasing maps.

**Definition 1.1.** A *simplicial set* is a contravariant functor

$$X : \Delta^{op} \to \textsc{Sets}.$$

We denote SSets the category of simplicial sets.

To give examples of simplicial sets, one can use the following description of the category $\Delta$ by generators and relations.

**Proposition 1.1.** *The morphisms*

$$\begin{array}{lll} d^i : [n-1] \to [n] & 0 \le i \le n & \textit{(cofaces)} \\ s^j : [n+1] \to [n] & 0 \le j \le n & \textit{(codegeneracies)} \end{array}$$

*given by*

$$d^i([0, \ldots, n-1]) = [0, \ldots, i-1, i+1, \ldots, n]$$

*and*

$$s^j([0, \ldots, n+1]) = [0, \ldots, j, j, \ldots, n]$$

*fulfill the so-called cosimplicial identities*

$$\begin{cases} d^j d^i &=& d^i d^{j-1} & \textit{if } i < j \\ s^j d^i &=& d^i s^{j-1} & \textit{if } i < j \\ s^j d^j &=& 1 = s^j d^{j+1} & \\ s^j d^i &=& d^{i-1} s^j & \textit{if } i > j+1 \\ s^j s^i &=& s^i s^{j+1} & \textit{if } i \le j \end{cases}$$

*and give a set of generators and relations for the category $\Delta$.*

Define the $n$ simplex to be $\Delta^n := \mathrm{Hom}(-, [n])$. Yoneda's lemma imply that for every simplicial set $X$, one has

$$\mathrm{Hom}(\Delta^n, X) \cong X_n.$$

The boundary $\partial\Delta^n$ is defined by

$$(\partial\Delta^n)_m := \{f : [m] \to [n],\ \mathrm{im}(f) \neq [n]\}$$

and the $k$-th horn $\Lambda^n_k \subset \partial\Delta^n$ by

$$(\Lambda^n_k)_m := \{f : [m] \to [n],\ k \notin \mathrm{im}(f)\}.$$

The category SSets has all limits and colimits (defined component-wise) and also internal homomorphisms defined by

$$\underline{\mathrm{Hom}}(X, Y) : [n] \mapsto \mathrm{Hom}_{\Delta_n}(X \times \Delta_n, Y \times \Delta_n) = \mathrm{Hom}(X \times \Delta^n, Y).$$

The geometric realization $|\Delta^n|$ is defined to be

$$|\Delta^n| := \{(t_0, \dots, t_n) \in [0,1]^{n+1}, \sum t_i = 1\}.$$

It defines a covariant functor $|.| : \Delta \to \mathrm{TOP}$ with maps $\theta_* : |\Delta^n| \to |\Delta^m|$ for $\theta : [n] \to [m]$ given by

$$\theta_*(t_0, \dots, t_m) = (s_0, \dots, s_n)$$

where

$$s_i = \begin{cases} 0 & \text{if} \quad \theta^{-1}(i) = \emptyset \\ \sum_{j \in \theta^{-1}(i)} t_j & \text{if} \quad \theta^{-1}(i) \neq \emptyset. \end{cases}$$

The geometric realization of a general simplicial set $X$ is the colimit

$$|X| = \operatorname{colim}_{\Delta^n \to X} |\Delta^n|$$

indexed by the category of maps $\Delta^n \to X$ for varying $n$. The $\infty$-groupoid (also called the singular simplex) of a given topological space $Y$ is the simplicial set

$$\Pi_\infty(Y) : [n] \mapsto \mathrm{Hom}_{\mathrm{TOP}}(|\Delta^n|, Y).$$

The geometric realization and $\infty$-groupoid functors are adjoint meaning that

$$\mathrm{Hom}_{\mathrm{TOP}}(|X|, Y) \cong \mathrm{Hom}_{\mathrm{SSETS}}(X, \Pi_\infty(Y)).$$

**Definition 1.2.** The *simplicial cylinder* of a given simplicial set is defined as $\mathrm{Cyl}(X) := X \times \Delta^1$. Let $f, g : X \to Y$ be morphisms of simplicial sets. A *homotopy* between $f$ and $g$ is a factorization

$$\begin{array}{ccc} X & & \\ \downarrow{\scriptstyle i_0} & \searrow^{f} & \\ \mathrm{Cyl}(X) & \xrightarrow{h} & X \\ \uparrow{\scriptstyle i_1} & \nearrow_{g} & \\ X & & \end{array}$$

Fibrations of simplicial sets are defined as maps that have the right lifting property with respect to all the standard inclusions $\Lambda^n_k \subset \Delta^n$, $n > 0$.

**Definition 1.3.** A morphism $p : X \to Y$ of simplicial sets is called a *fibration* (or a *Kan fibration*) if in every commutative square

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ \Delta^n & \longrightarrow & Y \end{array},$$

the dotted arrow exists to as to make the two triangle commutative. A simplicial set $X$ is called *fibrant* (or a *Kan complex*) if the projection map $X \to \{*\}$ is a fibration.

Let $X$ be a simplicial set. Its set of connected components is defined as the quotient

$$\pi_0(X) := X_0/\sim$$

by the equivalence relation $\sim$ generated by the relation $\sim_1$, called "being connected by a path", i.e., $x_0 \sim_1 x_1$ if there exists $\gamma : \Delta^1 \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

**Lemma 1.1.** *The functor* $\pi_0 : \mathrm{SS}\textsc{ets} \to \textsc{Sets}$ *commutes with direct products.*

*Proof.* This follows from the fact that $\pi_0(X)$ identifies naturally to the set of connected components $\pi_0(|X|)$ of the geometric realization, and geometric realization commutes with products. □

The above lemma implies that simplicial sets may be equiped with a new category structure $\mathrm{SS}\textsc{ets}^{\pi_0}$, whose morphisms are given by elements in $\pi_0(\underline{\mathrm{Hom}}(X, Y))$, and whose composition is induced by the internal composition map

$$\underline{\mathrm{Hom}}(X, Y) \times \underline{\mathrm{Hom}}(Y, Z) \to \underline{\mathrm{Hom}}(X, Z).$$

**Definition 1.4.** A morphism $f : X \to Y$ of simplicial sets is called a *weak equivalence* if for all fibrant object $Z$, the natural map

$$\pi_0(\underline{\mathrm{Hom}}(f, Z)) : \pi_0(\underline{\mathrm{Hom}}(Y, Z)) \to \pi_0(\underline{\mathrm{Hom}}(X, Z))$$

is bijective.

The following theorem is due to Quillen [Qui67].

**Theorem 1.1.** *The category of simplicial sets is equiped with a model category structure such that*

1. *weak equivalences are weak equivalences of simplicial sets,*
2. *fibrations are Kan fibrations,*
3. *cofibrations are injections.*

**Definition 1.5.** The above defined model structure is called the model structure of higher groupoids, and denoted $^{\infty}\mathbf{Grpd}$.

Let $X$ be a fibrant simplicial set and $x : \Delta^0 \to X$ be a base point of $X$. For $n \geq 1$, define the homotopy group $\pi_n(X, x)$ as the set of homotopy classes of maps $f : \Delta^n \to X$ that fit into a diagram

$$\begin{array}{ccc} \Delta^n & \xrightarrow{f} & X \\ \uparrow & & \uparrow \\ \partial\Delta^n & \longrightarrow & \Delta^0 \end{array} .$$

The geometric realization of such maps correspond to maps

$$f : S^n \to |X|.$$

**Theorem 1.2.** *Weak equivalences of simplicial sets may be defined as morphisms that induce isomorphisms on homotopy groups, for any choice of base point* $x : \Delta^0 \to X$.

**Theorem 1.3.** *The geometric realization and the $\infty$-groupoid functors induce a Quillen adjunction*

$$|-|: {}^{\infty}\mathbf{Grpd} \leftrightarrows \mathbf{Top} : \Pi_{\infty}$$

*whose derived version is an equivalence of categories*

$$\mathbb{L}|-|: {}^{\infty}\text{GRPD} \overset{\sim}{\leftrightarrows} h(\text{TOP}) : \mathbb{R}\Pi_{\infty}.$$

**Definition 1.6.** An $\infty$-groupoid is called *$k$-connected* if it has no homotopy in degrees smaller or equal to $k$. It is called an *$n$-type* if it has no homotopy in degrees strictly larger than $n$.

## 1.2 Stabilization

**Definition 1.7.** The *suspension* of a topological space $X$ is the quotient $SX$ of the topological space $X \times [0,1]$ by the relation that identifies all points $(x,0)$, $(x,1)$, and $(*,t)$ to a single point.

Suspension is a functor and if $f : X \to Y$ is a morphism, one gets a sequence

$$[X,Y] \overset{S}{\to} [SX,SY] \overset{S}{\to} [S^2X,S^2Y] \overset{S}{\to} \cdots$$

where $[X,Y] := \pi_0(\underline{\text{Hom}}(X,Y))$. If $X$ is a CW complex (or a smooth manifold, say) of dimension $n$, the sequence stabilizes after $n+2$ steps, i.e., the natural map

$$S : [S^kX, S^kX] \to [S^{k+1}X, S^{k+1}X]$$

is an isomorphism for $k \geq n+2$.

One may also define suspensions of *pointed* spaces by

$$\Sigma X := X \wedge S^1,$$

the smash product $\wedge$ being the product of pointed spaces. This suspension is adjoint to the loop space operation

$$\Omega X := \underline{\text{Hom}}(S^1, X).$$

A $k$-morphism in $\Pi_\infty(SX)$ corresponds to a $(k+1)$-morphism in $\Pi_\infty(X)$. For example, a point in $X$ gives a loop in $SX$.

**Definition 1.8.** A *properad* is a symmetric monoidal category $(\mathcal{P}, \otimes)$ whose objects are all of the form $x^{\otimes n}$ for $x$ a given object. The *operations* of $\mathcal{P}$ with $n$ entries and one output are defined by

$$\mathcal{P}(n) := \underline{\text{Hom}}_{\mathcal{P}}(x^{\otimes n}, x).$$

An algebra on $\mathcal{P}$ with values in a symmetric monoidal category $(\mathcal{C}, \otimes)$ is a symmetric monoidal functor

$$A : (\mathcal{P}, \otimes) \to (\mathcal{C}, \otimes).$$

One may also define $\infty$-properads and algebras on them in a similar way, using $\infty$-categories, to be defined latter. Properads also have nontrivial operations with multiple outputs, that are not constructed by a direct sum of one output operations. We will not consider them and thus restrict to the situation of so-called *operads*.

**Definition 1.9.** The little $k$-disc $\infty$-operad $\mathcal{E}_k$ has generating object the unit $k$-dimensional disc, with tensor product the disjoint union, and morphisms given by topological embeddings.

The fact that it is not a usual operad comes from the fact that its composition is associative only up to homotopy, and not on the nose. It is an operad enriched in the homotopy category $h(\text{TOP})$.

**Theorem 1.4.** *If $X$ is pointed and $(k-1)$-connected, then*

$$\Omega^k \Sigma^k X \cong X.$$

The following theorem is due to Stasheff, Boardman-Vogt or May. It allows to find a "purely algebraic" characterization of $k$-connected spaces.

**Theorem 1.5.** *If $X$ is a homotopy type, $\Omega^k X$ is naturally equiped with an (homotopy) action of $\mathcal{E}_k$. Moreover, if $Y$ is equiped with an (homotopy) action of $\mathcal{E}_k$, then there exists $X$ such that*

$$Y = \Omega^k X.$$

In the above theorem, the space $X$ is often denoted $B^k Y$ and called a $k$-delooping of $Y$. Delooping, when it exists, is equivalent to suspension, i.e., $B^k Y \cong \Sigma^k Y$.

# 2 Higher categories

## 2.1 Strict and weak higher categories

We first give a simple definition of a notion of $n$-category, following Simpson's book [Sim10]. One can find a finer axiomatic in [Sim01].

**Definition 2.1.** A theory of $n$-categories is supposed to have a notion of sum and product. A 0-category is a set. An *$n$-category* $\mathcal{A}$ is given by the following data:

1. (OB) a set $\text{Ob}(\mathcal{A})$ of *objects*, compatible with sums and products of $n$-categories.
2. (MOR) for each pair $(x, y)$ of objects, an $(n-1)$-category $\underline{\text{Mor}}(x, y)$. One defines

$$\underline{\text{Mor}}(\mathcal{A}) := \coprod_{x,y \in \text{Ob}(\mathcal{A})} \underline{\text{Mor}}(x, y)$$

   and by induction $\underline{\text{Mor}}^i(\mathcal{A}) := \underline{\text{Mor}}(\underline{\text{Mor}}(\dots(\mathcal{A})))$ for $0 \leq i \leq n$. One has $\underline{\text{Mor}}^0(\mathcal{A}) := \mathcal{A}$ and $\underline{\text{Mor}}^n(\mathcal{A})$ is a set. Denote $\text{Mor}(\mathcal{A}) = \text{Ob}(\underline{\text{Mor}}(\mathcal{A}))$. By construction, one has source and target maps

$$s_i, t_i : \text{Mor}^i(\mathcal{A}) \to \text{Mor}^{i-1}(\mathcal{A})$$

   that satisfy

$$s_i s_{i+1} = s_i t_{i+1},\ t_i s_{i+1} = t_i t_{i+1}.$$

3. (ID) for each $x \in \text{Ob}(\mathcal{A})$, there should be a natural element $1_x \in \text{Mor}(x, x)$. This gives morphisms

$$e_i : \text{Mor}^i(\mathcal{A}) \to \text{Mor}^{i+1}(\mathcal{A})$$

   such that $s_{i+1} e_i(u) = u$ and $t_{i+1} e_i(u) = u$.

4. (EQUIV) on each set $\mathrm{Mor}^i(\mathcal{A})$, there is an equivalence relation $\sim$ compatible with the source and target maps. The induced equivalence relation on $\mathrm{Mor}^i(x, y)$ is also denoted $\sim$.
5. (COMP) for any $0 < i \leq n$ and any three $i-1$-morphisms $u$, $v$ and $w$ sharing the same source and target, there is a well-defined composition map

$$(\mathrm{Mor}^i(u, v)/\sim) \times (\mathrm{Mor}^i(v, w)/\sim) \to (\mathrm{Mor}^i(u, w)/\sim)$$

which is associative and has the classes of identity morphisms as left and right units.
6. (EQC) Equivalence and composition are compatible: for any $0 \leq i < n$ and $u, v \in \mathrm{Mor}^i(u, v)$, sharing the same source and target, then $u \sim v$ if and only if there exists $f \in \mathrm{Mor}^{i+1}(u, v)$ and $g \in \mathrm{Mor}^{i+1}(v, u)$ such that $f \circ g \sim 1_u$ and $g \circ f \sim 1_v$. This allows one to define the category $\tau_{\leq 1}\underline{\mathrm{Mor}}^i(u, v)$ with objects $\mathrm{Mor}^i(u, v)$ and morphisms between $w, z \in \mathrm{Mor}^i(u, v)$ given by equivalence classes in $\mathrm{Mor}^{i+1}(w, z)/\sim$.

An $n$-category is called *strict* if its composition laws all lift on morphisms (meaning that all compositions maps are well defined and strictly associative). We will also sometimes call an $n$-category *weak* if it is not strict.

The main problem of higher category is to find the proper notion of weakening associativity, by replacing the equality

$$x(yz) = (xy)z$$

by an isomorphism, for example if we generalize monoids to monoidal categories, we get an isomorphism

$$x \otimes (y \otimes z) \to (x \otimes y) \otimes z$$

that must fulfil a higher coherence condition. The problem is that these higher coherence conditions are not easy to define when one goes to higher dimension. Simplicial and homotopical methods give tools that make this generalization systematic.

## 2.2 $^{\infty}$1-categories

**Definition 2.2.** Let $\mathcal{C}$ be a category. The *nerve* of $\mathcal{C}$ is the simplicial set $N(\mathcal{C})$ whose $n$-simplices are

$$N(\mathcal{C})_n := \mathrm{Hom}_{\mathrm{CAT}}([n], \mathcal{C}),$$

where $[n]$ is the linearly ordered set $\{0, \ldots, n\}$. More concretely,

1. $N(\mathcal{C})_0$ is the set of objects of $\mathcal{C}$,
2. $N(\mathcal{C})_1$ is the set of morphisms in $\mathcal{C}$, and
3. $N(\mathcal{C})_n$ is the set of families

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n$$

of composable arrows.

To define the simplicial structure of $N(\mathcal{C})$, one only needs to define maps

$$\begin{array}{lll} d_i : N(\mathcal{C})_n \to N(\mathcal{C})_{n-1}, & \text{for } 0 \leq i \leq n & \text{(faces)} \\ s_j : N(\mathcal{C})_n \to N(\mathcal{C})_{n+1}, & \text{for } 0 \leq j \leq n & \text{(degeneracies)} \end{array}$$

satisfying the simplicial identities, described in Section **??**. The faces $d_i$ are given by composition

$$x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i-1}} x_{i-1} \xrightarrow{f_{i+1} \circ f_i} x_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_n} x_n$$

of two consecutive arrows and the degeneracies $s_i$ by insertion of identities

$$x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_i} x_i \xrightarrow{\mathrm{id}_{x_i}} x_i \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_n} x_n.$$

Let $a_i : [1] \to [k]$ be the morphisms in the simplicial category $\Delta$ given by $a^i(0) = i$ and $a^i(1) = i + 1$ for $0 \leq i \leq k - 1$ and let $a_i$ be the corresponding maps in $\Delta^{op}$. Denote $X$ the simplicial set $N(\mathcal{C})(a_i)$. The map

$$X(a_i) = X_k \to X_1$$

simply sends a family of $k$ composable arrows to the $i$-th one. Their fiber product give the so-called *Segal maps*

$$\varphi_k : X_k \to X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1,$$

and by definition of $X_k$, these maps are bijections.

**Proposition 2.1.** *Let* Cat *be the category of small categories. The nerve functor*

$$\begin{array}{rccc} N : & \mathrm{Cat} & \to & \mathrm{SSets} \\ & C & \mapsto & [n] \mapsto \mathrm{Hom}_{\mathrm{Cat}}([n], C) \end{array}$$

*is a fully faithful embedding, with essential image the subcategory of simplicial sets whose Segal maps are bijections. Moreover, it has a left adjoint $\tau_1$, called the* fundamental category *functor.*

*Proof.* The full faithfulness is clear from the above description of the nerve. The existence of an adjoint to the nerve functor follows formally from the completeness of the categories in play (see [GZ67]). We only describe it explicitely. Let $X$ be a simplicial set. Two elements $f$ and $g$ of $X_1$ are called composable if there exists $h \in X_2$ such that $h_{[0,1]} = f$ and $h_{[1,2]} = g$. Their composition is then given by $h_{[0,2]}$. We put on $X_1$ the smallest equivalence relation generated by this composition, that is stable by composition. We define $\tau_1(X)$ as the category whose objects are elements in $X_0$, whose morphisms are families $(f_1, \ldots, f_n)$ of elements in $X_1$ that are composable, quotiented by the composition equivalence relation. □

**Definition 2.3.** A *quasi-fibration* of simplicial sets is a morphism $p : X \to Y$ such that every commutative square

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ \Delta^n & \longrightarrow & Y \end{array},$$

with $0 < k < n$ can be completed by a dotted arrow so as to make the two triangles commutative. A *categorical equivalence* of simplicial sets is a morphism $f : X \to Y$ that induces an equivalence of categories

$$\tau_1(f) : \tau_1(X) \to \tau_1(Y).$$

A *quasi-category* is a simplicial set $X$ such that the morphism $X \to \{*\}$ is a quasi-fibration. More precisely, it is a simplicial set such that for $0 < k < n$, there exists a dotted arrow rendering the diagram

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

commutative. An $\infty$-*groupoid* is a quasi-category such that the dotted arrow in the above diagram exists of $0 \leq k \leq n$.

Remark that an $\infty$-groupoid is exactly the same as a fibrant simplicial set for the model structure ${}^{\infty}\mathbf{Grpd}$ (i.e., a homotopy type), and a quasi-category such that the dotted arrow rendering the diagram

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

commutative is unique is simply a nerve.

**Lemma 2.1.** *A simplicial set $\mathcal{C}$ is a quasi-category if and only if the projection*

$$\underline{\mathrm{Hom}}_{\mathrm{SS}\text{ETS}}(\Delta^2, \mathcal{C}) \longrightarrow \underline{\mathrm{Hom}}_{\mathrm{SS}\text{ETS}}(\Lambda^2_1, \mathcal{C})$$

*is a trivial fibration.*

We now give an intrinsic formulation of the fact that the collection of $\infty$-categories form an $\infty$-category.

**Theorem 2.1.** *There exists a model structure on* SSETS*, denoted* ${}^{\infty}\mathbf{Cat}$*, whose fibrations are quasi-fibrations, whose cofibrations are monomorphisms, and whose weak equivalences are categorical equivalences. Its fibrant objects are quasi-categories.*

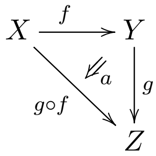
*Proof.* We refer to [Joy11], and [Lur09b] for a complete proof. □

**Definition 2.4.** An $\infty$-category is an object of the homotopy category of ${}^{\infty}\mathbf{Cat}$.

**Definition 2.5.** If $\mathcal{C}$ is a quasi-category representing an $\infty$-category, one calls

1. elements in $\mathcal{C}_0$ its *objects*,
2. elements in $\mathcal{C}_1$ its *morphisms*,
3. elements in $\mathcal{C}_2$ its *compositions*.

If $\mathcal{C}$ is a quasi-category, one may interpret elements in $\mathcal{C}_2$ as triangular 2-morphisms




in the corresponding $\infty$-category, encoding compositions. Remark that the arrow $g \circ f$ is only unique up to equivalence, and that the 2-morphism $a$ is invertible only up to a 3-morphism, that may be drawn as a tetrahedron.

**Proposition 2.2.** *The identity functors*

$$^{\infty}\mathbf{Grpd} \leftrightarrows {}^{\infty}\mathbf{Cat}$$

*induce a Quillen adjunction between the two given model categories on* SSETS. *The corresponding adjunction on homotopy categories is also denoted*

$$i : {}^{\infty}\mathrm{GRPD} \leftrightarrows {}^{\infty}\mathrm{CAT} : \tau_0,$$

*and the functor $i$ is a fully faithful embedding.*

**Proposition 2.3.** *Let $\mathcal{C}$ be a quasi-category and $X$ be a simplicial set. The simplicial set*

$$\underline{\mathrm{Mor}}_{Q\mathrm{CAT}}(X, \mathcal{C}) := \underline{\mathrm{Hom}}_{\mathrm{SSETS}}(X, \mathcal{C})$$

*is a quasi-category. Moreover, if $F : X \to Y$ or $G : \mathcal{C} \to \mathcal{D}$ are categorical equivalences, then*

$$\underline{\mathrm{Mor}}(F, \mathcal{C}) : \underline{\mathrm{Mor}}_{Q\mathrm{CAT}}(X, \mathcal{C}) \to \underline{\mathrm{Mor}}_{Q\mathrm{CAT}}(Y, C)$$

*and*

$$\underline{\mathrm{Mor}}(X, G) : \underline{\mathrm{Mor}}_{Q\mathrm{CAT}}(X, \mathcal{C}) \to \underline{\mathrm{Mor}}_{Q\mathrm{CAT}}(X, \mathcal{D})$$

*are categorical equivalences.*

The above proposition implies that $^{\infty}\mathrm{CAT}$ form an $\infty$-category.

**Definition 2.6.** Let $\mathcal{C}$ be a quasi-category, and $x$ and $y$ be two objects of $\mathcal{C}$. The simplicial set of morphisms from $x$ to $y$ is defined by the fiber product

$$\begin{array}{ccc} \underline{\mathrm{Mor}}_{\mathcal{C}}(x, y) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathrm{SSETS}}(\Delta^1, \mathcal{C}) \\ \downarrow & & \downarrow{\scriptstyle (s,t)} \\ \Delta_0 & \xrightarrow{(x,y)} & \mathcal{C} \times \mathcal{C} \end{array}$$

**Proposition 2.4.** *The simplicial set $\underline{\mathrm{Mor}}_{\mathcal{C}}(x, y)$ is a Kan complex. There exists a morphism of simplicial sets*

$$\circ : \underline{\mathrm{Mor}}_{\mathcal{C}}(x, y) \times \underline{\mathrm{Mor}}_{\mathcal{C}}(y, z) \to \underline{\mathrm{Mor}}_{\mathcal{C}}(x, z)$$

*that gives an associative composition law in the category $^{\infty}\mathrm{GRPD}$, unique up to isomorphism.*

**Definition 2.7.** Let $\mathcal{C}$ and $I$ be two $\infty$-categories. A limit functor (resp. colimit functor) is a right (resp. left) adjoint to the constant $\infty$-functor

$$\mathcal{C} \cong \underline{\mathrm{Mor}}_{\infty\mathrm{CAT}}(*, \mathcal{C}) \to \underline{\mathrm{Mor}}_{\infty\mathrm{CAT}}(I, \mathcal{C}).$$

## 2.3 Monoidal structures

**Definition 2.8.** Let $(\mathcal{T}_{Mon}, \times)$ be the category with finite (and empty) products opposite to that of finitely generated free monoids. We denote $x$ the free monoid on one element. A 1-*monoidal* ${}^{\infty}n$*-category* is a functor

$$\mathcal{C} : (\mathcal{T}_{Mon}, \times) \to ({}^{\infty}n\text{CAT}, \times)$$

that commutes with finite products, meaning that the natural morphism

$$\mathcal{C}(x^{\times n}) \to \mathcal{C}(x)^{\times n}$$

is a homotopy equivalence for all $n$. We denote ${}^{\infty}n\text{CAT}_k$ the ${}^{\infty}n$-category of $k$-monoidal ${}^{\infty}n$-categories, defined by the following inductive method: a $k$*-monoidal* ${}^{\infty}n$*-category* is a functor

$$\mathcal{C} : (\mathcal{T}_{Mon}, \times) \to ({}^{\infty}n\text{CAT}_{k-1}, \times)$$

that commutes with finite products.

We may already see ${}^{\infty}n\text{CAT}$ as a monoidal ${}^{\infty}n$-category.

**Definition 2.9.** A symmetric monoidal ${}^{\infty}n$-category is a product preserving $\infty$-functor

$$\mathcal{C} : (\mathcal{T}_{\text{CMon}}, \times) \to ({}^{\infty}n\text{CAT}, \times)$$

from the category with finite products opposite to that of free finitely generated commutative monoids to higher categories.

Remark that both $\mathcal{E}_k$ and ${}^{\infty}n\text{CAT}$ may be seen as symmetric monoidal $\infty$-categories.

**Definition 2.10.** An $\mathcal{E}_k$-algebra in ${}^{\infty}n\text{CAT}$ is a symmetric monoidal $\infty$-functor

$$\mathcal{C} : (\mathcal{E}_k, \coprod) \to ({}^{\infty}n\text{CAT}, \times).$$

**Theorem 2.2** (Delooping for higher categories)**.** *There is a natural equivalence*

$${}^{\infty}n\text{CAT}_k \xrightarrow{\sim} \text{ALG}_{\mathcal{E}_k}({}^{\infty}n\text{CAT}, \times).$$

## 2.4 Stabilization

The following theorem is due to Simpson [Sim10].

**Theorem 2.3.** *There is a natural equivalence*

$$S^k : {}^{\infty}n\text{CAT}_k \overset{\sim}{\leftrightarrows} {}^{\infty}(n+k)\text{CAT}_{k-con} : B^k$$

*of $k$-monoidal $n$-categories with $k$-connected higher categories (that have only* 1 *morphism up to degree $k$).*

*Example* 2.1. We now illustrate the above general definition by a simple example, that is related to the general stabilization theorem, called the Heckmann-Hilton argument. Suppose given a set with two compatible unitary and associative multiplications denoted $\otimes$ and by concatenation. One then has

$$\begin{array}{rcl} \alpha\otimes\beta & = & (1\alpha)\otimes(\beta 1) \\ & = & (1\otimes\beta)(\alpha\otimes 1) \\ & = & \beta\alpha \\ & = & (\beta\otimes 1)(1\otimes\alpha) \\ & = & (\beta 1)\otimes(1\alpha) \\ & = & \beta\otimes\alpha, \end{array}$$

This shows that both structure coincide and are commutative.

**Theorem 2.4** (Stabilization)**.** *If $k \geq n+2$, the suspension functor*

$$S : {}^{\infty}n\mathrm{CAT}_k \xrightarrow{\sim} {}^{\infty}n\mathrm{CAT}_{k+1}$$

*is an equivalence.*

Here is the table given in the original article [BD95].

| | $n=0$ | $n=1$ | $n=2$ |
|---|---|---|---|
| $k=0$ | sets | categories | 2-categories |
| $k=1$ | monoids | monoidal categories | monoidal 2-categories |
| $k=2$ | commutative monoids | braided monoidal categories | braided monoidal 2-categories |
| $k=3$ | ‘’ | symmetric monoidal categories | weakly involutory monoidal 2-categories |
| $k=4$ | ‘’ | ‘’ | strongly involutory monoidal 2-categories |
| $k=5$ | ‘’ | ‘’ | ‘’ |

21. Semistrict $k$-tuply monoidal $n$-categories

# 3 Topological quantum field theories

All theories that we will encounter can be classified by using the following useful definition, that generalizes Lawvere's 2-categorical notion.

**Definition 3.1.** A doctrine is an ${}^{\infty}n$-category $\mathcal{D}$. A theory for the doctrine $\mathcal{D}$ is an object $\mathcal{T}$ of $\mathcal{D}$. A model for a theory $\mathcal{T}_1$ in a theory $\mathcal{T}_2$ is a morphism

$$M : \mathcal{T}_1 \to \mathcal{T}_2.$$

For example, we already discussed the doctrine FPCAT of finite product categories, that correspond to algebraic theories, and the theory of monoids $\mathcal{T}_{Mon}$ in FPCAT. Linear theories are given by theories for the doctrine MonCAT of monoidal categories. One may also work with the doctrine SymMonCAT of symmetric monoidal categories. This gives operadic and properadic theories, for which $\mathcal{E}_k$ is an example. Similarly, we have defined the doctrine of $k$-monoidal $n$-categories $^{\infty}n\mathrm{CAT}_k$, and also $^{\infty}n$SymMonCAT.

It seems useful to always say what kind of theories we are describing.

We will now present other higher categorical doctrines and theories, called the theory of extended topological field theory and the theory of tangled topological field theory. These can be completely described as free theories on one object in doctrines related to $^{\infty}n$SymMonCAT and $^{\infty}n\mathrm{CAT}_k$ respectively. We will also discuss briefly their relations.

## 3.1 Cobordism monoidal category

One defines $n\mathrm{COB}_n$ as the $^{\infty}n$-category with objects oriented points, morphisms given by 1-manifolds with boundary, 2-morphisms given by 2-manifolds with corners, and so on. This is a symmetric monoidal $n$-category with duals. An extended topological quantum field theory is a symmetric monoidal functor from it to another symmetric monoidal $^{\infty}n$-category with duals.

## 3.2 Cobordism and Morse theory

Generators and relations for $n\mathrm{COB}_n$ over $(n-1)\mathrm{COB}_{n-1}$ are given by a kind of generalized/parametrized Morse theory. Generators of it are given by handle attachments, through usual morse theory and relations are given by handle slides and cancellations, using cerf theory of parametrized morse functions.

## 3.3 Tangle hypothesis

An $n$-tangle in $k$-dimension (i.e., entrelac) is an $n$-manifold with corners embedded in $[0,1]^{n+k}$ so that the dimension $k$ corners of the manifold are mapped into the subset of $[0,1]^{n+k}$ for which the last $j$ coordinates are either 0 or 1.

**Theorem 3.1** (Tangle hypothesis)**.** *The $n$-category $n\mathrm{TANGLES}_k$ of framed $n$-tangles in $n+k$ dimensions is $(n+k)$-equivalent to the free $k$-tuply monoidal $n$-category with duals on one object.*

In particular, the theory of 1-tangles in 3-dimensions gives us the theory

$$2\mathrm{CAT}_1^{duals} = \mathrm{MonCAT}_{braided}^{dual}$$

of braided monoidal categories with duals, that are essentially equivalent to the so-called quantum groups. The corresponding topological quantum field theories are usually with values in the symmetric monoidal category of vector spaces, or else in other braided monoidal categories.

See next page for the picture in the original article [BD95].

## 3.4 Extended TQFT

One may embed $n\textsc{Cob}_n$ into $n\textsc{Tangles}_k$ for some $k \geq n+2$. The point is that for $k$ big enough, the stabilization theorem shows that

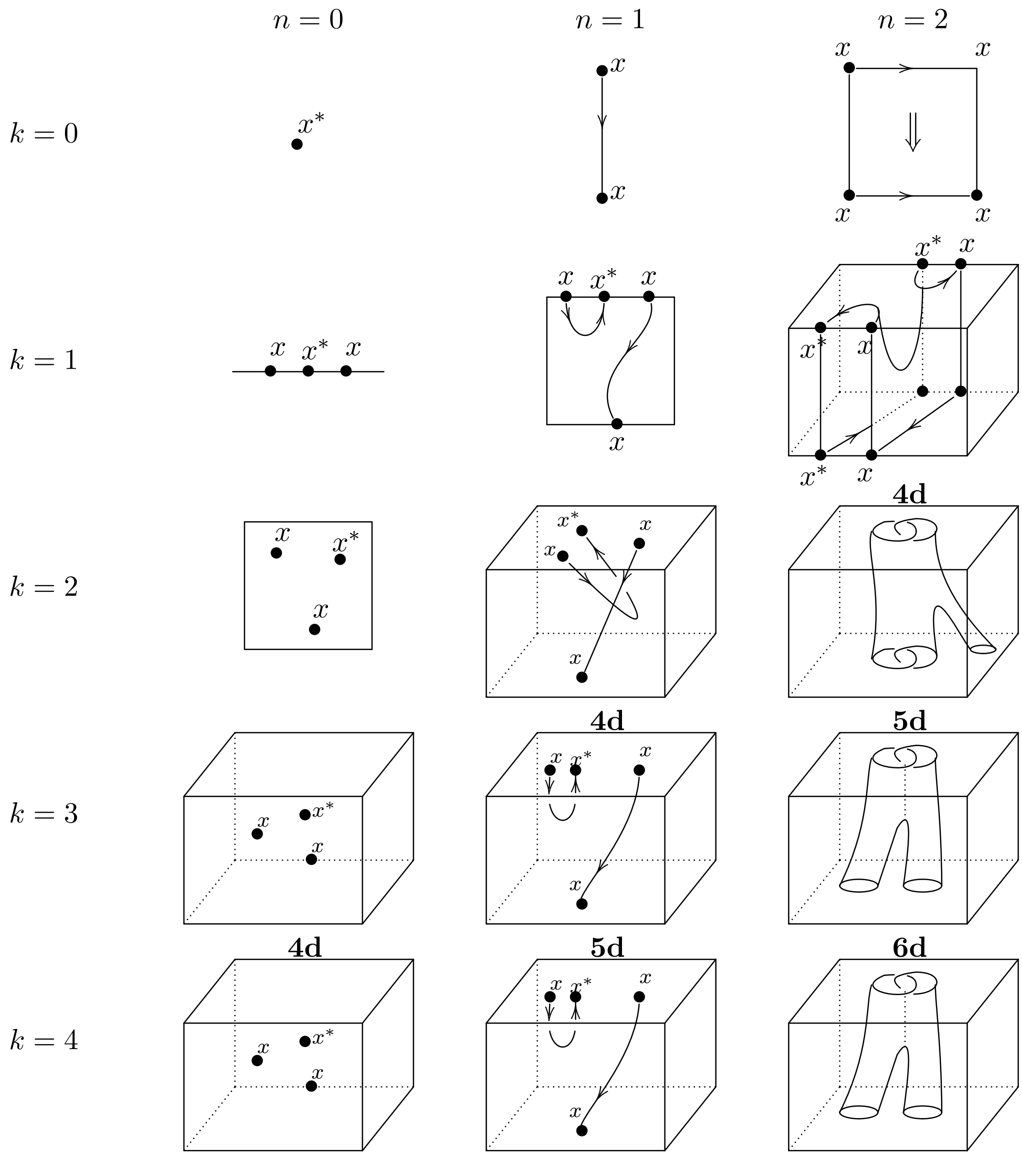
$$S : n\textsc{Tangles}_k \to n\textsc{Tangles}_{k+1}$$

is an equivalence.

**Theorem 3.2** (Extended TQFT hypothesis)**.** *The natural functor*

$$n\textsc{Cob}_n \to n\textsc{Tangles}_k$$

*is an equivalence for k big enough. The category* $n\textsc{Cob}_n$ *is the free symmetric (i.e., stable) monoidal category with duality on one object.*



30. Examples of $n$-morphisms in $C_{n,k}$, drawn as $n$-tangles in $n+k$ dimensions

### 3.5 Deformation quantization and generalized center

The use of refined notions of noncommutativity allows to define various notions of deformation quantizations.

One may deform a symmetric monoidal $n$-category to a $k$-monoidal $n$-category. For example, this is what is done for field theories in [PTVV11], using the model of dg-categories for linear $^{\infty}1$-categories.

**Definition 3.2.** Let $\mathcal{C}$ be a $k$-tuply monoidal $n$-category. The generalized center $Z(\mathcal{C})$ of $\mathcal{C}$ is the largest sub-$(n+k+1)$-category of $(n+k)\text{Cat}$ having $\mathcal{C}$ as its only object, $1_{\mathcal{C}}$ as its only morphism, and so on up to only one $k$-morphism. It is (equivalent to) a $(k+1)$-tuply monoidal $n$-category.

In the case of a braided monoidal category $\mathcal{C}$ of representations of a Hopf algebra $H$, the center is the category of representations of the quantum double of $H$. The quantum group associated to a given group is naturally a quotient of its quantum double.

# References


[BD95] J. C. Baez and J. Dolan. Higher-dimensional Algebra and Topological Quantum Field Theory. In *eprint arXiv:q-alg/9503002*, page 3002, March 1995.

[GZ67] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967.

[Joy11] André Joyal. Notes on quasi-categories, 2011.

[Lur09a] J. Lurie. On the Classification of Topological Field Theories. *ArXiv e-prints*, May 2009.

[Lur09b] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[PTVV11] T. Pantev, B. Toen, M. Vaquie, and G. Vezzosi. Quantization and Derived Moduli Spaces I: Shifted Symplectic Structures. *ArXiv e-prints*, November 2011.

[Qui67] Daniel G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin, 1967.

[Sim01] C. Simpson. Some properties of the theory of n-categories. *ArXiv Mathematics e-prints*, October 2001.

[Sim10] C. T. Simpson. Homotopy theory of higher categories. *ArXiv e-prints*, January 2010.