GROTHENDIECK ∞-GROUPOIDS,

AND STILL ANOTHER DEFINITION OF ∞-CATEGORIES

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Abstract. The aim of this paper is to present a simplified version of the notion of ∞-groupoid developed by Grothendieck in "Pursuing Stacks" and to introduce a definition of ∞-categories inspired by Grothendieck's approach.

Introduction

The precise definition of Grothendieck ∞-groupoids [5, sections 1-13] has been presented in [7]. In this paper, we give a slightly simplified version of this notion, and a variant leading to a definition of (weak) ∞-categories [8], very close to Batanin's operadic definition [2]. The precise relationship between these two notions is investigated by Ara in [1].

The basic intuition leading to the definition of a ∞-groupoid is presented as follows by Grothendieck (for a ∞-groupoid F, with set of i-cells F_i): "*Intuitively, it means that whenever we have* two *ways of associating to a finite family* $(u_i)_{i\in I}$ *of objects of an* ∞*-groupoid,* $u_i \in F_{n(i)}$*, subjected to a "standard" set of relations on the* u_i*'s, an element of some* F_n*, in terms of the* ∞*-groupoid structure only, then we have automatically a "homotopy" between these built-in in the very structure of the* ∞*-groupoid, provided it makes at all sense to ask for one . . .*" [5, section 9]. This leads him to the notion of *coherator*, category $\mathbf{C}$ endowed with a "universal ∞-cogroupoid", a ∞-groupoid being a presheaf on $\mathbf{C}$ satisfying some left exactness conditions, improperly called Segal conditions in the literature. In particular, Grothendieck ∞-groupoids define an algebraic structure species, and the category of ∞-groupoids is locally presentable.

The notion of a ∞-groupoid depends on the choice of a coherator. Two different coherators give rise in general to non-equivalent categories of ∞-groupoids. Nevertheless, the two notions of ∞-groupoid are expected to be equivalent in some subtler way. Grothendieck illustrates this fact as follows: "*Roughly saying, two different mathematicians, working independently on the conceptual problem I had in mind, assuming they both wind up with some explicit definition, will almost certainly get non-equivalent definitions – namely with non-equivalent categories of* (*set-valued, say*) ∞*-groupoids! And, secondly and as importantly, that* this ambiguity however is an irrelevant one. *To make this point a little clearer, I could say that a third mathematician, informed of the work of both, will readily think out a functor or rather a pair of functors, associating to any structure of Mr. X one of Mr. Y and conversely, in such a way that by composition of the two, we will associate to a* X*-structure* (T *say*) *another* T'*, which will not be isomorphic to* T *of course, but endowed with a canonical* ∞*-equivalence* (*in the sense of Mr. X*) $T \underset{\infty}{\simeq} T'$*, and the same on the Mr. Y side. Most probably, a fourth mathematician, faced with the same situation as the third, will get his own pair of functors to reconcile Mr. X*

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and Mr. Y, which very probably won't be equivalent (*I mean isomorphic*) *to the previous one. Here however a fifth mathematician, informed about this new perplexity, will probably show that the two Y-structures U and U', associated by his two colleagues to an X-structure T, while not isomorphic, also admit however a canonical ∞-equivalence between U and U'* (*in the sense of the Y-theory*)*. I could go on with a sixth mathematician, confronted with the same perplexity as the previous one, who winds up with another ∞-equivalence between U and U'* (*without being informed of the work of the fifth*)*, and a seventh reconciling them by discovering an ∞-equivalence between these equivalences. The story of course is infinite, I better stop with seven mathematicians, ...*" [5, section 9].

One of the reasons of Grothendieck's interest in ∞-groupoids is that (weak) ∞-groupoids are conjectured to modelize homotopy types (it's well known that *strict* ∞-groupoids don't): "*Among the things to be checked is of course that when we localize the category of ∞-groupoids with respect to morphisms which are "weak equivalences" in a rather obvious sense* (N.B. – *the definition of the π_i's of an ∞-groupoid is practically trivial!*)*, we get a category equivalent to the usual homotopy category* Hot." [5, section 12]. This conjecture is still *not* proven, for any definition of ∞-groupoid giving rise to an *algebraic* structure species, although some progress has been done in this direction by Cisinski [4]. It becomes tautological if we *define* ∞-groupoids as being Kan complexes or topological spaces! But the categories of such are *not* locally presentable. In a letter to Tim Porter, Grothendieck clearly explains that this was not the kind of definition he was looking for: "*my main point is that your suggestion that Kan complexes are "the ultimate in lax ∞-groupoids" does not in any way meet with what I am really looking for, and this for a variety of reasons, ...*"[6].

One of the peculiarities of Grothendieck's definition of ∞-groupoids is that this notion is *not* a particular case of a concept of lax ∞-category. Nevertheless, it was realized in [8] that a slight modification of the notion of coherator gives rise to such a concept. This new formalization of lax ∞-categories is very close, although not exactly equivalent, to the notion introduced by Batanin [2]. From a technical point of view, the basic difference is that the first is based on universal algebra, whereas the second on a generalization of the notion of operads, the globular operads. The precise relationship between the two concepts is studied in [1].

In the first section, Grothendieck's definition of ∞-groupoids is presented (in a two-page slightly simplified form), introducing the notion of "coherator for a theory of ∞-groupoids". Some examples of such coherators are given. The aim of the very long (too long?) last subsection 1.7 is to convince the reader of the pertinence of Grothendieck's concept, and define some structural maps in ∞-groupoids, useful in the next section.

In section 2, homotopy groups and weak equivalences between ∞-groupoids are introduced. The pair of adjoint functors "classifying space" of a ∞-groupoid and "fundamental ∞-groupoid" of a topological space are defined. Grothendieck's conjecture is presented.

In section 3, the only really original part of this paper, an interpretation of the notion of coherator for a theory of ∞-groupoids, in terms of lifting properties and weak factorization systems, is given.

In the last section, the definition of ∞-categories of [8] is presented, introducing the notion of "coherator for a theory of ∞-categories". The reader mainly interested by this definition can read directly this section after 1.4 and skip everything in between.

In appendix A, a technical result used in section 3 is proved.

1. Grothendieck ∞-groupoids

1.1. The category of globes. The *globular category* or *category of globes* is the category $\mathbb{G}$ generated by the graph

$$\mathbf{D}_0 \underset{\boldsymbol{\tau}_1}{\overset{\boldsymbol{\sigma}_1}{\rightrightarrows}} \mathbf{D}_1 \underset{\boldsymbol{\tau}_2}{\overset{\boldsymbol{\sigma}_2}{\rightrightarrows}} \cdots \underset{\boldsymbol{\tau}_{i-1}}{\overset{\boldsymbol{\sigma}_{i-1}}{\rightrightarrows}} \mathbf{D}_{i-1} \underset{\boldsymbol{\tau}_i}{\overset{\boldsymbol{\sigma}_i}{\rightrightarrows}} \mathbf{D}_i \underset{\boldsymbol{\tau}_{i+1}}{\overset{\boldsymbol{\sigma}_{i+1}}{\rightrightarrows}} \cdots ,$$

under the *coglobular* relations

$$\boldsymbol{\sigma}_{i+1}\boldsymbol{\sigma}_i = \boldsymbol{\tau}_{i+1}\boldsymbol{\sigma}_i \quad \text{and} \quad \boldsymbol{\sigma}_{i+1}\boldsymbol{\tau}_i = \boldsymbol{\tau}_{i+1}\boldsymbol{\tau}_i \ , \qquad i \geqslant 1 \ .$$

For every i, j, such that $0 \leqslant i \leqslant j$, define

$$\boldsymbol{\sigma}_i^j = \boldsymbol{\sigma}_j \cdots \boldsymbol{\sigma}_{i+2}\boldsymbol{\sigma}_{i+1} \quad \text{and} \quad \boldsymbol{\tau}_i^j = \boldsymbol{\tau}_j \cdots \boldsymbol{\tau}_{i+2}\boldsymbol{\tau}_{i+1} \ ,$$

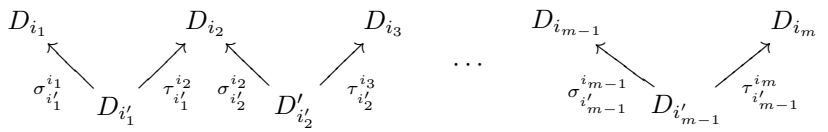
and observe that

$$\mathsf{Hom}_{\mathbb{G}}(\mathbf{D}_i, \mathbf{D}_j) = \begin{cases} \{\boldsymbol{\sigma}_i^j, \boldsymbol{\tau}_i^j\} \ , & \text{if } i < j \ , \\ \{1_{\mathbf{D}_i}\} \ , & \text{if } i = j \ , \\ \varnothing \ , & \text{else.} \end{cases}$$

1.2. Globular sums. Let C be a category, $\mathbb{G} \to C$ a functor, and let D_i, σ_i, τ_i, σ_i^j, τ_i^j be the image in C of $\mathbf{D}_i$, $\boldsymbol{\sigma}_i$, $\boldsymbol{\tau}_i$, $\boldsymbol{\sigma}_i^j$, $\boldsymbol{\tau}_i^j$ respectively. A *standard iterated amalgamated sum*, or more simply a *globular sum*, of *length* m, in C is an iterated amalgamated sum of the form

$$(D_{i_1}, \sigma_{i'_1}^{i_1}) \amalg_{D_{i'_1}} (\tau_{i'_1}^{i_2}, D_{i_2}, \sigma_{i'_2}^{i_2}) \amalg_{D_{i'_2}} \cdots \amalg_{D_{i'_{m-1}}} (\tau_{i'_{m-1}}^{i_m}, D_{i_m}) \ ,$$

colimit of the diagram



where $m \geqslant 1$, and for every k, $1 \leqslant k < m$, i'_k is strictly smaller then i_k and i_{k+1}. Such a globular sum is completely determined by the *table of dimensions*

$$\begin{pmatrix} i_1 & & i_2 & & \cdots & & i_m \\ & i'_1 & & i'_2 & \cdots & i'_{m-1} & \end{pmatrix} ,$$

and will be simply denoted

$$D_{i_1} \amalg_{D_{i'_1}} D_{i_2} \amalg_{D_{i'_2}} \cdots \amalg_{D_{i'_{m-1}}} D_{i_m} \ .$$

1.3. Globular extensions. A category C, endowed with a functor $\mathbb{G} \to C$, is called a *globular extension* if globular sums exist in C. For example, *any* functor from $\mathbb{G}$ to a cocomplete category defines a globular extension. A morphism from a globular extension to another is a functor under $\mathbb{G}$, commuting with globular sums. There exists a universal globular extension $\mathbb{G} \to \boldsymbol{\Theta}_0$, called a *globular completion of* $\mathbb{G}$, satisfying the following universal property: for every globular extension $\mathbb{G} \to C$, there exists a morphism of globular extensions $\boldsymbol{\Theta}_0 \to C$, unique up to unique natural isomorphism (inducing the identity on objects coming from $\mathbb{G}$). This universal globular extension, defined up to equivalence of categories, can be constructed, for example, by taking the closure by globular sums of $\mathbb{G}$ embedded by the Yoneda functor in the category $\widehat{\mathbb{G}}$ of *globular sets* (or ∞*-graphs*), *i.e.* presheaves on $\mathbb{G}$. The objects of $\boldsymbol{\Theta}_0$ are *rigid*, *i.e.* have no non-trivial automorphisms, and there is a combinatorial description of the category $\boldsymbol{\Theta}_0$ in terms of planar trees [1, 3, 8], leading to a *skeletal* incarnation of $\boldsymbol{\Theta}_0$ (such that isomorphic objects are equal), the objects being in bijection with tables of dimensions. In the sequel we choose, once and for all, such a skeletal model of $\boldsymbol{\Theta}_0$.

1.4. Coherators for a theory of ∞-groupoids. Let C be a category, $\mathbb{G} \to C$ a functor, and let D_i, σ_i, τ_i be the image in C of $\mathbf{D}_i$, $\boldsymbol{\sigma}_i$, $\boldsymbol{\tau}_i$ respectively. A *pair of parallel arrows* in C is a pair (f,g) of arrows $f,g : D_i \to X$ in C such that either $i=0$, or $i>0$ and $f\sigma_i = g\sigma_i$, $f\tau_i = g\tau_i$. A *lifting* of such a pair (f,g) is an arrow $h : D_{i+1} \to X$ such that $f = h\sigma_{i+1}$, $g = h\tau_{i+1}$. A *coherator for a theory of ∞-groupoids*, or more simply a Gr-*coherator*, is a globular extension $\mathbb{G} \to \mathbf{C}$ satisfying the following two conditions:

a) Every pair of parallel arrows in $\mathbf{C}$ has a lifting in $\mathbf{C}$.

b) There exists a "tower" of globular extensions (called *tower of definition* of the Gr-coherator $\mathbf{C}$) with colimit $\mathbf{C}$

$$\mathbb{G} \longrightarrow \mathbf{C}_0 \longrightarrow \mathbf{C}_1 \longrightarrow \cdots \longrightarrow \mathbf{C}_n \longrightarrow \mathbf{C}_{n+1} \longrightarrow \cdots \longrightarrow \mathbf{C} \simeq \varinjlim \mathbf{C}_n \quad ,$$

where for every $n \geqslant 0$, $\mathbf{C}_n$ is a small category, $\mathbf{C}_n \to \mathbf{C}_{n+1}$ a morphism of globular extensions, and satisfying the following properties:

- b_0) $\mathbb{G} \to \mathbf{C}_0$ is a globular completion;
- b_n) for every $n \geqslant 0$, there exists a family of pairs of parallel arrows in $\mathbf{C}_n$ such that $\mathbf{C}_{n+1}$ is the universal globular extension obtained from $\mathbf{C}_n$ by formally adding a lifting for every pair in this family.

Condition (b_0) implies that the category $\mathbf{C}_0$ is equivalent to $\boldsymbol{\Theta}_0$. We will usually assume that $\mathbf{C}_0$ is *equal* to $\boldsymbol{\Theta}_0$ (and that the functor $\mathbb{G} \to \mathbf{C}_0$ is the canonical inclusion $\mathbb{G} \to \boldsymbol{\Theta}_0$). Condition (b_n) means, more precisely, that there exists a family $(\mathbf{f}_i, \mathbf{g}_i)_{i \in I_n}$ of pairs of parallel arrows in $\mathbf{C}_n$, and for every $i \in I_n$, a lifting $\mathbf{h}_i$ in $\mathbf{C}_{n+1}$ of the image of the pair $(\mathbf{f}_i, \mathbf{g}_i)$ in $\mathbf{C}_{n+1}$, satisfying the following universal property. For every globular extension $\mathbb{G} \to C$ and every morphism of globular extensions $\mathbf{C}_n \to C$, if for every $i \in I_n$ a lifting h_i of the image of the pair $(\mathbf{f}_i, \mathbf{g}_i)$ in C is given, then there exists a unique morphism of globular extensions $F : \mathbf{C}_{n+1} \to C$ such that for every i in I_n, $F(\mathbf{h}_i) = h_i$ and such that the triangle

$$\begin{array}{ccc} \mathbf{C}_n & \longrightarrow & \mathbf{C}_{n+1} \\ & \searrow & \downarrow F \\ & & C \end{array}$$

is commutative. It can be easily seen that the functors $\mathbf{C}_n \to \mathbf{C}_{n+1}$ induce bijections on the sets of objects, so that we can suppose that all categories $\mathbf{C}_n$ and $\mathbf{C}$ have same objects, indexed by tables of dimensions (or planar trees). Furthermore, it can be proved (*cf.* 2.6) that for any Gr-coherator $\mathbf{C}$, the induced functor $\boldsymbol{\Theta}_0 = \mathbf{C}_0 \to \mathbf{C}$ is faithful. In particular, the category $\mathbb{G}$ will be usually identified to a (non-full) subcategory of $\mathbf{C}$. It is conjectured that all functors $\mathbf{C}_n \to \mathbf{C}$ are faithful.

1.5. Grothendieck ∞-groupoids. Let $\mathbb{G} \to \mathbf{C}$ be a Gr-coherator. A ∞-*groupoid of type* $\mathbf{C}$, or more simply a ∞-$\mathbf{C}$-*groupoid*, is a presheaf $G : \mathbf{C}^{\mathrm{op}} \to \mathsf{Set}$ on $\mathbf{C}$ such that the functor $G^{\mathrm{op}} : \mathbf{C} \to \mathsf{Set}^{\mathrm{op}}$ preserves globular sums. In other terms, for any globular sum in $\mathbf{C}$, the canonical map

$$G(\mathbf{D}_{i_1} \amalg_{\mathbf{D}_{i'_1}} \cdots \amalg_{\mathbf{D}_{i'_{m-1}}} \mathbf{D}_{i_m}) \longrightarrow G(\mathbf{D}_{i_1}) \times_{G(\mathbf{D}_{i'_1})} \cdots \times_{G(\mathbf{D}_{i'_{m-1}})} G(\mathbf{D}_{i_m})$$

is a bijection, the right hand side being the *standard iterated fiber product* or *globular product*, limit of the diagram

$$\begin{array}{ccccccc} G(\mathbf{D}_{i_1}) & & G(\mathbf{D}_{i_2}) & & & & G(\mathbf{D}_{i_m}) \\ & {\scriptstyle G(\boldsymbol{\sigma}^{i_1}_{i'_1})} \searrow \quad \swarrow {\scriptstyle G(\boldsymbol{\tau}^{i_2}_{i'_1})} & & {\scriptstyle G(\boldsymbol{\sigma}^{i_2}_{i'_2})} \searrow & \cdots & \swarrow {\scriptstyle G(\boldsymbol{\tau}^{i_m}_{i'_{m-1}})} & \\ & G(\mathbf{D}_{i'_1}) & & G(\mathbf{D}_{i'_2}) & & G(\mathbf{D}_{i'_{m-1}}) & \end{array} \quad .$$

The *category of* ∞-$\mathbf{C}$-*groupoids* is the full subcategory of $\widehat{\mathbf{C}}$, category of presheaves on $\mathbf{C}$, whose objects are ∞-$\mathbf{C}$-groupoids.

1.6. Examples of Gr-coherators. There is a general method for constructing inductively Gr-coherators. Take $\mathbf{C}_0 = \mathbf{\Theta}_0$. Suppose that $\mathbf{C}_n$ is defined and choose a set E_n of pairs of parallel arrows in $\mathbf{C}_n$. Define $\mathbf{C}_{n+1}$ as the universal globular extension obtained by formally adding a lifting for each pair in E_n (an easy categorical argument shows that such a universal globular extension exists, is unique up to unique isomorphism, and that the functor $\mathbf{C}_n \to \mathbf{C}_{n+1}$ induces a bijection on the sets of objects [1, section 2.6]). Let $\mathbf{C}$ be the colimit $\mathbf{C} = \varinjlim \mathbf{C}_n$. For an arbitrary choice of the sets E_n, $\mathbf{C}$ need not be a Gr-coherator, as there is no reason for condition (a) in 1.4 to be satisfied. A sufficient (but not necessary) condition for $\mathbf{C}$ to be a Gr-coherator is that every pair of parallel arrows in $\mathbf{C}$ is the image of a pair in E_n, for some $n \geqslant 0$. Three important examples can be constructed (among many others) by this method.

1) **The canonical Gr-coherator $\mathbf{C} = \mathbf{C}_{\mathrm{can}}$.** This example is obtained by taking E_n to be the set of *all* pairs of parallel arrows in $\mathbf{C}_n$.

2) **The Batanin-Leinster Gr-coherator $\mathbf{C} = \mathbf{C}_{\mathrm{BL}}$.** It is obtained by defining E_n to be the set of pairs of parallel arrows in $\mathbf{C}_n$ that are not the image of a pair in $E_{n'}$, for some $n' < n$ [1, 4.1.4].

3) **The canonical reduced Gr-coherator $\mathbf{C} = \mathbf{C}_{\mathrm{red}}$.** It is constructed by taking E_n to be the set of pairs of parallel arrows in $\mathbf{C}_n$ that do not have already a lifting in $\mathbf{C}_n$.

It is easily seen that examples 1 and 2 satisfy the sufficient condition stated above. The example 3 does *not* satisfy this condition; nevertheless, it is clear that $\mathbf{C}_{\mathrm{red}}$ *is* a Gr-coherator. It is possible to put even more restrictive conditions on the sets E_n and still obtain a Gr-coherator. It seems that it is not possible to find a minimal way for choosing the sets E_n.

1.7. Some structural maps. Fix a Gr-coherator $\mathbf{C}$ and a ∞-$\mathbf{C}$-groupoid $G : \mathbf{C}^{\mathrm{op}} \to \mathsf{Set}$. The restriction of G to the subcategory $\mathbb{G}$ of $\mathbf{C}$ defines a ∞-graph, called the *underlying ∞-graph*:

$$G_0 \underset{t_1}{\overset{s_1}{\leftleftarrows}} G_1 \underset{t_2}{\overset{s_2}{\leftleftarrows}} \quad \cdots \quad \underset{t_{i-1}}{\overset{s_{i-1}}{\leftleftarrows}} G_{i-1} \underset{t_i}{\overset{s_i}{\leftleftarrows}} G_i \underset{t_{i+1}}{\overset{s_{i+1}}{\leftleftarrows}} \quad \cdots \ ,$$

where $G_i = G(\mathbf{D}_i)$, $s_i = G(\boldsymbol{\sigma}_i)$ and $t_i = G(\boldsymbol{\tau}_i)$. The elements of G_i are the *i-cells* of G, and s_i, t_i are the *source* and *target* maps respectively, satisfying the *globular relations*:

$$s_i s_{i+1} = s_i t_{i+1} \quad , \qquad t_i s_{i+1} = t_i t_{i+1} \ , \qquad i \geqslant 1 \ .$$

Operations and coherence arrows in the ∞-$\mathbf{C}$-groupoid G are defined using the existence of lifting arrows for pairs of parallel arrows in the Gr-coherator $\mathbf{C}$. In what follows, we give some examples of such structural maps of G (for more details see [1, section 4.2]). When there is no ambiguity, let's denote by

$$\mathbf{D}_{i_k} \xrightarrow{\mathbf{can}_k} \mathbf{D}_{i_1} \amalg_{\mathbf{D}_{i'_1}} \cdots \amalg_{\mathbf{D}_{i'_{m-1}}} \mathbf{D}_{i_m}$$

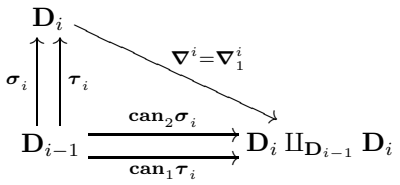
the canonical map of the k-th summand into a globular sum.

1.7.1. Level 1 binary composition. For every $i \geqslant 1$, the two composite arrows

$$\mathbf{D}_{i-1} \xrightarrow{\boldsymbol{\sigma}_i} \mathbf{D}_i \xrightarrow{\mathbf{can}_2} \mathbf{D}_i \amalg_{\mathbf{D}_{i-1}} \mathbf{D}_i \ ,$$

$$\mathbf{D}_{i-1} \xrightarrow{\boldsymbol{\tau}_i} \mathbf{D}_i \xrightarrow{\mathbf{can}_1} \mathbf{D}_i \amalg_{\mathbf{D}_{i-1}} \mathbf{D}_i$$

form a pair of parallel arrows in $\mathbf{C}$, therefore there is a lifting $\boldsymbol{\nabla}^i := \boldsymbol{\nabla}^i_1$ such that



$$\boldsymbol{\nabla}^i\boldsymbol{\sigma}_i = \mathbf{can}_2\boldsymbol{\sigma}_i \ ,$$
$$\boldsymbol{\nabla}^i\boldsymbol{\tau}_i = \mathbf{can}_1\boldsymbol{\tau}_i \ .$$

We deduce a map

$$G_i \times_{G_{i-1}} G_i \simeq G(\mathbf{D}_i \amalg_{\mathbf{D}_{i-1}} \mathbf{D}_i) \xrightarrow{G(\boldsymbol{\nabla}^i)} G(\mathbf{D}_i) = G_i \ ,$$

associating to each pair of i-cells (x, y) such that $s_i(x) = t_i(y)$ a i-cell

$$x * y := x \overset{i}{*} y := x \underset{1}{\overset{i}{*}} y := G(\boldsymbol{\nabla}^i)(x, y)$$

such that

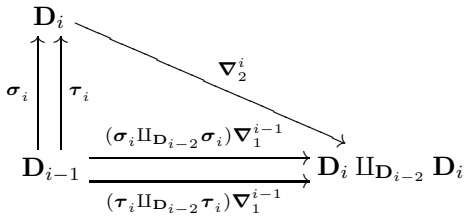
$$s_i(x * y) = s_i(y) \qquad \text{and} \qquad t_i(x * y) = t_i(x) \ .$$

This defines a "*vertical*" or "*level* 1" *composition* of i-cells. The lifting $\boldsymbol{\nabla}^i$ need not be unique, but if $\boldsymbol{\nabla}'^i$ is such another lifting (defining another vertical composition $*'$ on i-cells), then $(\boldsymbol{\nabla}^i, \boldsymbol{\nabla}'^i)$ is a pair of parallel arrows in $\mathbf{C}$, and there exists a lifting $\boldsymbol{\Gamma} : \mathbf{D}_{i+1} \to \mathbf{D}_i \amalg_{\mathbf{D}_{i-1}} \mathbf{D}_i$ such that $\boldsymbol{\Gamma}\,\boldsymbol{\sigma}_{i+1} = \boldsymbol{\nabla}^i$ and $\boldsymbol{\Gamma}\,\boldsymbol{\tau}_{i+1} = \boldsymbol{\nabla}'^i$, hence a map $c : G_i \times_{G_{i-1}} G_i \to G_{i+1}$, associating to each pair (x, y) of "composable" i-cells a "homotopy" $(i+i)$-cell $c(x, y)$ of source $x * y$ and target $x *' y$.

1.7.2. Level 2 binary composition. Given for every $i \geqslant 1$ a lifting $\boldsymbol{\nabla}^i = \boldsymbol{\nabla}^i_1$ as above, observe that, for $i \geqslant 2$, the two composite arrows

$$\mathbf{D}_{i-1} \xrightarrow{\boldsymbol{\nabla}^{i-1}_1} \mathbf{D}_{i-1} \amalg_{\mathbf{D}_{i-2}} \mathbf{D}_{i-1} \xrightarrow{\boldsymbol{\sigma}_i \amalg_{\mathbf{D}_{i-2}} \boldsymbol{\sigma}_i} \mathbf{D}_i \amalg_{\mathbf{D}_{i-2}} \mathbf{D}_i \ ,$$
$$\mathbf{D}_{i-1} \xrightarrow{\boldsymbol{\nabla}^{i-1}_1} \mathbf{D}_{i-1} \amalg_{\mathbf{D}_{i-2}} \mathbf{D}_{i-1} \xrightarrow{\boldsymbol{\tau}_i \amalg_{\mathbf{D}_{i-2}} \boldsymbol{\tau}_i} \mathbf{D}_i \amalg_{\mathbf{D}_{i-2}} \mathbf{D}_i$$

form a pair of parallel arrows in $\mathbf{C}$. Therefore there is a lifting $\boldsymbol{\nabla}^i_2$ such that



$$\boldsymbol{\nabla}^i_2\boldsymbol{\sigma}_i = (\boldsymbol{\sigma}_i \amalg_{\mathbf{D}_{i-2}} \boldsymbol{\sigma}_i)\boldsymbol{\nabla}^{i-1}_1 \ ,$$
$$\boldsymbol{\nabla}^i_2\boldsymbol{\tau}_i = (\boldsymbol{\tau}_i \amalg_{\mathbf{D}_{i-2}} \boldsymbol{\tau}_i)\boldsymbol{\nabla}^{i-1}_1 \ .$$

We deduce a map

$$G_i \times_{G_{i-2}} G_i \simeq G(\mathbf{D}_i \amalg_{\mathbf{D}_{i-2}} \mathbf{D}_i) \xrightarrow{G(\boldsymbol{\nabla}^i_2)} G(\mathbf{D}_i) = G_i \ ,$$

associating to each pair of i-cells (x, y) such that the iterated source of x in G_{i-2} is equal to the iterated target of y, a i-cell (level 2 composition of x and y)

$$x \underset{2}{\overset{i}{*}} y := G(\boldsymbol{\nabla}^i_2)(x, y)$$

such that

$$s_i(x \underset{2}{\overset{i}{*}} y) = s_i(x) \underset{1}{\overset{i-1}{*}} s_i(y) \qquad \text{and} \qquad t_i(x \underset{2}{\overset{i}{*}} y) = t_i(x) \underset{1}{\overset{i-1}{*}} t_i(y) \ .$$

1.7.3. Level l binary composition. The above construction can be iterated in order to obtain by induction on $l \geqslant 2$, for every $i \geqslant l$, a map $\boldsymbol{\nabla}^i_l : \mathbf{D}_i \to \mathbf{D}_i \amalg_{\mathbf{D}_{i-l}} \mathbf{D}_i$, lifting of the pair of parallel arrows $\big((\boldsymbol{\sigma}_i \amalg_{\mathbf{D}_{i-l}} \boldsymbol{\sigma}_i)\boldsymbol{\nabla}^{i-1}_{l-1}, (\boldsymbol{\tau}_i \amalg_{\mathbf{D}_{i-l}} \boldsymbol{\tau}_i)\boldsymbol{\nabla}^{i-1}_{l-1}\big)$, defining a map

$$G_i \times_{G_{i-l}} G_i \simeq G(\mathbf{D}_i \amalg_{\mathbf{D}_{i-l}} \mathbf{D}_i) \xrightarrow{G(\boldsymbol{\nabla}^i_l)} G(\mathbf{D}_i) = G_i \ ,$$

associating to each pair of i-cells (x, y) such that the iterated source of x in G_{i-l} is equal to the iterated target of y, a i-cell (level l composition of x and y)

$$x \overset{i}{\underset{l}{*}} y := G(\boldsymbol{\nabla}^i_l)(x, y)$$

such that

$$s_i(x \overset{i}{\underset{l}{*}} y) = s_i(x) \overset{i-1}{\underset{l-1}{*}} s_i(y) \qquad \text{and} \qquad t_i(x \overset{i}{\underset{l}{*}} y) = t_i(x) \overset{i-1}{\underset{l-1}{*}} t_i(y) \ .$$

1.7.4. Level 1 m-ary composition. There are many more general compositions in the structure of the ∞-$\mathbf{C}$-groupoid G. For example, let m be an integer, $m \geqslant 2$. For every $i \geqslant 1$, the two composite arrows

$$\mathbf{D}_{i-1} \xrightarrow{\boldsymbol{\sigma}_i} \mathbf{D}_i \xrightarrow{\mathbf{can}_m} \mathbf{D}_i \amalg_{\mathbf{D}_{i-1}} \cdots \amalg_{\mathbf{D}_{i-1}} \mathbf{D}_i \ ,$$

$$\mathbf{D}_{i-1} \xrightarrow{\boldsymbol{\tau}_i} \mathbf{D}_i \xrightarrow{\mathbf{can}_1} \mathbf{D}_i \amalg_{\mathbf{D}_{i-1}} \cdots \amalg_{\mathbf{D}_{i-1}} \mathbf{D}_i \ ,$$

with target the globular sum of length m, form a pair of parallel arrows in $\mathbf{C}$, hence a lifting

$$\boldsymbol{\nabla}^{i,m}_1 : \mathbf{D}_i \longrightarrow \mathbf{D}_i \amalg_{\mathbf{D}_{i-1}} \cdots \amalg_{\mathbf{D}_{i-1}} \mathbf{D}_i \ ,$$

inducing a map

$$G_i \times_{G_{i-1}} \cdots \times_{G_{i-1}} G_i \xrightarrow{\overset{i,m}{*} := G(\boldsymbol{\nabla}^{i,m}_1)} G_i \ .$$

This map defines a m-ary composition, associating to each "composable" m-uple $(x_1, \ldots, x_m)$ of i-cells, a i-cell

$$\overset{i,m}{*}(x_1, \ldots, x_m) = G(\boldsymbol{\nabla}^{i,m}_1)(x_1, \ldots, x_m)$$

such that

$$s_i\big(\overset{i,m}{*}(x_1, \ldots, x_m)\big) = s_i(x_m) \qquad \text{and} \qquad t_i\big(\overset{i,m}{*}(x_1, \ldots, x_m)\big) = t_i(x_1) \ .$$

1.7.5. Associativity constraint for level 1 binary composition. For every $i \geqslant 1$, observe that the two composite arrows

$$\mathbf{D}_i \xrightarrow{\boldsymbol{\nabla}^i} \mathbf{D}_i \amalg_{\mathbf{D}_{i-1}} \mathbf{D}_i \xrightarrow{\boldsymbol{\nabla}^i \amalg_{\mathbf{D}_{i-1}} 1_{\mathbf{D}_i}} \mathbf{D}_i \amalg_{\mathbf{D}_{i-1}} \mathbf{D}_i \amalg_{\mathbf{D}_{i-1}} \mathbf{D}_i \ ,$$

$$\mathbf{D}_i \xrightarrow{\boldsymbol{\nabla}^i} \mathbf{D}_i \amalg_{\mathbf{D}_{i-1}} \mathbf{D}_i \xrightarrow{1_{\mathbf{D}_i} \amalg_{\mathbf{D}_{i-1}} \boldsymbol{\nabla}^i} \mathbf{D}_i \amalg_{\mathbf{D}_{i-1}} \mathbf{D}_i \amalg_{\mathbf{D}_{i-1}} \mathbf{D}_i$$

(where $\boldsymbol{\nabla}^i = \boldsymbol{\nabla}^i_1$ is as in 1.7.1) form a pair of parallel arrows in $\mathbf{C}$, therefore there exists a lifting

$$\boldsymbol{\alpha} := \boldsymbol{\alpha}^i := \boldsymbol{\alpha}^i_1 : \mathbf{D}_{i+1} \longrightarrow \mathbf{D}_i \amalg_{\mathbf{D}_{i-1}} \mathbf{D}_i \amalg_{\mathbf{D}_{i-1}} \mathbf{D}_i \ .$$

We deduce a map

$$G_i \times_{G_{i-1}} G_i \times_{G_{i-1}} G_i \xrightarrow{a := a^i_1 := G(\boldsymbol{\alpha}^i_1)} G_{i+1} \ ,$$

associating to each triple of "composable" i-cells (x, y, z) an *associativity constraint* $(i+1)$-cell $a_{x,y,z}$ such that (in the notations of 1.7.1)

$$s_{i+1}(a_{x,y,z}) = (x * y) * z \qquad \text{and} \qquad t_{i+1}(a_{x,y,z}) = x * (y * z) \ .$$

1.7.6. Associativity constraint for level 2 binary composition. The construction of associativity constraints for higher-level compositions becomes more complicated. For example, for the level 2 composition, observe (in the notations of 1.7.2) that

$$\big((\boldsymbol{\nabla}_2^i \amalg_{\mathbf{D}_{i-2}} 1_{\mathbf{D}_i})\boldsymbol{\nabla}_2^i,\, (1_{\mathbf{D}_i} \amalg_{\mathbf{D}_{i-2}} \boldsymbol{\nabla}_2^i)\boldsymbol{\nabla}_2^i\big)\ , \quad i \geqslant 2\ ,$$

is *not* a pair of parallel arrows as

$$\begin{aligned}(\boldsymbol{\nabla}_2^i \amalg_{\mathbf{D}_{i-2}} 1_{\mathbf{D}_i})\boldsymbol{\nabla}_2^i\boldsymbol{\sigma}_i &= (\boldsymbol{\sigma}_i \amalg_{\mathbf{D}_{i-2}} \boldsymbol{\sigma}_i \amalg_{\mathbf{D}_{i-2}} \boldsymbol{\sigma}_i)(\boldsymbol{\nabla}_1^{i-1} \amalg_{\mathbf{D}_{i-2}} 1_{\mathbf{D}_{i-1}})\boldsymbol{\nabla}_1^{i-1}\\ &\neq (\boldsymbol{\sigma}_i \amalg_{\mathbf{D}_{i-2}} \boldsymbol{\sigma}_i \amalg_{\mathbf{D}_{i-2}} \boldsymbol{\sigma}_i)(1_{\mathbf{D}_{i-1}} \amalg_{\mathbf{D}_{i-2}} \boldsymbol{\nabla}_1^{i-1})\boldsymbol{\nabla}_1^{i-1} = (1_{\mathbf{D}_i} \amalg_{\mathbf{D}_{i-2}} \boldsymbol{\nabla}_2^i)\boldsymbol{\nabla}_2^i\boldsymbol{\sigma}_i\end{aligned}$$

and similarly with $\boldsymbol{\sigma}_i$ replaced by $\boldsymbol{\tau}_i$. In order to be able to define an associativity constraint for level 2 composition, verify that the two following composite arrows

$$\begin{aligned}&\mathbf{D}_i \xrightarrow{\boldsymbol{\nabla}_1^i} \mathbf{D}_i \amalg_{\mathbf{D}_{i-1}} \mathbf{D}_i \xrightarrow{((\boldsymbol{\tau}_i \amalg_{\mathbf{D}_{i-2}} \boldsymbol{\tau}_i \amalg_{\mathbf{D}_{i-2}} \boldsymbol{\tau}_i)\boldsymbol{\alpha}_1^{i-1},\, (\boldsymbol{\nabla}_2^i \amalg_{\mathbf{D}_{i-2}} 1_{\mathbf{D}_i})\boldsymbol{\nabla}_2^i)} \mathbf{D}_i \amalg_{\mathbf{D}_{i-2}} \mathbf{D}_i \amalg_{\mathbf{D}_{i-2}} \mathbf{D}_i,\\ &\mathbf{D}_i \xrightarrow{\boldsymbol{\nabla}_1^i} \mathbf{D}_i \amalg_{\mathbf{D}_{i-1}} \mathbf{D}_i \xrightarrow{((1_{\mathbf{D}_i} \amalg_{\mathbf{D}_{i-2}} \boldsymbol{\nabla}_2^i)\boldsymbol{\nabla}_2^i,\, (\boldsymbol{\sigma}_i \amalg_{\mathbf{D}_{i-2}} \boldsymbol{\sigma}_i \amalg_{\mathbf{D}_{i-2}} \boldsymbol{\sigma}_i)\boldsymbol{\alpha}_1^{i-1})} \mathbf{D}_i \amalg_{\mathbf{D}_{i-2}} \mathbf{D}_i \amalg_{\mathbf{D}_{i-2}} \mathbf{D}_i\end{aligned}$$

form a pair of parallel arrows. Hence a lifting

$$\boldsymbol{\alpha}_2^i : \mathbf{D}_{i+1} \longrightarrow \mathbf{D}_i \amalg_{\mathbf{D}_{i-2}} \mathbf{D}_i \amalg_{\mathbf{D}_{i-2}} \mathbf{D}_i\ ,$$

inducing a map

$$G_i \times_{G_{i-2}} G_i \times_{G_{i-2}} G_i \xrightarrow{a_2^i := G(\boldsymbol{\alpha}_2^i)} G_{i+1}\ ,$$

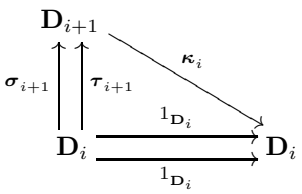
associating to each triple of i-cells (x, y, z) "composable" over G_{i-2} an *associativity constraint* $(i+1)$-cell $a^i_{2;x,y,z}$ such that (in the notations of 1.7.1, 1.7.2 and 1.7.5)

$$\begin{aligned}s_{i+1}(a^i_{2;x,y,z}) &= a^{i-1}_{1;\,t_i(x),t_i(y),t_i(z)} \overset{i}{\underset{1}{*}} \Big((x \overset{i}{\underset{2}{*}} y) \overset{i}{\underset{2}{*}} z\Big)\ ,\\ t_{i+1}(a^i_{2;x,y,z}) &= \Big(x \overset{i}{\underset{2}{*}} (y \overset{i}{\underset{2}{*}} z)\Big) \overset{i}{\underset{1}{*}} a^{i-1}_{1;\,s_i(x),s_i(y),s_i(z)}\ .\end{aligned}$$

It is left as an exercise to the reader to proceed to the construction of associativity constraints for higher-level compositions.

1.7.7. Pentagon and exchange constraints. Similarly, Mac Lane's pentagon and Godement's exchange rule give rise to "higher" constraints, defined by choosing suitable pairs of parallel arrows (see [1, 4.2.4]).

1.7.8. Units. Let $i \geqslant 0$. The most trivial pair of parallel arrows is $(1_{\mathbf{D}_i}, 1_{\mathbf{D}_i})$. It gives rise to a lifting $\boldsymbol{\kappa}_i$ such that



$$\boldsymbol{\kappa}_i\boldsymbol{\sigma}_{i+1} = 1_{\mathbf{D}_i} = \boldsymbol{\kappa}_i\boldsymbol{\tau}_{i+1}\ .$$

It defines a map

$$k_i := G(\boldsymbol{\kappa}_i) : G_i \longrightarrow G_{i+1}\ ,$$

associating to each i-cell x a *unit* $(i+1)$-cell $\mathsf{id}_x := k_i(x)$ such that

$$s_{i+1}(\mathsf{id}_x) = x = t_{i+1}(\mathsf{id}_x)$$

(the name of unit and the notation being justified by what follows).

1.7.9. Unit constraints. Let $i \geqslant 1$. Observe (using the notations of 1.7.1 and 1.7.8) that

$$\big((\boldsymbol{\tau}_i\boldsymbol{\kappa}_{i-1}, 1_{\mathbf{D}_i})\boldsymbol{\nabla}^i, 1_{\mathbf{D}_i}\big) \qquad \text{and} \qquad \big((1_{\mathbf{D}_i}, \boldsymbol{\sigma}_i\boldsymbol{\kappa}_{i-1})\boldsymbol{\nabla}^i, 1_{\mathbf{D}_i}\big)$$

are pairs of parallel arrows, hence liftings $\boldsymbol{\lambda}_i, \boldsymbol{\rho}_i : \mathbf{D}_{i+1} \to \mathbf{D}_i$ such that

$$\boldsymbol{\lambda}_i\boldsymbol{\sigma}_{i+1} = (\boldsymbol{\tau}_i\boldsymbol{\kappa}_{i-1}, 1_{\mathbf{D}_i})\boldsymbol{\nabla}^i \ , \qquad \boldsymbol{\lambda}_i\boldsymbol{\tau}_{i+1} = 1_{\mathbf{D}_i} \ ,$$

$$\boldsymbol{\rho}_i\boldsymbol{\sigma}_{i+1} = (1_{\mathbf{D}_i}, \boldsymbol{\sigma}_i\boldsymbol{\kappa}_{i-1})\boldsymbol{\nabla}^i \ , \qquad \boldsymbol{\rho}_i\boldsymbol{\tau}_{i+1} = 1_{\mathbf{D}_i} \ .$$

We deduce maps

$$G_i \xrightarrow{l_i := G(\boldsymbol{\lambda}_i)} G_{i+1} \quad , \qquad G_i \xrightarrow{r_i := G(\boldsymbol{\rho}_i)} G_{i+1}$$

associating to each i-cell x a *left*, respectively *right*, *unit constraint* $(i+1)$-cell $l_i(x)$, respectively $r_i(x)$, such that (in the notations of 1.7.1 and 1.7.8)

$$s_{i+1}(l_i(x)) = \mathsf{id}_{t_i(x)} \underset{1}{\overset{i}{*}} x \ , \qquad t_{i+1}(l_i(x)) = x \ ,$$

$$s_{i+1}(r_i(x)) = x \underset{1}{\overset{i}{*}} \mathsf{id}_{s_i(x)} \ , \qquad t_{i+1}(r_i(x)) = x \ .$$

1.7.10. Triangle constraint. In a similar way a *triangle* higher constraint can be defined, involving associativity, left and right unit constraints (see [1, 4.2.4]).

1.7.11. Level 1 inverse. For every $i \geqslant 1$, $(\boldsymbol{\tau}_i, \boldsymbol{\sigma}_i)$ is a pair of parallel arrows, hence a lifting $\boldsymbol{\omega}^i := \boldsymbol{\omega}^i_1 : \mathbf{D}_i \to \mathbf{D}_i$ such that

$$\boldsymbol{\omega}^i\boldsymbol{\sigma}_i = \boldsymbol{\tau}_i \qquad \text{and} \qquad \boldsymbol{\omega}^i\boldsymbol{\tau}_i = \boldsymbol{\sigma}_i \ .$$

We deduce a map $w^i := w^i_1 := G(\boldsymbol{\omega}^i_1) : G_i \to G_i$, associating to a i-cell an "*inverse*" i-cell $x^{-1} := w^i_1(x)$ such that

$$s_i(x^{-1}) = t_i(x) \qquad \text{and} \qquad t_i(x^{-1}) = s_i(x) \ .$$

(the name of "inverse" and the notation being justified by what follows).

1.7.12. Level 1 inverse constraints. Let $i \geqslant 1$. Observe (using the notations of 1.7.1, 1.7.8 and 1.7.11) that

$$\big((\boldsymbol{\omega}^i, 1_{\mathbf{D}_i})\boldsymbol{\nabla}^i, \boldsymbol{\sigma}_i\boldsymbol{\kappa}_{i-1}\big) \qquad \text{and} \qquad \big((1_{\mathbf{D}_i}, \boldsymbol{\omega}^i)\boldsymbol{\nabla}^i, \boldsymbol{\tau}_i\boldsymbol{\kappa}_{i-1}\big)$$

are pairs of parallel arrows, hence liftings $\boldsymbol{\gamma}_i, \boldsymbol{\delta}_i : \mathbf{D}_{i+1} \to \mathbf{D}_i$ such that

$$\boldsymbol{\gamma}_i\boldsymbol{\sigma}_{i+1} = (\boldsymbol{\omega}^i, 1_{\mathbf{D}_i})\boldsymbol{\nabla}^i \ , \qquad \boldsymbol{\gamma}_i\boldsymbol{\tau}_{i+1} = \boldsymbol{\sigma}_i\boldsymbol{\kappa}_{i-1} \ ,$$

$$\boldsymbol{\delta}_i\boldsymbol{\sigma}_{i+1} = (1_{\mathbf{D}_i}, \boldsymbol{\omega}^i)\boldsymbol{\nabla}^i \ , \qquad \boldsymbol{\delta}_i\boldsymbol{\tau}_{i+1} = \boldsymbol{\tau}_i\boldsymbol{\kappa}_{i-1} \ .$$

We deduce maps

$$G_i \xrightarrow{g_i := G(\boldsymbol{\gamma}_i)} G_{i+1} \ , \qquad G_i \xrightarrow{d_i := G(\boldsymbol{\delta}_i)} G_{i+1} \ ,$$

associating to each i-cell x a *left*, respectively *right*, *inverse constraint* $(i+1)$-cell $g_i(x)$, respectively $d_i(x)$, such that (in the notations of 1.7.1, 1.7.8 and 1.7.11)

$$s_{i+1}(g_i(x)) = x^{-1} \underset{1}{\overset{i}{*}} x \ , \qquad t_{i+1}(g_i(x)) = \mathsf{id}_{s_i(x)} \ ,$$

$$s_{i+1}(d_i(x)) = x \underset{1}{\overset{i}{*}} x^{-1} \ , \qquad t_{i+1}(d_i(x)) = \mathsf{id}_{t_i(x)} \ .$$

1.7.13. Level 2 inverse. Let $i \geqslant 2$. Observe (using the notations of 1.7.11) that $(\boldsymbol{\sigma}_i\boldsymbol{\omega}^{i-1}_1, \boldsymbol{\tau}_i\boldsymbol{\omega}^{i-1}_1)$ is a pair of parallel arrows, hence a lifting $\boldsymbol{\omega}^i_2 : \mathbf{D}_i \to \mathbf{D}_i$ defining a map $w^i_2 := G(\boldsymbol{\omega}^i_2) : G_i \to G_i$, associating to a i-cell x a "level 2 inverse" i-cell $w^i_2(x)$ such that

$$s_i(w^i_2(x)) = w^{i-1}_1(s_i(x)) \qquad \text{and} \qquad t_i(w^i_2(x)) = w^{i-1}_1(t_i(x)) \ .$$

1.7.14. Level l inverse. The above construction can be iterated in order to obtain by induction on $l \geqslant 2$, for every $i \geqslant l$, a map $\boldsymbol{\omega}^i_l : \mathbf{D}_i \to \mathbf{D}_i$, lifting of the pair of parallel arrows $(\boldsymbol{\sigma}_i\boldsymbol{\omega}^{i-1}_{l-1}, \boldsymbol{\tau}_i\boldsymbol{\omega}^{i-1}_{l-1})$, defining a map $w^i_l := G(\boldsymbol{\omega}^i_l) : G_i \to G_i$, associating to a i-cell x a "level l inverse" i-cell $w^i_l(x)$ such that

$$s_i(w^i_l(x)) = w^{i-1}_{l-1}(s_i(x)) \qquad \text{and} \qquad t_i(w^i_l(x)) = w^{i-1}_{l-1}(t_i(x)) \ .$$

It is left as an exercise to the reader to define level l inverse constraints relating level l inverse with level l binary composition.

2. Grothendieck's conjecture.

2.1. The homotopy groups of a ∞-groupoid. Fix a Gr-coherator $\mathbf{C}$ and a ∞-$\mathbf{C}$-groupoid $G : \mathbf{C}^{\mathrm{op}} \to \mathsf{Set}$. We will freely use the notations of the structural maps introduced in the previous section.

For every $i \geqslant 0$, we define a *homotopy* relation $\sim_i$ between i-cells of G by

$$x \sim_i y \quad \overset{\text{def}}{\iff} \quad \exists h \in G_{i+1} \quad s_{i+1}(h) = x \ , \quad t_{i+1}(h) = y \ .$$

The homotopy relation is an equivalence relation:

a) **Reflexivity.** Let x be a i-cell. We have (*cf.* 1.7.8):

$$s_{i+1}(\mathsf{id}_x) = x \qquad \text{and} \qquad t_{i+1}(\mathsf{id}_x) = x \quad ,$$

therefore $x \sim_i x$.

b) **Symmetry.** Let x, y be two i-cells such that $x \sim_i y$. By definition, there exists a $(i+1)$-cell h such that $s_{i+1}(h) = x$ and $t_{i+1}(h) = y$, therefore (*cf.* 1.7.11)

$$s_{i+1}(h^{-1}) = t_{i+1}(h) = y \qquad \text{and} \qquad t_{i+1}(h^{-1}) = s_{i+1}(h) = x \ ,$$

and hence $y \sim_i x$.

c) **Transitivity.** Let x, y, z be three i-cells such that $x \sim_i y$ and $y \sim_i z$. By definition, there exist two $(i+1)$-cells h, k such that

$$s_{i+1}(h) = x \ , \quad t_{i+1}(h) = y \ , \quad s_{i+1}(k) = y \ , \quad t_{i+1}(k) = z \ .$$

In particular, (k, h) is an element of the (globular) fiber product $G_{i+1} \times_{G_i} G_{i+1}$, so $k * h$ is defined and (*cf.* 1.7.1)

$$s_{i+1}(k * h) = s_{i+1}(h) = x \qquad \text{and} \qquad t_{i+1}(k * h) = t_{i+1}(k) = z \ ,$$

which proves that $x \sim_i z$.

Two i-cells x, y are called *homotopic* if $x \sim_i y$. We denote by $\overline{G}_i$ the quotient of the set G_i of i-cells by the homotopy equivalence relation $\sim_i$. We define the *set of connected components* of the ∞-$\mathbf{C}$-groupoid G as the set $\Pi_0(G) := \overline{G}_0 := G_0/\sim_0$.

Suppose now that $i \geqslant 1$ and observe that if x, y are two homotopic i-cells, then the globular relations imply that $s_i(x) = s_i(y)$ and $t_i(x) = t_i(y)$. Therefore the maps $s_i, t_i : G_i \to G_{i-1}$ induce maps $\overline{s}_i, \overline{t}_i : \overline{G}_i \to G_{i-1}$. On the other hand, the equivalence relation $\sim_i$ is compatible with the composition $* = \overset{i}{\underset{1}{*}}$. Let us prove for example that if x_1, x_2 are two homotopic i-cells, then for every $\overset{1}{i}$-cell y with target the common source of x_1 and x_2, the i-cells $x_1 * y$ and $x_2 * y$ are homotopic. By definition, there exists a $(i+1)$-cell h such that $s_{i+1}(h) = x_1$ and $t_{i+1}(h) = x_2$. If we denote by h' the $(i+1)$-cell

$$h' = h \overset{i+1}{\underset{2}{*}} \mathsf{id}_y \ ,$$

then we have (*cf.* 1.7.2 and 1.7.8)

$$s_{i+1}(h') = s_{i+1}(h) \overset{i}{\underset{1}{*}} s_{i+1}(\mathsf{id}_y) = x_1 \overset{i}{\underset{1}{*}} y \ \text{ and } \ t_{i+1}(h') = t_{i+1}(h) \overset{i}{\underset{1}{*}} t_{i+1}(\mathsf{id}_y) = x_2 \overset{i}{\underset{1}{*}} y \ ,$$

and hence $x_1 * y$ and $x_2 * y$ are homotopic. Therefore the map $\overset{i}{\underset{1}{*}} : G_i \times_{G_{i-1}} G_i \to G_i$ induces a map $\overline{G}_i \times_{G_{i-1}} \overline{G}_i \to \overline{G}_i$. The existence of the associativity constraint and

the unit constraints (*cf.* 1.7.5 and 1.7.9) implies that this composition defines a category $\varpi_i(G)$ with object set G_{i-1}, arrow set $\overline{G}_i$, and target and source maps $\overline{s}_i$, $\overline{t}_i$. The existence of the inverse constraints (*cf.* 1.7.12) implies that this category is a groupoid. The remark at the end of 1.7.1 proves that this groupoid is independent of the choice of the lifting $\boldsymbol{\nabla}_1^i$ defining the composition $\underset{1}{\overset{i}{*}}$. For more details, see [1], propositions 4.3.2, 4.3.3.

Let x be a 0-cell of G, $x \in G_0$, and i an integer, $i \geqslant 1$. The i-th *homotopy group of* G *at* x is the group

$$\pi_i(G;x) := \mathsf{Hom}_{\varpi_i(G)}(k(x),k(x)) \ ,$$

where using the notations of 1.7.8, $k = k_{i-2}\dots k_1 k_0$. In order to justify this definition, it should be verified that this group is, up to canonical isomorphism, independent of the choice of the lifting arrows $\boldsymbol{\kappa}_j$, $0 \leqslant j \leqslant i-2$, giving rise to the maps k_j.

2.2. Weak equivalences of ∞-groupoids. Fix a Gr-coherator $\mathbf{C}$. A morphism $f : G \to G'$ of ∞-$\mathbf{C}$-groupoids is called a *weak equivalence* or a ∞-*equivalence* if the following two conditions are satisfied:

a) the map $\Pi_0(f) : \Pi_0(G) \to \Pi_0(G')$, induced by f, is a bijection;
b) for every $i \geqslant 1$ and every 0-cell x of G, the map

$$\pi_i(f;x) : \pi_i(G;x) \to \pi_i(G';f(x)) \ ,$$

induced by f, is an isomorphism of groups.

2.3. Grothendieck's conjecture (weak form). *For every* Gr*-coherator* $\mathbf{C}$*, the localization of the category of* ∞-$\mathbf{C}$*-groupoids by the* ∞*-equivalences is equivalent to the homotopy category of* CW*-complexes.*

For a strategy for proving this conjecture, see [7]. A more precise form of this conjecture is given below (*cf.* 2.8).

2.4. The topological realization of the category of globes. For $i \geqslant 0$, let D_i be the i-dimensional topological disk

$$D_i = \{x \in \mathbb{R}^i \mid \|x\| \leq 1\} \ ,$$

where $\|x\|$ denotes the Euclidian norm of x, and for $i > 0$,

$$\sigma_i, \tau_i : D_{i-1} \hookrightarrow D_i$$

the inclusions defined by

$$\sigma_i(x) = \big(x, -\sqrt{1-\|x\|^2}\,\big) \ , \qquad \tau_i(x) = \big(x, \sqrt{1-\|x\|^2}\,\big) \ , \qquad x \in D_{i-1} \ .$$

The maps σ_i and τ_i factorize through the $(i-1)$-dimensional sphere, boundary of D_i,

$$S_{i-1} = \partial D_i = \{x \in \mathbb{R}^i \mid \|x\| = 1\} \ ,$$

identifying D_{i-1} to the lower respectively upper hemisphere of S_{i-1}. As the maps σ_i, τ_i, $i > 0$, satisfy the coglobular relations, the assignment

$$\mathbf{D}_i \longmapsto D_i \ , \quad \boldsymbol{\sigma}_i \longmapsto \sigma_i \ , \quad \boldsymbol{\tau}_i \longmapsto \tau_i$$

defines a functor $\mathbb{G} \to \mathsf{Top}$ from the category of globes (*cf.* 1.1) to the category of topological spaces. It is easy to verify that this functor is faithful, identifying $\mathbb{G}$ to the (non-full) subcategory of Top with objects the disks D_i, $i \geqslant 0$, and morphisms generated by the inclusions σ_i, τ_i, $i > 0$.

2.5. Topological spaces as a globular extension. As the category Top of topological spaces is cocomplete, Top endowed with the functor $\mathbb{G} \to \mathsf{Top}$, defined above, is a globular extension (*cf.* 1.3). By the universal property of $\boldsymbol{\Theta}_0$, there exists a morphism of globular extensions $\boldsymbol{\Theta}_0 \to \mathsf{Top}$, unique up to unique natural isomorphism, extending the functor $\mathbb{G} \to \mathsf{Top}$. It is easy to verify that this functor

is faithful, identifying $\mathbf{\Theta}_0$ to the (non-full) subcategory of Top with objects globular sums of disks and morphisms continuous maps

$$X = D_{i_1} \amalg_{D_{i'_1}} D_{i_2} \amalg_{D_{i'_2}} \cdots \amalg_{D_{i'_{m-1}}} D_{i_m} \xrightarrow{\varphi} D_{j_1} \amalg_{D_{j'_1}} D_{j_2} \amalg_{D_{j'_2}} \cdots \amalg_{D_{j'_{n-1}}} D_{j_n} = Y$$

such that for every k, $1 \leqslant k \leqslant m$, there exists an integer l, $1 \leqslant l \leqslant n$, and a commutative square

$$\begin{array}{ccc} D_{i_k} & \xrightarrow{\psi} & D_{i_l} \\ {\scriptstyle \mathrm{can}_k}\downarrow & & \downarrow{\scriptstyle \mathrm{can}_l} \\ X & \xrightarrow[\varphi]{} & Y \end{array} \quad ,$$

with ψ the image of an arrow of $\mathbb{G}$, *i.e.* a composite of σ_i's or τ_i's. All objects of this subcategory are contractible spaces.

2.6. The fundamental ∞-groupoid of a space. Let $\mathbf{C}$ be a Gr-coherator. In order to associate functorially to every topological space a ∞-$\mathbf{C}$-groupoid, it's enough to define a ∞-*cogroupoid* of type $\mathbf{C}$ in Top, *i.e.* a functor

$$\mathbf{C} \longrightarrow \mathsf{Top}$$

preserving globular sums. The *fundamental* ∞-*groupoid* of a topological space X can then be defined as the composite of the opposite functor

$$\mathbf{C}^{\mathrm{op}} \longrightarrow \mathsf{Top}^{\mathrm{op}}$$

with the representable presheaf on Top defined by X

$$\mathsf{Hom}_{\mathsf{Top}}(\,?\,, X) : \mathsf{Top}^{\mathrm{op}} \longrightarrow \mathsf{Set}\ .$$

Let

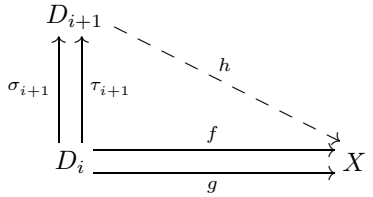
$$\mathbb{G} \longrightarrow \mathbf{\Theta}_0 = \mathbf{C}_0 \longrightarrow \mathbf{C}_1 \longrightarrow \cdots \longrightarrow \mathbf{C}_n \longrightarrow \mathbf{C}_{n+1} \longrightarrow \cdots \longrightarrow \mathbf{C} \simeq \varinjlim \mathbf{C}_n$$

be a tower of definition of the Gr-coherator $\mathbf{C}$. We have already defined a functor $\mathbf{\Theta}_0 \to \mathsf{Top}$ preserving globular sums. In order to prove that this functor can be extended to a functor $\mathbf{C} \to \mathsf{Top}$ preserving globular sums, *i.e.* to a morphism of globular extensions, it's enough to prove that for every $n \geqslant 0$, any morphism of globular extensions $\mathbf{C}_n \to \mathsf{Top}$ can be extended to a morphism of globular extensions $\mathbf{C}_{n+1} \to \mathsf{Top}$. Using the universal property of $\mathbf{C}_{n+1}$, it's enough to prove that for every globular sum X in Top, every $i \geqslant 0$, and every pair $(f, g) : D_i \to X$ of continuous maps such that if $i > 0$, we have

$$f\sigma_i = g\sigma_i \qquad \text{and} \qquad f\tau_i = g\tau_i\ ,$$

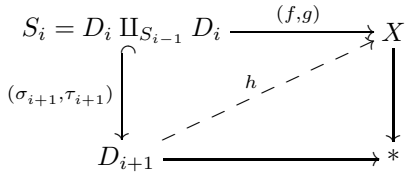
there exists a continuous map $h : D_{i+1} \to X$ such that

$$f = h\sigma_{i+1} \qquad \text{and} \qquad g = h\tau_{i+1}\ .$$



As X is a contractible space, the map $X \to *$ of X to the point is a trivial fibration. Therefore, as the inclusion $S_i \hookrightarrow D_{i+1}$ is a cofibration, the existence of such a map

h is a consequence of the lifting property of the following square

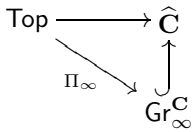


The functor $\mathbf{C} \to \mathsf{Top}$ defined this way is *not* independent of the choice of the relevant lifting maps h. Nevertheless, it is conjectured that this dependence is inessential (*cf.* 2.8). As the functor $\mathbf{\Theta}_0 \to \mathsf{Top}$ is faithful, the existence of the extension $\mathbf{C} \to \mathsf{Top}$ implies that the functor $\mathbf{\Theta}_0 \to \mathbf{C}$ is faithful. It is not known if the functors $\mathbf{C}_n \to \mathsf{Top}$ extending $\mathbf{\Theta}_0 \to \mathsf{Top}$ can be chosen faithful, which would imply the conjecture that the functors $\mathbf{C}_n \to \mathbf{C}$ are faithful (*cf.* 1.4).

2.7. The classifying space of a ∞-groupoid. Fix a Gr-coherator $\mathbf{C}$ and choose, as above, an extension $\mathbf{C} \to \mathsf{Top}$ of the canonical functor $\mathbf{\Theta}_0 \to \mathsf{Top}$, preserving globular sums. As the category Top is cocomplete, this functor induces a pair of adjoint functors

$$\widehat{\mathbf{C}} \longrightarrow \mathsf{Top}\ , \qquad \mathsf{Top} \longrightarrow \widehat{\mathbf{C}}$$

between the category of presheaves on $\mathbf{C}$ and the category of topological spaces. The functor $\mathsf{Top} \to \widehat{\mathbf{C}}$ factors through the full subcategory $\mathsf{Gr}_\infty^{\mathbf{C}}$ of $\widehat{\mathbf{C}}$, of ∞-$\mathbf{C}$-groupoids, associating to a topological space X its fundamental ∞-groupoid, defined in the previous section.



Therefore there is an induced pair of adjoint functors

$$\mathsf{Gr}_\infty^{\mathbf{C}} \xrightarrow{B} \mathsf{Top}\ , \qquad \mathsf{Top} \xrightarrow{\Pi_\infty} \mathsf{Gr}_\infty^{\mathbf{C}}\ ,$$

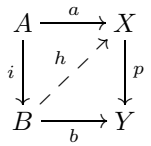
where B is the restriction of the functor $\widehat{\mathbf{C}} \to \mathsf{Top}$ to the subcategory $\mathsf{Gr}_\infty^{\mathbf{C}}$ of $\widehat{\mathbf{C}}$. For a ∞-$\mathbf{C}$-groupoid G, the topological space $B(G)$ is called the *classifying space* of G.

2.8. Grothendieck's conjecture (precise form). *The functors B and Π_∞ are compatible with the weak equivalences of ∞-groupoids and of spaces, and induce mutually quasi-inverse equivalences of the localized categories. Furthermore, different extensions $\mathbf{C} \to \mathsf{Top}$ of the functor $\mathbf{\Theta}_0 \to \mathsf{Top}$ induce isomorphic functors between the localized categories.*

2.9. The ∞-groupoid of an object in a model category. A similar construction can be done in any Quillen model category, such that *all objects are fibrant* ([5], section 12). For more details on this construction, see [1, section 4.4].

3. Coherators and weak factorization systems

3.1. Lifting properties. Fix a category $\mathcal{C}$. Recall that given two arrows $i : A \to B$ and $p : X \to Y$ of $\mathcal{C}$, *i has the left lifting property with respect to p*, or equivalently *p has the right lifting property with respect to i*, if for every commutative solid arrow square



there exists a dotted arrow such that the total diagram is commutative:

$$a = hi \qquad \text{and} \qquad b = ph \ .$$

If $\mathcal{F}$ is a class of arrows of $\mathcal{C}$, we denote by $l(\mathcal{F})$ (resp. $r(\mathcal{F})$) the class of arrows of $\mathcal{C}$ having the left (resp. right) lifting property with respect to all arrows in $\mathcal{F}$. The classes $l(\mathcal{F})$ and $r(\mathcal{F})$ are stable under composition and retracts. Sometimes arrows in $r(\mathcal{F})$ are called *$\mathcal{F}$-fibrations* and arrows in $\mathsf{cof}(\mathcal{F}) := l(r(\mathcal{F}))$, *$\mathcal{F}$-cofibrations*, and if $\mathcal{C}$ has a final object $*$ (resp. an initial object $\varnothing$), an object X of $\mathcal{C}$ is called *$\mathcal{F}$-fibrant* (resp. *$\mathcal{F}$-cofibrant*) if the map $X \to *$ (resp. $\varnothing \to X$) is a *$\mathcal{F}$-fibration* (resp. a *$\mathcal{F}$-cofibration*).

3.2. Weak factorization systems. A *weak factorization system* in $\mathcal{C}$ is a pair $(\mathcal{A}, \mathcal{B})$ of classes of arrows of $\mathcal{C}$ such that the following conditions are satisfied:

a) $\mathcal{A}$ and $\mathcal{B}$ are stable under retracts;
b) arrows in $\mathcal{A}$ have the left lifting property with respect to arrows in $\mathcal{B}$ (or equivalently arrows in $\mathcal{B}$ have the right lifting property with respect to arrows in $\mathcal{A}$);
c) every arrow f in $\mathcal{C}$ can be factored as $f = pi$, with $i \in \mathcal{A}$ and $p \in \mathcal{B}$.

It is well known that the conjunction of conditions (a) and (b) is equivalent to the conjunction of equalities

$$\mathcal{A} = l(\mathcal{B}) \qquad \text{and} \qquad \mathcal{B} = r(\mathcal{A}) \ .$$

3.3. Cellular maps. Let $\mathcal{C}$ be a cocomplete category and $\mathcal{F}$ a class of arrows in $\mathcal{C}$. A *$\mathcal{F}$-cellular* map in $\mathcal{C}$ is a map obtained as transfinite composition of pushouts of (small) sums of arrows in $\mathcal{F}$. The class of $\mathcal{F}$-cellular maps in $\mathcal{C}$ is denoted by $\mathsf{cell}(\mathcal{F})$. It is easy to verify that $\mathsf{cell}(\mathcal{F})$ is stable under pushouts, sums and transfinite composition, and that it contains isomorphisms. It is the smallest class containing $\mathcal{F}$ and stable under pushouts and transfinite composition. It is equal to the class of maps obtained by transfinite composition of pushouts of arrows in $\mathcal{F}$. There is an inclusion

$$\mathsf{cell}(\mathcal{F}) \subset \mathsf{cof}(\mathcal{F}) = l(r(\mathcal{F})) \ .$$

An object X of $\mathcal{C}$ is called *$\mathcal{F}$-cellular* if the map from the initial object of $\mathcal{C}$ to X is a $\mathcal{F}$-cellular map.

3.4. Set of arrows admissible for the small object argument. Let $\mathcal{C}$ be a cocomplete category and I a (small) set of arrows in $\mathcal{C}$. The set of arrows I is said to be *admissible for the small object argument*, or more simply *admissible*, if there exists a well ordered set J satisfying the following conditions:

a) J does *not* have a maximal element;
b) for every J-diagram $(X_j)_{j \in J}$ such that all maps $X_j \to X_{j'}$, $j \leqslant j'$, are I-cellular maps, and every arrow $A \to B$ in I, the canonical map

$$\varinjlim_j \mathsf{Hom}(A, X_j) \longrightarrow \mathsf{Hom}(A, \varinjlim_j X_j)$$

is a bijection.

For example, if all domains of maps in I are *α-presentable* for some regular cardinal α, then I is admissible. In particular if $\mathcal{C}$ is *locally presentable*, then *any* (small) set of arrows in $\mathcal{C}$ is admissible. Recall the small object argument:

Proposition 3.5. *Let $\mathcal{C}$ be a cocomplete category and I a (small) set of arrows in $\mathcal{C}$. If I is admissible for the small object argument, then every arrow f of $\mathcal{C}$ can be factored as $f = pi$, with $i \in \mathsf{cell}(I)$ and $p \in r(I)$. In particular,*

$$(\mathsf{cof}(I), r(I)) \qquad (\mathsf{cof}(I) = l(r(I)))$$

is a weak factorization system. Moreover, $\mathsf{cof}(I)$ *is the smallest class of arrows in* $\mathcal{C}$ *containing* I *stable under pushouts, transfinite composition and retracts, and it is equal to the class of retracts of arrows in* $\mathsf{cell}(I)$*.*

3.6. Finitely presentable objects. Recall that an object A of $\mathcal{C}$ is called *finitely presentable* if it is $\aleph_0$-presentable, *i.e.* if for every small *filtered* category J, and every functor $X : J \to \mathcal{C}$, the canonical map

$$\varinjlim_J \mathsf{Hom}(A, X) \longrightarrow \mathsf{Hom}(A, \varinjlim_J X)$$

is a bijection. When the domains of the arrows in a (small) set I are finitely presentable, I is in particular admissible, and the following lemma gives a more precise description of $\mathsf{cell}(I)$:

Lemma 3.7. *Let* $\mathcal{C}$ *be a cocomplete category and* I *a (small) set of arrows in* $\mathcal{C}$*. If the domains of the arrows in* I *are finitely presentable, then an arrow* $X \to Y$ *is in* $\mathsf{cell}(I)$ *if and only if there exists a sequence of maps*

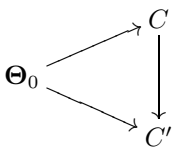
$$X = X_0 \xrightarrow{i_0} X_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{n-1}} X_n \xrightarrow{i_n} X_{n+1} \xrightarrow{i_{n+1}} \cdots \longrightarrow Y \simeq \varinjlim X_n \quad ,$$

identifying $X \to Y$ *to the canonical map* $X = X_0 \to \varinjlim X_n \simeq Y$*, such that for every* $n \geqslant 0$*, the arrow* i_n *is a pushout of a (small) sum of arrows in* I*.*

For a proof of this lemma, see appendix A, proposition A.6.

3.8. Let $\mathcal{F}$ be a class of arrows in $\mathcal{C}$ and λ an ordinal. We denote by $\mathsf{cell}_\lambda(\mathcal{F})$ the class of arrows obtained by λ-indexed transfinite composition of pushouts of (small) sums of arrows in $\mathcal{F}$. With this notation, the lemma 3.7 says that if I is a (small) set of arrows in $\mathcal{C}$, with finitely presentable domains, then $\mathsf{cell}(I) = \mathsf{cell}_\omega(I)$, where ω is the smallest countable ordinal. More generally, it can be shown that if I is a (small) set of arrows in $\mathcal{C}$, with α-presentable domains, for some regular cardinal α, then $\mathsf{cell}(I) = \mathsf{cell}_\lambda(I)$, where λ is the smallest ordinal of cardinality α.

3.9. Globular extensions under $\boldsymbol{\Theta}_0$. Let $\boldsymbol{\Theta}_0\backslash\mathsf{Cat}$ be the category of small categories under $\boldsymbol{\Theta}_0$, whose objects are pairs $(C, \boldsymbol{\Theta}_0 \to C)$, where C is a small category and $\boldsymbol{\Theta}_0 \to C$ a functor, and whose morphisms are (strictly) commutative triangles



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Denote by $\boldsymbol{\Theta}_0\backslash\mathsf{Gl}\text{-}\mathsf{Ext}$ the full subcategory of $\boldsymbol{\Theta}_0\backslash\mathsf{Cat}$ whose objects are pairs $(C, \boldsymbol{\Theta}_0 \to C)$ such that C, endowed with the composite functor

$$\mathbb{G} \longrightarrow \boldsymbol{\Theta}_0 \longrightarrow C \quad ,$$

is a globular extension and the functor $\boldsymbol{\Theta}_0 \to C$ a morphism of globular extensions (*i.e.* preserves globular sums). It is easy to see that any morphism of $\boldsymbol{\Theta}_0\backslash\mathsf{Gl}\text{-}\mathsf{Ext}$ preserves globular sums and therefore defines a morphism of the underlying globular extensions. In other terms, the category $\boldsymbol{\Theta}_0\backslash\mathsf{Gl}\text{-}\mathsf{Ext}$ is the category of globular extensions *under* the universal globular extension $\boldsymbol{\Theta}_0$. It's nothing else then the category of globular extensions endowed with a fixed choice of objects representing the globular sums, and morphisms of globular extensions compatible with these choices.

3.10. The left adjoint to the inclusion functor $\mathbf{\Theta}_0\backslash\mathsf{Gl}\text{-}\mathsf{Ext} \to \mathbf{\Theta}_0\backslash\mathsf{Cat}$. Let $(C, \mathbf{\Theta}_0 \to C)$ be any object of $\mathbf{\Theta}_0\backslash\mathsf{Cat}$. By a standard categorical construction [1, section 2.6], there exists a globular extension $\overline{C}$ with a morphism $C \to \overline{C}$ in $\mathbb{G}\backslash\mathsf{Cat}$

$$\begin{array}{ccccc} \mathbb{G} & \longrightarrow & \mathbf{\Theta}_0 & \longrightarrow & C \\ & \searrow & & & \downarrow \\ & & & & \overline{C} \end{array} \quad ,$$

satisfying the following universal property: for every morphism $C \to C'$ of globular extensions under $\mathbf{\Theta}_0$, there exists a unique morphism of globular extensions $\overline{C} \to C'$ such that the following triangle is commutative

$$\begin{array}{ccc} C & & \\ \downarrow & \searrow & \\ \overline{C} & \longrightarrow & C' \end{array} \quad .$$

Moreover, the functor $C \to \overline{C}$ induces a bijection on the sets of objects. It is easy to see that the assignment

$$(C, \mathbf{\Theta}_0 \to C) \longmapsto (\overline{C}, \mathbf{\Theta}_0 \to C \to \overline{C})$$

defines a functor $\mathbf{\Theta}_0\backslash\mathsf{Cat} \to \mathbf{\Theta}_0\backslash\mathsf{Gl}\text{-}\mathsf{Ext}$, which is a left adjoint of the full and faithful inclusion $\mathbf{\Theta}_0\backslash\mathsf{Gl}\text{-}\mathsf{Ext} \to \mathbf{\Theta}_0\backslash\mathsf{Cat}$, the canonical map $C \to \overline{C}$ corresponding to the unit of the adjunction. In particular, as the category $\mathbf{\Theta}_0\backslash\mathsf{Cat}$ is complete and cocomplete, the same holds for $\mathbf{\Theta}_0\backslash\mathsf{Gl}\text{-}\mathsf{Ext}$. Furthermore, one can easily verify that the inclusion $\mathbf{\Theta}_0\backslash\mathsf{Gl}\text{-}\mathsf{Ext} \to \mathbf{\Theta}_0\backslash\mathsf{Cat}$ preserves filtered colimits, and as a consequence the category $\mathbf{\Theta}_0\backslash\mathsf{Gl}\text{-}\mathsf{Ext}$ is locally presentable.

3.11. The generating cofibrations. Let $\mathbf{X}$ be an object of $\mathbf{\Theta}_0$ and i an integer, $i \geqslant 0$. Denote by $\mathbf{\Theta}_0[\mathbf{X}, i]$ the globular extension obtained from $\mathbf{\Theta}_0$ by formally adding two arrows

$$\mathbf{D}_i \underset{\mathbf{g}=\mathbf{g}_{(\mathbf{X},i)}}{\overset{\mathbf{f}=\mathbf{f}_{(\mathbf{X},i)}}{\rightrightarrows}} \mathbf{X}$$

satisfying (in case $i > 0$) the relations

$$\mathbf{f}\boldsymbol{\sigma}_i = \mathbf{g}\boldsymbol{\sigma}_i \ , \quad \mathbf{f}\boldsymbol{\tau}_i = \mathbf{g}\boldsymbol{\tau}_i \ ,$$

i.e. by formally adding a pair of parallel arrows, and $\mathbf{\Theta}_0[\mathbf{X}, i]'$ the globular extension obtained from $\mathbf{\Theta}_0[\mathbf{X}, i]$ by formal adjunction of a lifting $\mathbf{h} = \mathbf{h}_{(\mathbf{X},i)}$ for the pair $(\mathbf{f}_{(\mathbf{X},i)}, \mathbf{g}_{(\mathbf{X},i)})$. Define

$$I = \{\mathbf{\Theta}_0[\mathbf{X}, i] \longrightarrow \mathbf{\Theta}_0[\mathbf{X}, i]' \mid i \geqslant 0\,,\ \mathbf{X} \in \mathsf{Ob}\ \mathbf{\Theta}_0\} \ .$$

Observe that as $\mathbf{\Theta}_0$ is a small category, I is a (small) set. On the other hand, we have the following immediate lemma:

Lemma 3.12. *The categories $\mathbf{\Theta}_0[\mathbf{X}, i]$ (as well as $\mathbf{\Theta}_0[\mathbf{X}, i]'$), considered as categories under $\mathbf{\Theta}_0$, are finitely presentable objects of $\mathbf{\Theta}_0\backslash\mathsf{Gl}\text{-}\mathsf{Ext}$.*

In particular, the set I is admissible for the small object argument, and applying proposition 3.5, we get:

Proposition 3.13. *Every arrow f of $\mathbf{\Theta}_0\backslash\mathsf{Gl}\text{-}\mathsf{Ext}$ can be factored as $f = pi$, with $i \in \mathsf{cell}(I)$ and $p \in r(I)$. In particular,*

$$(\mathsf{cof}(I), r(I)) \qquad (\mathsf{cof}(I) = l(r(I)))$$

is a weak factorization system. Moreover, $\mathsf{cof}(I)$ *is the smallest class of arrows in* $\mathbf{\Theta}_0\backslash\mathsf{Gl}\text{-}\mathsf{Ext}$ *containing* I *stable under pushouts, transfinite composition and retracts, and it is equal to the class of retracts of arrows in* $\mathsf{cell}(I)$.

We can now state the principal result of this section, roughly saying that Gr-coherators are exactly the I-fibrant I-cellular objects of $\mathbf{\Theta}_0\backslash\mathsf{Gl}\text{-}\mathsf{Ext}$. More precisely:

Theorem 3.14. *An object* $(C, \mathbf{\Theta}_0 \to C)$ *of* $\mathbf{\Theta}_0\backslash\mathsf{Gl}\text{-}\mathsf{Ext}$ *is* I*-fibrant and* I*-cellular if and only if* C*, endowed with the composite map* $\mathbb{G} \to \mathbf{\Theta}_0 \to C$*, is a* Gr*-coherator.*

Proof. Let $(C, \mathbf{\Theta}_0 \to C)$ be an object of $\mathbf{\Theta}_0\backslash\mathsf{Gl}\text{-}\mathsf{Ext}$. Observe that, for $\mathbf{X}$ an object of $\mathbf{\Theta}_0$ and i an integer, $i \geqslant 0$, by definition of $\mathbf{\Theta}_0[\mathbf{X}, i]$, there is a bijection of the set $\mathsf{Hom}_{\mathbf{\Theta}_0\backslash\mathsf{Gl}\text{-}\mathsf{Ext}}(\mathbf{\Theta}_0[\mathbf{X}, i], C)$ with the set of pairs of parallel arrows in C, with domain $\mathbf{D}_i$ and codomain $\mathbf{X}$. Furthermore, a map from $\mathbf{\Theta}_0[\mathbf{X}, i]$ to C has an extension to $\mathbf{\Theta}_0[\mathbf{X}, i]'$ if and only if the corresponding pair of parallel arrows has a lifting.

On the other hand, it is easy to verify that if $(f_k, g_k)_{k\in K}$, $f_k, g_k : \mathbf{D}_{i_k} \to \mathbf{X}_k$, is a family of pairs of parallel arrows in C, the universal globular extension $C \to C'$ obtained from C by formally adding a lifting for every pair in this family, can be constructed by the following pushout square in $\mathbf{\Theta}_0\backslash\mathsf{Gl}\text{-}\mathsf{Ext}$

$$\begin{array}{ccc} \coprod_{k\in K} \mathbf{\Theta}_0[\mathbf{X}_k, i_k] & \longrightarrow & C \\ \downarrow & & \downarrow \\ \coprod_{k\in K} \mathbf{\Theta}_0[\mathbf{X}_k, i_k]' & \longrightarrow & C' \end{array} \quad ,$$

where the upper horizontal arrow is defined by the maps $\mathbf{\Theta}_0[\mathbf{X}_k, i_k] \to C$ corresponding to the pairs of parallel arrows (f_k, g_k), $k \in K$.

Suppose now that C is a Gr-coherator. Then the functor $\mathbf{\Theta}_0 \to C$ induces a bijection of the sets of objects, and the previous considerations imply that the condition (a) of the definition of a Gr-coherator means exactly that $(C, \mathbf{\Theta}_0 \to C)$ is a I-fibrant object of $\mathbf{\Theta}_0\backslash\mathsf{Gl}\text{-}\mathsf{Ext}$, and that the condition (b) of the definition implies that $(C, \mathbf{\Theta}_0 \to C)$ is I-cellular. The converse is an immediate consequence of lemma 3.7. □

3.15. Gr**-Pseudo-coherators.** A Gr*-pseudo-coherator* is a globular extension $(C, \mathbb{G} \to C)$ such that every pair of parallel arrows in C *with codomain a globular sum* has a lifting in C. As observed in the proof above, the condition (a) in the definition of a Gr-coherator can be replaced by the condition of being a Gr-pseudo-coherator. If $(C, \mathbb{G} \to C)$ is a globular extension with C a small category, and $\mathbf{\Theta}_0 \to C$ the unique (up to unique isomorphism) morphism of globular extensions defined by the universal property of $\mathbf{\Theta}_0$, then $(C, \mathbf{\Theta}_0 \to C)$ is an object of $\mathbf{\Theta}_0\backslash\mathsf{Gl}\text{-}\mathsf{Ext}$, and $(C, \mathbb{G} \to C)$ is a Gr-pseudo-coherator if an only if $(C, \mathbf{\Theta}_0 \to C)$ is a I-fibrant object of $\mathbf{\Theta}_0\backslash\mathsf{Gl}\text{-}\mathsf{Ext}$.

Proposition 3.16. *Let* $\mathbf{C}$ *be a* Gr*-coherator and* C *a* Gr*-pseudo-coherator. Then there exists a morphism of globular extensions from* $\mathbf{C}$ *to* C*.*

Proof. Choosing a small full subcategory of C containing globular sums, we can suppose that C is small. By the universal property of $\mathbf{\Theta}_0$, $\mathbf{C}$ and C define objects of $\mathbf{\Theta}_0\backslash\mathsf{Gl}\text{-}\mathsf{Ext}$, denoted also by $\mathbf{C}$ and C, the first being, in particular, a I-cellular object (*cf.* 3.14) and the second being I-fibrant (*cf.* 3.15). Therefore the proposition is a consequence of the lifting properties between I-cellular maps and I-fibrations.

$$\begin{array}{ccc} \varnothing & \longrightarrow & C \\ \downarrow & \nearrow & \downarrow \\ \mathbf{C} & \longrightarrow & * \end{array}$$

□

3.17. The Gr-pseudo-coherator Top. The considerations of paragraph 2.6 show that the category Top of topological spaces, endowed with the canonical functor $\mathbb{G} \to \mathsf{Top}$, defined by topological disks (*cf.* 2.4), is a Gr-pseudo-coherator. For $\mathbf{C}$ a Gr-coherator, proposition 3.16 gives a formal interpretation of the construction of the topological ∞-$\mathbf{C}$-cogroupoid $\mathbf{C} \to \mathsf{Top}$, explained in paragraph 2.6.

4. A definition of lax ∞-categories

4.1. Globular theories. A *globular theory* is a globular extension $(C, \mathbb{G} \to C)$ such that the morphism of globular extensions $\mathbf{\Theta}_0 \to C$, defined by the universal property of $\mathbf{\Theta}_0$, is faithful and induces a bijection of the sets of isomorphism classes of objects. Replacing C by an equivalent category, we can suppose, without loss of generality, that this functor induces a bijection of the sets of objects, as we will always assume in the sequel. We will identify $\mathbf{\Theta}_0$ to a (non-full) subcategory of C, and we will say that an arrow f of C is *globular* if f is in $\mathbf{\Theta}_0$. The arrow f will be called *algebraic* if for every decomposition $f = gf'$, with g globular, g is an identity. A morphism of globular theories is a morphism of the underlying globular extensions.

4.2. Generalized cosource and cotarget maps. Let

$$\mathbf{X} = \mathbf{D}_{i_1} \amalg_{\mathbf{D}_{i'_1}} \mathbf{D}_{i_2} \amalg_{\mathbf{D}_{i'_2}} \cdots \amalg_{\mathbf{D}_{i'_{m-1}}} \mathbf{D}_{i_m}$$

an object of $\mathbf{\Theta}_0$. The *dimension* of $\mathbf{X}$ is the integer $i = \max\{i_k \,|\, 1 \leqslant k \leqslant m\}$. If $i > 0$, we define an object $\boldsymbol{\partial}\mathbf{X}$ of $\mathbf{\Theta}_0$, of dimension $i-1$,

$$\boldsymbol{\partial}\mathbf{X} := \mathbf{D}_{\overline{i}_1} \amalg_{\mathbf{D}_{i'_1}} \mathbf{D}_{\overline{i}_2} \amalg_{\mathbf{D}_{i'_2}} \cdots \amalg_{\mathbf{D}_{i'_{m-1}}} \mathbf{D}_{\overline{i}_m} \ ,$$

where, for $1 \leqslant k \leqslant m$,

$$\overline{i}_k = \begin{cases} i_k \ , & i_k < i \ , \\ i-1 \ , & i_k = i \ , \end{cases}$$

and arrows $\boldsymbol{\sigma}_{\mathbf{X}}, \boldsymbol{\tau}_{\mathbf{X}} : \boldsymbol{\partial}\mathbf{X} \to \mathbf{X}$ of $\mathbf{\Theta}_0$

$$\boldsymbol{\sigma}_{\mathbf{X}} := \boldsymbol{\epsilon}_1 \amalg_{\mathbf{D}_{i'_1}} \boldsymbol{\epsilon}_2 \amalg_{\mathbf{D}_{i'_2}} \cdots \amalg_{\mathbf{D}_{i'_{m-1}}} \boldsymbol{\epsilon}_m \ , \quad \boldsymbol{\tau}_{\mathbf{X}} := \boldsymbol{\eta}_1 \amalg_{\mathbf{D}_{i'_1}} \boldsymbol{\eta}_2 \amalg_{\mathbf{D}_{i'_2}} \cdots \amalg_{\mathbf{D}_{i'_{m-1}}} \boldsymbol{\eta}_m \ ,$$

where, for $1 \leqslant k \leqslant m$,

$$\boldsymbol{\epsilon}_k = \begin{cases} 1_{\mathbf{D}_{i_k}} \ , & i_k < i \ , \\ \boldsymbol{\sigma}_{i_k} \ , & i_k = i \ , \end{cases} \qquad \boldsymbol{\eta}_k = \begin{cases} 1_{\mathbf{D}_{i_k}} \ , & i_k < i \ , \\ \boldsymbol{\tau}_{i_k} \ , & i_k = i \ . \end{cases}$$

4.3. Admissible pairs of arrows. Let C be a globular theory. A *pair of parallel arrows admissible for a theory of ∞-categories*, or more simply an *admissible pair* of arrows of C, is a pair (f, g) of parallel arrows $f, g : \mathbf{D}_i \to \mathbf{X}$ (*cf.* 1.4) such that either the arrows f and g are algebraic, or there exist decompositions

$$f = \boldsymbol{\sigma}_{\mathbf{X}} f' \qquad \text{and} \qquad g = \boldsymbol{\tau}_{\mathbf{X}} g'$$

with f' and g' algebraic. The property for a parallel pair of arrows (f, g) to be admissible is not symmetric: if (f, g) is admissible, (g, f) *need not* be admissible.

4.4. Coherators for a theory of ∞-categories. A *coherator for a theory of ∞-categories*, or more simply a Cat-*coherator*, is a globular theory $\mathbb{G} \to \mathbf{C}$ satisfying the following two conditions:

a) Every admissible pair of arrows in $\mathbf{C}$ has a lifting in $\mathbf{C}$.

b) There exists a "tower" of globular theories (called *tower of definition* of the Cat-coherator $\mathbf{C}$) with colimit $\mathbf{C}$

$$\mathbb{G} \longrightarrow \mathbf{C}_0 \longrightarrow \mathbf{C}_1 \longrightarrow \cdots \longrightarrow \mathbf{C}_n \longrightarrow \mathbf{C}_{n+1} \longrightarrow \cdots \longrightarrow \mathbf{C} \simeq \varinjlim \mathbf{C}_n \quad ,$$

where for every $n \geqslant 0$, $\mathbf{C}_n \to \mathbf{C}_{n+1}$ is a morphism of globular theories, satisfying the following properties:

- b_0) $\mathbb{G} \to \mathbf{C}_0$ is a globular completion;
- b_n) for every $n \geqslant 0$, there exists a family of admissible pairs of arrows in $\mathbf{C}_n$ such that $\mathbf{C}_{n+1}$ is the universal globular extension obtained from $\mathbf{C}_n$ by formally adding a lifting for every pair in this family.

Condition (b_0) implies that the canonical functor from $\mathbf{\Theta}_0$ to $\mathbf{C}_0$ is an isomorphism. We will usually suppose that $\mathbf{C}_0$ is *equal* to $\mathbf{\Theta}_0$. Condition (b_n) means, more precisely, that there exists a family $(\mathbf{f}_i, \mathbf{g}_i)_{i \in I_n}$ of admissible pairs of arrows in $\mathbf{C}_n$, and for every $i \in I_n$, a lifting $\mathbf{h}_i$ in $\mathbf{C}_{n+1}$ of the image of the pair $(\mathbf{f}_i, \mathbf{g}_i)$ in $\mathbf{C}_{n+1}$, satisfying the following universal property. For every globular extension $\mathbb{G} \to C$ and every morphism of globular extensions $\mathbf{C}_n \to C$, if for every $i \in I_n$ a lifting h_i of the image of the pair $(\mathbf{f}_i, \mathbf{g}_i)$ in C is given, then there exists a unique morphism of globular extensions $F : \mathbf{C}_{n+1} \to C$ such that for every i in I_n, $F(\mathbf{h}_i) = h_i$ and such that the triangle

$$\begin{array}{ccc} \mathbf{C}_n & \longrightarrow & \mathbf{C}_{n+1} \\ & \searrow & \downarrow F \\ & & C \end{array}$$

is commutative.

4.5. Lax ∞-categories. Let $\mathbb{G} \to \mathbf{C}$ be a Cat-coherator. A *∞-category of type* $\mathbf{C}$, or more simply a *∞-$\mathbf{C}$-category*, is a presheaf $F : \mathbf{C}^{\mathrm{op}} \to \mathsf{Set}$ on $\mathbf{C}$ such that the functor $F^{\mathrm{op}} : \mathbf{C} \to \mathsf{Set}^{\mathrm{op}}$ preserves globular sums. In other terms, for any globular sum in $\mathbf{C}$, the canonical map

$$F(\mathbf{D}_{i_1} \amalg_{\mathbf{D}_{i'_1}} \cdots \amalg_{\mathbf{D}_{i'_{m-1}}} \mathbf{D}_{i_m}) \longrightarrow F(\mathbf{D}_{i_1}) \times_{F(\mathbf{D}_{i'_1})} \cdots \times_{F(\mathbf{D}_{i'_{m-1}})} F(\mathbf{D}_{i_m})$$

is a bijection, the right hand side being the globular product, limit of the diagram

$$\begin{array}{ccccccc} F(\mathbf{D}_{i_1}) & & F(\mathbf{D}_{i_2}) & & & & F(\mathbf{D}_{i_m}) \\ {\scriptstyle F(\boldsymbol{\sigma}^{i_1}_{i'_1})}\searrow & & \swarrow{\scriptstyle F(\boldsymbol{\tau}^{i_2}_{i'_1})} \quad \searrow{\scriptstyle F(\boldsymbol{\sigma}^{i_2}_{i'_2})} & & \cdots & & \swarrow{\scriptstyle F(\boldsymbol{\tau}^{i_m}_{i'_{m-1}})} \\ & F(\mathbf{D}_{i'_1}) & & F(\mathbf{D}_{i'_2}) & & F(\mathbf{D}_{i'_{m-1}}) & \end{array} \quad .$$

The *category of ∞-$\mathbf{C}$-categories* is the full subcategory of $\widehat{\mathbf{C}}$, category of presheaves on $\mathbf{C}$, whose objects are ∞-$\mathbf{C}$-categories.

4.6. Examples of Cat-coherators. As in the case of Gr-coherators, there is a general method for constructing inductively Cat-coherators. Take $\mathbf{C}_0 = \mathbf{\Theta}_0$. Suppose that $\mathbf{C}_n$ is defined and choose a set E_n of admissible pairs of arrows in $\mathbf{C}_n$. Define $\mathbf{C}_{n+1}$ as the universal globular extension obtained by formally adding a lifting for each pair in E_n, and $\mathbf{C}$ as the colimit $\mathbf{C} = \varinjlim \mathbf{C}_n$. In order for $\mathbf{C}$ to be a Cat-coherator, one has to verify first that the globular extensions $\mathbf{C}_n$, $n \geqslant 0$, are globular theories, and secondly that $\mathbf{C}$ satisfies the condition (a) in 4.4. The first of these conditions is conjectured to be always true, and is proved in [1, section 5.4] with a mild hypothesis always verified in the examples: it is enough that E_0 contains all the pairs $(1_{\mathbf{D}_i}, 1_{\mathbf{D}_i})$, $i \geqslant 0$. For the second condition to be verified, it is sufficient (but not necessary) that every admissible pair of arrows in $\mathbf{C}$ is the image of a pair in E_n, for some $n \geqslant 0$. Three important examples can be constructed (among many others) by this method.

1) **The canonical Cat-coherator.** This example is obtained by taking E_n to be the set of *all* admissible pairs of arrows in $\mathbf{C}_n$.

2) **The Batanin-Leinster Cat-coherator.** It is obtained by defining E_n to be the set of admissible pairs of arrows in $\mathbf{C}_n$ that are not the image of a pair in $E_{n'}$, for some $n' < n$ [1, 4.1.4].

3) **The canonical reduced Cat-coherator.** It is constructed by taking E_n to be the set of admissible pairs of arrows in $\mathbf{C}_n$ that do not have already a lifting in $\mathbf{C}_n$.

It is easily seen that examples 1 and 2 satisfy the sufficient condition stated above. The example 3 does *not* satisfy this condition; nevertheless, it is clear that it defines a Cat-coherator. It is possible to put even more restrictive conditions on the sets E_n and still obtain a Cat-coherator. It seems that it is not possible to find a minimal way for choosing the sets E_n.

4.7. Structural maps. Let $\mathbf{C}$ be a Cat-coherator. Those of the structural maps for ∞-groupoids defined in paragraph 1.7 that do not concern inverses (*i.e.* 1.7.1-1.7.10) exist equally well in ∞-$\mathbf{C}$-categories. To see this, one has to verify that the pairs of parallel arrows giving rise to these structural maps are admissible. This is proved in [1, 4.2.7], under the mild hypothesis considered above. It is conjectured that it is always true.

Appendix A. Proof of lemma 3.7

A.1. Let $\mathcal{C}$ be a cocomplete category and $\mathcal{F}$ a class of arrows of $\mathcal{C}$. Denote $\mathsf{imds}(\mathcal{F})$ the class of pushouts of (small) sums of arrows in $\mathcal{F}$. It is easy to verify that $\mathsf{imds}(\mathcal{F})$ is stable under pushouts and sums, and that it contains isomorphisms. It is the smallest class containing $\mathcal{F}$ and stable under pushouts and sums. There is an inclusion

$$\mathsf{imds}(\mathcal{F}) \subset \mathsf{cell}(\mathcal{F}) \ .$$

A.2. A commutative square

$$\mathcal{D} = \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

in $\mathcal{C}$ is called $\mathcal{F}$-special if there exists a commutative square

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X' & \xrightarrow[f']{} & Y' \end{array} \quad ,$$

with i in $\mathsf{imds}(\mathcal{F})$, such that the square

$$\begin{array}{ccc} A \amalg X & \xrightarrow{i \amalg f} & B \amalg Y \\ {\scriptstyle (a,x)}\downarrow & & \downarrow{\scriptstyle (b,y)} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

is cocartesian (a pushout square). The property of being a $\mathcal{F}$-special square is a property of a square in an *oriented* plane: if $\mathcal{D}$ is $\mathcal{F}$-special the square

$$\begin{array}{ccc} X & \xrightarrow{x} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow[y]{} & Y' \end{array}$$

need not be $\mathcal{F}$-special.

Lemma A.3. *Let $\mathcal{C}$ be a category, consider a commutative diagram in $\mathcal{C}$*

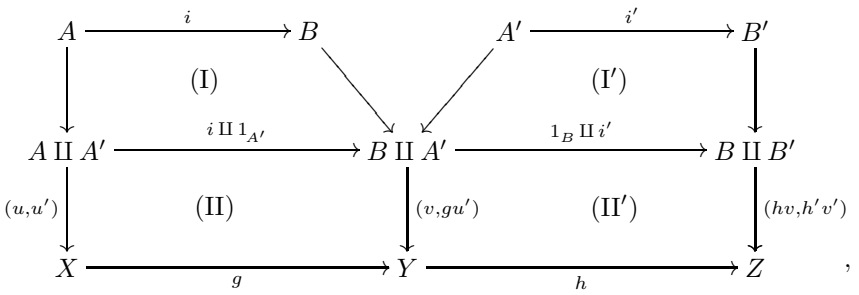
$$\begin{array}{ccccc} & & A & \xrightarrow{i} & B \\ & & {\scriptstyle u}\downarrow & (1) & \downarrow{\scriptstyle v} \\ A' & \xrightarrow{u'} & X & \xrightarrow{g} & Y \\ {\scriptstyle i'}\downarrow & (1') & {\scriptstyle g'}\downarrow & (2) & \downarrow{\scriptstyle h} \\ B' & \xrightarrow[v']{} & Y' & \xrightarrow[h']{} & Z \end{array}$$

and the induced commutative square

$$\begin{array}{ccc} A \amalg A' & \xrightarrow{i \amalg i'} & B \amalg B' \\ {\scriptstyle (u,u')}\downarrow & (3) & \downarrow{\scriptstyle (hv,h'v')} \\ X & \xrightarrow[hg=h'g']{} & Z \end{array} .$$

If (1) *and* (1′) *are cocartesian (pushout) squares, then the square* (3) *is cocartesian if and only if the square* (2) *is cocartesian.*

Proof. Consider the following commutative diagram



where the squares (I) and (I′) are cocartesian. Suppose that the squares (1) and (1′) are cocartesian. As the squares (I) and (II) ∘ (I) = (1) are cocartesian, so is (II). Therefore the square (3) = (II′) ∘ (II) is cocartesian if and only if (II′) is cocartesian. On the other hand as (I′) is cocartesian, (II′) is cocartesian if and only if (II′) ∘ (I′) = (2) ∘ (1′) is cocartesian. Finally, as (1′) is cocartesian, (2) ∘ (1′) is cocartesian if and only if (2) is cocartesian, which proves the lemma. □

Lemma A.4. *Let $\mathcal{C}$ be a cocomplete category and $\mathcal{F}$ a class of arrows of $\mathcal{C}$. A commutative square in $\mathcal{C}$*

$$\mathcal{D} = \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

is $\mathcal{F}$-special if and only if there exists a commutative diagram

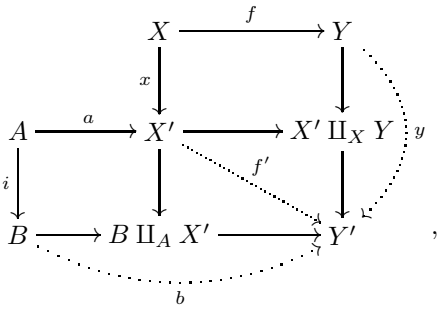
$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X' & \xrightarrow[f']{} & Y' \end{array} \quad ,$$

with i in $\mathsf{imds}(\mathcal{F})$, such that the induced square

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow & & \downarrow{\scriptstyle b} \\ X' \amalg_X Y & \longrightarrow & Y' \end{array}$$

(where $A \to X' \amalg_X Y$ is the composite map $A \xrightarrow{a} X' \xrightarrow{\mathrm{can}} X' \amalg_X Y$ and $X' \amalg_X Y \to Y'$ is the canonical map defined by the commutative square $\mathcal{D}$) is cocartesian.

Proof. This lemma is an immediate consequence of the previous one, applied to the following solid commutative diagram



defined in the obvious way. □

Lemma A.5. *Let $\mathcal{C}$ be a cocomplete category and $\mathcal{F}$ a class of arrows of $\mathcal{C}$.*

a) *Every cocartesian square in $\mathcal{C}$ is $\mathcal{F}$-special.*

b) *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

be a $\mathcal{F}$-special square. If f is in $\mathsf{imds}(\mathcal{F})$, then f' is in $\mathsf{imds}(\mathcal{F})$, too.

c) *The class of $\mathcal{F}$-special squares is stable under "vertical" transfinite composition. More precisely, let J be a well-ordered set, with minimal element 0, and*

$$X : J \longrightarrow \mathcal{A}r(\mathcal{C}) \ , \quad j \longmapsto X_j = \ X_{j,0} \to X_{j,1}$$

a functor from J to the category of arrows of $\mathcal{C}$, satisfying the following conditions:

i) *if $j+1$ is the successor of j in J, then the square*

$$\begin{array}{ccc} X_{j,0} & \longrightarrow & X_{j,1} \\ \downarrow & & \downarrow \\ X_{j+1,0} & \longrightarrow & X_{j+1,1} \end{array}$$

is $\mathcal{F}$-special;

ii) *if $j \neq 0$ is not the successor of an element of J, then $X_j \simeq \varinjlim_{j'<j} X_{j'}$.*

Then the square

$$\begin{array}{ccc} X_{0,0} & \longrightarrow & X_{0,1} \\ \downarrow & & \downarrow \\ \varinjlim_{j\in J} X_{j,0} & \longrightarrow & \varinjlim_{j\in J} X_{j,1} \end{array}$$

is $\mathcal{F}$-special.

Proof. The assertions (a) and (b) are tautological. Let's prove (c). Suppose given a functor $X : J \to \mathcal{A}r(\mathcal{C})$ as in (c). By hypothesis (i), if $j+1$ is the successor of some j in J, there exists a commutative square

$$\begin{array}{ccc} A_j & \xrightarrow{i_j} & B_j \\ \downarrow & & \downarrow \\ X_{j+1,0} & \longrightarrow & X_{j+1,1} \end{array} \quad ,$$

with i_j in $\mathsf{imds}(\mathcal{F})$, such that the square

$$\begin{array}{ccc} A_j \amalg X_{j,0} & \longrightarrow & B_j \amalg X_{j,1} \\ \downarrow & & \downarrow \\ X_{j+1,0} & \longrightarrow & X_{j+1,1} \end{array}$$

is cocartesian. For every j in J, *choose* such a commutative square. We will prove by transfinite induction on j that the "evident" commutative square

$$(*_j) \qquad \begin{array}{ccc} \left(\coprod_{j'<j} A_{j'}\right) \amalg X_{0,0} & \longrightarrow & \left(\coprod_{j'<j} B_{j'}\right) \amalg X_{0,1} \\ \downarrow & & \downarrow \\ X_{j,0} & \longrightarrow & X_{j,1} \end{array}$$

is cocartesian, which will prove the assertion by stability of cocartesian squares under colimits. For $j = 0$, there is nothing to prove. Suppose that the square $(*_j)$ is cocartesian for some j in J, and let $j+1$ be the successor of j. Consider the commutative diagram

$$\begin{array}{ccc} \left(\coprod_{j'<j} A_{j'}\right) \amalg A_j \amalg X_{0,0} & \longrightarrow & \left(\coprod_{j'<j} B_{j'}\right) \amalg B_j \amalg X_{0,1} \\ \downarrow & & \downarrow \\ A_j \amalg X_{j,0} & \longrightarrow & B_j \amalg X_{j,1} \\ \downarrow & & \downarrow \\ X_{j+1,0} & \longrightarrow & X_{j+1,1} \end{array} \quad .$$

The upper square is cocartesian as sum of two cocartesian squares, and the lower one is cocartesian by hypothesis. So the composite square $(*_{j+1})$ is cocartesian. Finally if $j \neq 0$ is not the successor of an element of J, and if for every $j' < j$, $(*_{j'})$ is cocartesian, then $(*_j)$ is cocartesian by stability of cocartesian squares under colimits. □

Proposition A.6. *Let $\mathcal{C}$ be a cocomplete category and I a (small) set of arrows of $\mathcal{C}$. If the domains of the arrows in I are finitely presentable, then* (in the notations of 3.8)

$$\mathsf{cell}_\omega(I) = \mathsf{cell}(I) \ .$$

Proof. The inclusion

$$\mathsf{cell}_\omega(I) \subset \mathsf{cell}(I)$$

being trivial, it's enough to prove the opposite inclusion. Recall that $\mathsf{cell}(I)$ is equal to the class of maps obtained by transfinite composition of pushouts of arrows in I (*cf.* 3.3). So, let J be a well-ordered set, with minimal element 0, and

$$(X_j)_{j\in J}\ ,\quad (X_{j'} \to X_j)_{j'\leqslant j}$$

a J-indexed inductive system with values in $\mathcal{C}$ (functor $J \to \mathcal{C}$), satisfying the following two conditions:

a) if $j+1$ is the successor of j in J, then there exists a cocartesian (pushout) square

$$\begin{array}{ccc} A_j & \xrightarrow{a_j} & X_j \\ {\scriptstyle i_j}\downarrow & & \downarrow \\ B_j & \xrightarrow[b_j]{} & X_{j+1} \end{array}$$

with i_j in I;

b) if $j \neq 0$ is not the successor of an element of J, then $X_j \simeq \varinjlim_{j'<j} X_{j'}$.

We have to prove that the canonical map $X_0 \to \varinjlim_{j\in J} X_j$ is in $\mathsf{cell}_\omega(I)$.

In the sequel, we suppose that for every $j \in J$ we have *chosen* a cocartesian square as in (a).

We will define by transfinite induction a J-indexed inductive system

$$(Y_j)_{j\in J}\ ,\quad (Y_{j'} \to Y_j)_{j'\leqslant j}\ ,$$

where for every $j \in J$,

$$Y_j = \quad Y_{j,0} \to Y_{j,1} \to Y_{j,2} \to \cdots \to Y_{j,n} \to Y_{j,n+1} \to \cdots$$

is a sequence of maps in $\mathcal{C}$, endowed with an isomorphism $\varinjlim_{n\in\mathbb{N}} Y_{j,n} \xrightarrow{\sim} X_j$, natural in j, satisfying the following conditions:

0) for every $j \in J$, $Y_{j,0} = X_0$ and $Y_{j',0} \to Y_{j,0}$, $j' \leqslant j$, is the identity, and for every $n \in \mathbb{N}$, $Y_{0,n} \to Y_{0,n+1}$ is in $\mathsf{imds}(I)$;

1) if $j+1$ is the successor of j in J, then for every $n \in \mathbb{N}$,

$$\begin{array}{ccc} Y_{j,n} & \longrightarrow & Y_{j,n+1} \\ \downarrow & & \downarrow \\ Y_{j+1,n} & \longrightarrow & Y_{j+1,n+1} \end{array}$$

is a I-special square;

2) if $j \neq 0$ is not the successor of an element of J, then $Y_j \simeq \varinjlim_{j'<j} Y_{j'}$.

This will prove the proposition. Indeed, if we set

$$Y := \varinjlim_{j\in J} Y_j = \ \Big(Y_0 := \varinjlim_{j\in J} Y_{j,0} \longrightarrow Y_1 := \varinjlim_{j\in J} Y_{j,1} \longrightarrow Y_2 := \varinjlim_{j\in J} Y_{j,2} \longrightarrow \cdots\Big)\ ,$$

then by lemma A.5, (b), (c), for every $n \in \mathbb{N}$, $Y_n \to Y_{n+1}$ will be in $\mathsf{imds}(I)$ and therefore the canonical map

$$X_0 \simeq \varinjlim_{j\in J} Y_{j,0} = Y_0 \longrightarrow \varinjlim_{n\in\mathbb{N}} Y_n = \varinjlim_{n\in\mathbb{N}} \varinjlim_{j\in J} Y_{j,n} \simeq \varinjlim_{j\in J} \varinjlim_{n\in\mathbb{N}} Y_{j,n} \simeq \varinjlim_{j\in J} X_j$$

will be in $\mathsf{cell}_\omega(I)$.

So let's construct such an inductive system. Define Y_0 by

$$Y_0 := \quad X_0 \xrightarrow{=} X_0 \xrightarrow{=} X_0 \longrightarrow \cdots \longrightarrow X_0 \xrightarrow{=} X_0 \longrightarrow \cdots \ ,$$

$\varinjlim_{n\in\mathbb{N}} Y_{0,n} = \varinjlim_{n\in\mathbb{N}} X_0 \xrightarrow{\sim} X_0$being the evident isomorphism. If $j \neq 0$ is not the successor of an element of J, and if $Y_{j'}$ and the isomorphism $\varinjlim_{n\in\mathbb{N}} Y_{j',n} \xrightarrow{\sim} X_{j'}$are defined for $j' < j$, define the sequence Y_j by $Y_j := \varinjlim_{j'<j} Y_{j'}$, and the isomorphism $\varinjlim_{n\in\mathbb{N}} Y_{j,n} \xrightarrow{\sim} X_j$as the colimit of the isomorphisms $\varinjlim_{n\in\mathbb{N}} Y_{j',n} \xrightarrow{\sim} X_{j'}$. Suppose now that

$$Y_j = \quad Y_{j,0} \to Y_{j,1} \to Y_{j,2} \to \cdots \to Y_{j,n} \to Y_{j,n+1} \to \cdots \ , \quad \varinjlim_{n\in\mathbb{N}} Y_{j,n} \xrightarrow{\sim} X_j$$

are defined, and let's define the sequence Y_{j+1}, where $j+1$ is the successor of j, and construct the dotted part of the diagram

$$\begin{array}{ccccccccc} Y_{j,0} & \longrightarrow & Y_{j,1} & \longrightarrow & \cdots & \longrightarrow & Y_{j,n} & \longrightarrow & Y_{j,n+1} \longrightarrow \cdots \ , \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ Y_{j+1,0} & \dashrightarrow & Y_{j+1,1} & \dashrightarrow & \cdots & \dashrightarrow & Y_{j+1,n} & \dashrightarrow & Y_{j+1,n+1} \dashrightarrow \cdots \ , \end{array} \qquad \begin{array}{ccc} \varinjlim_{n\in\mathbb{N}} Y_{j,n} & \xrightarrow{\sim} & X_j \\ \downarrow & & \downarrow \\ \varinjlim_{n\in\mathbb{N}} Y_{j+1,n} & \dashrightarrow^{\sim} & X_{j+1} \ . \end{array}$$

Recall that we have a cocartesian (pushout) square

$$(*) \qquad \begin{array}{ccc} A_j & \xrightarrow{a_j} & X_j \\ {\scriptstyle i_j}\downarrow & & \downarrow \\ B_j & \xrightarrow[b_j]{} & X_{j+1} \end{array}$$

with i_j in I. As A_j is of finite presentation, there exists $n_0 \in \mathbb{N}$ such that a_j factorizes through Y_{j,n_0}, and we can choose a minimal such n_0.

$$\begin{array}{ccc} & & Y_{j,n_0} \\ & \nearrow & \downarrow{\scriptstyle \mathrm{can}} \\ A_j & & \varinjlim Y_{j,n} \\ & {\scriptstyle a_j}\searrow & \downarrow{\scriptstyle \wr} \\ & & X_j \end{array}$$

For every $n \leqslant n_0$, set $Y_{j+1,n} := Y_{j,n}$, the arrow $Y_{j,n} \to Y_{j+1,n}$ being the identity. The remaining part of the diagram is defined by constructing the following solid diagram of cocartesian (pushout) squares

$$\begin{array}{ccccccccc} A_j & \longrightarrow & Y_{j,n_0} & \longrightarrow & Y_{j,n_0+1} & \longrightarrow & Y_{j,n_0+2} & \longrightarrow & \cdots \\ {\scriptstyle i_j}\downarrow & & \downarrow & \nearrow & \downarrow & & \downarrow & & \\ B_j & \longrightarrow & Z & \longrightarrow & Y_{j+1,n_0+1} & \longrightarrow & Y_{j+1,n_0+2} & \longrightarrow & \cdots \ , \end{array}$$

the map $Y_{j+1,n_0} = Y_{j,n_0} \to Y_{j+1,n_0+1}$ being the above dotted arrow. The isomorphism $\varinjlim_{n\in\mathbb{N}} Y_{j+1,n} \xrightarrow{\sim} X_{j+1}$ is deduced from the limit cocartesian square

$$\begin{array}{ccc} A_j & \longrightarrow & \varinjlim_{n>n_0} Y_{j,n} \simeq \varinjlim_{n\in\mathbb{N}} Y_{j,n} \\ {\scriptstyle i_j}\downarrow & & \downarrow \\ B_j & \longrightarrow & \varinjlim_{n>n_0} Y_{j+1,n} \simeq \varinjlim_{n\in\mathbb{N}} Y_{j+1,n} \ , \end{array}$$

the cocartesian square $(*)$, and the isomorphism $\varinjlim_{n\in\mathbb{N}} Y_{j,n} \xrightarrow{\sim} X_j$. It remains to prove that the squares

$$\begin{array}{ccc} Y_{j,n} & \longrightarrow & Y_{j,n+1} \\ \downarrow & & \downarrow \\ Y_{j+1,n} & \longrightarrow & Y_{j+1,n+1} \end{array}$$

are I-special. For $n \neq n_0$ these squares are cocartesian, hence I-special by lemma A.5, (a). If $n = n_0$, this is a consequence of lemma A.4 and the above construction. □

The same proof with only minor changes shows the more general result stated in 3.8.

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